UNIVERSITY OF Southampton
School of Physics and Astronomy

HIDDeν
Hunting Invisibles: Dark sectors, Dark matter and Neutrinos

Modular Symmetries for Flavour Physics

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Standard Model

Gauge bosons

$$SU(3)_C \times SU(2)_L \times U(1)_Y$$

Fermions

$$L_e = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \quad e_R, \quad \begin{pmatrix} u_L \\ d_L \end{pmatrix}^{r,b,g}, \quad u_R^{r,b,g}, \quad d_R^{r,b,g}$$

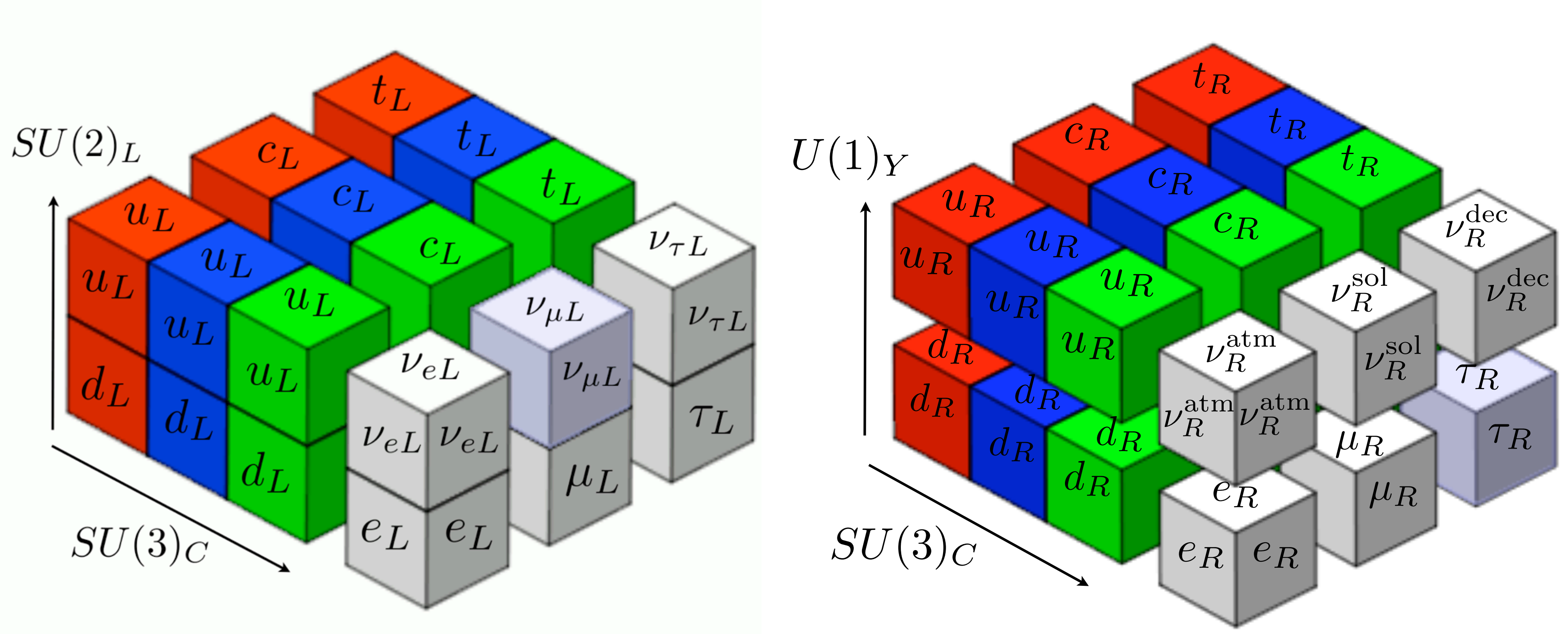
$$L_\mu = \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \quad \mu_R, \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix}^{r,b,g}, \quad c_R^{r,b,g}, \quad s_R^{r,b,g}$$

$$L_\tau = \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \quad \tau_R, \quad \begin{pmatrix} t_L \\ b_L \end{pmatrix}^{r,b,g}, \quad t_R^{r,b,g}, \quad b_R^{r,b,g}$$

Hypercharges

$$Y = -\frac{1}{2}, \quad -1, \quad \frac{1}{6}, \quad \frac{2}{3}, \quad -\frac{1}{3}$$

With RHNs



Higgs

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$$

$$\frac{1}{2}$$

EWSB

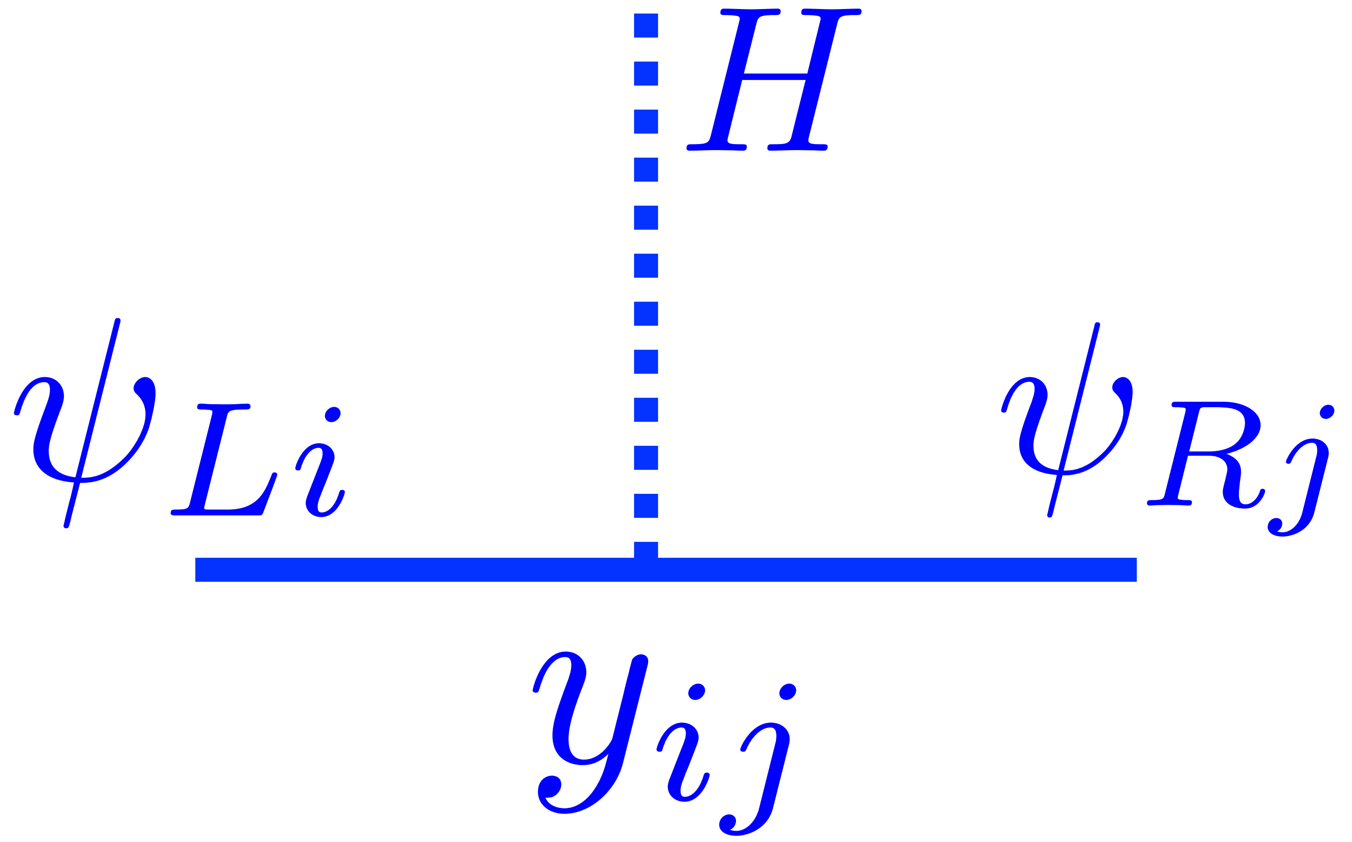
$$\langle h^0 \rangle = v/\sqrt{2}$$

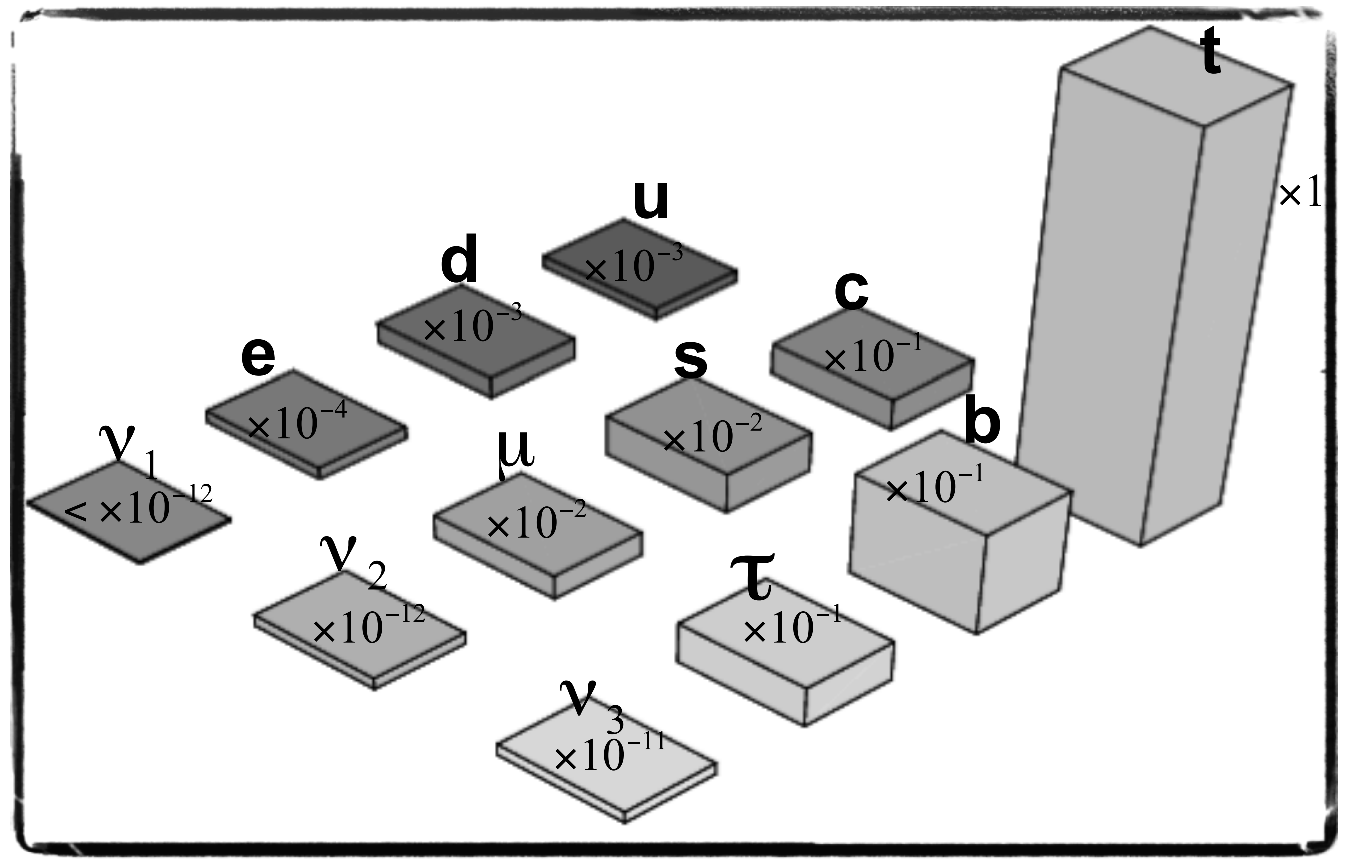
$$SU(2)_L \times U(1)_Y \to U(1)_Q$$

$$Q = T_{3L} + Y$$

Yukawa couplings

$$y_{ij} H \overline{\psi}_{Li} \psi_{Rj}$$

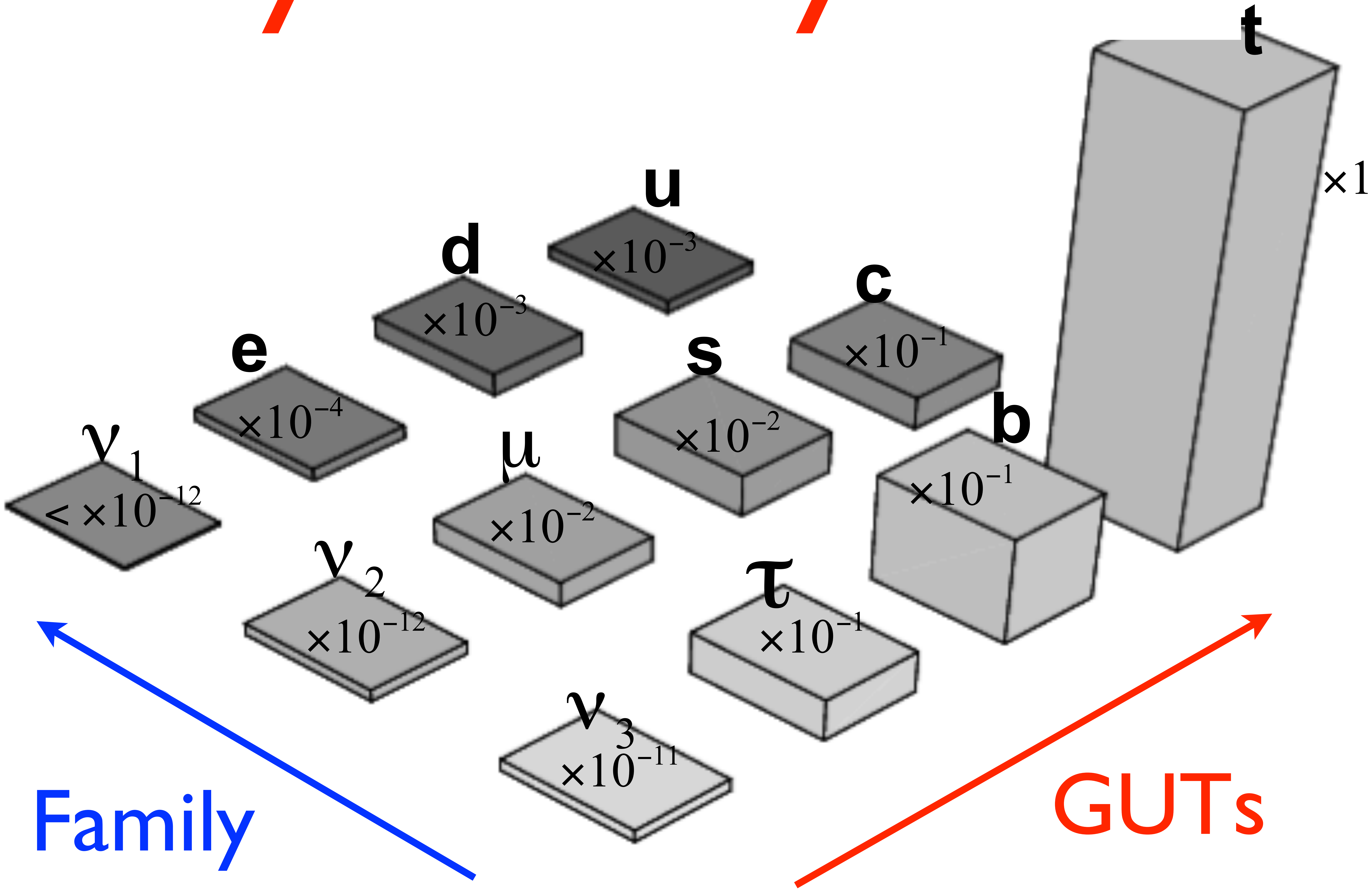




Ideally we would like all Yukawas to be of order unity, so what is going on?

Is there a symmetry at work?

