Free fields and discrete symmetries in κ-Minkowski

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Motivation

- We are interested in possible residual effects of quantum gravity in the flat spacetime limit

$$\text{General Relativity} \xrightarrow[\text{flat limit}]{} \text{Special Relativity}$$

$$\downarrow \kappa ?$$

$$\text{Quantum Gravity?} \xrightarrow[\text{flat limit}]{?} \textbf{Deformed SR?}$$

- Certain models (e.g. topological QG in 2+1 dimensions, some spin foam models) predict effective noncommutativity of spacetime:

$$[x^\mu, x^\nu] \neq 0$$

- There are many different models of noncommutative spacetime - e.g. Snyder spacetime, Moyal-Weyl spacetime, κ-Minkowski, ρ-Minkowski...
- The attractive feature of κ-Minkowski is that it admits a relativistically invariant length/energy/mass scale (characterized by κ), which is a recurring theme across many approaches to QG

G. Amelino-Camelia, "Quantum-Spacetime Phenomenology," Living Rev. Rel. (2013) [arXiv:0806.0339 [gr-qc]]
A. Addazi, J. Alvarez-Muniz, R. Alves Batista, G. Amelino-Camelia, V. Antonelli, M. Arzano, M. Asorey, J. L. Atteia, S. Bahamonde and F. Bajardi, *et al.* "Quantum gravity phenomenology at the dawn of the multi-messenger era—A review," Prog. Part. Nucl. Phys. (2022) [arXiv:2111.05659 [hep-ph]]

Motivation

- Our aim is to look for phenomenological - perhaps testable - implications
- For this, we need to investigate how these deformed symmetries would affect physical processes - hence, **field theory**
- The focus will be on Noether analysis, CPT and their mutual relationship (Jost-Whiteman-Greenberg theorem has been shown not to hold in noncommutative spacetime[1], so Lorentz invariance does not necessarily imply CPT symmetry)
- Before we delve into it, let's go over some basic definitions and properties of κ-Poincaré and κ-Minkowski

[1] A. Bevilacqua, J. Kowalski-Glikman and W. Wislicki, "κ-deformed complex scalar field: Conserved charges, symmetries, and their impact on physical observables," Phys. Rev. D (2022) [arXiv:2201.10191 [hep-th]]

κ-Poincaré and κ-Minkowski

- κ-Minkowski is characterized by the commutation relations[2]

$$[x^0, x^j] = \frac{i}{\kappa} x^j, \qquad [x^i, x^j] = 0$$

- As $\kappa \to \infty$, commutative spacetime is restored (this principle extends to virtually all formulas presented here)
- We will be focusing on the deformed symmetry group/algebra - the κ-Poincaré **quantum group**/**Hopf algebra**[3]
- Since this structure is at the core of most of the results, let us briefly review the Hopf algebra structure (at least the relevant parts)

[2] S. Majid and H. Ruegg, "Bicrossproduct structure of kappa Poincare group and noncommutative geometry," Phys. Lett. B (1994) [arXiv:hep-th/9405107 [hep-th]]

[3] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, "Q deformation of Poincare algebra," Phys. Lett. B (1991)

Translations as a Hopf algebra

- Let's think of the translation sector P of "ordinary" Poincaré , generated by ∂_μ
- A Hopf algebra is an algebra + **additional structure**
- The **algebra** sector is completely intuitive - the product μ allows us to take $\partial_\mu \partial_\nu$ (**commutative**!)
- The **coalgebra sector** is most easily understood through the action on **products**:

$$\partial_\mu(\phi\psi) = (\partial_\mu \phi)\psi + \phi(\partial_\mu \psi)$$

  in other words, the coproduct Δ is

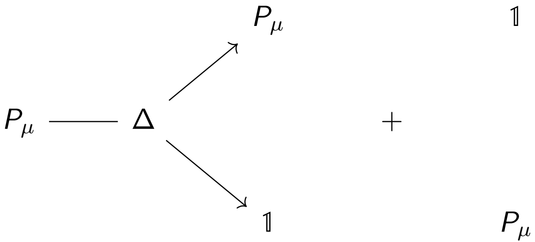
$$\Delta P_\mu = P_\mu \otimes \mathbb{1} + \mathbb{1} \otimes P_\mu$$

  (**cocommutative**)
- Now instead of thinking of a multiplicative inverse, we consider the **antipode** map S, which satisfies

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \mu \circ (\mathrm{id} \otimes S) \circ \Delta = 0, \qquad S(\mathbb{1}) = \mathbb{1}$$

