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GENERALIZED COTANGENT GEOMETRY AND ITS APPLICATIONS IN QUANTUM GRAVITY

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CONTENTS

1. INTRODUCTION

Quantum Gravity Theories

- Attempts of **unification** of QFT and GR: string theory, loop quantum gravity, supergravity, causal set theory...
- In most of them a **minimal length** appears $\Longrightarrow$ Planck length (l_P)??
- **BORN'S IDEA:** maybe a curved momentum space leads to noncommutative spacetime, avoiding the divergencies of QFT $\rightarrow$ QG?
- **Cotangent bundle geometries** can describe deformed relativistic kinematics of DSR

1. INTRODUCTION

Doubly Special Relativity (DSR)

- Kinematics of SR are deformed by including a **high-energy scale** Λ
- Deformed dispersion relation

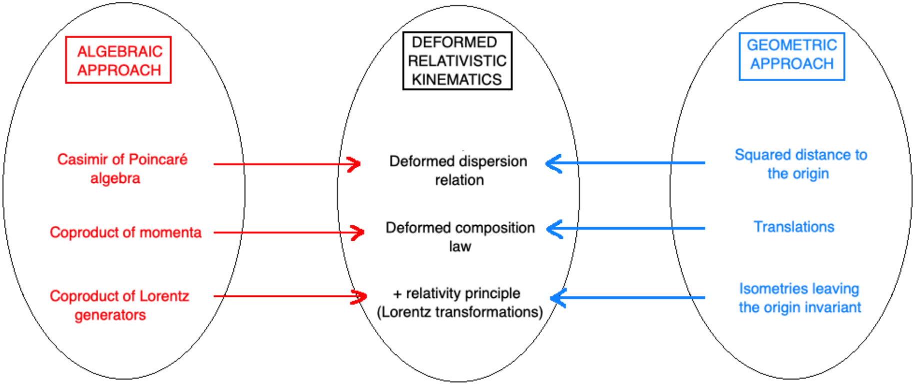
$$C(k) = k_0^2 - \vec{k}^2 + \frac{k_0^3}{\Lambda} + \ldots = m^2$$

- Deformed conservation laws (composition law of momenta)

$$\text{Total momentum} = (p \oplus q)_\mu = p_\mu + q_\mu + \frac{p_\mu q_0}{\Lambda} + \ldots$$

- Dispersion relation and conservation law compatible with **relativity principle** $\rightarrow$ deformed Lorentz transformations

1. INTRODUCTION



1. INTRODUCTION

Deformed kinematics from geometric elements (tetrad, connection, metric)

- Starting with a **maximally symmetric momentum metric** $g_{\mu\nu}(k)$
- Computing the **Casimir** by using

$$C(k) = f^{\mu}(k) g_{\mu\nu}(k) f^{\nu}(k)\,, \qquad f^{\mu}(k) := \frac{1}{2}\frac{\partial C(k)}{\partial k_{\mu}}$$

- Computing the **composition law** and tetrad through

$$g_{\mu\nu}\left(p\oplus q\right) = \frac{\partial \left(p\oplus q\right)_{\mu}}{\partial q_{\rho}} g_{\rho\sigma}(q) \frac{\partial \left(p\oplus q\right)_{\nu}}{\partial q_{\sigma}}; \qquad e^{\mu}{}_{\nu}(p) := \left.\frac{\partial \left(p\oplus q\right)_{\nu}}{\partial q_{\mu}}\right|_{q\to 0}$$

- Compute the **Lorentz transformations** by means of

$$g_{\mu\nu}(k') = \frac{\partial k'_{\mu}}{\partial k_{\rho}} g_{\rho\sigma}(k) \frac{\partial k'_{\nu}}{\partial k_{\sigma}} \quad \text{where} \quad k'_{\mu} = k_{\mu} + \epsilon_{\alpha\beta}\mathcal{J}^{\alpha\beta}_{\mu}$$

[1] J.M. Carmona, J.L. Cortés and J.J Relancio. *Phys. Rev. D 100* (2019)
[2] J.J. Relancio and S. Liberati. *Phys. Rev. D 101* (2020)