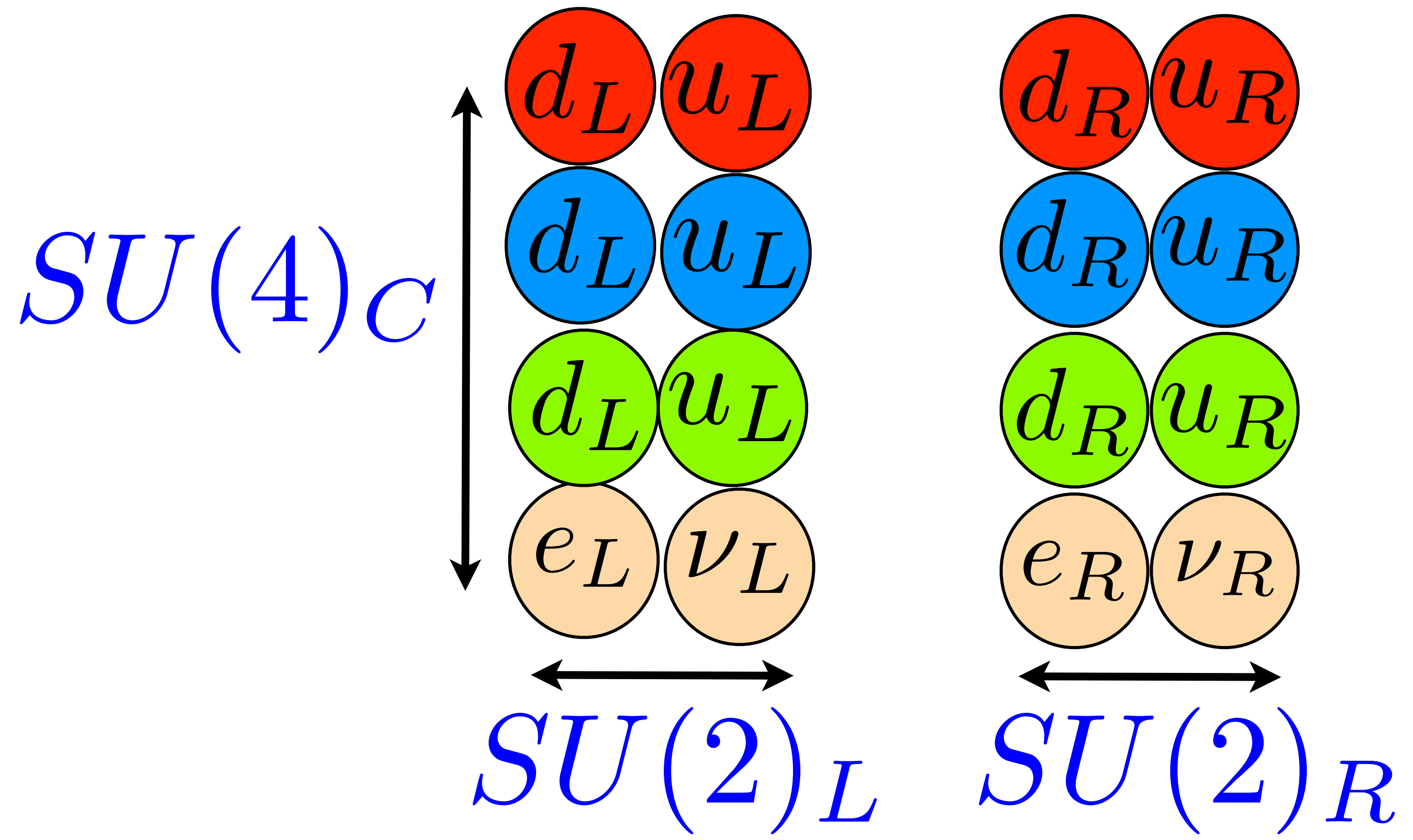
Symmetry



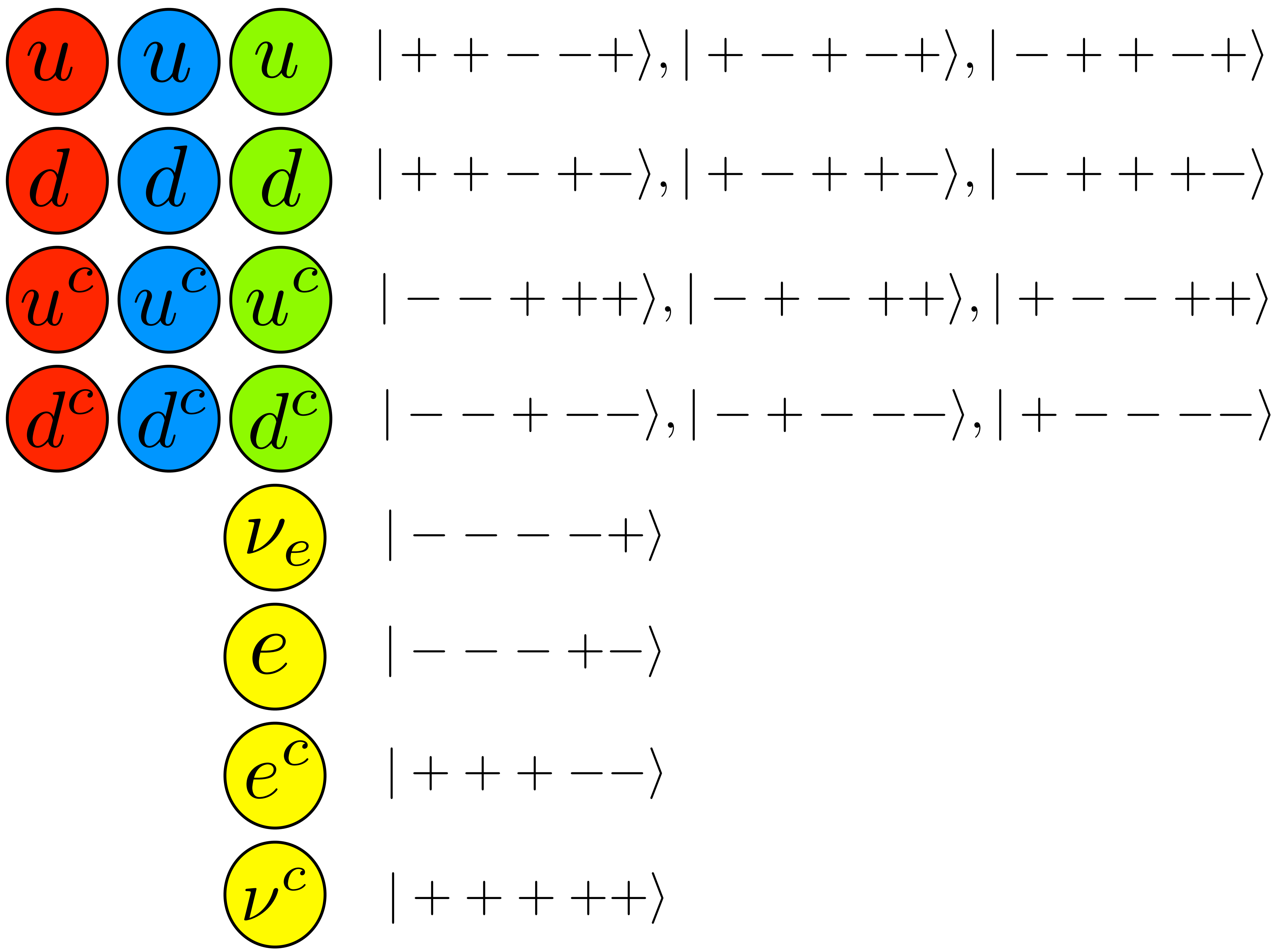
t
×1
u
×10⁻³
d
×10⁻³
c
×10⁻¹
e
×10⁻⁴
s
×10⁻²
b
×10⁻¹
ν1
< ×10⁻¹²
μ
×10⁻²
ν2
×10⁻¹²
τ
×10⁻¹
ν3
×10⁻¹¹
Family
GUTs

GUTs

Pati-Salam, 1974



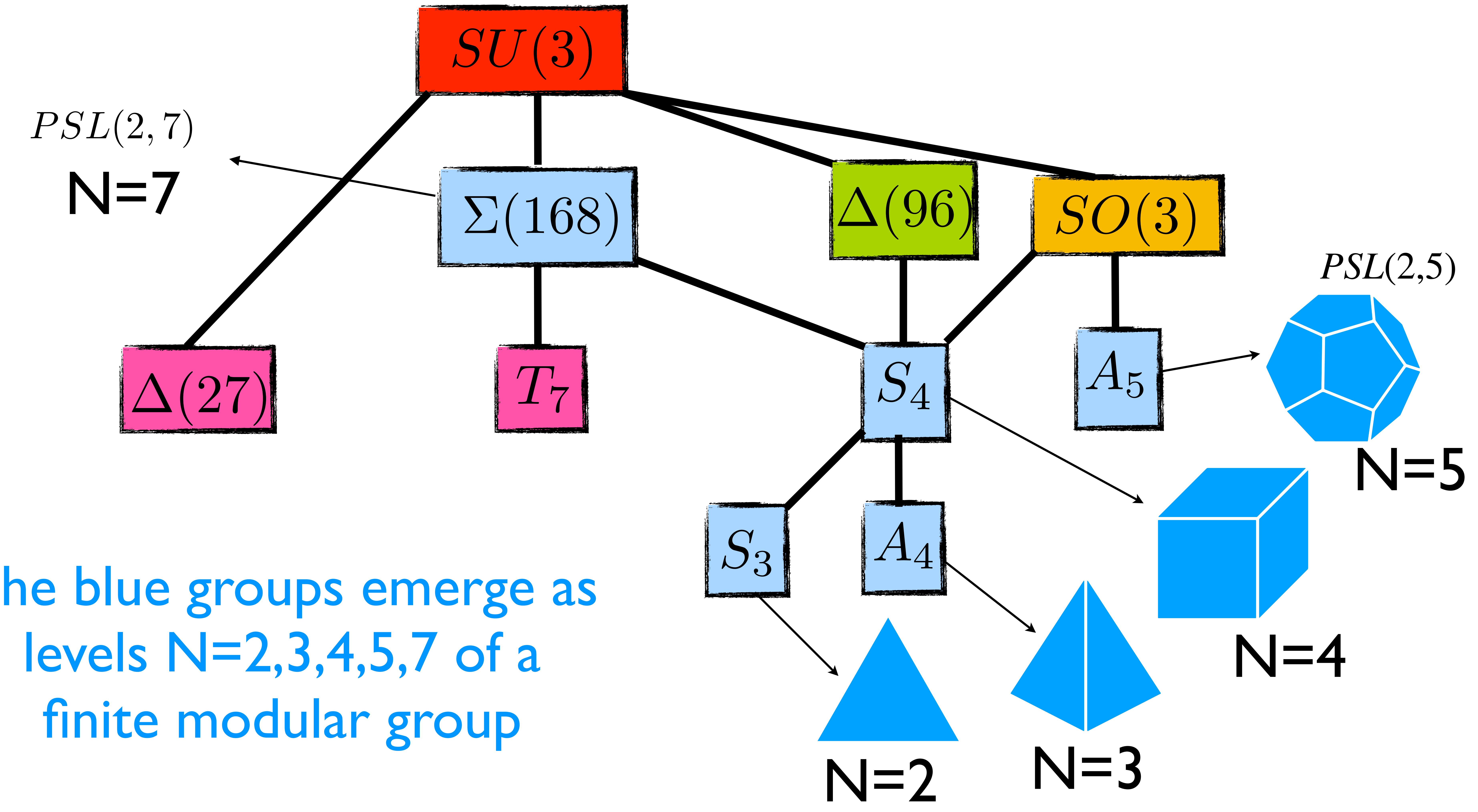
SO(10), Fritzsch-Minkowski, 1975



SU(5), Georgi-Glashow, 1974

$$F = \begin{pmatrix} d^c_r \\ d^c_b \\ d^c_g \\ e^- \\ -\nu_e \end{pmatrix}_L , \qquad T = \begin{pmatrix} 0 & u^c_g & -u^c_b & u_r & d_r \\ . & 0 & u^c_r & u_b & d_b \\ . & . & 0 & u_g & d_g \\ . & . & . & 0 & e^c \\ . & . & . & . & 0 \end{pmatrix}_L$$

Family Symmetry

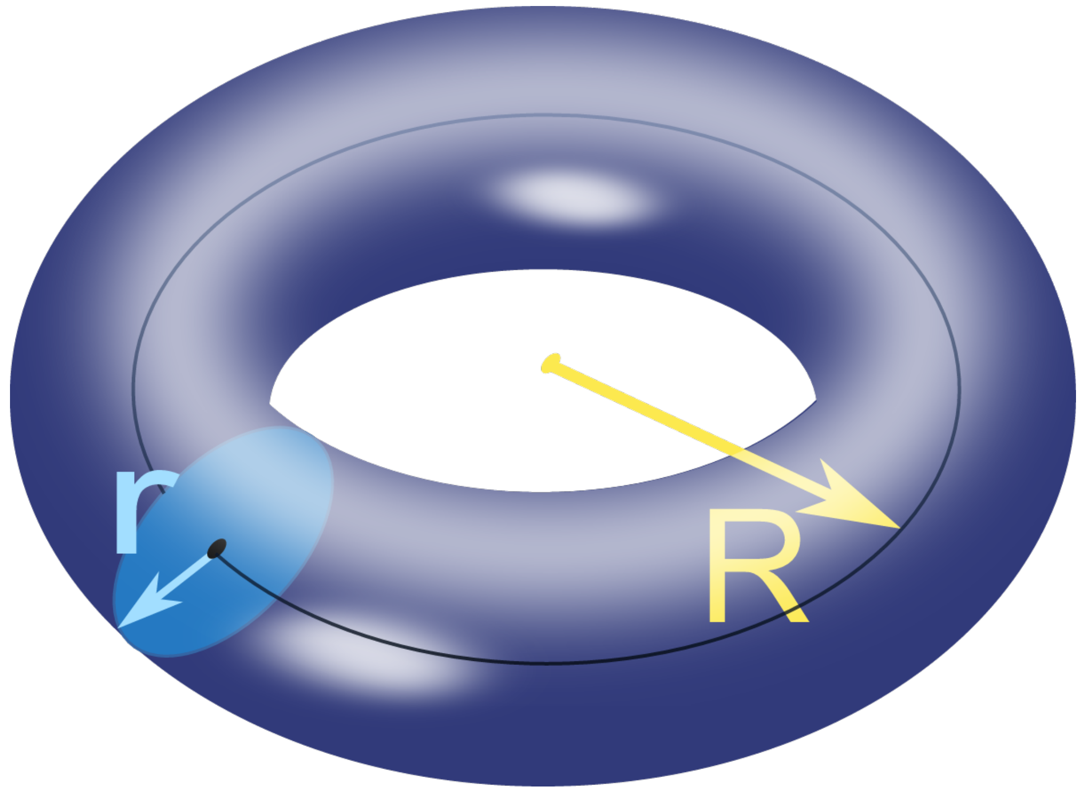


The blue groups emerge as levels N=2,3,4,5,7 of a finite modular group

For reviews see e.g. T.Kobayashi and M.Tanimoto, 2307.03384, G.J.Ding and S.F.K., 2311.09282

