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Revisiting noncommutative spacetimes from the relative locality principle

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Quantum Gravity Theories

- Attempts of unification: string theory, loop quantum gravity, supergravity, causal set theory...
- In most of them a minimal length appears $\Longrightarrow$ Planck length (l_P)??
- This is closely related to an energy scale $\Longrightarrow$ Planck energy (Λ)??
- Problem: there are no experimental evidences of a fundamental QGT

Spacetime: the last frontier

- Classical spacetime $\rightarrow$ "quantum" spacetime
- Symmetries? $\rightarrow$ LI should be broken/deformed at Planckian scales
- New effects $\rightarrow$ Micro black holes creation?
- Spacetime can be regarded as a "foam"

Simple κ-Poincaré kinematics [Carmona et al., 2019]

- Simple composition law of momenta

$$(p \oplus q)_\mu = p_\mu + (1 + p_0/\Lambda)\; q_\mu$$

- Lorentz transformations for left momentum

$$\mathcal{J}^{0j}_{L\,0}(p) = -p_j(1 + p_0/\Lambda)\,, \qquad \mathcal{J}^{ij}_{L\,k}(p) = \delta^j_k\, p_i - \delta^i_k\, p_j\,,$$
$$\mathcal{J}^{ij}_{L\,0}(p) = 0\,, \qquad \mathcal{J}^{0j}_{L\,k} = -\delta^j_k \left(p_0 + \left(p_0^2 - \vec{p}^2\right)/2\Lambda\right) - p_j p_k/\Lambda$$

- Lorentz transformations for right momentum

$$\begin{aligned}
\mathcal{J}^{0i}_{R\,0}(p,q) &= (1 + p_0/\Lambda)\, \mathcal{J}^{0i}_{L\,0}(q)\,, \\
\mathcal{J}_j^{\;0i}(p,q) &= -\,(1 + p_0/\Lambda)\, \mathcal{J}^{0i}_{L\,j}(q) + \left(\delta^i_j \vec{p}\cdot\vec{q} - p_j q_i\right)/\Lambda\,, \\
\mathcal{J}^{ij}_{R\,0}(p,q) &= 0\,, \qquad \mathcal{J}^{ij}_{R\,k}(p,q) = \mathcal{J}^{ij}_{L\,k}(q)
\end{aligned}$$

Simple κ-Poincaré kinematics

- The relativity principle is satisfied since

$$(p \oplus q)'_\mu = \left(p' \oplus \tilde{q}\right)_\mu$$

where

$$p'_\mu = p_\mu + \epsilon_{\alpha\beta} \mathcal{J}^{\alpha\beta}_{L\,\mu}(p)\,, \qquad \tilde{q}_\mu = q_\mu + \epsilon_{\alpha\beta} \mathcal{J}^{\alpha\beta}_{R\,\mu}(p, q)$$

This is equivalent to

$$\mathcal{J}^{\alpha\beta}_\mu(p \oplus q) = \frac{\partial (p \oplus q)_\mu}{\partial p_\nu} \mathcal{J}^{\alpha\beta}_{L\,\nu} + \frac{\partial (p \oplus q)_\mu}{\partial q_\nu} \mathcal{J}^{\alpha\beta}_{R\,\nu}$$

- Definition of total Lorentz generator $J^{\mu\nu}$

$$J^{\mu\nu} := y^\lambda \mathcal{J}^{\alpha\beta}_{L\,\lambda}(p) + z^\lambda \mathcal{J}^{\alpha\beta}_{R\,\lambda}(p, q)$$

