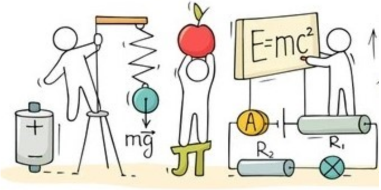
Towards generalized group structures for changes of quantum reference frames (QRFs): the twisted Poincaré case



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Partially joint work with
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Based on:

G.F., F. Lizzi [arXiv:2507.05758] + forthcoming papers

Plan

Introduction

A RF $\mathfrak{R}$ is usually built using a macroscopic body with $N \gg 1$ atoms. Then typically "collective" observables like the position $\mathbf{X}$ of its CM and its total momentum $\mathbf{P}$ are not significantly affected by the observation of a system S or another RF $\mathfrak{R}'$ with respect to (wrt) $\mathfrak{R}$. That's why RFs are usually idealized as classical.
But the ultimate quantum nature of these bodies will spoil their classical (i.e. idealized) properties, via UR, etc; particularly manifest if $N \sim O(1)$.

Can we formulate a consistent theory of QRFs?

The idea of QRFs was first proposed by Aharonov & Susskind 1967, Aharonov & Kaufherr 1984. Ever since many hundreds papers.

"Quantum mechanics and the covariance of physical laws in quantum reference frames", by F. Giacomini, E. Castro-Ruiz, Č. Brukner, Nat. Commun. **10**, 494 (2019), is particularly significant.

They mostly use "Relational Quantum Mechanics" (Rovelli 1996, Loveridge et al 2018, Höhn et al. 2021,...): $\nexists$ unique "absolute" state of a system S; rather, $\exists$ one state relative to each observer.
Consequently, a composite system can be in an entangled state wrt QRF $\mathfrak{R}$, a factorized state wrt QRF $\mathfrak{R}'$.

Use of spacetime observables relative to QRFs can heal QFT divergences:

- Chandrasekaran, Longo, Penington, Witten, JHEP02(2023)082;
- E. Witten, JHEP03(2024)077;
- Fewster, Janssen, Loveridge, Rejzner, Waldron, Comm. Math. Phys. 402 (2024), 1-41;
- De Vuyst, Eccles, Höhn, Kirklin, arXiv:2405.00114; arXiv:2412.15502;

propose operational frameworks for local measurements of QFs on a symmetric background wrt a QRF: under suitable assumptions the algebra of (relative) observables is a type II factor (instead of type III_1), i.e. has a semifinite, or even finite, (instead of an infinite) trace, what allows e.g. computing entropy.

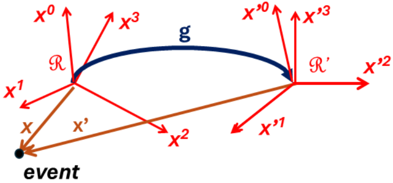
The approach to investigate properties of a QRF can be:

1. bottom-up: start from quantum properties of its microscopic constituents, operationally measuring spacetime coords wrt it.
2. top-down: study which classical properties of RFs are compatible with their quantum nature, or must be generalized, and how

Here we adopt 2., focusing on the group structure of changes of RFs.

Preliminaries, paradoxes for CRFs. Need generalized groups

Changes of classical reference frames (RF) $g : \mathfrak{R} \mapsto \mathfrak{R}'$ in space(time) make up a Lie group G:
the product gg' is the composition of g, g';
the unit is $\mathbf{1} : \mathscr{R} \mapsto \mathscr{R}$;
the inverse of g is $g^{-1} : \mathfrak{R}' \mapsto \mathfrak{R}$.



g sharply specifies where $\mathfrak{R}$ is located and how it moves wrt $\mathfrak{R}'$.