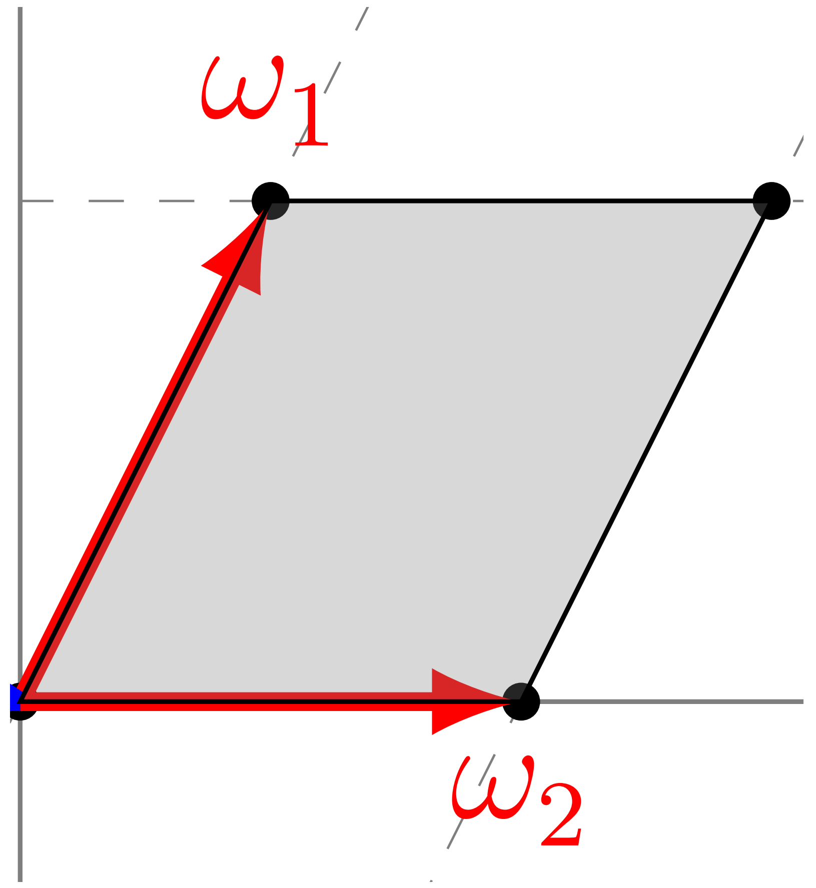
Modular Symmetry

6d with 2 compact dimensions

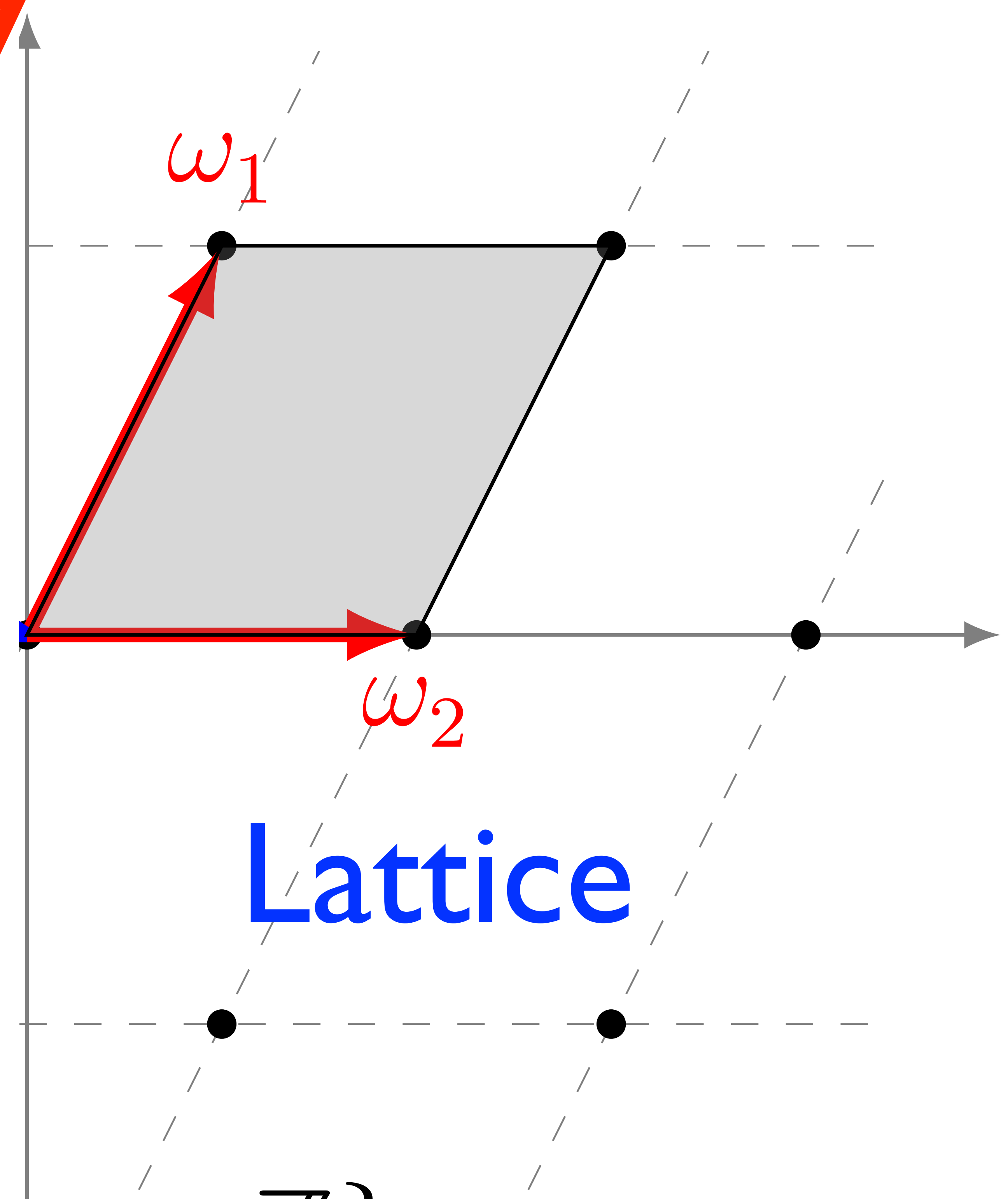


Torus

$$T^2 = \mathbb{C}/\Lambda_{\omega_1,\omega_2}$$



Parallelogram



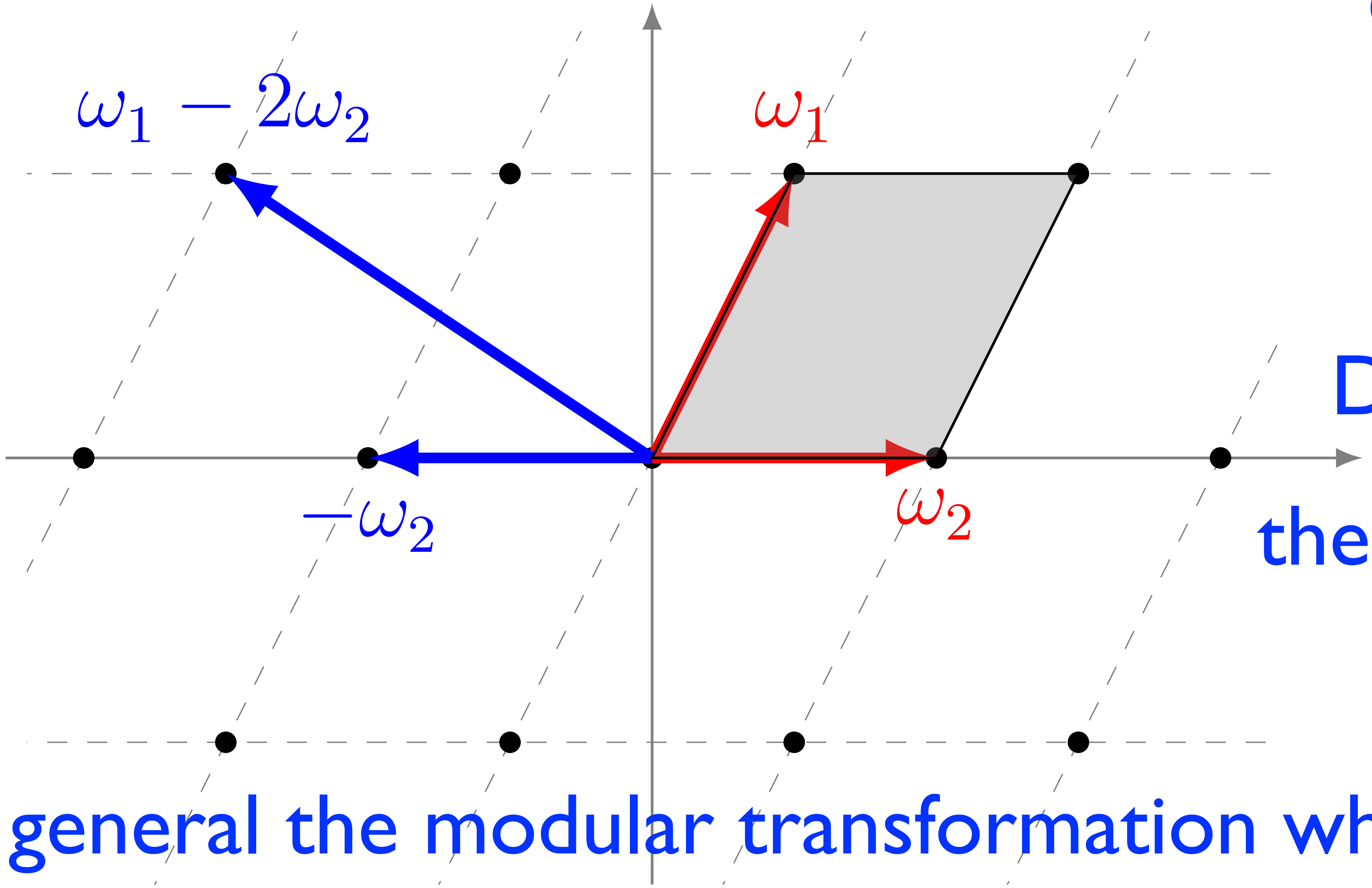
Lattice

$$\Lambda_{\omega_1,\omega_2} = \{m\omega_1 + n\omega_2, m, n \in \mathbb{Z}\}$$

Modular symmetry = "geometrical symmetry of the torus"

Modular Symmetry

J.Lauer, J.Mas and H.P.Nilles 1989,
S.Ferrara, D.Lust, A.D.Shapere and S.Theisen, 1989
G.Altarelli and F.Feruglio, hep-ph/0512103
R.de Adelhart Toorop, F.Feruglio and C.Hagedorn, 1112.1340



Certain new choice of lattice basis preserves the parallelogram area

E.g. $\{\omega_1, \omega_2\} \longrightarrow \{-\omega_2, \omega_1 - 2\omega_2\}$

Define **modulus field** $\tau \equiv \omega_1/\omega_2$

then equivalent to $\tau \to -1/(\tau - 2)$

giving a torus with same area.

In general the modular transformation which preserves the torus area is

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

Integers a,b,c,d

$$ad - bc = 1$$

Preserves torus area

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Infinite group

$$\Gamma \equiv SL(2, \mathbb{Z})$$

Generators

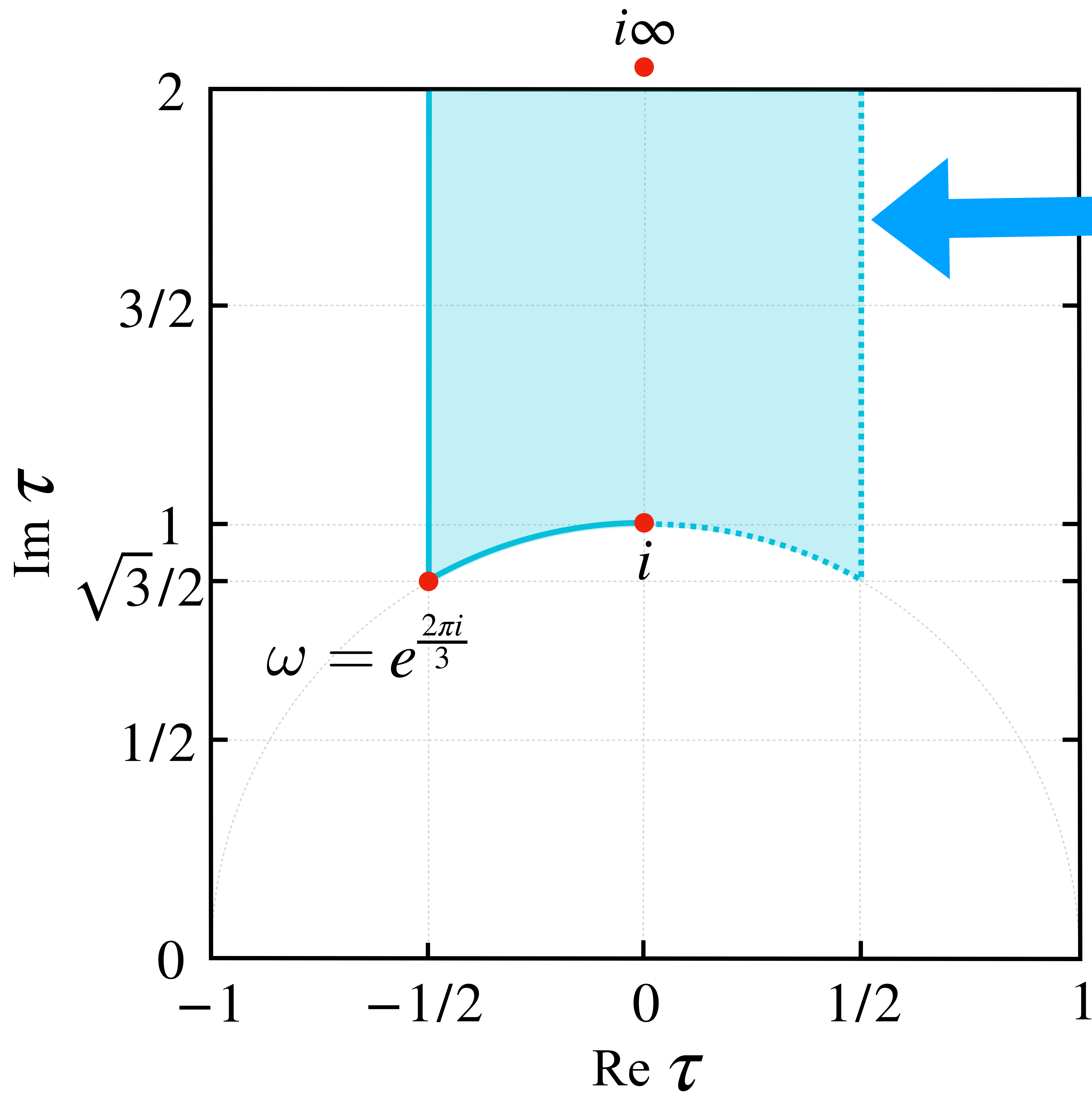
$$S: \ \tau \mapsto -\frac{1}{\tau}, \qquad T: \ \tau \mapsto \tau + 1$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S^2 = -\mathbb{1}_2$$
$$(ST)^3 = \mathbb{1}$$

F.Feruglio, 1706.08749

Fundamental Complex Domain of τ and Fixed Points



Any point can be moved into fundamental domain by applying

$$S:\ \tau \mapsto -\frac{1}{\tau}\,, \qquad T:\ \tau \mapsto \tau + 1$$

Fixed Points

$$\gamma_0 \tau_0 = \tau_0\,.$$

τ_0	γ_0
i	S
$e^{2\pi i/3}$	ST, S^2
$i\infty$	T, S^2

e.g. $S\tau_S = \tau_S \longrightarrow \tau_S = i$

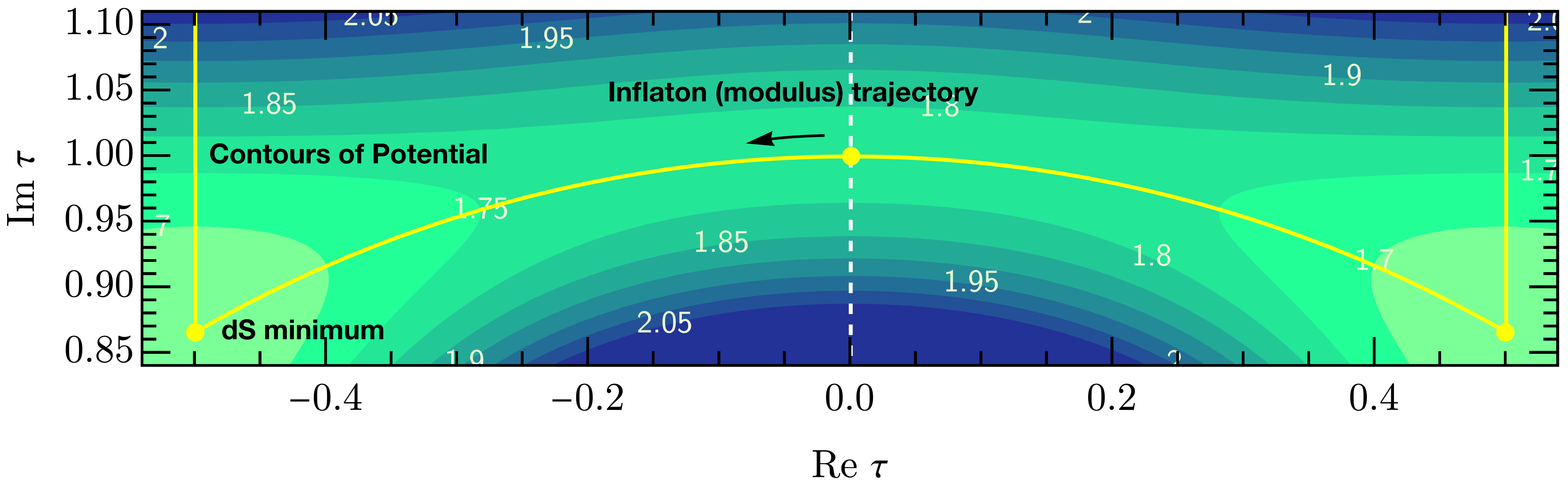
Invariant under $S:\ \tau \mapsto -\frac{1}{\tau}$

G.J.Ding, S.Y.Jiang and W.Zhao, 2405.06497 S.F.K. and X.Wang, 2405.08924

Modular Invariant Hilltop Inflation

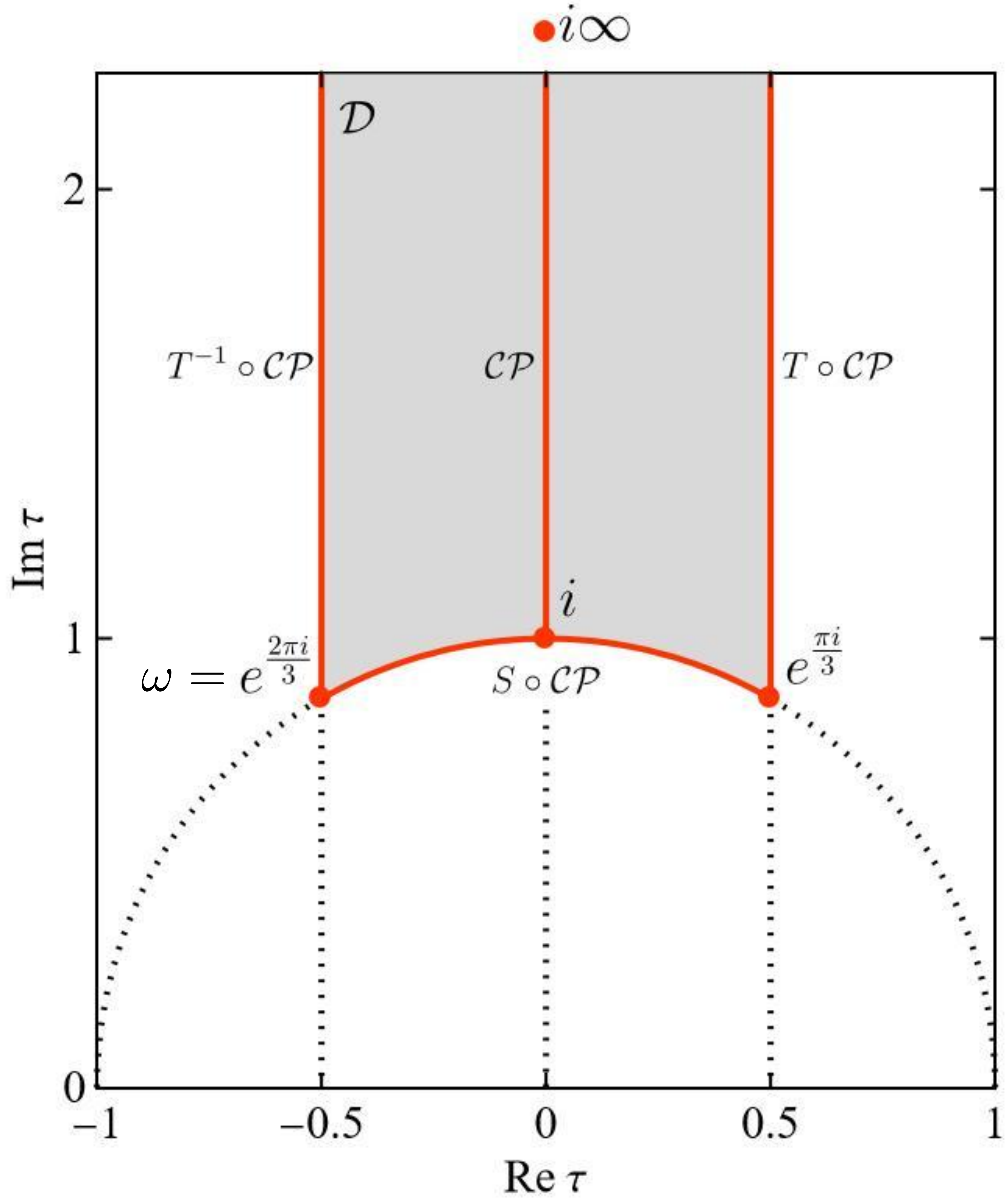
$$\mathcal{K}(\tau,\overline{\tau},S,\overline{S}) = K(S,\overline{S}) - 3\log(2\,\mathrm{Im}\,\tau) \qquad \mathcal{W}(\tau,S) = \Lambda_W^3 \frac{\Omega(S)H(\tau)}{\eta^6(\tau)} \qquad H(\tau) = (j(\tau)-1728)^{m/2} j(\tau)^{n/3} \mathcal{P}(j(\tau))$$

Klein



CP boundary

Gui-Jun Ding, S.F.K, Lu, Weng, 2505.12916



Red lines are the CP boundary, invariant under the residual symmetry shown

$$\tau \xrightarrow{\mathcal{CP}} -\tau^*$$

In a suitable basis, CP restricts all coupling constants to be real, then CP is broken only by τ

F.Feruglio, 1706.08749

From Infinite to Finite Modular Symmetry

$$\Gamma \equiv SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\}$$ Infinite $S^2 = -\mathbb{I} \quad (ST)^3 = \mathbb{I}$

$$\overline{\Gamma} \equiv PSL(2,\mathbb{Z}) \cong SL(2,\mathbb{Z})/\{\mathbb{1}_2, -\mathbb{1}_2\}$$ Infinite $S^2 = \mathbb{I} \quad (ST)^3 = \mathbb{I}$

Principle congruence subgroups

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$ Infinite

$$\overline{\Gamma}(N) = \Gamma(N)/\{\mathbb{1}_2, -\mathbb{1}_2\}$$ Infinite

G/N = all elements in G that are NOT in N (sorta)

$$\Gamma'_N \equiv SL(2,\mathbb{Z})/\Gamma(N) \equiv \Gamma/\Gamma(N)$$ Finite (double cover) $S^2 = -\mathbb{I} \quad T^N = \mathbb{I}$

$$\Gamma_N \equiv PSL(2,\mathbb{Z})/\overline{\Gamma}(N) \equiv \overline{\Gamma}/\overline{\Gamma}(N) \equiv PSL(2,N)$$ Finite $S^2 = \mathbb{I} \quad T^N = \mathbb{I}$

