Karlsruhe Institute of Technology

Particle Physics Phenomenology after the Higgs Discovery

Institute for Theoretical Physics

Interpolating Amplitudes

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Les Houches

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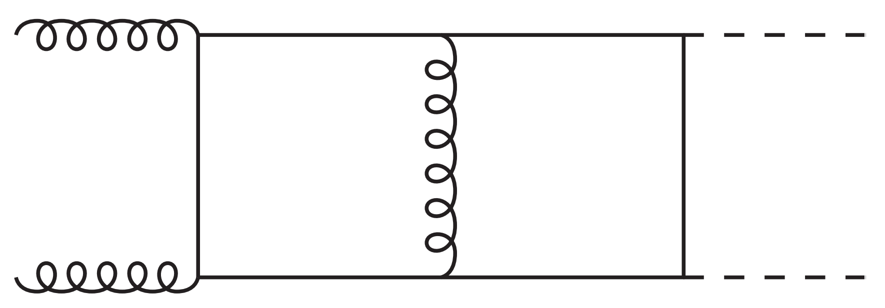
Why (I am interested in) amplitude interpolation

NLO QCD corrections to HH and HJ production

Borowka, Greiner, Heinrich, Jones, MK, Schlenk, Schubert, Zirke 1604.06447
Jones, MK, Luisoni 1802.00349



based on numerical integration of 2-loop integrals

- slow, runtime per phase-space point: - median 2h on GPU
  - up to 2d, may not reach desired precision
- for fixed-order results: optimized phase-space sampling based on unweighed LO event
  $\rightarrow$ can generate FO results using only 665 phase-space points
  $\rightarrow$ can not be directly interfaced to parton-shower MC,

$\rightarrow$ amplitude interpolation

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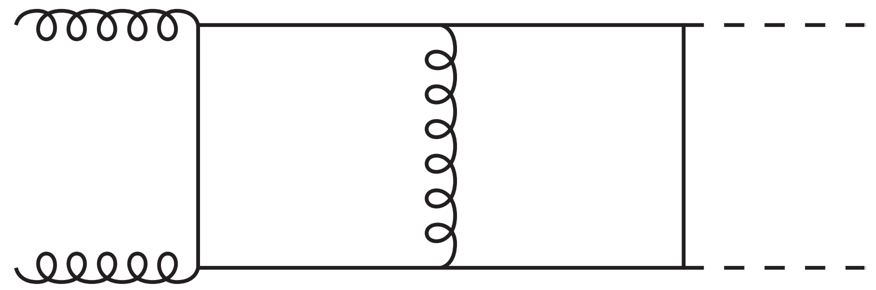

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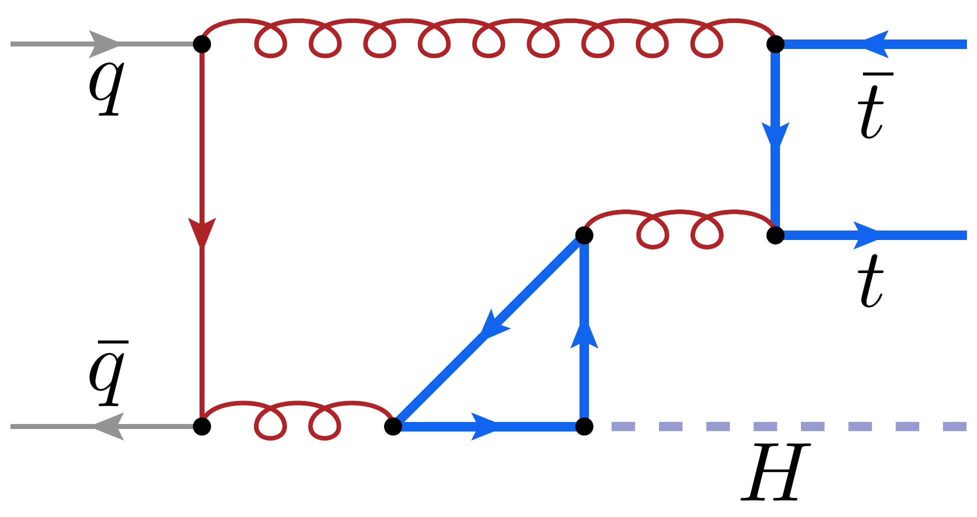
Current project: 2-loop corrections to $t\bar{t}H$ production