Let x, x' resp. be the (sets of) spacetime coordinates of a generic event wrt $\mathfrak{R}, \mathfrak{R}'$; g determines the 1-to-1 map $x \mapsto x'$. The latter induces a map (*passive transf.*) between the dynamical variables used by $\mathfrak{R}, \mathfrak{R}'$ to describe a physical system S; e.g. for scalar fields the map

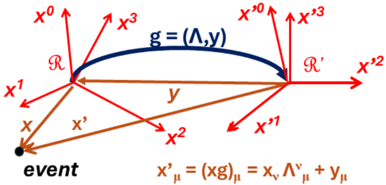
$$\gamma(g) : \varphi \mapsto \varphi'$$

is determined by the eq. $\varphi(x) \stackrel{!}{=} \varphi'[x'(x)]$.

Enforcing this map assumes $\mathfrak{R}'$ has: i) got information about the description of S by $\mathfrak{R}$; ii) sharply determined g, i.e. how $\mathfrak{R}$ moves wrt $\mathfrak{R}'$.

Cartesian coords wrt inertial RFs on $X \equiv$NR/ Minkowski spacetime: $x'_\mu = (xg)_\mu \equiv x_\nu \Lambda^\nu_\mu + y_\mu$, $g \equiv (\Lambda, y) \in G \equiv$ Galilei/Poincaré group,

$$\begin{aligned} &\gamma(g) : \varphi \mapsto \varphi' \equiv \varphi \triangleleft g, \\ &\varphi'(x') \equiv \varphi\left(x' g^{-1}\right). \end{aligned} \qquad (1)$$

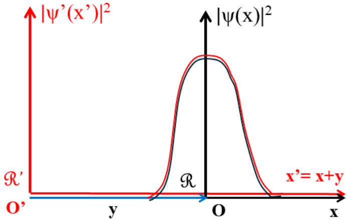


These maps apply also if S is quantum, e.g. a 0-spin elementary particle:

$$\hat{x} \mapsto \hat{x}' = \hat{x}\Lambda + y, \qquad \hat{p} \mapsto \hat{p}' = \hat{p}\Lambda, \qquad \rho \mapsto \rho', \qquad \psi \mapsto \psi'. \qquad (2)$$

All pure (resp. mixed) states ρ ($\equiv$density operator) wrt $\mathfrak{R}$ are mapped into pure (resp. mixed) states ρ' wrt $\mathfrak{R}'$.

The wavefunctions $\psi(x) = {}_{\mathfrak{R}}\langle x | \Psi \rangle_{\mathfrak{R}}$, $\psi'(x') = {}_{\mathfrak{R}'}\langle x' | \Psi' \rangle_{\mathfrak{R}'}$ of S wrt resp. $\mathfrak{R}, \mathfrak{R}'$ fulfill $|\psi'(x')|^2 = |\psi(x)|^2$, and by Wigner Thm can be chosen so that $\psi'(x') = \psi(x)$.

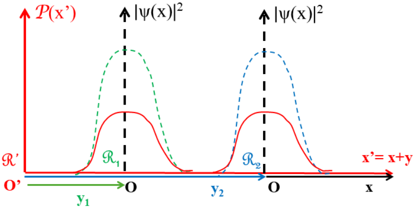


However, if $\mathfrak{R}'$ has a coarse (i.e., probabilistic) knowledge about $\mathscr{R}$, then a pure state $\rho = |\Psi\rangle_{\mathfrak{R}\mathfrak{R}}\langle\Psi|$ is mapped into a mixed state ρ'.

For instance, if $\mathfrak{R}'$ knows exactly Λ, i.e. the (relative) orientation and velocity of $\mathfrak{R}$, and that the origins' displacement is y_1, y_2 with probabilities $1/2$, then

$$\rho' = \tfrac{1}{2}\,|\Psi'_{y_1}\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle\Psi'_{y_1}| + \tfrac{1}{2}\,|\Psi'_{y_2}\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle\Psi'_{y_2}|$$

$$\mathscr{P}(x') = \tfrac{1}{2}\left|\psi'_{y_1}(x')\right|^2 + \tfrac{1}{2}\left|\psi'_{y_2}(x')\right|^2;$$



here $\psi'_y(x') = \psi(x'-y)$, $\mathscr{P}(x') = \mathrm{Tr}\left(|x'\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle x'|\rho'\right)$ is the probability density to find the particle at position x' wrt $\mathfrak{R}'$.