F.Feruglio, 1706.08749

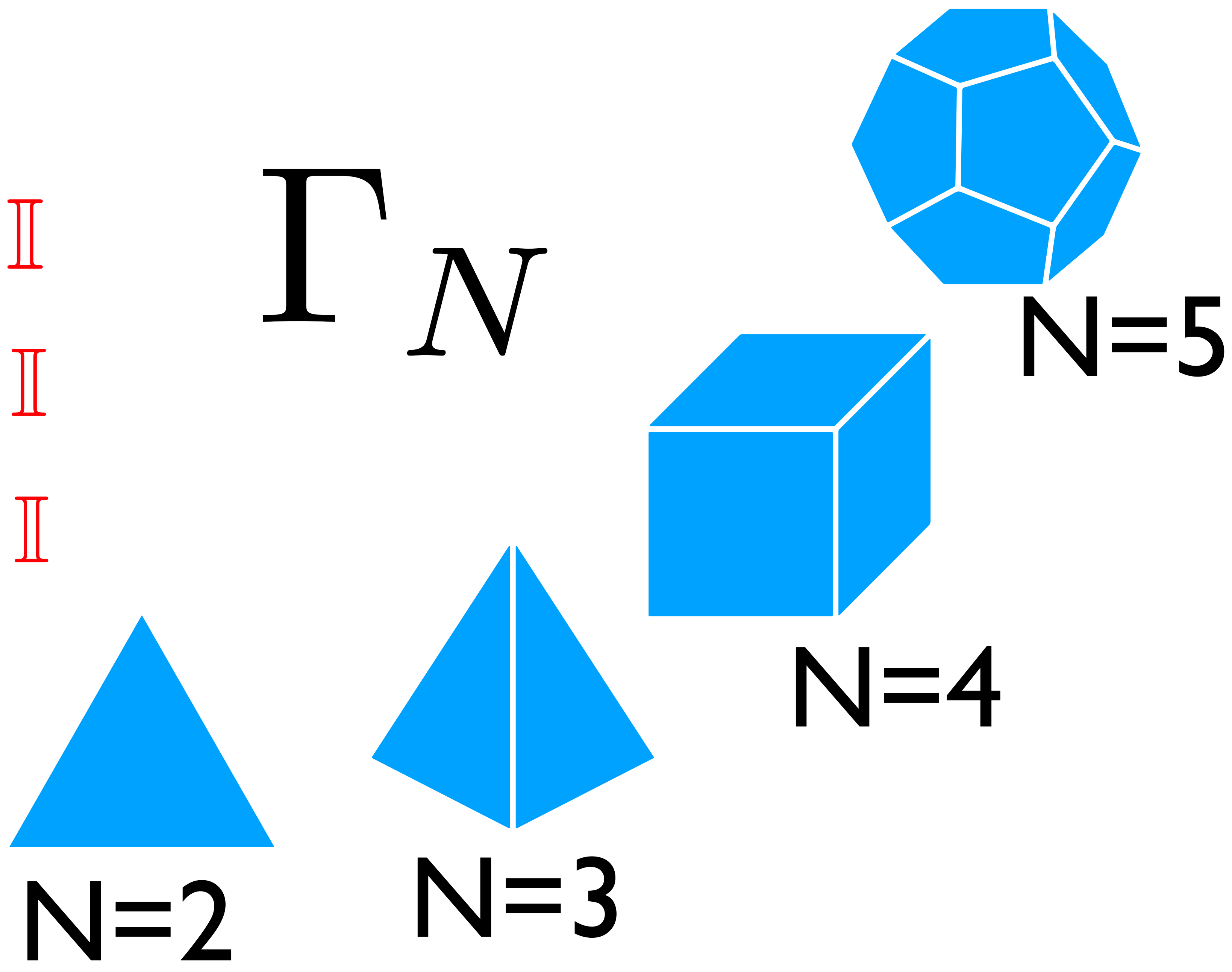
Examples

Level

N	Γ_N	$\lvert\Gamma_N\rvert$	Γ'_N	$\lvert\Gamma'_N\rvert$
2	S_3	6	S_3	6
3	A_4	12	T'	24
4	S_4	24	S'_4	48
5	A_5	60	A'_5	120
6	$S_3 \times A_4$	72	$S_3 \times T'$	144
7	$PSL(2,7) \cong \Sigma(168)$	168	$SL(2,7)$	336

$$S^2 = \mathbb{I}$$
$$T^N = \mathbb{I}$$
$$(ST)^3 = \mathbb{I}$$

$$\Gamma_N$$



Yukawa coupling

$$Y(\tau)\phi_1\phi_2\phi_3$$

Fields transform under modular symmetry with weight k

$$\phi_1 \to (c\tau + d)^{k_1}\rho_1(\gamma)\phi_1$$

Weights must add up to zero so the Yukawa carries weight

$$-k_Y = k_1 + k_2 + k_3$$

Yukawa coupling is a **modular form**

$$Y(\tau) \to Y(\gamma\tau) = (c\tau + d)^{k_Y}\rho_{\mathbf{r}_Y}(\gamma)Y(\tau)$$

$$\rho_{\mathbf{r}_Y} \times \rho_1 \times \rho_2 \times \rho_3 = 1 + \ldots$$

F.Feruglio, 1706.08749

Level N=3: $\Gamma_3 \sim A_4$

Yukawa couplings involving twisted states whose modular weights do not add up to zero are modular forms

$q \equiv e^{i2\pi\tau}$

A_4 triplet **3**
Weight k_Y=2

$$Y_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + \ldots \\ -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + 14q^4 + \ldots) \\ -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + 8q^4 + \ldots) \end{pmatrix}$$

In the fundamental domain $|q| \leq e^{-\sqrt{3}\pi} \simeq 0.0043 \ll 1$

Possibility of hierarchical Yukawa couplings

$$\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + \mathcal{O}(q) \\ -6q^{1/3} + \mathcal{O}(q) \\ -18q^{2/3} + \mathcal{O}(q) \end{pmatrix}$$

Lepton Model at level N=3

F.Feruglio, 1706.08749

	L	e^c_3	e^c_2	e^c_1	N^c	$H_{u,d}$
A_4	$\mathbf{3}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$
k_I	1	1	1	1	1	0

Weinberg operator $\frac{1}{\Lambda}(H_u H_u \; LL \; Y)$

A4 rep: **3 3 3**

Modular weights k : 1 1 2

Neutrino mass matrix

$$m_\nu = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \frac{v_u^2}{\Lambda}$$

(plus seesaw)

Holomorphic superpotential for charged leptons

$$W_e = \alpha e_1^c (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}} H_d + \beta e_2^c (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}'} H_d + \gamma e_3^c (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}''} H_d$$

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3}_S \oplus \mathbf{3}_A$$

$$= \alpha e_1^c (L_1 Y_1 + L_2 Y_3 + L_3 Y_2) H_d + \beta e_2^c (L_3 Y_3 + L_1 Y_2 + L_2 Y_1) H_d + \gamma e_3^c (L_2 Y_2 + L_3 Y_1 + L_1 Y_3) H_d \, .$$

Charged lepton mass matrix

$$M_e = \begin{pmatrix} \alpha Y_1(\tau) & \alpha Y_3(\tau) & \alpha Y_2(\tau) \\ \beta Y_2(\tau) & \beta Y_1(\tau) & \beta Y_3(\tau) \\ \gamma Y_3(\tau) & \gamma Y_2(\tau) & \gamma Y_1(\tau) \end{pmatrix} v_d$$

Mass hierarchies unexplained

Best Fit

$\mathrm{Re}(\tau)$	$\mathrm{Im}(\tau)$
0.049	2.264

β/α	γ/α
210.767	3581.720

S.J.D. King and S.F.K, 2002.00969

Introduce a weighton

	L	e_3^c	e_2^c	e_1^c	N^c	$H_{u,d}$	ϕ
A_4	$\mathbf{3}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$
k_I	1	0	-1	-3	1	0	1

$$W_{driv} = \chi(Y_{\mathbf{1}}^{(4)} \frac{\phi^4}{M_{fl}^2} - M^2)$$

$$\tilde{\phi} \equiv \frac{\langle\phi\rangle}{M_{fl}} \sim (M/M_{fl})^{1/2}$$

small parameter

$$W_e = \alpha_e e_1^c \tilde{\phi}^4 (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}} H_d + \beta_e e_2^c \tilde{\phi}^2 (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}'} H_d + \gamma_e e_3^c \tilde{\phi} (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}''} H_d$$

$$Y_e = \begin{pmatrix} \alpha_e \tilde{\phi}^4 Y_1 & \alpha_e \tilde{\phi}^4 Y_3 & \alpha_e \tilde{\phi}^4 Y_2 \\ \beta_e \tilde{\phi}^2 Y_2 & \beta_e \tilde{\phi}^2 Y_1 & \beta_e \tilde{\phi}^2 Y_3 \\ \gamma_e \tilde{\phi} Y_3 & \gamma_e \tilde{\phi} Y_2 & \gamma_e \tilde{\phi} Y_1 \end{pmatrix}$$

Natural explanation of charged lepton mass hierarchy

$$m_e : m_\mu : m_\tau = \alpha_e \tilde{\phi}^4 : \beta_e \tilde{\phi}^2 : \gamma_e \tilde{\phi}$$

Unlike the FN flavon, the weighton phi does not break the flavour symmetry

S.J.D. King and S.F.K, 2002.00969

Quark Sector with weighton

	Q	d^c_3	d^c_2	d^c_1	u^c_3	u^c_2	u^c_1	$H_{u,d}$	ϕ
A_4	$\mathbf{3}$	$\mathbf{1'}$	$\mathbf{1''}$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{1''}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
k_I	1	0, 2, 4	−2	−3	5, 3, 1	−1, 2, 4	−3	0	1

$$\tilde{\phi} \equiv \frac{\langle \phi \rangle}{M_{fl}} \sim (M/M_{fl})^{1/2}$$

small parameter

E.g. $k_{d^c_{3,2,1}} = 0, -2, -3$ $\quad k_{u^c_{3,2,1}} = 5, -1, -3$

$$W_d = \alpha_d d^c_1 \tilde{\phi}^4 (QY^{(2)}_{\mathbf{3}})_{\mathbf{1}} H_d + \beta_d d^c_2 \tilde{\phi}^3 (QY^{(2)}_{\mathbf{3}})_{\mathbf{1'}} H_d + \gamma_d d^c_3 \tilde{\phi} (QY^{(2)}_{\mathbf{3}})_{\mathbf{1''}} H_d$$

$$W_u = \alpha_u u^c_1 \tilde{\phi}^4 (QY^{(2)}_{\mathbf{3}})_{\mathbf{1}} H_u + \beta_u u^c_2 \tilde{\phi}^2 (QY^{(2)}_{\mathbf{3}})_{\mathbf{1'}} H_u + \gamma^I_u u^c_3 (QY^{(6)}_{\mathbf{3},I})_{\mathbf{1''}} H_u + \gamma^{II}_u u^c_3 (QY^{(6)}_{\mathbf{3},II})_{\mathbf{1''}} H_u$$

$$m_d : m_s : m_b \sim \tilde{\phi}^4 : \tilde{\phi}^3 : \tilde{\phi}$$

$$m_u : m_c : m_t \sim \tilde{\phi}^4 : \tilde{\phi}^2 : 1$$

Natural explanation of quark mass hierarchy

Good fits to mass and mixing for certain cases

Gui-Jun Ding, S.F.K, Lu, Weng, 2505.12916

General analysis with a weighton

$$\mathcal{W} = \alpha\, \tilde{\phi}^I \left(\psi_1^c \psi Y_{\boldsymbol{r}_1}^{(k_1)} H_{u/d}\right)_{\mathbf{1}} + \beta\, \tilde{\phi}^J \left(\psi_2^c \psi Y_{\boldsymbol{r}_2}^{(k_2)} H_{u/d}\right)_{\mathbf{1}} + \gamma\, \tilde{\phi}^K \left(\psi_3^c \psi Y_{\boldsymbol{r}_3}^{(k_3)} H_{u/d}\right)_{\mathbf{1}}$$

level N=3 independent of modular weights $|q| \le e^{-\sqrt{3}\pi} \simeq 0.0043 \ll 1$

$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde{\phi}$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta_{12}^\psi, \theta_{23}^\psi, \theta_{13}^\psi)$
$\mathbf{3} \otimes \mathbf{3}$	$I = J = K$	P_{132}	$(\tilde{\phi}^I, \tilde{\phi}^I, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$
$\mathbf{3} \otimes (\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1})$	$I \ge J \ge K$	P_{231}	$(\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}} \tilde{\phi}^{2(I-K)} + \lvert q\rvert^{\frac{2}{3}})$
	$J \ge K \ge I$	P_{312}	$(\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}} \tilde{\phi}^{2(J-I)} + \lvert q\rvert^{\frac{2}{3}})$
	$K \ge I \ge J$	P_{123}	$(\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}} \tilde{\phi}^{2(K-J)} + \lvert q\rvert^{\frac{2}{3}})$
	$J \ge I \ge K$	P_{321}	$(\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}} \tilde{\phi}^{2(I-K)} + \lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}})$
(cont'd…)	$I \ge K \ge J$	P_{213}	$(\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}} \tilde{\phi}^{2(K-J)} + \lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$K \ge J \ge I$	P_{132}	$(\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}} \tilde{\phi}^{2(J-I)} + \lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}})$

Similar tables for N=4,5

Weightons control mass hierarchy

q-expansion suppresses mixing

(N=3 cont'd...)

Gui-Jun Ding, S.F.K, Lu, Weng, 2505.12916

$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde\phi$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta^\psi_{12}, \theta^\psi_{23}, \theta^\psi_{13})$
$\mathbf{3}\otimes(\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1})$	$I\geq J\geq K$	P_{231}	$\left(\tilde\phi^I, \tilde\phi^J, \tilde\phi^K\right)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(I-K)} + \lvert q\rvert^{\frac{2}{3}})$
	$J\geq K\geq I$	P_{312}	$(\tilde\phi^J, \tilde\phi^K, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(J-I)} + \lvert q\rvert^{\frac{2}{3}})$
	$K\geq I\geq J$	P_{123}	$(\tilde\phi^K, \tilde\phi^I, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-J)} + \lvert q\rvert^{\frac{2}{3}})$
	$J\geq I\geq K$	P_{321}	$(\tilde\phi^J, \tilde\phi^I, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(I-K)} + \lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$I\geq K\geq J$	P_{213}	$(\tilde\phi^I, \tilde\phi^K, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-J)} + \lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq J\geq I$	P_{132}	$(\tilde\phi^K, \tilde\phi^J, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(J-I)} + \lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}})$
$\mathbf{3}\otimes(\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1})$	$I\geq J\geq K, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J \geq \tilde\phi^I$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I\geq J\geq K, \tilde\phi^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^I, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J\geq K\geq I$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^K, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-I)}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq I\geq J$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^I, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J\geq I\geq K$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^I, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I\geq K\geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^K \geq \tilde\phi^I$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I\geq K\geq J,\ \tilde\phi^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^K$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^I, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K\geq J\geq I$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(J-I)}, \lvert q\rvert^{\frac{1}{3}})$
$\mathbf{3}\otimes(\mathbf{1}''\oplus\mathbf{1}\oplus\mathbf{1})$	$I\geq J\geq K,\ \tilde\phi^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^I, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(I-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$I\geq J\geq K,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J \geq \tilde\phi^I$	P_{231}	$(\tilde\phi^I, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(I-K)} + \lvert q\rvert^{\frac{2}{3}})$
	$J\geq K\geq I$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K\geq I\geq J$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^I, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$J\geq I\geq K$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^I, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$I\geq K\geq J,\ \tilde\phi^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^K$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^I, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(I-J)}, \lvert q\rvert^{\frac{1}{3}})$
	$I\geq K\geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^K \geq \tilde\phi^I$	P_{231}	$(\tilde\phi^I, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(I-J)} + \lvert q\rvert^{\frac{2}{3}})$
	$K\geq J\geq I$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3}\otimes(\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1})$	$I\geq J\geq K$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J\geq K\geq I$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-I)}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq I\geq J,\ \tilde\phi^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^I$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \tilde\phi^K, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-J)}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq I\geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^I \geq \tilde\phi^K$	P_{123}	$(\tilde\phi^K, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-J)})$
	$J\geq I\geq K$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^I, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I\geq K\geq J$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \tilde\phi^K, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq J\geq I,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J \geq \tilde\phi^K$	P_{123}	$(\tilde\phi^K, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-I)})$
	$K\geq J\geq I,\ \tilde\phi^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$

$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde\phi$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta^\psi_{12}, \theta^\psi_{23}, \theta^\psi_{13})$
$\mathbf{3}\otimes(\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1})$	$I\geq J\geq K$	P_{321}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^I, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(J-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$J\geq K\geq I$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^K, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K\geq I\geq J,\ \tilde\phi^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^I$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^I, \tilde\phi^K, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K\geq I\geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^I \geq \tilde\phi^K$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J\geq I\geq K$	P_{321}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^I, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(I-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$I\geq K\geq J$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^I, \tilde\phi^K, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K\geq J\geq I,\ \tilde\phi^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^K, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K\geq J\geq I,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J \geq \tilde\phi^K$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3}\otimes(\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}')$	$I\geq J\geq K,\ \tilde\phi^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^I, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I\geq J\geq K,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J \geq \tilde\phi^I$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J\geq K\geq I$	P_{132}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^K, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-I)}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq I\geq J$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^I, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J\geq I\geq K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^I, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I\geq K\geq J,\ \tilde\phi^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^I, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I\geq K\geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^K \geq \tilde\phi^I$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K\geq J\geq I$	P_{132}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(J-I)}, \lvert q\rvert^{\frac{1}{3}})$
$\mathbf{3}\otimes(\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}')$	$I\geq J\geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J\geq K\geq I$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-I)}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq I\geq J,\ \tilde\phi^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^I$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \tilde\phi^K, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-J)}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq I\geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^I \geq \tilde\phi^K$	P_{312}	$(\tilde\phi^K, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J\geq I\geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^I, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I\geq K\geq J$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^I, \tilde\phi^K, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-J)}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq J\geq I,\ \tilde\phi^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-I)}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq J\geq I,\ \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J \geq \tilde\phi^K$	P_{312}	$(\tilde\phi^K, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3}\otimes(\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1})$	$I\geq J\geq K$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^I, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3}\otimes(\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}')$	$I\geq J\geq K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^I, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3}\otimes(\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}'')$	$I\geq J\geq K$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^I, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3}\otimes(\widehat{\mathbf{2}}\oplus\mathbf{1})$	$I=J\geq K$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K\geq I=J$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3}\otimes(\widehat{\mathbf{2}}\oplus\mathbf{1}')$	$I=J\geq K$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(J-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq I=J$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3}\otimes(\widehat{\mathbf{2}}\oplus\mathbf{1}'')$	$I=J\geq K$	P_{312}	$(\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(J-K)})$
	$K\geq I=J$	P_{231}	$(\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-J)})$
$\mathbf{3}\otimes(\widehat{\mathbf{2}}'\oplus\mathbf{1})$	$I=J\geq K$	P_{321}	$(\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(J-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$K\geq I=J$	P_{123}	$(\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{2(K-J)})$
$\mathbf{3}\otimes(\widehat{\mathbf{2}}'\oplus\mathbf{1}')$	$I=J\geq K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K\geq I=J$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$

(N=3 cont'd...)