Simple κ-Poincaré kinematics

- In terms of Poisson brackets

$$\begin{aligned} p'_\mu =& p_\mu + \epsilon_{\alpha\beta}\{p_\mu, J^{\alpha\beta}\} = p_\mu + \epsilon_{\alpha\beta}\mathcal{J}^{\alpha\beta}_{L\,\mu}(p)\,, \\ \tilde{q}_\mu =& q_\mu + \epsilon_{\alpha\beta}\{q_\mu, J^{\alpha\beta}\} = q_\mu + \epsilon_{\alpha\beta}\mathcal{J}^{\alpha\beta}_{R\,\mu}(p,q) \end{aligned}$$

with

$$\{a,b\} = \frac{\partial a}{\partial p_\mu}\frac{\partial b}{\partial y^\mu} - \frac{\partial b}{\partial p_\mu}\frac{\partial a}{\partial y^\mu} + \frac{\partial a}{\partial q_\mu}\frac{\partial b}{\partial z^\mu} - \frac{\partial b}{\partial q_\mu}\frac{\partial a}{\partial z^\mu}$$

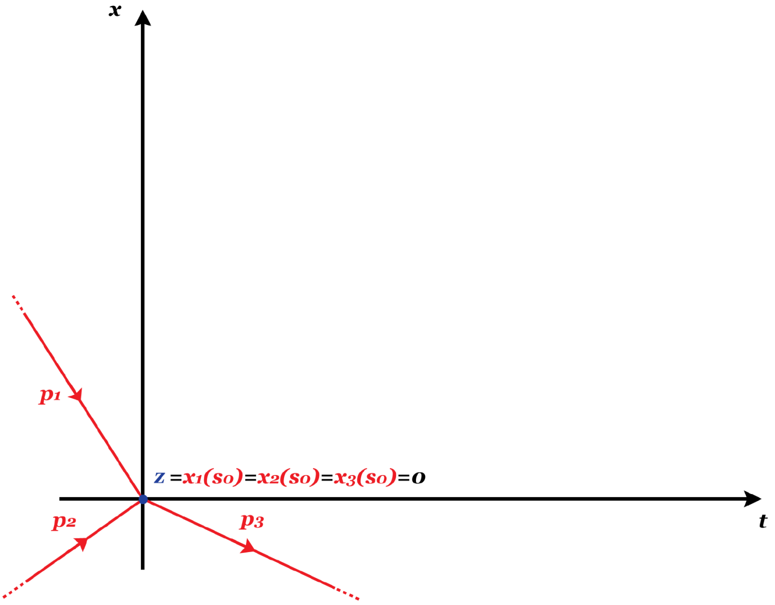
- The dispersion relation can be obtained from the Casimir (invariant under Lorentz transformations)

$$\{C(p), J^{\mu\nu}\} = \frac{\partial C(p)}{\partial p_\rho}\mathcal{J}^{\mu\nu}_{L\,\rho} = 0\,, \quad \{C(q), J^{\mu\nu}\} = \frac{\partial C(q)}{\partial q_\rho}\mathcal{J}^{\mu\nu}_{R\,\rho} = 0$$