More generally, if $\mathfrak{R}'$ knows that the origins' displacement is y with probability density $\widetilde{\rho}(y)$, then the state of S wrt $\mathfrak{R}'$ will be

$$\rho' = \int d^4y\, \widetilde{\rho}(y)\,|\Psi'_y\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle\Psi'_y|; \tag{4}$$

this is pure iff $\widetilde{\rho}(y) = \delta_a(y) \equiv \delta(y-a)$, for some $a \in \mathbb{R}^4$.

Thus, **purity of states is a frame-dependent notion!**

To explain the paradox: $\tilde{\rho}$ is a classical state (probability distribution) of $\mathfrak{R}$ wrt $\mathfrak{R}'$; it is mixed, and so is the state of $S \cup \mathfrak{R}$ wrt $\mathfrak{R}'$, iff $\widetilde{\rho} \neq \delta_a$.

More formally, regard y_μ, Λ_μ^ν as coordinate functions on G; by def. they take the values a_μ, λ_μ^ν when evaluated at the point $g \equiv (\lambda, a) \in G$. Associate to each $g \in G$ the projector $\mathscr{P}_g$ (on a suitable $\mathscr{H}_{\mathfrak{R}}^{ext}$) such that

$$y_\mu \mathscr{P}_g = a_\mu \mathscr{P}_g = \mathscr{P}_g y_\mu, \qquad \Lambda_\mu^\nu \mathscr{P}_g = \lambda_\mu^\nu \mathscr{P}_g = \mathscr{P}_g \Lambda_\mu^\nu.$$

We can write the most general state of $\mathfrak{R}$ (w.r.t. $\mathfrak{R}'$) in the form

$$\rho_{\mathfrak{R}} = \int_G dg\, \tilde{\rho}(g) \mathscr{P}_g, \tag{5}$$

where dg is the (left and right) G-invariant (Haar) measure on G.

We postulate that the state ρ_S' of S w.r.t. $\mathfrak{R}'$ is obtained from $\rho_{S\cup\mathfrak{R}}$ via

$$\rho_S' = \mathrm{tr}_{\mathscr{H}_{\mathfrak{R}}} \left[\mathscr{U}(\Lambda, y) \rho_{S\cup\mathfrak{R}} \mathscr{U}^\dagger(\Lambda, y)\right] = \int da\, \tilde{\rho}(g)\, U(g) \rho_S U^\dagger(g), \tag{6}$$

where $\mathscr{U}(\Lambda, y) \equiv e^{iL_{\nu\rho} \otimes (\ln\Lambda)_\mu^\nu \eta^{\mu\rho}} e^{ip^\mu \otimes y_\mu}$, $U(g) \equiv e^{i(\ln\lambda)_\mu^\nu \eta^{\mu\rho} L_{\nu\rho}} e^{ia_\mu p^\mu}$. In particular, if $\rho_{S\cup\mathfrak{R}} = \mathscr{P}_\Psi \otimes \rho_{\mathfrak{R}}$, $\mathscr{P}_\Psi =$ pure state of S w.r.t. $\mathfrak{R}$, then

$$\rho_S' = \int dg\, \tilde{\rho}(g)\, \mathscr{P}_{U(g)\Psi}. \tag{7}$$

ρ_S' is mixed unless $\rho_{\mathfrak{R}}$ is pure ($\tilde{\rho} = \delta_{\bar{g}}$ for some $\bar{g} \in G$; $\Rightarrow \rho_S' = \mathscr{P}_{U(\bar{g})\Psi}$).