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$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde{\phi}$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta_{12}^\psi, \theta_{23}^\psi, \theta_{13}^\psi)$
$\mathbf{3} \otimes (\widehat{\mathbf{2}}' \oplus \mathbf{1}'')$	$I = J \geq K$	P_{132}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\|q\|^{\frac{2}{3}}, \|q\|^{\frac{2}{3}} + \|q\|^{\frac{1}{3}} \tilde{\phi}^{2(J-K)}, \|q\|^{\frac{1}{3}})$
	$K \geq I = J$	P_{123}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}}, \|q\|^{\frac{2}{3}})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}}'' \oplus \mathbf{1})$	$I = J \geq K$	P_{321}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\|q\|^{\frac{2}{3}}, \|q\|^{\frac{2}{3}} + \|q\|^{\frac{1}{3}} \tilde{\phi}^{2(J-K)}, \|q\|^{\frac{1}{3}})$
	$K \geq I = J$	P_{321}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\|q\|^{\frac{2}{3}}, \|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}}'' \oplus \mathbf{1}')$	$I = J \geq K$	P_{123}	$(\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}}, \|q\|^{\frac{2}{3}} + \|q\|^{\frac{1}{3}} \tilde{\phi}^{2(J-K)})$
	$K \geq I = J$	P_{321}	$(\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\|q\|^{\frac{2}{3}}, \|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}}'' \oplus \mathbf{1}'')$	$I = J \geq K$	P_{312}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}}, \|q\|^{\frac{2}{3}})$
	$K \geq I = J$	P_{321}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\|q\|^{\frac{2}{3}}, \|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}})$
$(\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{I-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I})$
	$K \geq I \geq J$	P_{312}	$(\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K})$
	$I \geq K \geq J$	P_{132}	$(\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J})$
	$K \geq J \geq I$	P_{321}	$(\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I})$
$(\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^J \geq \tilde{\phi}^I$	P_{123}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^I, \|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K)$	$(\|q\|^{-\frac{1}{3}} \tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K})$
	$I \geq J \geq K,\ \tilde{\phi}^I \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^J$	P_{213}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$J \geq K \geq I$	P_{231}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I})$
	$K \geq I \geq J$	P_{312}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$I \geq K \geq J,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^K \geq \tilde{\phi}^I$	P_{132}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^I, \|q\|^{\frac{1}{3}} \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{-\frac{1}{3}} \tilde{\phi}^{I-K}, \tilde{\phi}^{K-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J})$
	$I \geq K \geq J,\ \tilde{\phi}^I \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^K$	P_{312}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$K \geq J \geq I$	P_{321}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\tilde{\phi}^{K-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I})$
$(\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K,\ \tilde{\phi}^I \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^J$	P_{213}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$I \geq J \geq K,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^J \geq \tilde{\phi}^I$	P_{123}	$(\tilde{\phi}^I, \|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I})$
	$K \geq I \geq J$	P_{312}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$I \geq K \geq J,\ \tilde{\phi}^I \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^K$	P_{312}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$I \geq K \geq J,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^K \geq \tilde{\phi}^I$	P_{132}	$(\tilde{\phi}^I, \|q\|^{\frac{1}{3}} \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \tilde{\phi}^{K-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J})$
	$K \geq J \geq I$	P_{321}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\tilde{\phi}^{K-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I})$

$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde{\phi}$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta_{12}^\psi, \theta_{23}^\psi, \theta_{13}^\psi)$
$(\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
	$K \geq I \geq J,\ \tilde{\phi}^K \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^I$	P_{132}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq I \geq J,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^I \geq \tilde{\phi}^K$	P_{312}	$(\tilde{\phi}^K, \|q\|^{\frac{1}{3}} \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \tilde{\phi}^{I-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K})$
	$I \geq K \geq J$	P_{132}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq J \geq I,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^J \geq \tilde{\phi}^K$	P_{321}	$(\tilde{\phi}^K, \|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^I)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I})$
	$K \geq J \geq I,\ \tilde{\phi}^K \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^J$	P_{231}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
$(\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
	$K \geq I \geq J,\ \tilde{\phi}^K \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^I$	P_{132}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq I \geq J,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^I \geq \tilde{\phi}^K$	P_{312}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^K, \|q\|^{\frac{1}{3}} \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{-\frac{1}{3}} \tilde{\phi}^{K-I}, \tilde{\phi}^{I-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K})$
	$I \geq K \geq J$	P_{132}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq J \geq I,\ \tilde{\phi}^K \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^J$	P_{231}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
	$K \geq J \geq I,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^J \geq \tilde{\phi}^K$	P_{321}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^K, \|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^I)$	$(\|q\|^{-\frac{1}{3}} \tilde{\phi}^{K-J}, \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I})$
$(\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}') \otimes \mathbf{3}$	$I \geq J \geq K,\ \tilde{\phi}^I \geq^{\frac{1}{3}} \tilde{\phi}^J$	P_{213}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$I \geq J \geq K,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^J \geq \tilde{\phi}^I$	P_{123}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^I, \|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K)$	$(\|q\|^{-\frac{1}{3}} \tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I})$
	$K \geq I \geq J$	P_{312}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$I \geq K \geq J,\ \tilde{\phi}^I \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^K$	P_{312}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$I \geq K \geq J,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^K \geq \tilde{\phi}^I$	P_{132}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^I, \|q\|^{\frac{1}{3}} \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{-\frac{1}{3}} \tilde{\phi}^{I-K}, \tilde{\phi}^{K-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-J})$
	$K \geq J \geq I$	P_{321}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\tilde{\phi}^{K-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I})$
$(\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}') \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
	$K \geq I \geq J,\ \tilde{\phi}^K \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^I$	P_{132}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq I \geq J,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^I \geq \tilde{\phi}^K$	P_{312}	$(\tilde{\phi}^K, \|q\|^{\frac{1}{3}} \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \tilde{\phi}^{I-J}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K})$
	$I \geq K \geq J$	P_{132}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{I-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq J \geq I,\ \tilde{\phi}^K \geq \|q\|^{\frac{1}{3}} \tilde{\phi}^J$	P_{231}	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^{J-K}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
	$K \geq J \geq I,\ \|q\|^{\frac{1}{3}} \tilde{\phi}^J \geq \tilde{\phi}^K$	P_{321}	$(\tilde{\phi}^K, \|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^I)$	$(\|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \tilde{\phi}^{J-I}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-I})$
$(\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^I, \|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \tilde{\phi}^{I-K})$
$(\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}') \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^I, \|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \tilde{\phi}^{I-K})$
$(\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'') \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^I, \|q\|^{\frac{1}{3}} \tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \tilde{\phi}^{I-K})$
$(\widehat{\mathbf{2}} \oplus \mathbf{1}) \otimes \mathbf{3}$	$I = J \geq K$	P_{213}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{J-K}, \tilde{\phi}^{J-K})$
	$K \geq I = J$	P_{312}	$(\|q\|^{\frac{2}{3}} \tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{K-J}, \tilde{\phi}^{K-J})$

(N=3 cont'd...)

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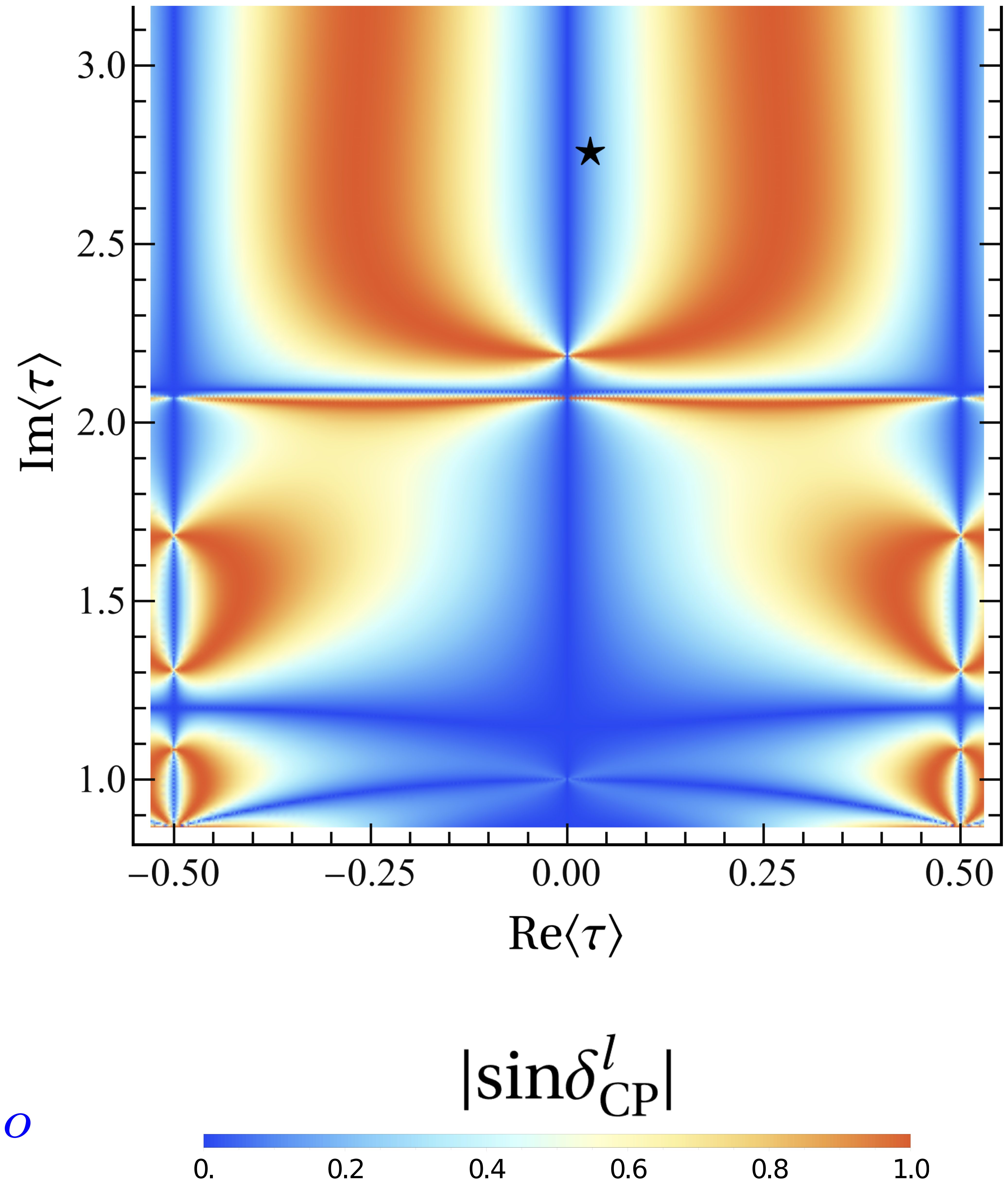
$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde\phi$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta^\psi_{12}, \theta^\psi_{23}, \theta^\psi_{13})$
$(\widehat{\mathbf{2}} \oplus \mathbf{1}') \otimes \mathbf{3}$	$I = J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \tilde\phi^{J-K})$
	$K \geq I = J$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$
$(\widehat{\mathbf{2}} \oplus \mathbf{1}'') \otimes \mathbf{3}$	$I = J \geq K$	P_{123}	$(\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K})$
	$K \geq I = J$	P_{312}	$(\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}' \oplus \mathbf{1}) \otimes \mathbf{3}$	$I = J \geq K$	P_{213}	$(\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K})$
	$K \geq I = J$	P_{312}	$(\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}' \oplus \mathbf{1}') \otimes \mathbf{3}$	$I = J \geq K$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \tilde\phi^{J-K})$
	$K \geq I = J$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}' \oplus \mathbf{1}'') \otimes \mathbf{3}$	$I = J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \tilde\phi^{J-K})$
	$K \geq I = J$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}'' \oplus \mathbf{1}) \otimes \mathbf{3}$	$I = J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \tilde\phi^{J-K})$
	$K \geq I = J$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}'' \oplus \mathbf{1}') \otimes \mathbf{3}$	$I = J \geq K$	P_{123}	$(\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K})$
	$K \geq I = J$	P_{321}	$(\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}'' \oplus \mathbf{1}'') \otimes \mathbf{3}$	$I = J \geq K$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \tilde\phi^{J-K})$
	$K \geq I = J$	P_{321}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$



For example...

$$L \sim (\mathbf{3}, 2)\ ,\quad e^c \sim (\mathbf{1}, -1)\ ,\quad \mu^c \sim (\mathbf{1}, 1)\ ,\quad \tau^c \sim (\mathbf{1''}, 3)\ ,\quad H_{u,d} \sim (\mathbf{1}, 0)\ ,$$
$$Q_L \sim (\mathbf{3}, k_{Q_L})\ ,\quad u^c \sim (\mathbf{1}, -k_{Q_L})\ ,\quad c^c \sim (\mathbf{1''}, 5 - k_{Q_L})\ ,\quad t^c \sim (\mathbf{1'}, 4 - k_{Q_L})\ ,$$
$$D^c_D \equiv \{d^c, s^c\} \sim \left(\widehat{\mathbf{2}}', 2 - k_{Q_L}\right)\ ,\quad b^c \sim (\mathbf{1}, 5 - k_{Q_L})\ ,\quad \phi \sim (\mathbf{1}, 1)\ .$$

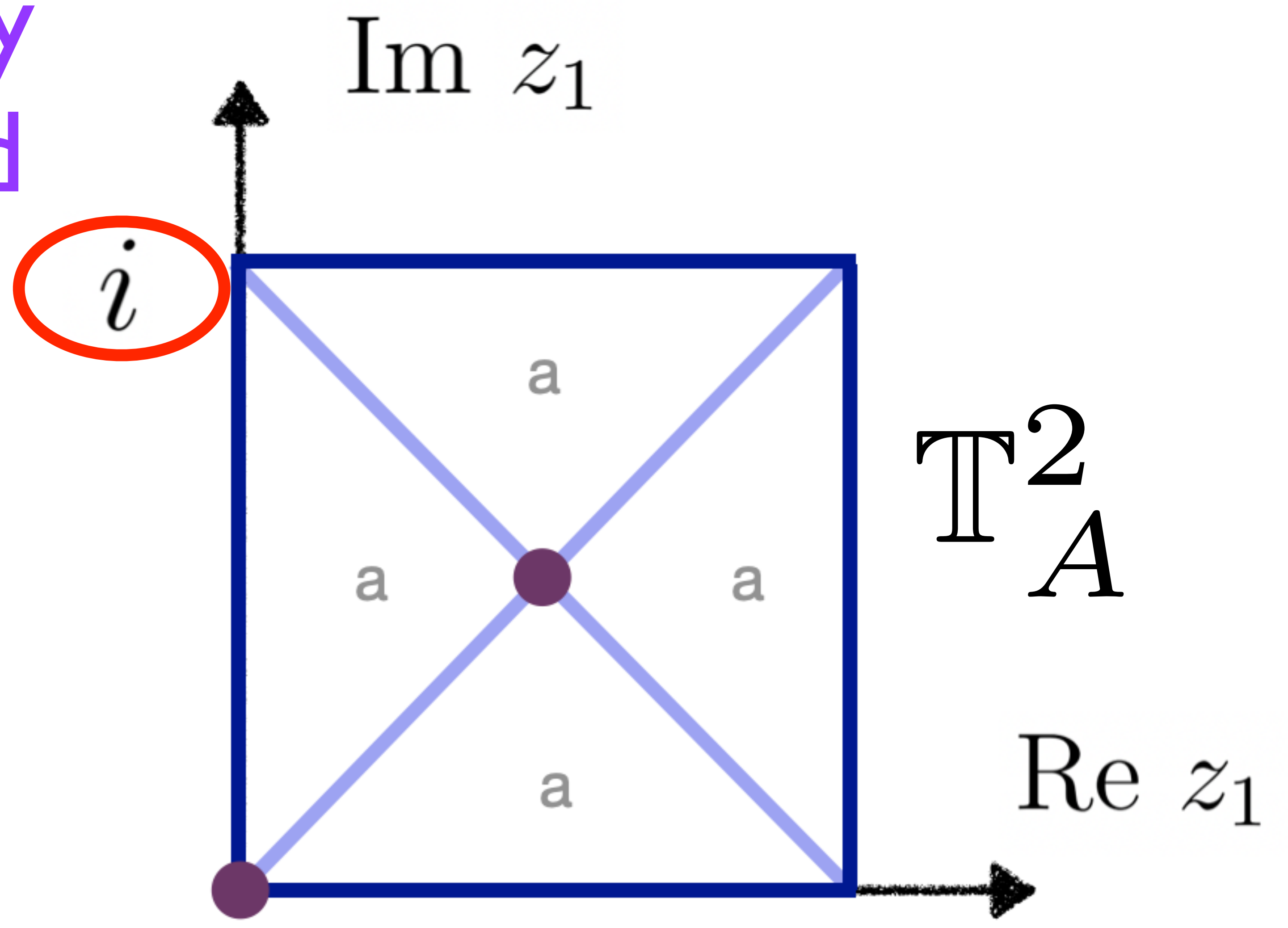
18 parameter fit to fermion mass and mixing data, predicts $\delta^l_{CP} = 190^o$

10d models $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$

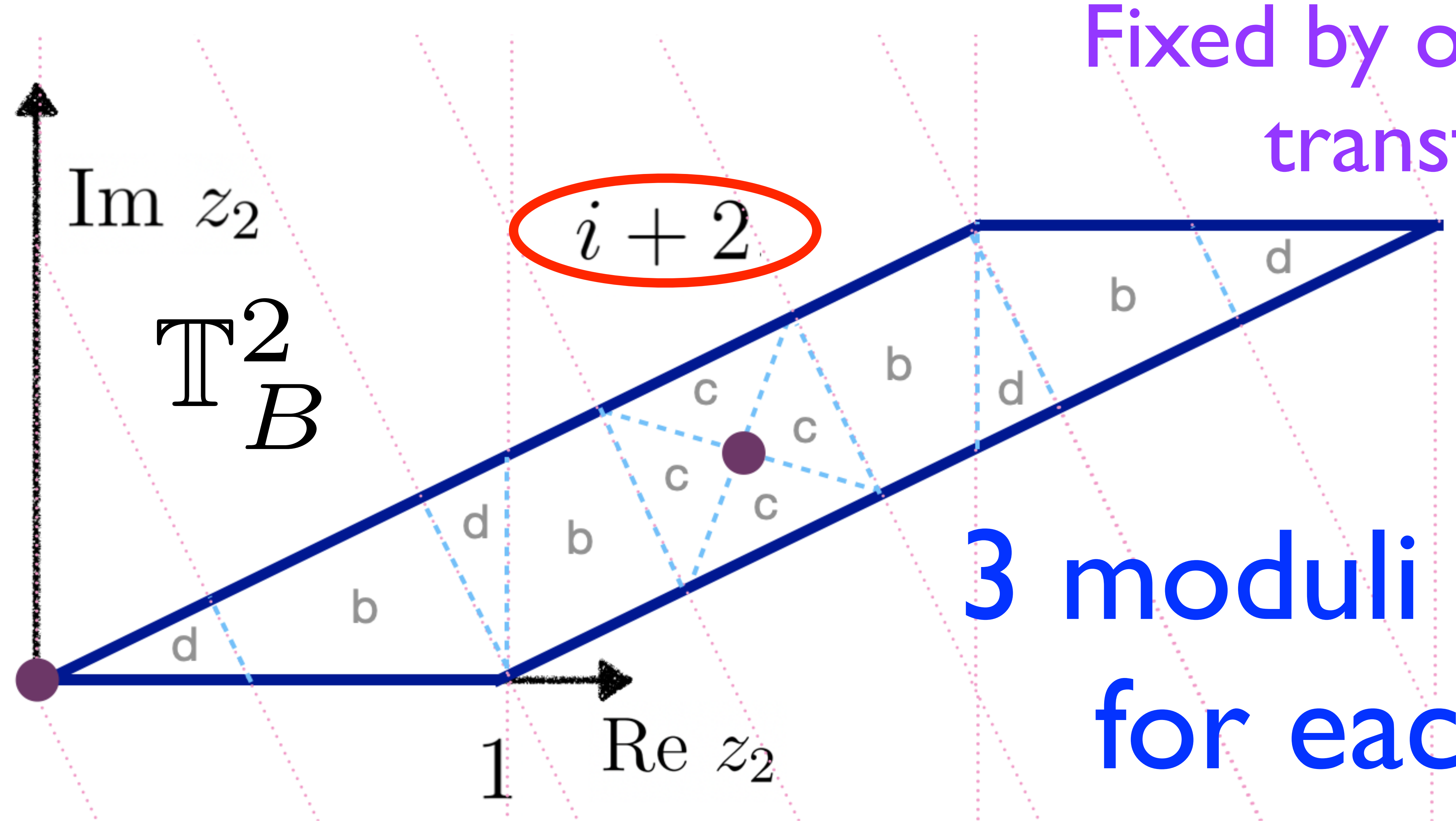
De Anda, S.F.K. 2312.09010, 2304.05958
M. Fischer, M. Ratz, J. Torrado and P. K. S. Vaudrevange 1209.3906