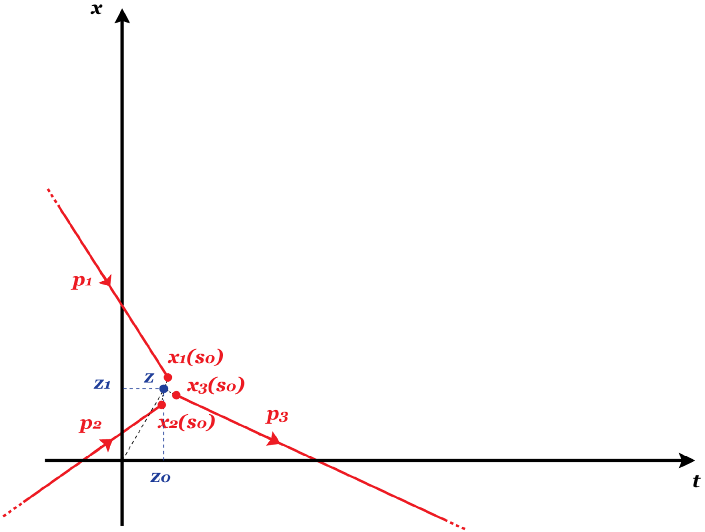
which is

$$C(k) = \frac{k_0^2 - \vec{k}^2}{1 + k_0/\Lambda}$$

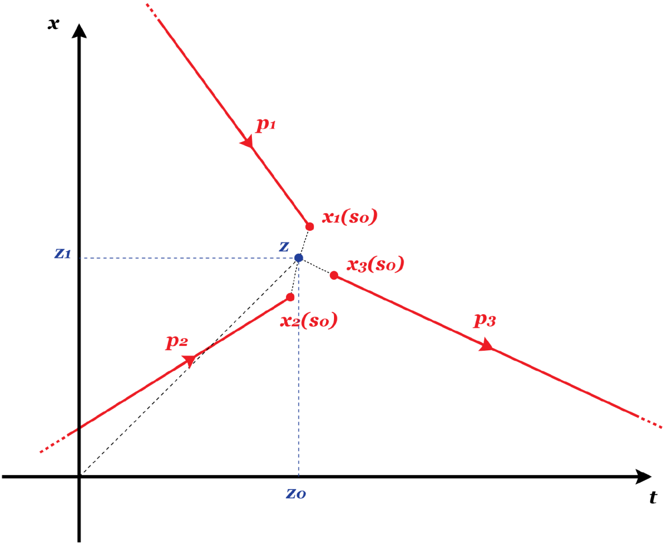
Relative locality



Relative locality



Relative locality



Implementation of locality

- From an action

$$S = \int_{-\infty}^{0} d\tau \sum_{i=1,2} \left[x^{\mu}_{-(i)}(\tau)\dot{p}^{-(i)}_{\mu}(\tau) + N_{-(i)}(\tau)\left[C(p^{-(i)}(\tau)) - m^2_{-(i)} \right] \right]$$

$$+ \int_{0}^{\infty} d\tau \sum_{j=1,2} \left[x^{\mu}_{+(j)}(\tau)\dot{p}^{+(j)}_{\mu}(\tau) + N_{+(j)}(\tau)\left[C(p^{+(j)}(\tau)) - m^2_{+(j)} \right] \right]$$

$$+ \xi^{\mu} \left[(p^{-(1)} \oplus p^{-(2)})_{\mu}(0) - (p^{+(1)} \oplus p^{+(2)})_{\mu}(0) \right]$$

one finds

$$x^{\mu}_{\pm(i)}(0) = \xi^{\nu} \frac{\partial (p^{\pm(1)} \oplus p^{\pm(2)})_{\nu}}{\partial p^{\pm(i)}_{\mu}}(0)$$

- When $\xi^{\mu} = 0$ the interaction is local $x^{\mu}_{-(i)}(0) = x^{\mu}_{+(j)}(0) = 0$

Recovering locality [Carmona et al., 2018, Carmona et al., 2020, Relancio, 2021]

- Locality of interactions can be recovered by introducing noncommutative space–time coordinates

$$\tilde{y}_L^{\alpha} = y^{\mu}\,\varphi^{(L)\alpha}_{(L)\mu}(p,q) + z^{\mu}\,\varphi^{(L)\alpha}_{(R)\mu}(p,q)$$
$$\tilde{z}_R^{\alpha} = y^{\mu}\,\varphi^{(R)\alpha}_{(L)\mu}(p,q) + z^{\mu}\,\varphi^{(R)\alpha}_{(R)\mu}(p,q)$$

- When there is only one momentum, one recovers the noncommutativity of one particle

$$\tilde{x}^{\mu} = x^{\lambda}\varphi^{\mu}_{\lambda}(k)$$

- The condition to have an event defined by the interaction is

$$\varphi^{\mu}_{\nu}(p\oplus q) = \frac{\partial\,(p\oplus q)_{\mu}}{\partial p_{\nu}}\varphi^{(L)\alpha}_{(L)\nu}(p,q) + \frac{\partial\,(p\oplus q)_{\mu}}{\partial q_{\nu}}\varphi^{(L)\alpha}_{(R)\nu}(p,q)$$
$$= \frac{\partial\,(p\oplus q)_{\mu}}{\partial p_{\nu}}\varphi^{(R)\alpha}_{(L)\nu}(p,q) + \frac{\partial\,(p\oplus q)_{\mu}}{\partial q_{\nu}}\varphi^{(R)\alpha}_{(R)\nu}(p,q)$$