The set of (classical) states of $\mathfrak{R}$ wrt $\mathfrak{R}'$ becomes a semigroup if we define the product by convolution. Sticking to translations,

$$(\widetilde{\rho}_1 * \widetilde{\rho}_2)(y) = \int d^4b \, \widetilde{\rho}_1(b) \, \widetilde{\rho}_2(y-b); \tag{8}$$

δ_0 plays the role of unit element. A mixed state $\widetilde{\rho}$ (e.g. $\widetilde{\rho} = \frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$) has no inverse. Only pure states have: the inverse of δ_a is δ_{-a}. Hence the group G can be identified with the set of pure states.

Instead of endowing the set of states with the structure of a (semi)group, one can encode the group structure of G in the *Hopf algebra structure* of $Fun(G)$. This is more convenient, because it allows to replace $Fun(G)$ by a noncommutative algebra, as we may need for dealing with Quantum Reference Frames (QRFs; i.e. RF whose ultimate quantum nature cannot be ignored) and for describing symmetries of a NC spacetime.

Below I consider some NC deformations of $X =$ Minkowski space and $G =$ Poincaré group P, ideally relating inertial QRFs; $-\eta =$diag$(-1,1,1,1)$.

Why NC spacetime?

Idea of noncommutative (NC) spacetime is rather old [Heisenberg].
Possible motivations:

1. framework where to reconcile the principles of QM and GR;
2. inthrinsic regularization mechanism of UV divergences in QFT (Heisenberg's motivation);
3. due to the quantum nature of RFs **(new!)**;
4. effective description in some low energy regime of string theory (e.g. D3-brane with a large B-field) or LQG (in flat spacetime limit).

1. In usual QFT no universal minimum for the localization Δx of events: $\Delta x \sim \hbar/\Delta p$ can be reduced at will by increasing the energy of the probe. On the other hand (argument due to [Bronstein,Mead,Wheeler]), by GR the energy concentration should not cause the formation of a black hole

$$\Rightarrow \qquad \Delta x \gtrsim l_p \quad \text{(Planck length)}. \tag{9}$$

Doplicher, Fredenhagen & Roberts [DFR95] propose more sophisticated bounds, and noncommuting x_i that could naturally imply such bounds.

Simplest NC spacetime: "Moyal", i.e. **constant commutators**

$$[\hat{x}_\mu, \hat{x}_\nu] = i\mathbf{1}\theta_{\mu\nu}, \tag{10}$$

$\theta_{\mu\nu} = -\theta_{\nu\mu}$. Theoretical laboratory to investigate QM, QFT on NC spaces. Note that (10) are translation invariant, not Lorentz-invariant.

Algebra $\widehat{\mathscr{X}}$ of functions on Moyal space: generated by $\mathbf{1}, \hat{x}_\mu$ fulfilling (10), with $\theta_{\mu\nu} \in \mathbb{R}$ (suitably extended). In [DFR95] $\theta_{\mu\nu} \in \mathscr{Z}(\widehat{\mathscr{X}})$ is dynamical.

Various inequivalent approaches to QFT on Moyal spaces. I would divide them according to: quantization approach, spacetime symmetries.

1. Path-integral quantization on Moyal-Euclidean spacetime: T. Filk , M. Douglas, A.S. Schwarz, N. Nekrasov, N. Seiberg, E. Witten, S. Minwalla, M. Van Raamsdonk, J. Gomis, L. Alvarez-Gaume, T. Mehen, M. Vazquez-Mozo, ...,R. Oeckl, J. Wess, P. Aschieri, P. Schupp, R.J. Szabo, M. Dimitrijevic,..., H. Grosse, R. Wulkenhaar,...
2. Field=operator-valued, Moyal-Minkowski spacetime. Quantization: canonical; or á la Wightman; ... DFR, Bahns, Piacitelli, Chaichian, Balachandran et al, Aschieri, Lizzi,Vitale, Abe, Zahn, GF & Wess,...