1. INTRODUCTION

Different kinematics from the same metric

- *Particular example*:
  1. De Sitter metric
  $$g_{\mu\nu}(k) = \eta_{\mu\nu} + k_\mu k_\nu/\Lambda^2$$
  2. **Snyder** kinematic's isometry generators[3]
  $$\mathcal{T}_S^\lambda = \sqrt{1+\frac{\bar{k}^2}{\Lambda^2}}\,\frac{\partial}{\partial k_\lambda}\,, \qquad \mathcal{J}^{\mu\nu} = k_\rho(\delta^\nu_\lambda \eta^{\mu\rho} - \delta^\mu_\lambda \eta^{\nu\rho})\frac{\partial}{\partial k_\lambda}\,,$$
  satisfying
  $$[\mathcal{T}_S^\alpha, \mathcal{T}_S^\beta] = \frac{\mathcal{J}^{\alpha\beta}}{\Lambda^2}\,, \qquad [\mathcal{T}_S^\alpha, \mathcal{J}^{\beta\gamma}] = \eta^{\alpha\beta}\mathcal{T}_S^\gamma - \eta^{\alpha\gamma}\mathcal{T}_S^\beta\,,$$
  $$[\mathcal{J}^{\alpha\beta}, \mathcal{J}^{\gamma\delta}] = \eta^{\beta\gamma}\mathcal{J}^{\alpha\delta} - \eta^{\alpha\gamma}\mathcal{J}^{\beta\delta} - \eta^{\beta\delta}\mathcal{J}^{\alpha\gamma} + \eta^{\alpha\delta}\mathcal{J}^{\beta\gamma}$$
  3. Deformed composition law
  $$(p\oplus q)^S_\mu = p_\mu\left(\sqrt{1+\frac{q^2}{\Lambda^2}} + \frac{p_\mu \eta^{\mu\nu} q_\nu}{\Lambda^2\left(1+\sqrt{1+p^2/\Lambda^2}\right)}\right) + q_\mu$$
  4. **Noncommutativity** of the spacetime coordinates $\rightarrow [x^\mu, x^\nu] = i\mathcal{J}^{\mu\nu}/\Lambda$

[3] M.V. Battisti and S. Meljanac. *Phys. Rev. D 82* (2010)

1. INTRODUCTION

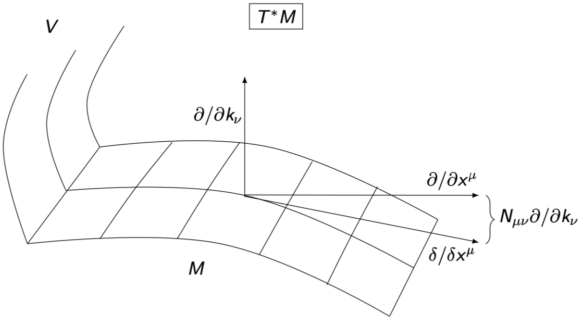
Different kinematics from the same metric

- *Particular example*:
  1. De Sitter metric
$$g_{\mu\nu}(k) = \eta_{\mu\nu} + k_\mu k_\nu/\Lambda^2$$
  2. **$\boldsymbol{\kappa}$-Poincaré** kinematic's isometry generators[4]
$$\mathcal{T}^\mu_\kappa = \mathcal{T}^\mu_S + n_\alpha \frac{\mathcal{J}^{\mu\alpha}}{\Lambda}\,, \qquad \mathcal{J}^{\mu\nu} = k_\rho(\delta^\nu_\lambda \eta^{\mu\rho} - \delta^\mu_\lambda \eta^{\nu\rho})\frac{\partial}{\partial k_\lambda}\,,$$
satisfying ($n_\mu := (1,0,0,0)$)
$$[\mathcal{T}^\alpha_\kappa, \mathcal{T}^\beta_\kappa] = \frac{n_\gamma}{\Lambda}\left(\mathcal{T}^\alpha_\kappa \eta^{\beta\gamma} - \mathcal{T}^\beta_\kappa \eta^{\alpha\gamma}\right)\,,$$
$$[\mathcal{T}^\alpha_S, \mathcal{J}^{\beta\gamma}] = \eta^{\alpha\beta}\mathcal{T}^\gamma_\kappa - \eta^{\alpha\gamma}\mathcal{T}^\beta_\kappa + \frac{n_\delta}{\Lambda}\left(\eta^{\delta\beta}\mathcal{J}^{\alpha\gamma} - \eta^{\delta\gamma}\mathcal{J}^{\alpha\beta}\right)$$
$$[\mathcal{J}^{\alpha\beta}, \mathcal{J}^{\gamma\delta}] = \eta^{\beta\gamma}\mathcal{J}^{\alpha\delta} - \eta^{\alpha\gamma}\mathcal{J}^{\beta\delta} - \eta^{\beta\delta}\mathcal{J}^{\alpha\gamma} + \eta^{\alpha\delta}\mathcal{J}^{\beta\gamma}$$
  3. Deformed composition law
$$(p\oplus q)^\kappa_\mu = p_\mu\left(\sqrt{1+\frac{q^2}{\Lambda^2}}+\frac{q_0}{\Lambda}\right)+q_\mu+\boldsymbol{n_\mu}\left[\frac{\sqrt{1+p^2/\Lambda^2}-p_0/\Lambda}{1-\vec{p}^2/\Lambda^2}\left(q_0+\frac{q_\alpha\eta^{\alpha\beta}p_\beta}{\Lambda}\right)-q_0\right]$$
  4. **Noncommutativity** of the spacetime coordinates $\rightarrow [x^0, x^i] = -ix^i/\Lambda$