3 factorable tori, SUSY preserving

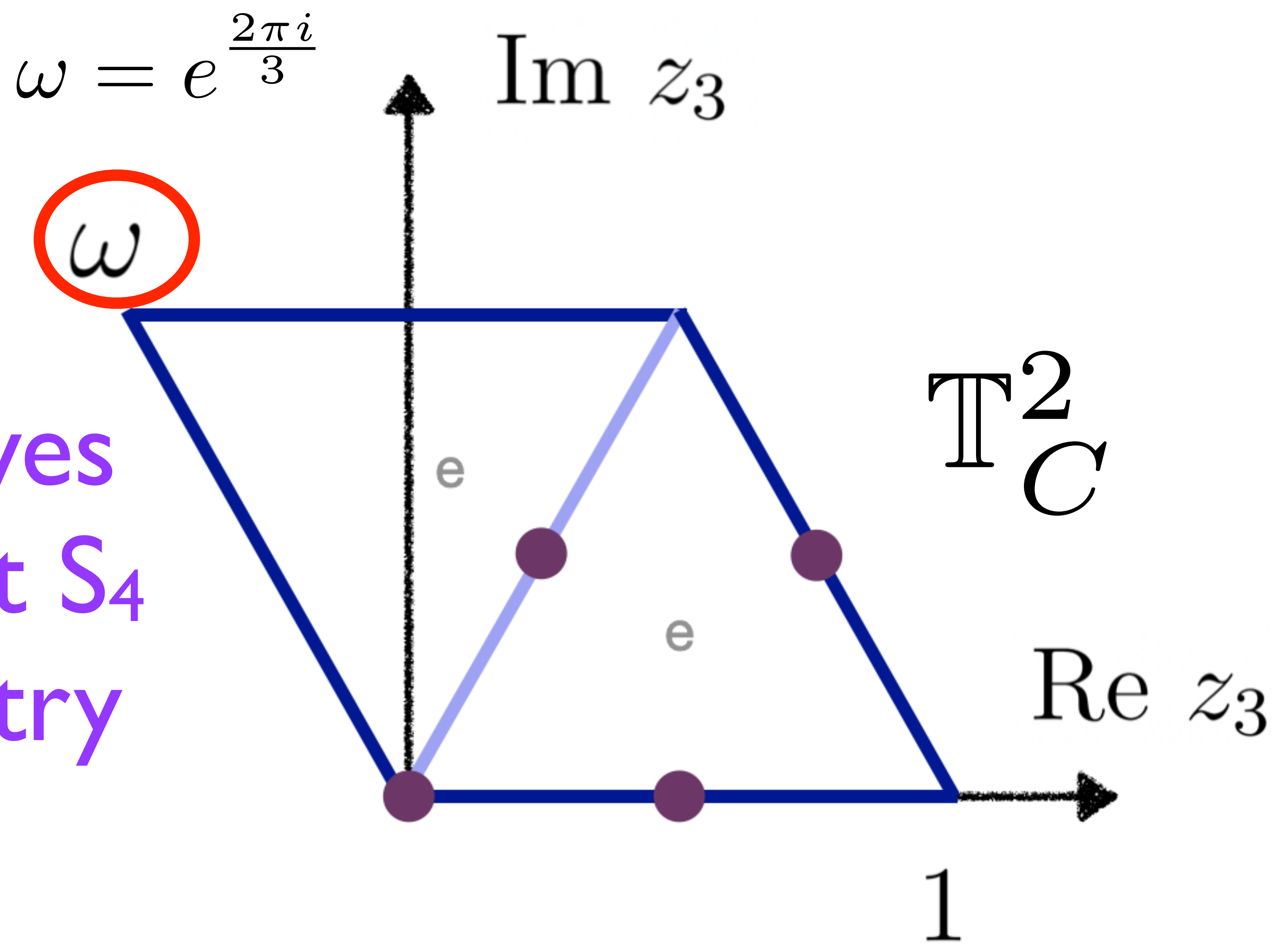
Fixed by orbifold



Fixed by orbifold up to T transformation



3 moduli fields, one for each torus



Preserves remnant S_4 symmetry

Lattice vectors for each torus are $(1, \tau_i)$

$$\tau_1 = i,\ \tau_2 = i+2,\ \tau_3 = \omega$$

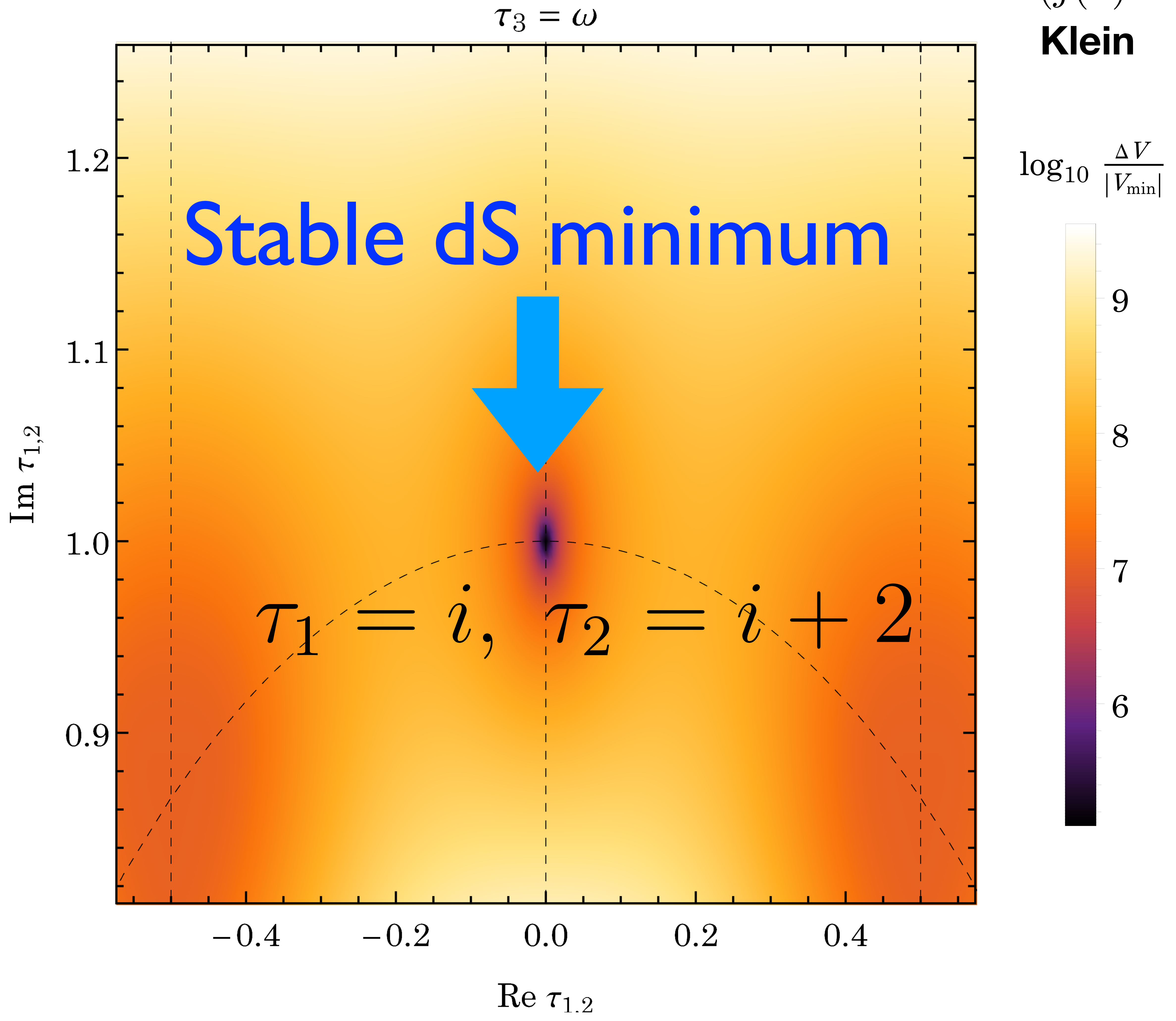
which define 3 fixed moduli

S.F.K. and X.Wang,2310.10369

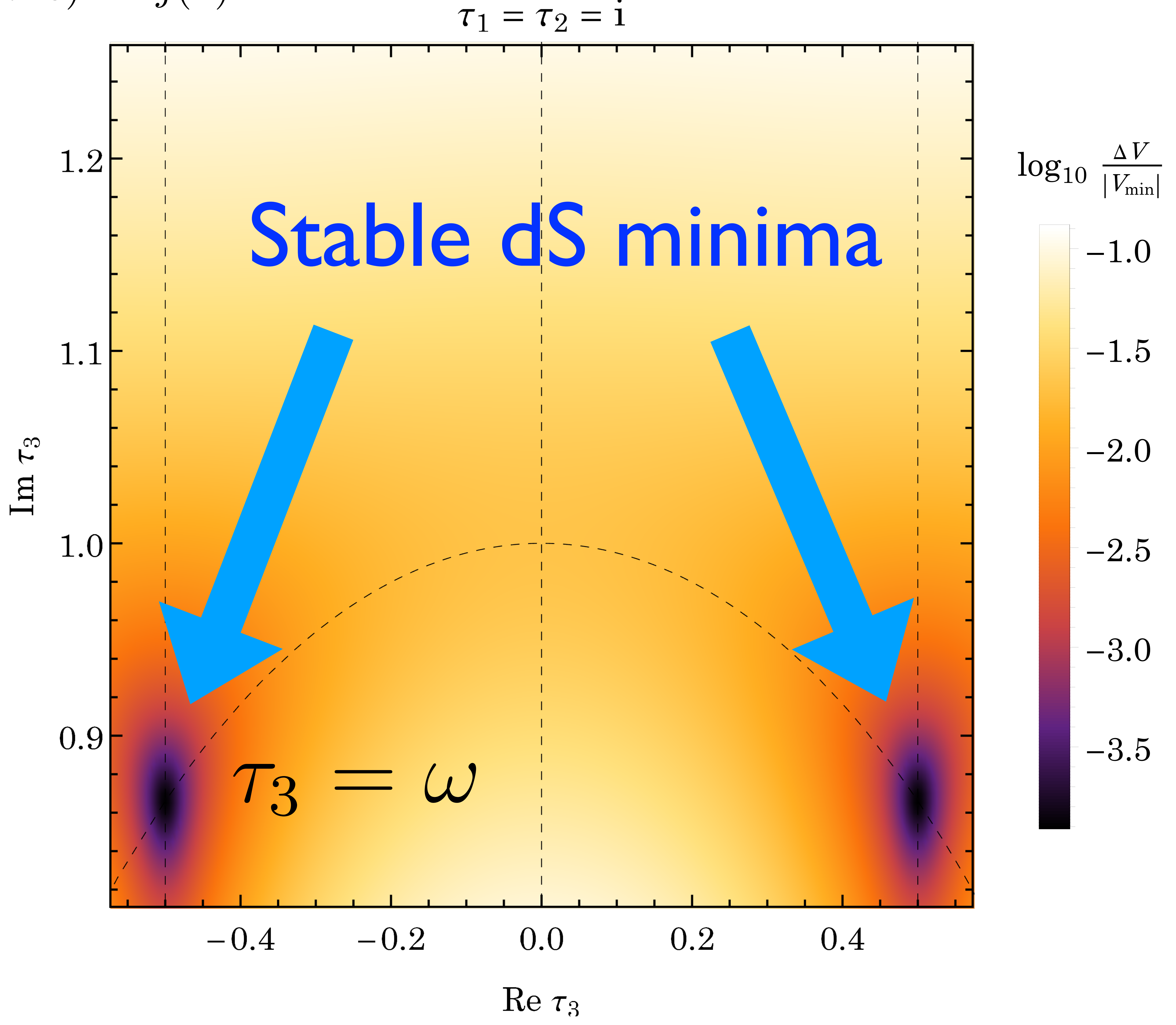
Modulus Stabilisation in the multiple modulus framework

$$\mathcal{K}(\tau_i,\overline{\tau}_i,S,\overline{S}) = \Lambda_K^2\{K(S,\overline{S}) - \log[(2\,\mathrm{Im}\,\tau_1)(2\,\mathrm{Im}\,\tau_2)(2\,\mathrm{Im}\,\tau_3)]\} \qquad \mathcal{W}(\tau_i,S) = \frac{\Lambda_W^3\Omega(S)[H^{(m_1,n_1)}(\tau_1)+H^{(m_2,n_2)}(\tau_2)+H^{(m_3,n_3)}(\tau_3)]}{\eta^2(\tau_1)\eta^2(\tau_2)\eta^2(\tau_3)}$$

$$H^{(m_i,n_i)} = (j(\tau)-1728)^{m_i/2}j(\tau)^{n_i/3}$$



Klein



De Anda, S.F.K. 2312.09010, 2304.05958

SU(5) orbifold GUT in 10d

$(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$

Field	$SU(5)$	S_4^A	S_4^B	S_4^C	k_A	k_B	k_C	Loc
F	$\mathbf{5}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	0	0	$\mathbb{T}^2_C$
T_1	$\mathbf{10}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	1	$\mathbb{T}^2_C$
T_2	$\mathbf{10}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	1/2	$\mathbb{T}^2_C$
T_3	$\mathbf{10}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	$\mathbb{T}^2_C$
N_a^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	-4	0	$\mathbb{T}^2_B$
N_s^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	-2	0	0	$\mathbb{T}^2_A$
H_u	$\mathbf{5}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	Bulk
H_d	$\overline{\mathbf{5}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	1/2	Bulk
H_{45}	$\mathbf{45}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	1/2	Bulk
$H_{\overline{45}}$	$\overline{\mathbf{45}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	Bulk
Φ_{BC}	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	0	0	0	Bulk
Φ_{AC}	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{3}$	0	0	0	Bulk
ξ_F	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$-5/2$	$\mathbb{T}^2_C$
ξ_T	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$-1/2$	$\mathbb{T}^2_C$

Weightons

10d Lagrangian

$$
\begin{aligned}
\mathcal{L}^{(0)}_{10d} = & \left(\frac{y^u_{33}}{\Lambda^5}T_3T_3 + \frac{y^u_{23}}{\Lambda^7}\xi_T T_2 T_3\right) H_u \delta^6(z) \\
& + \left(\frac{y^u_{22}}{\Lambda^9}\xi_T^2 T_2T_2 + \frac{y^u_{13}}{\Lambda^9}\xi_T^2 T_1 T_3 + \frac{y^u_{12}}{\Lambda^{11}}(\xi_T^3 + \xi_F^3) T_1 T_2\right) H_u \delta^6(z) \\
& + \left(\frac{Y_a}{\Lambda^9} F N_a^c \Phi_{BC} + \frac{Y_s}{\Lambda^9} F N_s^c \Phi_{AC}\right) H_u \delta^6(z) \\
& + \left(\frac{Y_{5\tau}}{\Lambda^7}\xi_F F T_3 + \frac{Y'_{5\tau}}{\Lambda^9}\xi_F \xi_T F T_2 + \frac{Y''_{5\tau}}{\Lambda^{11}}\xi_F \xi_T^2 F T_1\right) H_{5d}\delta^6(z) \\
& + \left(\frac{Y_{5\mu}}{\Lambda^9}\xi_F^2 F T_2 + \frac{Y'_{5\mu}}{\Lambda^{11}}\xi_F^2 \xi_T F T_1 + \frac{Y_{5e}}{\Lambda^{11}}\xi_F^3 F T_1\right) H_{5d}\delta^6(z) \\
& + (H^{(0)}_{5d} \to H^{(0)}_{45d} \text{ terms}) \\
& + \frac{M_a}{2} N_a^c N_a^c \delta^2(z_1)\delta^2(z_3) + \frac{M_s}{2} N_s^c N_s^c \delta^2(z_2)\delta^2(z_3),
\end{aligned}
$$

De Anda, S.F.K. 2312.09010, 2304.05958

4d Lagrangian

$$\mathcal{L}_{4d}^{(0)} = \left(y_{33}^u T_3^{(0)} T_3^{(0)} + y_{23}^u \tilde{\xi}_T T_2^{(0)} T_3^{(0)} + y_{22}^u \tilde{\xi}_T^2 T_2^{(0)} T_2^{(0)} + y_{13}^u \tilde{\xi}_T^2 T_1^{(0)} T_3^{(0)} + y_{12}^u \tilde{\xi}_{T,F}^3 T_1^{(0)} T_2^{(0)}\right) H_u^{(0)}$$
$$+ \left(Y_a \tilde{\Phi}_{BC} F^{(0)} N_a^{c(0)} + Y_s \tilde{\Phi}_{AC} F^{(0)} N_s^{c(0)}\right) H_u^{(0)}$$
$$+ \left(Y_{5\tau} \tilde{\xi}_F F^{(0)} T_3^{(0)} + Y'_{5\tau} \tilde{\xi}_F \tilde{\xi}_T F^{(0)} T_2^{(0)} + Y''_{5\tau} \tilde{\xi}_F \tilde{\xi}_T^2 F^{(0)} T_1^{(0)}\right.$$
$$\left. + Y_{5\mu} \tilde{\xi}_F^2 F^{(0)} T_2^{(0)} + Y'_{5\mu} \tilde{\xi}_F^2 \tilde{\xi}_T F^{(0)} T_1^{(0)} + Y_{5e} \tilde{\xi}_F^3 F^{(0)} T_1^{(0)}\right) H_{5d}^{(0)} \quad + (H_{5d}^{(0)} \to H_{45d}^{(0)} \text{ terms})$$
$$+ \frac{1}{2} M_a N_a^{c(0)} N_a^{c(0)} + \frac{1}{2} M_s N_s^{c(0)} N_s^{c(0)}.$$