Various problems, some interesting features.
E.g. in 1: causality violation, non-unitarity (for $\theta_{0i} \neq 0$), UV-IR mixing of divergences, non-renormalizability, claimed changes of statistics, etc.
Some problems may arise because naively deformed Euclidean Feynman rules are not justified by a Wick rotation.

Standard or deformed Poincaré covariance? ...?

Doplicher-Fredenhagen-Roberts, *et* Bahns, Piacitelli,...: since 1995:
First canonical quantization of the free fields. $\theta_{\mu\nu} \mapsto Q_{\mu\nu}$ central Lorentz tensor (obeying some conditions), becoming on each irrep a set of fixed constants $\theta_{\mu\nu}$ (joint spectrum of $Q_{\mu\nu}$). $\Rightarrow$ Poincaré-covariant.
But with interacting fields Lorentz covariance is sooner or later lost.
Doplicher's speculations: $Q_{\mu\nu}$ finally related to v.e.v. of $R_{\mu\nu}$, in turn influenced by matter quantum fields through quantum eq.s of motion.

Oeckl 2000, Chaichian *et al* 2004, Wess 2004, Koch *et al* 2004:
(10) are not Poincaré -invariant; *but "twisted Poincaré"* invariant.

Then attempts to construct twisted Poincaré covariant quantum fields started: Chaichian *et al*, Balachandran *et al*, Lizzi-Vitale, Abe, Zahn, F.-Wess, F.,... Our framework.

The Hopf algebra $\left(H \equiv Fun\ (\mathrm{P}), \varepsilon, \Delta, S\right)$

$$x_\mu \mapsto x'_\mu = (xg)_\mu = x_\nu \Lambda^\nu_\mu + y_\mu \equiv x_\nu \otimes \Lambda^\nu_\mu + \mathbf{1} \otimes y_\mu =: \Delta^r(x_\mu). \tag{11}$$

Regard: $\mathbf{1}, x_\mu$ as generators of $\mathscr{X} := Fun(X)$; $\mathbf{1}_H, \Lambda^\nu_\mu, y_\mu$ as generators of the algebra $H \equiv Fun\ (\mathrm{P})$ of functions on P. The transf. rule (11) is extended to all of $\mathscr{X}$ an algebra map (i.e. $\Delta^r(fg) = \Delta^r(f)\Delta^r(g)$, etc.), the *coaction* $\Delta^r : \mathscr{X} \to \mathscr{X} \otimes H, \quad f(x) \mapsto f(x') =: \left[\Delta^r(f)\right](x)$.
The group structure of P is encoded in the *counit* $\varepsilon : H \to \mathbb{C}$, *coproduct* $\Delta : H \to H \otimes H$, *antipode* $S : H \to H$, defined on the generators by

$$\begin{aligned} &\varepsilon(\Lambda^\nu_\mu) = \delta^\nu_\mu, \quad \Delta(\Lambda^\nu_\mu) = \Lambda^\nu_\rho \otimes \Lambda^\rho_\mu, \qquad S(\Lambda^\nu_\mu) = (\eta \Lambda^T \eta)^\nu_\mu \equiv {\Lambda^{-1}}^\nu_\mu, \\ &\varepsilon(y_\mu) = 0, \qquad \Delta(y_\mu) = y_\nu \otimes \Lambda^\nu_\mu + \mathbf{1}_H \otimes y_\mu, \quad S(y_\mu) = -y_\nu {\Lambda^{-1}}^\nu_\mu, \end{aligned} \tag{12}$$

which resp. give the identical, (twice) repeated, inverse change of frame. ε, Δ, S are extended as (anti-)algebra maps; fulfill many properties, e.g.