Translations as a Hopf algebra

Schematically:



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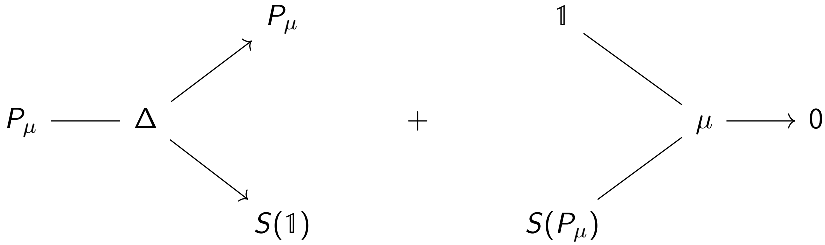

$$\mu(P_\mu \otimes S(\mathbb{1}) + \mathbb{1} \otimes S(P_\mu)) = P_\mu \cdot 1 + 1 \cdot S(P_\mu) = 0$$

Translations as a Hopf algebra

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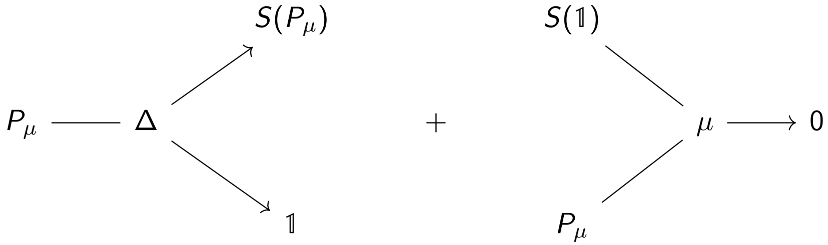
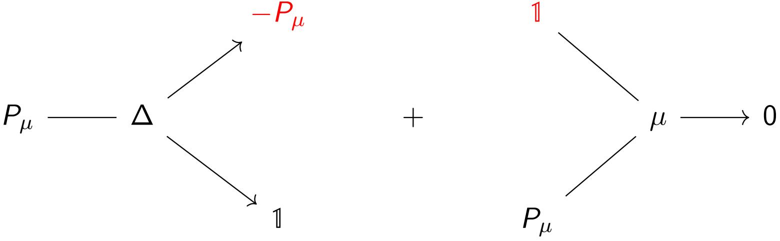


$$\mu(S(P_\mu) \otimes \mathbb{1} + S(1) \otimes P_\mu) = S(P_\mu) \cdot 1 + 1 \cdot P_\mu = 0$$

Translations as a Hopf algebra

This is of course trivially solved by $S(P_\mu) = -P_\mu$:



$$\mu(S(P) \otimes \mathbb{1} + S(\mathbb{1}) \otimes P) = -P \cdot 1 + 1 \cdot P = 0$$

Hopf algebras in a nutshell

- The product can be **noncommutative**
- The coproduct can be **noncocommutative**
- By extension, the antipode can be funky
- As long as some axioms are satisfied (eg. associativity, coassociativity, unit/counit properties...), we can have all kinds of funky structures

κ-Poincaré in the classical basis

- In the classical basis[4][5], the κ-Poincaré algebra is just a central extension of Poincaré with the central element P_4.
- The coalgebra, on the other hand, is **noncocommutative**, with the translation coproducts:

$$\Delta P_0 = P_0 \otimes \frac{P_+}{\kappa} + \frac{\vec{P}}{P_+} \otimes \vec{P} + \frac{\kappa}{P_+} \otimes P_0$$

$$\Delta P_j = P_j \otimes \frac{P_+}{\kappa} + \mathbb{1} \otimes P_j$$

$$\Delta P_4 = P_4 \otimes \frac{P_+}{\kappa} - \frac{\vec{P}}{P_+} \otimes \vec{P} - \frac{\kappa}{P_+} \otimes P_0$$

  where $P_+ = P_0 + P_4$

- Their antipodes are:

$$S(P_0) = -P_0 + \frac{\vec{P}^2}{P_+} \qquad S(P_j) = -\frac{\kappa}{P_+} P_j \qquad S(P_4) = P_4$$

[4] P. Kosinski, J. Lukierski, P. Maslanka and J. Sobczyk, "The Classical basis for kappa deformed Poincare (super)algebra and the second kappa deformed supersymmetric Casimir," Mod. Phys. Lett. A (1995) [arXiv:hep-th/9412114 [hep-th]]

[5] L. Freidel, J. Kowalski-Glikman and S. Nowak, "Field theory on kappa-Minkowski space revisited: Noether charges and breaking of Lorentz symmetry," Int. J. Mod. Phys. A (2008) [arXiv:0706.3658 [hep-th]]