Yukawa couplings are modular forms at 3 fixed points

$$Y_a = Y_{\mathbf{3}}^{(4)}(i+2) = y_a(0,1,-1)^T,$$
$$Y_s = Y_{\mathbf{3}}^{(2)}(i) = y_s(1, 1+\sqrt{6}, 1-\sqrt{6})^T,$$
$$Y_\tau = Y_{\mathbf{3}}^{(2)}(\omega) = y_\tau(0,1,0)^T,$$
$$Y_\mu = Y_{\mathbf{3}}^{(4)}(\omega) = y_\mu(0,0,1)^T,$$
$$Y_e = Y_{\mathbf{3}I}^{(6)}(\omega) = y_e(1,0,0)^T.$$

$$\omega = e^{\frac{2\pi i}{3}}$$

G.J.Ding, S.F.K., X.G.Liu and J.N.Lu, 1910.03460

	τ	$Y_{\mathbf{3}}^{(2)}(\tau)$, $Y_{\mathbf{3,I}}^{(6)}(\tau)$	$Y_{\mathbf{3}}^{(4)}(\tau)$, $Y_{\mathbf{3'}}^{(6)}(\tau)$
τ_1	i	$(1, 1+\sqrt{6}, 1-\sqrt{6})$	$(1, -\frac{1}{2}, -\frac{1}{2})$
	$i+1$	$(1, -\frac{\omega}{3}(1+i\sqrt{2}), -\frac{\omega^2}{3}(1+i\sqrt{2}))$	$(0, 1, -\omega)$
τ_2	$i+2$	$(1, \frac{1}{3}(-1+i\sqrt{2}), \frac{1}{3}(-1+i\sqrt{2}))$	$(0, 1, -1)$
	$i+3$	$(1, \omega(1+\sqrt{6}), \omega(1-\sqrt{6}))$	$(1, -\frac{\omega}{2}, -\frac{\omega^2}{2})$

	τ	$Y_{\mathbf{3}}^{(2)}(\tau)$	$Y_{\mathbf{3}}^{(4)}(\tau)$, $Y_{\mathbf{3'}}^{(4)}(\tau)$	$Y_{\mathbf{3,II}}^{(6)}(\tau)$, $Y_{\mathbf{3'}}^{(6)}(\tau)$
τ_3	ω	$(0,1,0)$	$(0,0,1)$	$(1,0,0)$
	$\omega+1$	$(1, 1, -\frac{1}{2})$	$(1, -\frac{1}{2}, 1)$	$(1, -2, -2)$
	$\omega+2$	$(1, -\frac{\omega^2}{2}, \omega)$	$(1, \omega^2, -\frac{\omega}{2})$	$(1, -2\omega^2, -2\omega)$
	$\omega+3$	$(1, \omega, -\frac{\omega^2}{2})$	$(1, -\frac{\omega}{2}, \omega^2)$	$(1, -2\omega, -2\omega^2)$

I.de Medeiros Varzielas, S.F.K., M.Levy, 2211.00654, 2309.15901

De Anda, S.F.K. 2312.09010, 2304.05958

$$M_u = \begin{pmatrix} 0 & y^u_{12}\tilde{\xi}^3_{T,F}e^{i\phi_{u1}} & y^u_{13}\tilde{\xi}^2_T \\ y^u_{12}\tilde{\xi}^3_{T,F}e^{i\phi_{u1}} & y^u_{22}\tilde{\xi}^2_T & y^u_{23}\tilde{\xi}_T e^{i\phi_{u2}} \\ y^u_{13}\tilde{\xi}^2_T & y^u_{23}\tilde{\xi}_T e^{i\phi_{u2}} & y^u_{33} \end{pmatrix} v_u$$

$$m_u \sim \tilde{\xi}^4_{T,F} v_u,\ m_c \sim \tilde{\xi}^2_T v_u,\ m_t \sim v_u$$

$$M_d = \begin{pmatrix} y_{d11}\tilde{\xi}^3_F & y_{d12}\tilde{\xi}^2_F\tilde{\xi}_T & y_{d13}\tilde{\xi}_F\tilde{\xi}^2_T \\ 0 & y_{d22}\tilde{\xi}^2_F & y_{d23}\tilde{\xi}_F\tilde{\xi}_T e^{i\phi_{d2}} \\ 0 & 0 & y_{d33}\tilde{\xi}_F \end{pmatrix} v_d,$$

LR convention

CKM mixing from M_d

$$M_e = \begin{pmatrix} y_{e11}\tilde{\xi}^3_F & 0 & 0 \\ y_{e21}\tilde{\xi}^2_F\tilde{\xi}_T & y_{e22}\tilde{\xi}^2_F & 0 \\ y_{e31}\tilde{\xi}_F\tilde{\xi}^2_T & y_{e32}\tilde{\xi}_F\tilde{\xi}_T e^{i\phi_{d1}} & y_{e33}\tilde{\xi}_F \end{pmatrix} v_d,$$

No PMNS mixing from M_e

$$m_d \sim m_e \sim \tilde{\xi}^3_F v_d$$

$$m_s \sim m_\mu \sim \tilde{\xi}^2_F v_d$$

$$m_b \sim m_\tau \sim \tilde{\xi}_F v_d$$

Dirac neutrino matrix

$$M_D = \begin{pmatrix} 0 & y_s\tilde{\Phi}_{AC} \\ y_a\tilde{\Phi}_{BC} & y_s\tilde{\Phi}_{AC}(1-\sqrt{6}) \\ -y_a\tilde{\Phi}_{BC} & y_s\tilde{\Phi}_{AC}(1+\sqrt{6}) \end{pmatrix} v_u, \quad M_N = \begin{pmatrix} M_a & 0 \\ 0 & M_s \end{pmatrix}$$

Heavy Majorana neutrino matrix → Seesaw mechanism

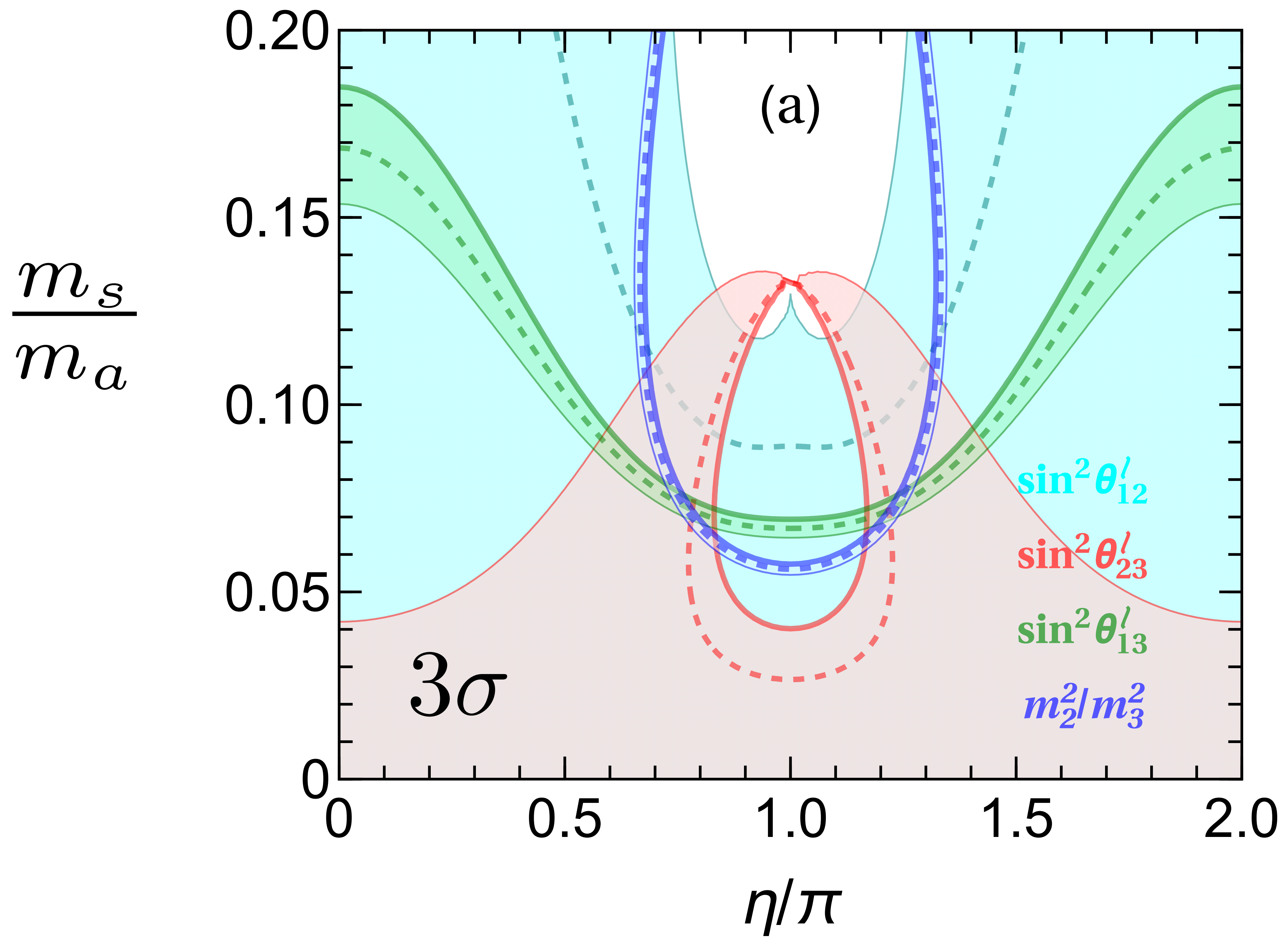
Seesaw mechanism gives 3 parameter neutrino mass matrix

$$m_\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + m_s e^{i\eta} \begin{pmatrix} 1 & 1-\sqrt{6} & 1+\sqrt{6} \\ 1-\sqrt{6} & 7-2\sqrt{6} & -5 \\ 1+\sqrt{6} & -5 & 7+2\sqrt{6} \end{pmatrix}$$

"Littlest Seesaw"

G.J.Ding, S.F.K, X.G.Liu and J.N.Lu,1910.03460

G.J.Ding, S.F.K. and C.Y.Yao, 2103.16311



De Anda, SFK 2304.05958

	without SK atmospheric data	
	NuFit $\pm 1\sigma$	Model
$\theta_{12}/°$	$33.41^{+0.75}_{-0.72}$	34.34
$\theta_{23}/°$	$49.1^{+1.0}_{-1.3}$	48.31
$\theta_{13}/°$	$8.54^{+0.11}_{-0.12}$	8.54
$\delta/°$	197^{+42}_{-25}	284
$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.41^{+0.21}_{-0.20}$	7.42
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.511^{+0.028}_{-0.021}$	2.510
$\frac{m_a}{10^{-3}\ \mathrm{eV}}$		31.47
$\frac{m_b}{10^{-3}\ \mathrm{eV}}$	$\frac{m_{\beta\beta}}{10^{-3}\ eV}$	2.28
η/π		1.24
χ^2	Three parameter fit	6.3

6 measured observables

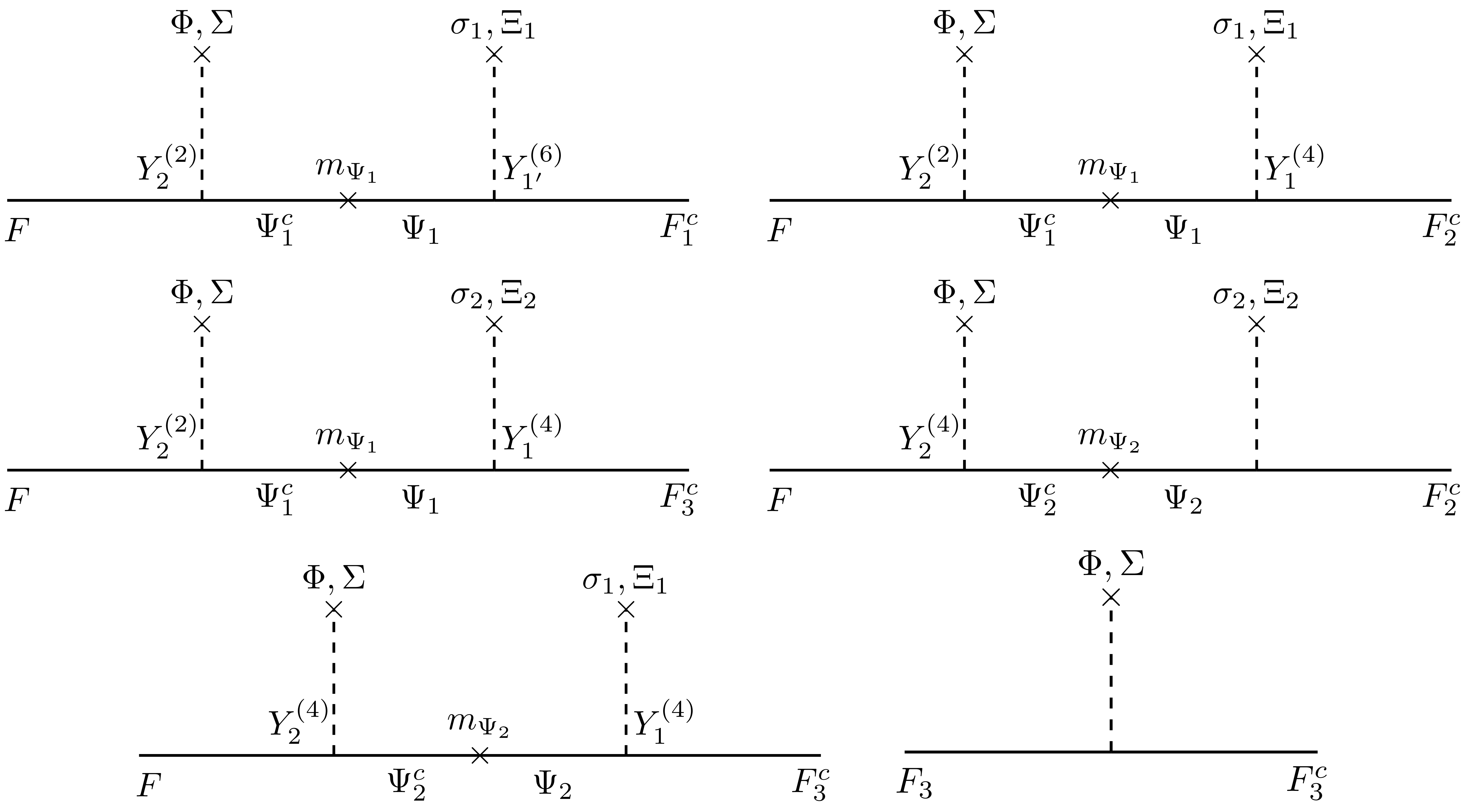
3 input parameters

A.E.Carcamo Hernandez, I.de Medeiros Varzielas, S.F.K., K.N.Vishnudath, 2503.14610

Modular seesaw mechanism for light families

No Pati-Salam N=2

Weightons	$SU(4)_C$	$SU(2)_L$	$SU(2)_R$	S_3	k
$F=(F_1,F_2)$	$\mathbf{4}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	0
F_3	$\mathbf{4}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1'}$	-1
F_1^c	$\overline{\mathbf{4}}$	$\mathbf{1}$	$\overline{\mathbf{2}}$	$\mathbf{1}$	1
F_2^c	$\overline{\mathbf{4}}$	$\mathbf{1}$	$\overline{\mathbf{2}}$	$\mathbf{1'}$	-1
F_3^c	$\overline{\mathbf{4}}$	$\mathbf{1}$	$\overline{\mathbf{2}}$	$\mathbf{1'}$	1
Ψ_1	$\mathbf{4}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1'}$	-2
Ψ_2	$\mathbf{4}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1'}$	-4
Ψ_1^c	$\overline{\mathbf{4}}$	$\mathbf{1}$	$\overline{\mathbf{2}}$	$\mathbf{1'}$	2
Ψ_2^c	$\overline{\mathbf{4}}$	$\mathbf{1}$	$\overline{\mathbf{2}}$	$\mathbf{1'}$	4
$S^c=(S_1^c,S_2^c)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	1
S_3^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1'}$	1
Φ	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$	0
χ_R	$\overline{\mathbf{4}}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	2
Σ	$\mathbf{15}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$	0
σ_1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	7
σ_2	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	5
Ξ_1	$\mathbf{15}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	7
Ξ_2	$\mathbf{15}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	5



Seesaw like diagrams for light families

Third family renormalisable

Mass matrices have a seesaw-like structure

$$\widetilde{M}_u = M_c^{(u)} - M_a^{(u)} M_U^{-1} M_b^{(u)}\,,$$
$$\widetilde{M}_e = M_c^{(e)} - M_a^{(e)} M_E^{-1} M_b^{(e)}\,,$$

Renormalisable
Third family

Seesaw-like
Light families

$$\widetilde{M}_d = M_c^{(d)} - M_a^{(d)} M_D^{-1} M_b^{(d)}\,;$$
$$\widetilde{M}_\nu = M_c^{(\nu)} - M_a^{(\nu)} M_N^{-1} M_b^{(\nu)}\,.$$

$$M_U = M_D = M_E = M_N = \begin{pmatrix} m_{\Psi_1} & 0 \\ 0 & m_{\Psi_2} \end{pmatrix}$$

Vector-like masses

Neutrino masses arise from a double seesaw

$$\frac{1}{2}\begin{pmatrix} \nu & \nu^c & S^c \end{pmatrix}\begin{pmatrix} 0_{3\times3} & \widetilde{M}_\nu & 0_{3\times3} \\ \widetilde{M}_\nu^T & 0_{3\times3} & M_R \\ 0_{3\times3} & M_R^T & M_S \end{pmatrix}\begin{pmatrix} \nu \\ \nu^c \\ S^c \end{pmatrix} \Rightarrow M_\nu = \widetilde{M}_\nu M_R^{-1} M_S M_R^{-1} \widetilde{M}_\nu^T$$

Best Fit

A.E.Carcamo Hernandez, I.de Medeiros Varzielas, S.F.K., K.N.Vishnudath, 2503.14610

13 real scales and VEVs

20 complex Yukawa couplings

Input Parameters	Best fit value for NH of light neutrinos
τ	$-0.03224 \; + \; i\; 1.86682$
m_{Ψ_1}, m_{Ψ_2} (GeV)	2.38×10^{12}, 8.71×10^{12}
v_1, v_2 (GeV)	245.62150, 3.95665
$v_{\Sigma_1}, v_{\Sigma_2}$ (GeV)	8.37732, 10.01214
$v_{\sigma_1}, v_{\sigma_2}$ (GeV)	6.12×10^{12}, 4.64×10^{12}
v_{Ξ_1}, v_{Ξ_2} (GeV)	7.90×10^{12}, 3.53×10^{12}
v_R (GeV)	1.23×10^{11}
M_1, M_2 (GeV)	1.0×10^{12}, 6.20×10^{8}
x_1, x_2, x_3	$3.53382\; e^{i\; 2.23402}$, $2.93598\; e^{i\; 3.85910}$, $0.20033\; e^{i\; 2.15046}$
y_1, y_2, y_3	$0.27120\; e^{i\; 5.93154}$, $0.46397\; e^{i\; 5.47007}$, $0.98750\; e^{i\; 2.24996}$
w_1, w_2, w_3, w_4, w_5	$0.75168\; e^{i\; 0.96442}$, $0.20136\; e^{i\; 2.04578}$, $1.37663\; e^{i\; 5.91647}$, $0.23368\; e^{i\; 0.86247}$, $2.62639\; e^{i\; 5.87924}$
z_1, z_2, z_3, z_4, z_5	$2.37549\; e^{i\; 0.67600}$, $0.25164\; e^{i\; 1.45687}$, $2.46850\; e^{i\; 4.02943}$, $0.32353\; e^{i\; 0.15040}$, $3.54392\; e^{i\; 6.12983}$
$\gamma_1, \gamma_2, \gamma_3, \gamma_4$	$0.20302\; e^{i\; 3.09154}$, $1.99746\; e^{i\; 6.05522}$, $3.42165\; e^{i\; 0.22612}$, $1.96501\; e^{i\; 5.93289}$