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta^r = \mathrm{id}\,, \quad (\Delta \otimes \mathrm{id}\,) \circ \Delta^r = (\mathrm{id} \otimes \Delta^r) \circ \Delta^r. \tag{13}$$

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Transf. (11) preserves $[x_\mu, x_\nu] = 0$. Does it preserve $[\hat{x}_\mu, \hat{x}_\nu] = i\mathbf{1}\theta_{\mu\nu}$?
Yes, if we "quantize" H, i.e. make it a NC Hopf algebra $\hat{H}$, such that $[\hat{x}'_\mu, \hat{x}'_\nu] = i\mathbf{1}\theta_{\mu\nu}$ holds as well $\Rightarrow$ **all inertial QRFs are equivalent!**

The Hopf algebra $\left(\hat{H}\equiv Fun_\theta(\mathrm{P}),\varepsilon,\Delta,S\right)$

$$\hat{x}_\mu \mapsto \hat{x}'_\mu = \hat{x}_\nu \Lambda^\nu_\mu + \hat{y}_\mu \equiv \hat{x}_\nu \otimes \Lambda^\nu_\mu + \mathbf{1}\otimes\hat{y}_\mu =: \Delta^r(\hat{x}_\mu). \tag{11}$$

Regard: $\mathbf{1},\hat{x}_\mu$ as generators of $\widehat{\mathscr{X}}$; $\mathbf{1}_H,\Lambda^\nu_\mu,\hat{y}_\mu$ as generators of the algebra $\widehat{H}=Fun_\theta(\mathrm{P})$. The transf. rule (11) is extended to all of $\widehat{\mathscr{X}}$ an algebra map (i.e. $\Delta^r(fg)=\Delta^r(f)\Delta^r(g)$, etc.), the *coaction* $\Delta^r:\widehat{\mathscr{X}}\to\widehat{\mathscr{X}}\otimes\widehat{H}$, $\quad f(\hat{x})\mapsto f(\hat{x}')$. The *counit* $\varepsilon:\widehat{H}\to\mathbb{C}$, *coproduct* $\Delta:\widehat{H}\to\widehat{H}\otimes\widehat{H}$, *antipode* $S:\widehat{H}\to\widehat{H}$, defined on the generators by

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&\varepsilon(\hat{y}_\mu)=0, && \Delta(\hat{y}_\mu)=\hat{y}_\nu\otimes\Lambda^\nu_\mu+\mathbf{1}_H\otimes\hat{y}_\mu, && S(\hat{y}_\mu)=-\hat{y}_\nu\Lambda^{-1}{}^\nu_\mu,
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$$(\mathrm{id}\otimes\varepsilon)\circ\Delta^r=\mathrm{id}\,,\quad (\Delta\otimes\mathrm{id}\,)\circ\Delta^r=(\mathrm{id}\otimes\Delta^r)\circ\Delta^r. \tag{13}$$

Transf. (11) preserves $\quad[\hat{x}_\mu,\hat{x}_\nu]=i\mathbf{1}\theta_{\mu\nu}\quad$ if [Oeckl 2000]:

$$[\Lambda^\rho_\mu,\cdot]=0,\quad \Lambda^\mu_\rho\Lambda^\nu_\sigma\eta^{\rho\sigma}=\eta^{\mu\nu}\mathbf{1}_H,\quad [\hat{y}_\mu,\hat{y}_\nu]=i(\theta_{\mu\nu}\mathbf{1}_H-\theta_{\rho\sigma}\Lambda^\rho_\mu\Lambda^\sigma_\nu). \tag{14}$$

Regular representation of $\hat{H}$; coherent states for QRFs

Abbreviating $\chi := \mathbf{1}_H\theta - \Lambda^T\theta\Lambda$, $\hat{H}$ is generated by $\Lambda^\nu_\mu, \hat{y}_\mu$ fulfilling

$$[\Lambda^\nu_\mu, \cdot\,] = 0, \qquad \Lambda\eta\Lambda^T = \mathbf{1}_H\eta, \qquad [\hat{y}_\mu, \hat{y}_\nu] = i\,\chi_{\mu\nu}. \tag{15}$$