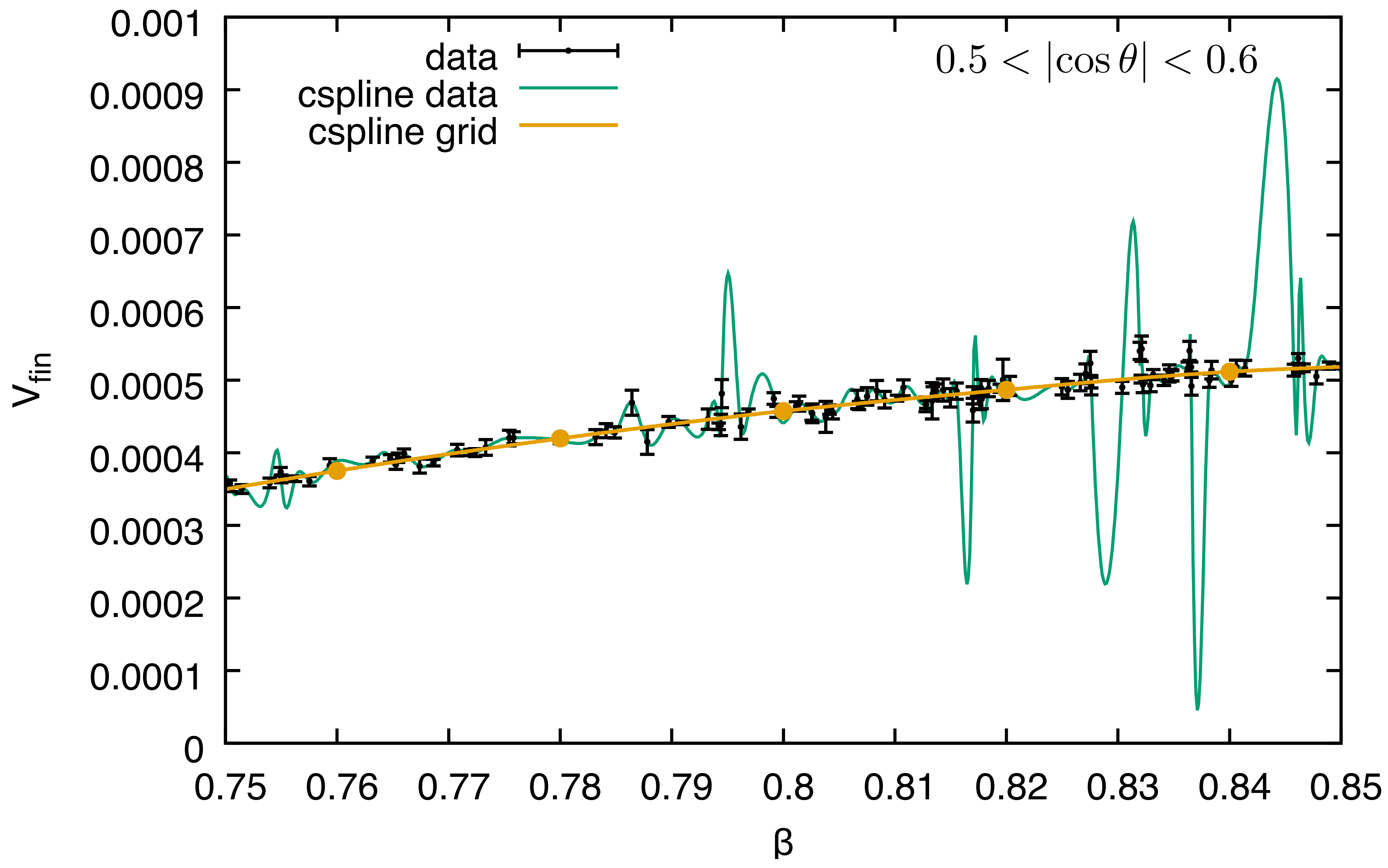
- first results: Agarwal, Heinrich, Jones, MK, Klein, Lang, Magerya, Olsson 2402.03301
  - N_f - contributions to $q\bar{q} \rightarrow t\bar{t}H$,
  - only 1d and 2d splices in phase space
- full 2-loop corrections & pheno applications
  $\rightarrow$ need integration over 5-dimensional phase-space
  $\rightarrow$ construct interpolation framework; try to minimize number of amplitude results required