[4] A. Borowiec and A. Pachol. *J. Phys. A 43* (2010)

Definition (Cotangent bundle)

The cotangent bundle T^*M is a structure (M, π, T_p^*M) formed by the union of all the cotangent spaces T_p^*M placed at each point p of a smooth 4-dimensional manifold M. The bundle projection is $\pi : T^*M \to M$.



$$T_{(x,k)}T^*M = \mathrm{V}_{(x,k)} \oplus \mathrm{H}_{(x,k)} = \mathrm{span}\left\{\bar{\partial}^\mu = \partial/\partial k_\mu\right\} \oplus \mathrm{span}\left\{\delta_\mu = \partial_\mu + N_{\nu\mu}(x,k)\bar{\partial}^\nu\right\}$$

$$\mathcal{G} = g_{\mu\nu}(x,k)dx^\mu dx^\nu + g^{\mu\nu}(x,k)\delta k_\mu \delta k_\nu, \qquad \delta k_\mu = dk_\mu - N_{\nu\mu}(x,k)\,dx^\nu$$

[5] R. Miron et al. *The Geometry of Hamilton and Lagrange Spaces* (2001).

NEW DEVELOPMENTS IN GENERALIZED HAMILTON SPACES

Difficulty → one cannot determine neither the nonlinear coefficients nor the horizontal affine connection directly from the metric, contrary to what happens in Hamilton spaces

1. Defining the **affine coefficients** in momentum space from

$$C_\rho{}^{\mu\nu}(x,k) = -\frac{1}{2} g_{\rho\sigma}\left(\bar{\partial}^\mu g^{\sigma\nu}(x,k) + \bar{\partial}^\nu g^{\sigma\mu}(x,k) - \bar{\partial}^\sigma g^{\mu\nu}(x,k)\right)$$

2. Obtaining the squared geodesic distance in momentum space (which will be identified with the **Hamiltonian** $\mathcal{H}$) from the geodesic equations

$$\ddot{k}_\mu - C_\mu{}^{\nu\sigma}(x,k)\dot{k}_\nu \dot{k}_\sigma = 0$$

3. Computing the **nonlinear coefficients** from

$$N_{\mu\nu} = -\frac{1}{4}\left(\{g_{\mu\nu}, \mathcal{H}\} + g_{\mu\rho}\bar{\partial}^\rho \partial_\nu \mathcal{H} + g_{\nu\rho}\bar{\partial}^\rho \partial_\mu \mathcal{H}\right)$$

4. Determining the **affine coefficients** in spacetime through

$$H^\rho{}_{\mu\nu}(x,k) = \bar{\partial}^\rho N_{\mu\nu}(x,k)$$

[6] J.J. Relancio and L. Santamaria-Sanz arXiv:2407.18819 (2024)

CONSTRUCTION OF THE METRIC

- Trajectories in GR are described by **geodesic equations**
- One obtains the same result from **Hamilton equations** with Casimir

$$C(x,k) \;=\; k_\mu a^{\mu\nu}(x) k_\nu$$

- It is possible to pass **from SR to GR** by replacing $\boxed{k_\mu \mapsto \bar{k}_\alpha(x) = k_\mu e^\mu_\alpha(x)}$ so that

$$C(x,k) = \bar{k}_\alpha \eta^{\alpha\beta} \bar{k}_\beta$$

CONSTRUCTION OF THE METRIC

- We move **from DSR to DGR** by replacing $\mathcal{H}(k) \mapsto \mathcal{H}(\bar{k})$ and hence the metric goes from $\boxed{g(k) \to g(x,k)}$