It can be faithfully represented on the space V of smooth functions $f(y,\lambda)$ of real commuting variables y_μ, λ^ν_μ fulfilling $\lambda\eta\lambda^T = \eta$, e.g. by

$$\Lambda^\nu_\mu\, f(y,\lambda) = \lambda^\nu_\mu\, f(y,\lambda), \qquad \hat{y}_\mu\, f(y,\lambda) = \left(y_\mu + \frac{i}{2}\chi_{\mu\rho}\frac{\partial}{\partial y_\rho}\right) f(y,\lambda); \tag{16}$$

reducible representation of $\hat{H}$ with the correct commutative limit $\theta \to 0$.
In fact, rhs(16b) is the star-product $y_\mu \star f(y,\Lambda)$ induced by the twist $\mathscr{F} = \exp[\frac{i}{2}\theta_{\mu\rho}p^\mu \otimes p^\rho]$, leading to $\hat{y}_\mu \star \hat{y}_\nu - \hat{y}_\nu \star \hat{y}_\mu = i\chi_{\mu\nu}$.
After a suitable orthogonal transf. $\hat{\mathrm{y}}_\mu = \hat{y}_\rho Q^\rho_\mu(\lambda)$ only nontrivial (15c) are

$$\left[\hat{\mathrm{y}}_0, \hat{\mathrm{y}}_1\right] = -\left[\hat{\mathrm{y}}_1, \hat{\mathrm{y}}_0\right] = i\beta, \qquad \left[\hat{\mathrm{y}}_2, \hat{\mathrm{y}}_3\right] = -\left[\hat{\mathrm{y}}_3, \hat{\mathrm{y}}_2\right] = i\gamma, \tag{17}$$

where β, γ are λ-dependent linear combinations of the $\theta_{\mu\nu}$; (16) becomes

$$\hat{\mathrm{y}}_0 = \mathrm{y}_0 + \frac{i\beta}{2}\frac{\partial}{\partial \mathrm{y}_1}, \quad \hat{\mathrm{y}}_1 = \mathrm{y}_1 - \frac{i\beta}{2}\frac{\partial}{\partial \mathrm{y}_0}, \quad \hat{\mathrm{y}}_2 = \mathrm{y}_2 + \frac{i\gamma}{2}\frac{\partial}{\partial \mathrm{y}_3}, \quad \hat{\mathrm{y}}_3 = \mathrm{y}_3 - \frac{i\gamma}{2}\frac{\partial}{\partial \mathrm{y}_2}.$$

$$b_1 = \frac{\hat{y}_0 + i\hat{y}_1}{\sqrt{2\beta}}, \quad b_1^\dagger = \frac{\hat{y}_0 - i\hat{y}_1}{\sqrt{2\beta}}, \quad b_2 = \frac{\hat{y}_2 + i\hat{y}_3}{\sqrt{2\gamma}}, \quad b_2^\dagger = \frac{\hat{y}_2 - i\hat{y}_3}{\sqrt{2\gamma}} \tag{18}$$

are ladder operators fulfilling the CCR

$$[b_1, b_2] = [b_1^\dagger, b_2^\dagger] = 0, \qquad [b_a, b_b^\dagger] = \delta_{ab}. \tag{19}$$

One can easily show that for all $\alpha \equiv (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^4$ the

$$\psi_\alpha(y, \lambda) = \sqrt{\frac{4}{\pi_2 \beta \gamma}} \exp\left[-\frac{(y_0 - \alpha_0)^2 + (y_1 - \alpha_1)^2}{\beta(\lambda)} - \frac{(y_2 - \alpha_2)^2 + (y_3 - \alpha_3)^2}{\gamma(\lambda)} \right] \tag{20}$$