Bresó, Heinrich, Margery, Olsson 2412.09534



HH — Grid interpolation

Heinrich, Jones, MK, Luisoni, Vryonidou 1703.09252



2-dimensional grid interpolation $(\hat{s},\, \hat{t})$

Problems during construction of grid:
- interpolation can enhance numerical uncertainties
- input data not evaluated on equidistant grid points

Details of grid interpolation:

- interpolation done in 2 steps:
  1. choose equidistant grid points, estimate result at each grid point with least-square fit to linear function of amplitude results in vicinity
  2. Clough-Tocher interpolation (as implemented in SciPy) to estimate amplitude at arbitrary sampling points

  → reduces sensitivity to uncertainties of input-data points
- input parameters $x = f(\beta(\hat{s})), \quad c_\theta = |\cos\theta| = \left|\frac{\hat{s} + 2\hat{t} - 2m_H^2}{\hat{s}\beta(\hat{s})}\right|,$ with $\beta = \left(1 - \frac{4m_H^2}{\hat{s}}\right)^{\frac{1}{2}}$

  → nearly uniform distribution of phase space points in $(x, c_\theta) \in [0,1]^2$ if $f(\beta)$ chosen according to cumulative distribution of points in original calculation

HJ [Campillo Aveleira, Heinrich, MK, Kunz 2409.05728]
amplitude regression using neural network (MLP),
avoid divergences in IRC limits by multiplication of amplitude with $\beta^2(1-\beta)(1-\cos^2\theta)$ where $\beta = \frac{s - m_H^2}{s + m_H^2}$

Interpolating Amplitudes [Bresó, Heinrich, Margery, Olsson 2412.09534]

Methods:

- Polynomial interpolation
- B-splines
- Sparse grids
- Machine Learning (MLP, L-GATr)

Test functions:

- f_1 : $q\bar{q} \to t\bar{t}H$ @ $0L$
- f_2 : $q\bar{q} \to t\bar{t}H$ @ $1L$, with Coulomb singularity subtracted
- f_3 : $gg \to t\bar{t}H$ @ $0L$
- f_4 : $gg \to t\bar{t}H$ @ $1L$, with Coulomb singularity subtracted:
- f_5 : $gg \to Hg$ @ $1L$

$$a_4 = 2\mathrm{Re}\left[\langle \mathcal{M}_0^{ggt\bar{t}H} | \mathcal{M}_1^{ggt\bar{t}H} \rangle - \frac{\pi^2}{\beta_{t\bar{t}}} \langle \mathcal{M}_0^{ggt\bar{t}H} | \mathbf{T}_{t\bar{t}} | \mathcal{M}_0^{ggt\bar{t}H} \rangle\right]$$

Approximation error based on L^1-norm, target for $\varepsilon < 1\,\%$:

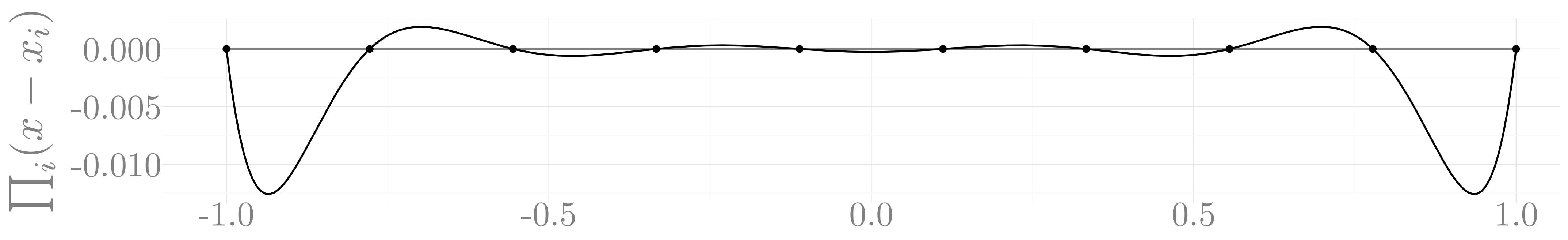
$$L^1[f] = \left(\int |f(\vec{x})|^p \, \mathrm{d}\vec{x} \right)^{1/p}$$

$$\varepsilon = \frac{||\widetilde{f} - f||_1}{||f||_1}, \quad \text{where} \quad f(\vec{x}) \equiv a(\vec{x}) \underbrace{\frac{1}{2\hat{s}} \left| \frac{\mathrm{d}(\Phi, \rho)}{\mathrm{d}\vec{x}} \right|}_{\equiv \text{weight } w(\vec{x})}$$