The Weyl map

- We can take advantage of the fact that the Fourier transform is well defined for functions of noncommutative coordinates (which we will briefly denote by $\hat{x}$) to construct the following map:

$$\mathcal{W} : \hat{f}(\hat{x}) \xrightarrow[\text{nc. Fourier transform}]{} \tilde{f}(p) \xrightarrow[\text{inverse Fourier transform}]{} f(x)$$

- $\mathcal{W}$ is not unique - it depends on the chosen kernel of the Fourier transform - but the choice is inconsequential. We choose it to be such that

$$\mathcal{W}(\exp(ip_j\hat{x}^j)\exp(ip_0\hat{x}^0)) = \exp(ip_\mu x^\mu)$$

- The Weyl map gives rise to the $\star$-**product**, which allows us to capture noncommutativity while still using commutative variables:

$$f \star g = \mathcal{W}(\mathcal{W}^{-1}(f)\mathcal{W}^{-1}(g))$$

- The $\star$-product replaces all products of functions of x in the deformed theory

Scalar field

- In the classical basis, for any action we've come up with, the **mass-shell** condition remains **undeformed**: $p^2 = m^2$
- We can thus take a look at the mode expansion of a scalar field in κ-Minkowski ($\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$):

$$\phi(x) = \int \frac{d^3 p}{2\omega_{\mathbf{p}} p_4/\kappa} \left[a_{\mathbf{p}} e^{-ipx} + b^{\dagger}_{S(\mathbf{p})} e^{-iS(p)x} \right]$$

- Compared to the undeformed case:

$$\phi(x) = \int \frac{d^3 p}{2\omega_{\mathbf{p}}} \left[a_{\mathbf{p}} e^{-ipx} + b^{\dagger}_{\mathbf{p}} e^{ipx} \right]$$

  we see the basic features of κ-deformed theory: a change in the integration measure which comes from a second "shell" condition: $p_4 = \sqrt{m^2 + \kappa^2}$ and the splitting of momentum space into **two distinguished sectors: p and $S(\mathbf{p})$**

Scalar field

- Given the nature of the $\star$-product, there are many possible choices for the scalar field Lagrangian:
  - $S(\partial_\mu)\phi^\dagger \star \partial^\mu \phi + m^2 \phi^\dagger \star \phi$
  - $\partial^\mu \phi \star S(\partial_\mu)\phi^\dagger + m^2 \phi \star \phi^\dagger$
  - $\partial^\mu \phi^\dagger \star S(\partial_\mu)\phi + m^2 \phi^\dagger \star \phi$
  - $\phi^\dagger \star \partial_\mu \partial^\mu \phi + m^2 \phi^\dagger \star \phi$
  - ...

  which are, in general, not equivalent
- From the perspective of Poincaré symmetries, they display one common property: in all the conserved charges, they assign **momentum p to one species of particle and $-S(p)$ to the other**, e.g.

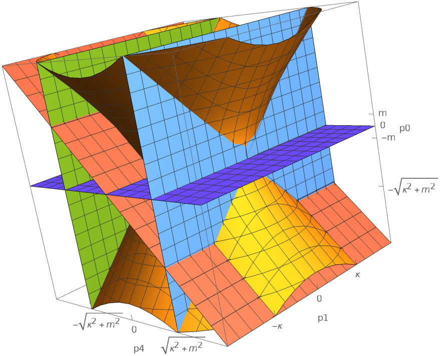
$$\mathcal{P}_\mu = \int \frac{d^3p}{2\omega_{\mathbf{p}}} \left[\frac{p_+^3}{\kappa^3} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} p_\mu - b_{S(\mathbf{p})}^\dagger b_{S(\mathbf{p})} S(p_\mu) \right]$$

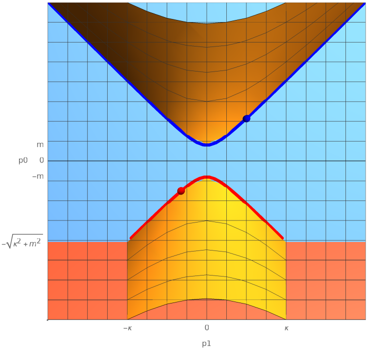
  This assignment depends solely on the **ordering** in the Lagrangian.

TA, "Complex scalar field in κ-Minkowski noncommutative spacetime," [arXiv:2505.12115 [hep-th]].