General Poisson brackets

We start by considering the most general Poisson-Lie algebra in the two-particle system

$$\{\tilde{y}_L^\mu, \tilde{y}_L^\nu\} = \frac{c_L^L}{\Lambda}\left(\tilde{y}_L^\mu\, n^\nu - \tilde{y}_L^\nu\, n^\mu\right) + \frac{c_R^L}{\Lambda}\left(\tilde{z}_R^\mu\, n^\nu - \tilde{z}_R^\nu\, n^\mu\right) + \frac{1}{\Lambda^2} D^{\mu\nu}_{L\,\lambda\sigma} J^{\lambda\sigma}$$

$$\{\tilde{y}_L^\mu, \tilde{z}_R^\nu\} = C^{\mu\nu}_{L\,\xi}\, \tilde{y}_L^\xi - C^{\mu\nu}_{R\,\xi}\, \tilde{z}_R^\xi + \frac{1}{\Lambda^2} D^{\mu\nu}_{\lambda\sigma} J^{\lambda\sigma}$$

$$\{\tilde{z}_R^\mu, \tilde{z}_R^\nu\} = \frac{c_L^R}{\Lambda}\left(\tilde{y}_L^\mu\, n^\nu - \tilde{y}_L^\nu\, n^\mu\right) + \frac{c_R^R}{\Lambda}\left(\tilde{z}_R^\mu\, n^\nu - \tilde{z}_R^\nu\, n^\mu\right) + \frac{1}{\Lambda^2} D^{\mu\nu}_{R\,\lambda\sigma} J^{\lambda\sigma}$$

$$\{J^{\mu\nu}, \tilde{y}_L^\nu\} = \eta^{\nu\rho}\tilde{y}_L^\mu - \eta^{\mu\rho}\tilde{y}_L^\nu + \frac{1}{\Lambda} E^{\mu\nu\rho}_{L\,\lambda\sigma} J^{\lambda\sigma}$$

$$\{J^{\mu\nu}, \tilde{z}_R^\nu\} = \eta^{\nu\rho}\tilde{z}_R^\mu - \eta^{\mu\rho}\tilde{z}_R^\nu + \frac{1}{\Lambda} E^{\mu\nu\rho}_{R\,\lambda\sigma} J^{\lambda\sigma}$$

$$\{J^{\mu\nu}, J^{\rho\sigma}\} = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}$$

with C's, D's and E's are constructed with $\eta_{\mu\nu}$, δ^μ_ν, and a fixed vector n^μ (time-, light- or space-like)

General Poisson brackets

Imposing Jacobi identities, we find different solutions

$$\{\tilde{y}_L^\mu, \tilde{y}_L^\nu\} = \frac{\lambda_1}{\Lambda}\left(\tilde{y}_L^\mu\, n^\nu - \tilde{y}_L^\nu\, n^\mu\right) - \frac{\alpha\,\lambda_1^2}{\Lambda^2} J^{\mu\nu}\,,$$

$$\{\tilde{y}_L^\mu, \tilde{z}_R^\nu\} = \frac{\lambda_1}{\Lambda}\tilde{z}_R^\mu\, n^\nu - \frac{\lambda_2}{\Lambda}\tilde{y}_L^\nu\, n^\mu + \eta^{\mu\nu}\left(\frac{\lambda_2}{\Lambda}\tilde{y}_L^\alpha\, n_\alpha - \frac{\lambda_1}{\Lambda}\tilde{z}_R^\alpha\, n_\alpha\right) - \frac{\alpha\lambda_1\lambda_2}{\Lambda^2} J^{\mu\nu}\,,$$

$$\{\tilde{z}_R^\mu, \tilde{z}_R^\nu\} = \frac{\lambda_2}{\Lambda}\left(\tilde{z}_R^\mu\, n^\nu - \tilde{z}_R^\nu\, n^\mu\right) - \frac{\alpha\,\lambda_2^2}{\Lambda^2} J^{\mu\nu}\,,$$