Hierarchy of scales and VEVs

Yukawa couplings order unity

$$
\begin{aligned}
-\mathcal{W} \;=\; & y_1 Y_{\mathbf{2}}^{(2)}(\tau)\, F\Phi\Psi_1^c + y_2 Y_{\mathbf{2}}^{(4)}(\tau)\, F\Phi\Psi_2^c + y_3 F_3\Phi F_3^c \\
& + z_1 Y_{\mathbf{1'}}^{(6)}\Psi_1\sigma_1 F_1^c + z_2 Y_{\mathbf{1}}^{(4)}(\tau)\,\Psi_1\sigma_1 F_2^c + z_3 Y_{\mathbf{1}}^{(4)}(\tau)\,\Psi_1\sigma_2 F_3^c + z_4\Psi_2\sigma_2 F_2^c + z_5 Y_{\mathbf{1}}^{(4)}(\tau)\,\Psi_2\sigma_1 F_3^c \\
& + w_1 Y_{\mathbf{1'}}^{(6)}\Psi_1\Xi_1 F_1^c + w_2 Y_{\mathbf{1}}^{(4)}(\tau)\,\Psi_1\Xi_1 F_2^c + w_3 Y_{\mathbf{1}}^{(4)}(\tau)\,\Psi_1\Xi_2 F_3^c + w_4\Psi_2\Xi_2 F_2^c + w_5 Y_{\mathbf{1}}^{(4)}(\tau)\,\Psi_2\Xi_1 F_3^c \\
& + x_1 Y_{\mathbf{2}}^{(2)}(\tau)\, F\Sigma\Psi_1^c + x_2 Y_{\mathbf{2}}^{(4)}(\tau)\, F\Sigma\Psi_2^c + x_3 F_3\Sigma F_3^c + m_{\Psi_1}\Psi_1\Psi_1^c + m_{\Psi_2}\Psi_2\Psi_2^c \\
& + \gamma_1 Y_{\mathbf{2}}^{(4)}(\tau)\, F_1^c\chi_R S^c + \gamma_2 Y_{\mathbf{2}}^{(2)}(\tau)\, F_2^c\chi_R S^c + \gamma_3 Y_{\mathbf{2}}^{(4)}(\tau)\, F_3^c\chi_R S^c \\
& + \gamma_4 Y_{\mathbf{1}}^{(4)}(\tau)\, F_3^c\chi_R S_3^c + M_1 Y_{\mathbf{2}}^{(2)}(\tau)\,(S^c S^c)_{\mathbf{2}} + M_2 Y_{\mathbf{2}}^{(2)}(\tau)\,(S^c S_3^c) + \text{h.c.}.
\end{aligned}
$$

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Best Fit

	Low energy mass matrices, masses and mixing parameters
$\widetilde{M}_u$ (GeV)	$\begin{pmatrix} -0.00018e^{i(-2.57621)} & -0.01312e^{i2.50106} & 0.00174e^{i1.43459} \\ 0.00428e^{i0.66665} & -0.19406e^{i2.37489} & -0.06547e^{i(-2.04282)} \\ 0.00000e^{i0.00000} & 0.00000e^{i0.00000} & -108.38236e^{i2.24928} \end{pmatrix}$
$\widetilde{M}_d$ (GeV)	$\begin{pmatrix} 0.00016e^{i(-0.36728)} & 0.00339e^{i0.98476} & -0.00459e^{i(-2.87477)} \\ -0.00409e^{i2.87558} & -0.00659e^{i1.86300} & 0.15841e^{i0.40887} \\ 0.00000e^{i0.00000} & 0.00000e^{i0.00000} & -2.51228e^{i2.21622} \end{pmatrix}$
m_u, m_c, m_t (GeV)	0.00054, 0.2670, 172.69001
m_d, m_s, m_b (GeV)	0.00120, 0.0240, 4.180
$s^q_{12}, s^q_{23}, s^q_{13}, \delta^q_{CP}$	0.2250, 0.04182, 0.00370, 65.6750°
$\widetilde{M}_e$ (GeV)	$\begin{pmatrix} 0.00011e^{i0.67714} & -0.00335e^{i1.79184} & 0.02874e^{i(-0.74991)} \\ -0.00261e^{i(-2.36319)} & -0.10577e^{i(-2.70883)} & -0.64387e^{i2.36816} \\ 0.00000e^{i0.00000} & 0.00000e^{i0.00000} & 0.59512e^{i(-1.17142)} \end{pmatrix}$
M_ν (eV)	$\begin{pmatrix} 0.00013e^{i0.26925} & 0.00065e^{i(-1.25936)} & 0.00314e^{i0.96134} \\ 0.00065e^{i(-1.25936)} & -0.02407e^{i2.14357} & 0.01431e^{i(-0.64736)} \\ 0.00314e^{i0.96134} & 0.01431e^{i(-0.64736)} & 0.00000e^{i0.00000} \end{pmatrix}$
m_e, m_μ, m_τ (GeV)	0.00048, 0.10155, 1.77686
m_{ν_1}(eV), Δm^2_{sol}, Δm^2_{atm}(eV^2)	0.00276, 7.49×10^{-5}, 0.00251
$s^\nu_{12}, s^\nu_{23}, s^\nu_{13}$	0.55497, 0.68557, 0.14883
$\delta^\nu_{CP}, \alpha^M, \beta^M$	323.53560°, 194.46228°, 297.46063°

Modular symmetry suppresses first family masses

$$(\widetilde{M}_u)_{11} \approx -\frac{9e^{-6\pi\tau_I+2i\pi\tau_R}\left(e^{4\pi\tau_I}-16e^{2\pi(\tau_I+i\tau_R)}+576e^{4i\pi\tau_R}\right)(v_{\Sigma 1}x_1+v_1y_1)(v_{\Xi 1}w_1+v_{\Sigma 1}z_1)}{128M_1}$$

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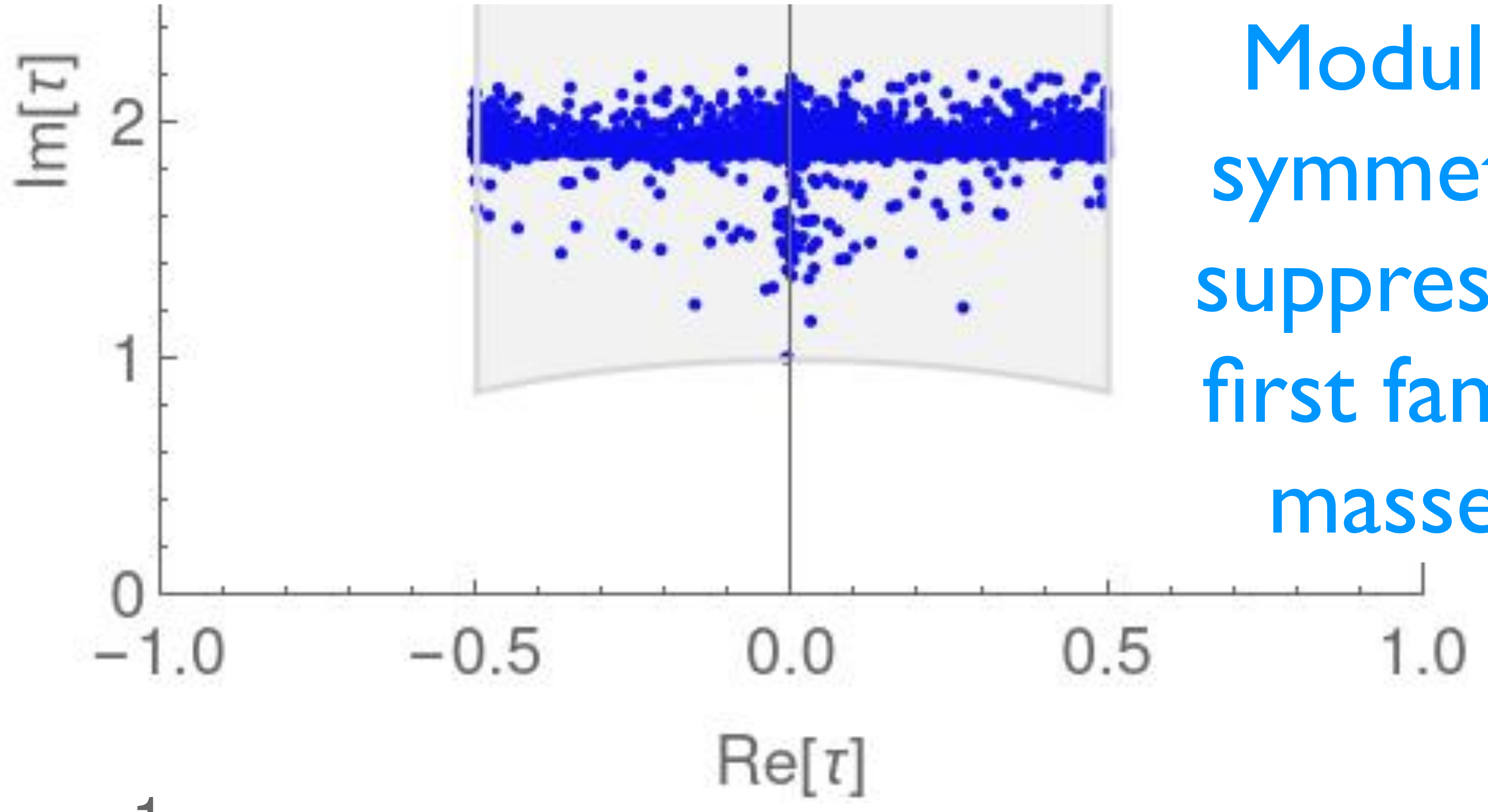
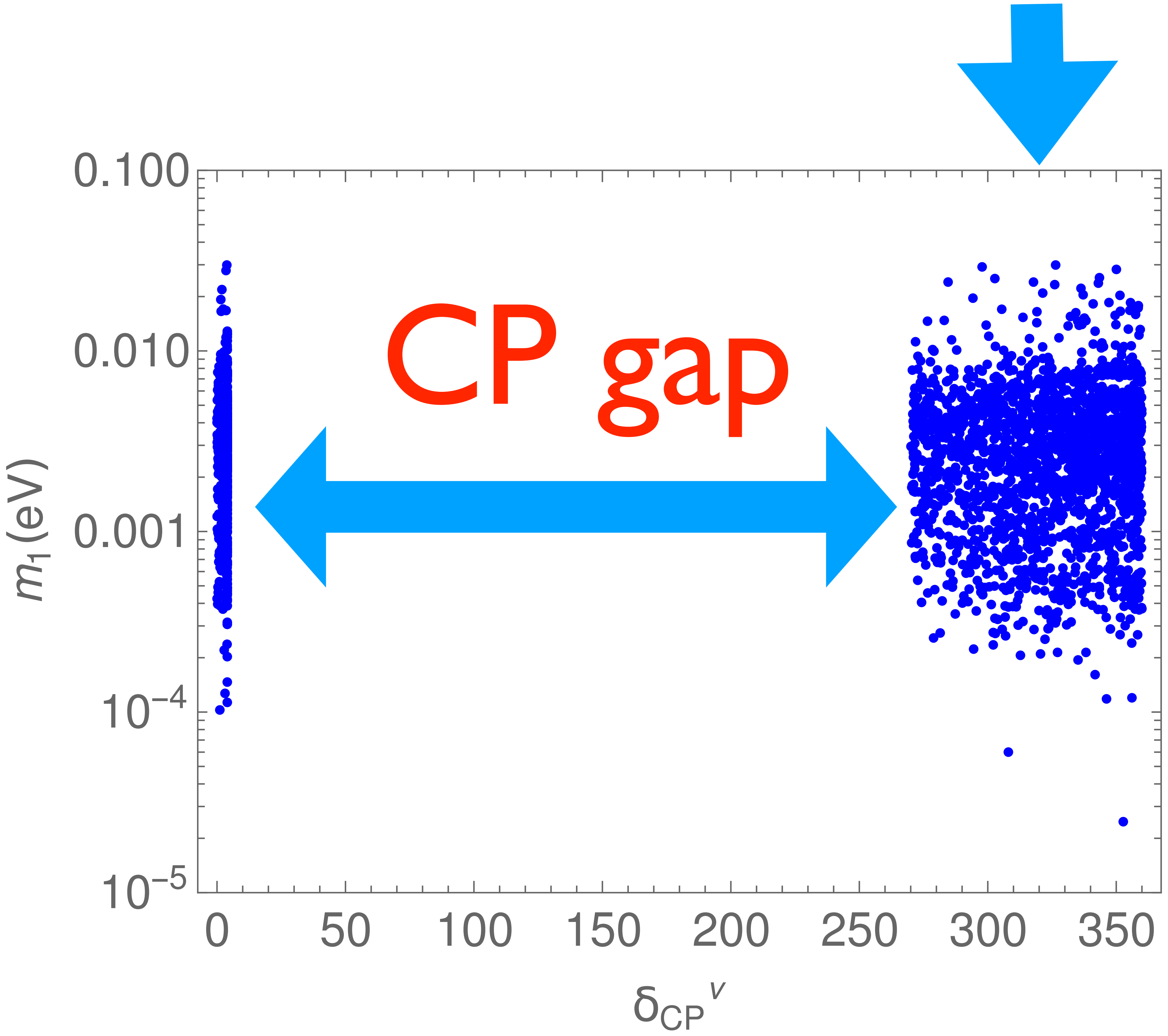
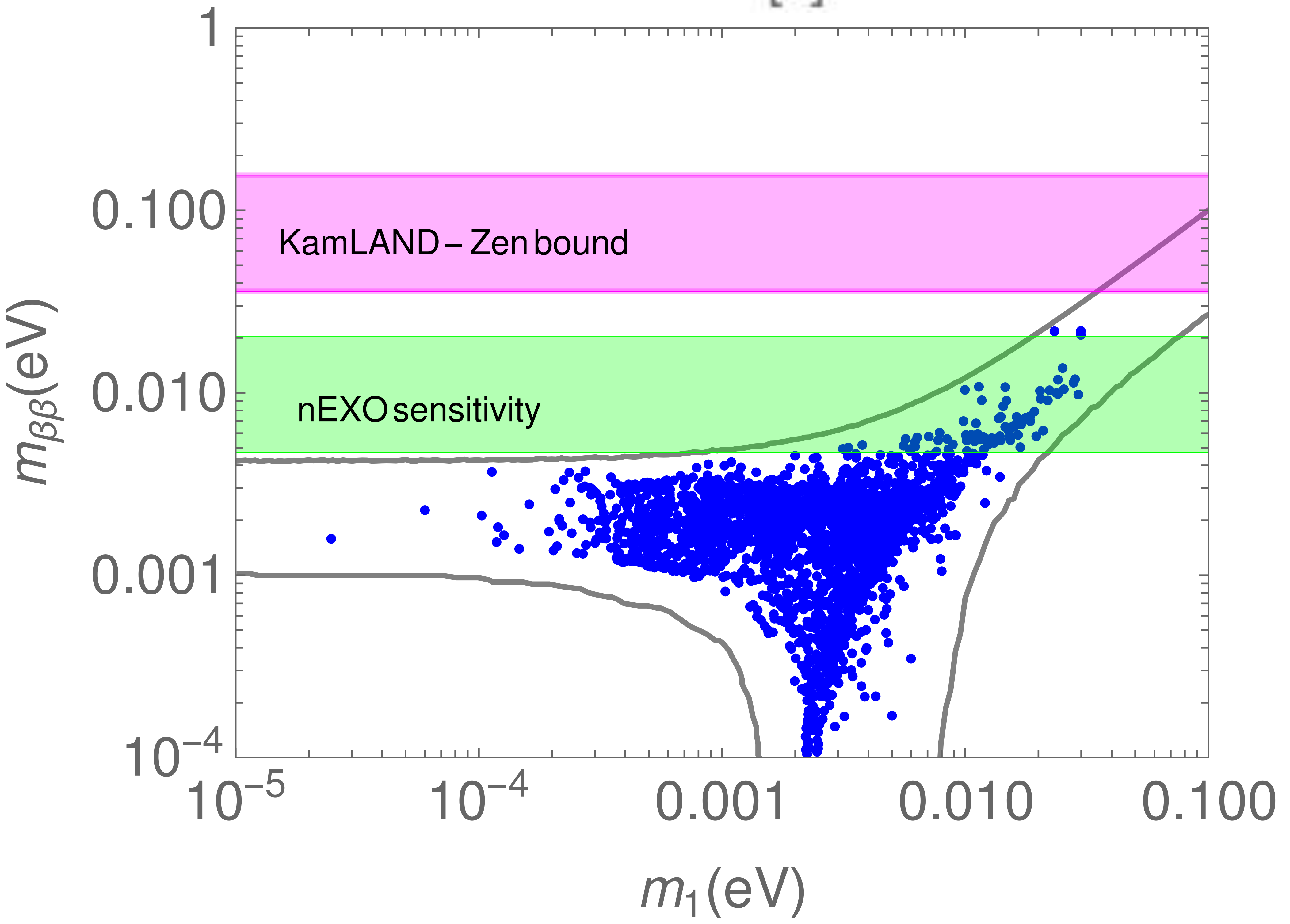
Numerical scans show a restricted range for leptonic CP phase



Modular symmetry suppresses first family masses





[Chen, Ramon-Sanchez, Ratz 1909.06910]

Unconstrained Kahler potential

$$\mathcal{K} = \underbrace{(-i\tau + i\bar{\tau})^{-k_\psi}\left(\psi^\dagger\psi\right)_{\mathbf{1}}}_{\text{minimal Kähler potential}} + \underbrace{\sum_{n,\boldsymbol{r}_1,\boldsymbol{r}_2} c^{(n,\boldsymbol{r}_1,\boldsymbol{r}_2)}(-i\tau + i\bar{\tau})^{-k_\psi+n}\left(\psi^\dagger Y_{\boldsymbol{r}_1}^{(n)\dagger} Y_{\boldsymbol{r}_2}^{(n)}\psi\right)_{\mathbf{1}}}_{\text{non-canonical terms}}$$

minimal Kähler potential

non-canonical terms
unsuppressed since τ is dimensionless

Canonical normalisation can lead to sizeable corrections

One solution is to introduce a flavour symmetry as well as modular symmetry, leading to so called "Eclectic Flavour Symmetry"

It may help to be Goofy

[Nilles et al, 2001.01736; 2004.05200]

A.Trautner, 2505.00099

Conclusions

- Flavour problem of Standard Model remains
- Symmetry may guide us - GUTs and Flavour Symmetry
- Modular Family Symmetry motivated by String theory
- Modulus field τ is the only "flavon" (minimal)
- Yukawa matrices in terms of the modulus field
- Introduce "weighton" to explain fermion mass hierarchies
- 10d model gives 3 moduli fields stabilised at fixed points
- SU(5) Orbifold GUT in 10d leads to predictive model
- First two families may be light due to modular **seesaw**
- Challenge to models from unconstrained Kahler potential