- $g(x,k)$ in DGR satisfies
  - -Hamilton equations $\Longleftrightarrow$ geodesic motion
  - -Distance in momentum space is conserved along horizontal curves
  - -Casimir as the square of such distance

- Properties of $g(x,k)$
  - -**Invariant under spacetime diffeomorphisms**
  - -If the starting momentum space is maximally symmetric we can define a **relativistic kinematics**

For autoparallel Hamiltonians

$$g_{\mu\nu}(x,k) = a_{\mu\nu}(x) f_1\left(\frac{k_\alpha a^{\alpha\beta} k_\beta}{\Lambda^2}\right) + \frac{1}{\Lambda^2} k_\mu k_\nu f_2\left(\frac{k_\alpha a^{\alpha\beta} k_\beta}{\Lambda^2}\right)$$

SPACETIME ISOMETRIES

- Transformations of the form

$$\boxed{x'^{\mu} = x'^{\mu}(x)\,, \qquad k'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} k_{\nu}}$$

implying

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}} g_{\mu\nu}(x', k') \frac{\partial x'^{\nu}}{\partial x^{\sigma}} = g_{\rho\sigma}(x, k)$$

- Metrics of the form

$$g_{\mu\nu}(x,k) = a_{\mu\nu}(x) f_1\left(\frac{k_\alpha a^{\alpha\beta} k_\beta}{\Lambda^2}\right) + \frac{1}{\Lambda^2} k_\mu k_\nu f_2\left(\frac{k_\alpha a^{\alpha\beta} k_\beta}{\Lambda^2}\right)$$

have the same number of spacetime isometries as $a_{\mu\nu}(x)$ in GR

[6] J.J. Relancio and L. Santamaria-Sanz arXiv:2407.18819 (2024)

MOMENTUM ISOMETRIES

- Transformations of the form

$$\boxed{x'^{\mu} = x\,, \qquad k'_{\mu} = k'_{\mu}(x,k)}$$

implying

$$g_{\mu\nu}(x,k') = \frac{\partial k'_{\mu}}{\partial k_{\rho}} g_{\rho\sigma}(x,k) \frac{\partial k'_{\nu}}{\partial k_{\sigma}}$$

- Isometries in momentum space when spacetime is also curved are the same ones as in flat spacetimes but replacing $\eta^{\mu\nu} \to a^{\mu\nu}$ as well as $n_{\alpha} \to Z_{\alpha} = n_{\lambda} e^{\lambda}{}_{\alpha}(x)$
- κ-Poincaré composition laws depends on the tetrad (observer) $\to$ Snyder kinematics are privileged

[6] J.J. Relancio and L. Santamaria-Sanz arXiv:2407.18819 (2024)

NONCOMMUTATIVITY OF SPACETIME

- First option: **generators of translations** in momentum space

$$\mathcal{X}^\alpha = \mathcal{T}^\alpha$$

- Second option: **tetrad of the cotangent bundle metric**

$$\mathcal{X}^\alpha = e^\alpha{}_\mu(x,k)\frac{\partial}{\partial k_\mu}\,, \quad \text{with} \quad e^\alpha{}_\mu(x,k)\eta_{\alpha\beta}e^\beta{}_\nu(x,k) = g_{\mu\nu}(x,k)$$

- κ-Poincaré noncommutativity depends on the tetrad (observer) $\rightarrow$ Snyder kinematics are privileged

$$[\mathcal{X}^\alpha_S, \mathcal{X}^\beta_S] = f(\bar{k}^2)\frac{\mathcal{J}^{\alpha\beta}}{\Lambda^2}\,,$$

[6] J.J. Relancio and L. Santamaria-Sanz arXiv:2407.18819 (2024)

CONCLUSIONS

- We have shown how to construct the geometrical structure of **generalized Hamilton spaces**
- **Applications to QG**. Lifting symmetries to curved spacetimes leads to a restriction of kinematics
- **Noncommutative spacetimes** can be induced in the cotangent bundle geometry framework
- **Snyder kinematics** are privileged from geometrical arguments
- **Future application** to the study of phenomenological consequences of DSR, such as time delays of massless particles with different energies

Thanks for your attention!

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