$$\{J^{\mu\nu}, \tilde{y}_L^\rho\} = \eta^{\nu\rho}\tilde{y}_L^\mu - \eta^{\mu\rho}\tilde{y}_L^\nu + \frac{\lambda_1}{\Lambda}\left(n^\mu J^{\nu\rho} - n^\nu J^{\mu\rho}\right)\,,$$

$$\{J^{\mu\nu}, \tilde{z}_R^\rho\} = \eta^{\nu\rho}\tilde{z}_R^\mu - \eta^{\mu\rho}\tilde{z}_R^\nu + \frac{\lambda_2}{\Lambda}\left(n^\mu J^{\nu\rho} - n^\nu J^{\mu\rho}\right)$$

where $\alpha = n^\mu n_\mu$ ($\alpha = 1, 0, -1$ for time-, light- and space-like cases, respectively)

General Poisson brackets

- We obtained a bi-parametric algebra, corresponding to the $R^{6,2} \rtimes o(3,1)$ algebra, as can be seen from the following change of basis of the generators

$$\tilde{y}_L^\mu = y_L^\mu + \frac{\lambda_1}{\Lambda}\, n_\lambda J^{\mu\lambda}\,, \qquad \tilde{z}_R^\mu = z_R^\mu + \frac{\lambda_2}{\Lambda}\, n_\lambda J^{\mu\lambda}$$

- For $\lambda_1 = \lambda_2 = 1$ we find a symmetric case (natural for recovering the one-particle noncommutativity)
- It can be easily generalized for any number of particles

First attempt [Carmona et al., 2018]

- The noncommutativity is given by

$$
\begin{aligned}
\tilde{y}_L^\alpha =& y^\mu \, \varphi^{(L)\alpha}_{(L)\mu}(p,q) \\
\tilde{z}_R^\alpha =& z^\mu \, \varphi^{(R)\alpha}_{(R)\mu}(p,q)
\end{aligned}
$$

- Lorentz generators cannot appear in the space-time Poisson brackets $\rightarrow$ light-like case $\rightarrow$ κ-Minkowski
- The obtained composition law is symmetric and associative
- The (left and right) Lorentz transformations mix both momenta
- The kinematics cannot be reduced to that of SR

Second attempt [Carmona et al., 2020]

- The noncommutativity is given by

$$\begin{aligned} \tilde{y}_L^\alpha =& y^\mu \, \varphi^{(L)\alpha}_{(L)\mu}(p) + z^\mu \, \varphi^{(L)\alpha}_{(R)\mu}(q) \\ \tilde{z}_R^\alpha =& y^\mu \, \varphi^{(R)\alpha}_{(L)\mu}(p) + z^\mu \, \varphi^{(R)\alpha}_{(R)\mu}(q) \end{aligned}$$

- The obtained composition law is symmetric and associative
- The (left and right) Lorentz transformations do not mix both momenta
- The kinematics can be reduced to that of SR

Third attempt [Relancio, 2021]

- The noncommutativity is given by

$$\begin{aligned}\tilde{y}_L^\alpha =& y^\mu\, \varphi^{(L)\alpha}_{(L)\mu}(p,q) + z^\mu\, \varphi^{(L)\alpha}_{(R)\mu}(p,q)\\ \tilde{z}_R^\alpha =& y^\mu\, \varphi^{(R)\alpha}_{(L)\mu}(p,q) + z^\mu\, \varphi^{(R)\alpha}_{(R)\mu}(p,q)\end{aligned}$$

- There is an arbitrariness in both the composition law and Lorentz transformations
- Too difficult to work with this attempt $\rightarrow$ geometrical interpretation?

Conclusions

- Relative locality in DSR theories can be avoided by a noncommutative spacetime
- Noncommutativity of spacetime in a multiparticle system can be obtained from Poisson-Lie algebra involving space-time coordinates and Lorentz generators
- This algebra depends on the deformation
- It can be generalized for any number of particles
- The obtained kinematics are more restricted
- Future work: impose new physical or mathematical conditions to restrict the general implementation of locality

Thanks for your attention!

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