κ-momentum space

- The two momentum spaces differ primarily in that $S(\mathbf{p})$ **is bounded** by κ, while **p is unbounded**[6]
- This stems from the conditions $p_4 > 0$ and $p_+ > 0$ that have to be imposed[7]





[6] L. Freidel, J. Kowalski-Glikman and S. Nowak, "Field theory on kappa-Minkowski space revisited: Noether charges and breaking of Lorentz symmetry," Int. J. Mod. Phys. A (2008) [arXiv:0706.3658 [hep-th]]

[7] M. Arzano, J. Kowalski-Glikman and A. Walkus, "Lorentz invariant field theory on kappa-Minkowski space," Class. Quant. Grav. (2010) [arXiv:0908.1974 [hep-th]]

Dirac field

- The same formalism can be applied to Dirac fields:

$$\psi(x) = \int \frac{d^3p}{2\omega_{\mathbf{p}} p_4/\kappa} \left[u_s(\mathbf{p}) a^s_{\mathbf{p}} e^{-ipx} + v_s(-S(\mathbf{p})) b^{\dagger s}_{S(\mathbf{p})} e^{-iS(p)x} \right]$$

- The range of Lagrangians to choose from is even bigger, but their momentum properties are completely analogous to the scalar field - we again have the same two momentum spaces determined by ordering
- At face value, C-symmetry appears to be **fundamentally violated** in κ-field theory

TA, A. Bevilacqua, J. Kowalski-Glikman, G. Rosati and W. Wiślicki , "κ-deformed spin-1/2 field" (in preparation)

Charge conjugation

- It's tempting to just sum two C-related Lagrangians to recover the symmetry
- This, somewhat surprisingly, breaks Poincaré invariance of the theory
- The reason for this is related to how p and $S(p)$ transform under Lorentz. In κ-Poincaré , $S(p)$ is not a Lorentz vector, but it is an S(Lorentz) vector[8] - remember we're in a Hopf algebra!
- In effect, summing the Lagrangians gives us $p + S(p)$ momenta, but these don't transform as vectors under $L, S(L)$ or $L + S(L)$
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[8] M. Arzano and J. Kowalski-Glikman, "Deformed discrete symmetries," Phys. Lett. B (2016) [arXiv:1605.01181 [hep-th]]

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- C is actually incompatible with κ-Poincaré
- **Are we violating CPT?** (if P and T are defined naively, yes)

[10] M. Arzano and J. Kowalski-Glikman, "Deformed discrete symmetries," Phys. Lett. B (2016) [arXiv:1605.01181 [hep-th]]

Discrete symmetries

- While the definition of C is quite natural (assuming $C\phi C^{-1} = \phi^\dagger$ automatically fixes $Ca_{\mathbf{p}}C^{-1} = b_{S(\mathbf{p})}$), the action of PT leaves some room for artistic creativity
- For instance, for time reversal we can start from several different perspectives:
  - it reverses time: $(t, \mathbf{x}) \to (-t, \mathbf{x})$
  - it reverses momentum: $\mathbf{p} \to -\mathbf{p}$
  - a particle and its time reversed version should have zero total momentum
  - it conjugates plane waves: $Te^{ipx}T^{-1} = e^{-ipx} = \left(e^{ipx}\right)^\dagger$
  - it's antiunitary
  - ...
- In the deformed theory, these may lead to completely **different operations**
- C is already deformed - why not P and T too?

Recovering CPT

- If we focus on the **wave conjugation** property, we can actually obtain an operation that's compatible with C.
- Since in the deformed theory $(e^{ipx})^\dagger = e^{iS(p)x}$, if T conjugates plane waves, it replaces momenta with antipodes
- Moreover, since S is an algebra antihomomorphism, i.e. $S(f \star g) = S(g) \star S(f)$, this also effectively switches the order in the Lagrangian
- If P is undeformed, **CPT becomes a symmetry of the theory again!**

$$S(\partial_\mu)\phi^\dagger \star \partial^\mu \phi \xrightarrow{C} S(\partial_\mu)\phi \star \partial^\mu \phi^\dagger \xrightarrow{PT} S(\partial^\mu)\phi^\dagger \star \partial_\mu \phi$$

TA, A. Bevilacqua and G. Rosati, "Asymmetry in momentum space: restoring CPT invariance in κ-field theory" (in preparation)

Conclusions

- κ-field theory is, so far, self-consistent and Poincaré -invariant (unless it isn't)
- Charge conjugation symmetry is broken - great playground for phenomenology!
- *CPT* violation depends on exact definition - but **CPT symmetry is possible**

Thank you