are normalized coherent states, eigenvectors of b_1, b_2 with eigenvalues $z_1 \equiv (\alpha_0 + i\alpha_1)/\sqrt{\beta}$, $z_2 \equiv (\alpha_2 + i\alpha_3)/\sqrt{\gamma}$ respectively. They make up a family of Gaussian states centered around the 4 real parameters $\alpha \in \mathbb{R}^4$, which parametrize all possible classical translations. Correspondingly,

$$\langle \hat{y}_\mu \rangle = \alpha_\mu, \qquad \Delta\hat{y}_2 = \Delta\hat{y}_3 = \sqrt{\tfrac{\gamma}{2}}, \qquad \Delta\hat{y}_0 = \Delta\hat{y}_1 = \sqrt{\tfrac{\beta}{2}}, \tag{21}$$

$$\Delta\hat{y}_0 \Delta\hat{y}_1 = \tfrac{\beta}{2}, \qquad \Delta\hat{y}_2 \Delta\hat{y}_3 = \tfrac{\gamma}{2} \qquad \text{(saturated UR)}. \tag{22}$$

As $\lambda \to I$ (or $\theta \to 0$) the $\beta, \gamma \to 0$, and the RF change gets "classical":

$$\psi_\alpha(y, \lambda) \to \delta^{(4)}(y - \alpha). \tag{23}$$

A consequence is the relation, invariant under orthogonal transformations,

$$\sum_\mu \left(\Delta\hat{y}_\mu\right)^2 \quad = \quad \sqrt{2\mathrm{Pf}(\chi) - \tfrac{1}{2}\mathrm{tr}(\chi^2)}. \tag{24}$$

Rhs(24) depends on the specific Lorentz transformation λ. Sticking to $\lambda =$rotations, one can easily show the rotation-independent bound

$$\sum_\mu \left(\Delta\hat{y}_\mu\right)^2 \leq 2(e+b); \tag{25}$$

here e, b are the norms of the 3-vectors $\boldsymbol{e}, \boldsymbol{b}$ of components $e^i \equiv \theta_{0i}$, $b^i \equiv \frac{1}{2}\varepsilon^{ijk}\theta_{jk}$. More generally, one can show the bound

$$\sum_\mu \left(\Delta\hat{y}_\mu\right)^2 \leq 4\lambda_0^0\sqrt{2(e^2+b^2)}. \tag{26}$$

where $\lambda_0^0 = \gamma = 1/\sqrt{1-v^2/c^2}$, $v \equiv$ speed of the origin of $\mathfrak{R}$ wrt $\mathfrak{R}'$.

Summary and conclusions

Physical theories are covariant under changes of reference frames (RFs). An *ordinary* change $\mathfrak{R} \mapsto \mathfrak{R}'$ of *classical* RFs can be seen as a point g in a Lie group manifold G.

If the state of $\mathfrak{R}$ w.r.t. $\mathfrak{R}'$ is mixed (a statistical distribution on G), or more generally if $\mathfrak{R}$ or $\mathfrak{R}'$ are quantum RFs (i.e., use "clocks", "rulers" that are themselves quantum systems), then one can describe the associated "unsharp" changes of RFs only via some *generalized group* structure.

The notion of a Hopf algebra, and of its (co)representation, is a possible one, naturally associated with NC spacetimes. We have shown the first steps in formulating quantum theories on the NC ("Moyal") θ-Minkowski space, which is covariant the under the θ-Poincaré Hopf algebra (here formulated as NC algebra $\hat{H}$ of "functions on the group"); in particular, coherent states for $\hat{H}$ best approximate sharp changes of classical RFs.

We have also shown that the state of a generic system S can be pure relative to a RF $\mathfrak{R}$ and mixed relative to another one $\mathfrak{R}'$; in particular, if $\mathfrak{R}$ is in a thermal state w.r.t. $\mathfrak{R}'$, then a particle in a pure momentum eigenstate w.r.t. $\mathfrak{R}$ is in a (boosted) thermal state w.r.t. $\mathfrak{R}'$.

Thank you for your attention!