Interpolating Amplitudes [Bresó, Heinrich, Margery, Olsson 2412.09534]

Polynomial Interpolation (1D):

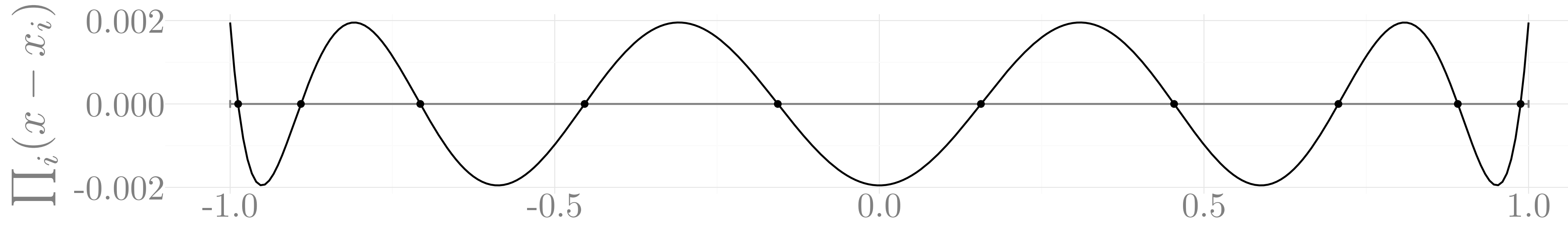
$$f(x) - \tilde{f}(x) = \frac{f^{(n)}(\xi)}{n!} \prod_{i=1}^{n} (x - x_i)$$



Runge Phenomenon can be avoided by choosing Chebyshev nodes

$$x_i = \cos\left(\frac{2i-1}{2n}\pi\right) \qquad \tilde{f}(x) = \sum_i c_i\, T_i(x)$$

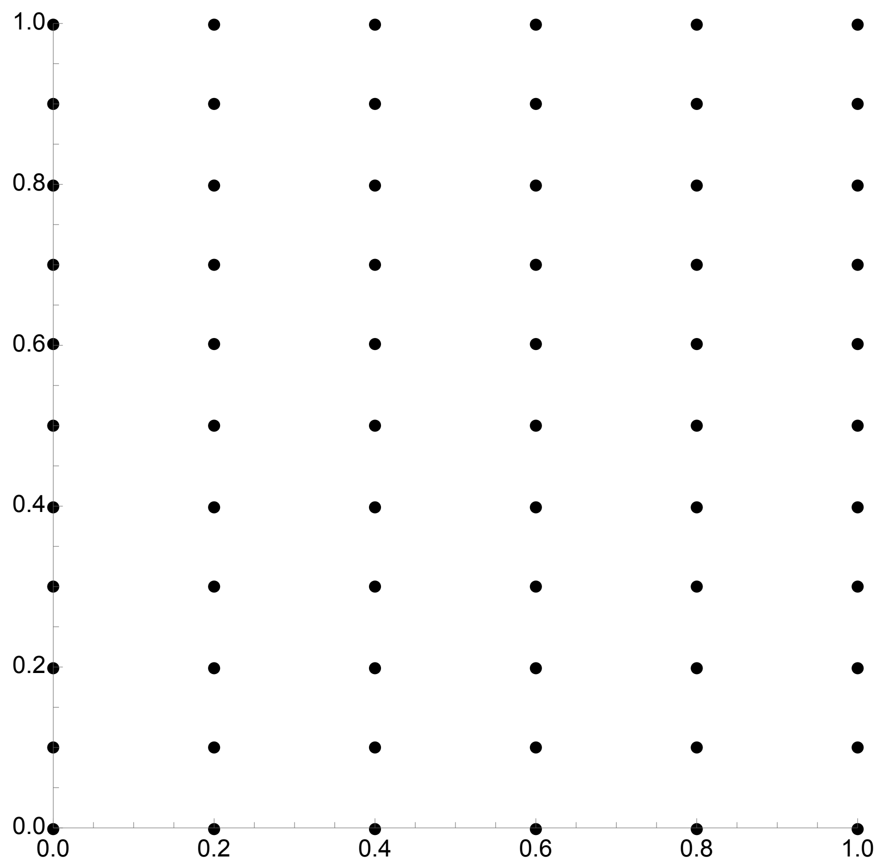
Chebyshev polynomials



Polynomial Interpolation (Multiple dimensions):

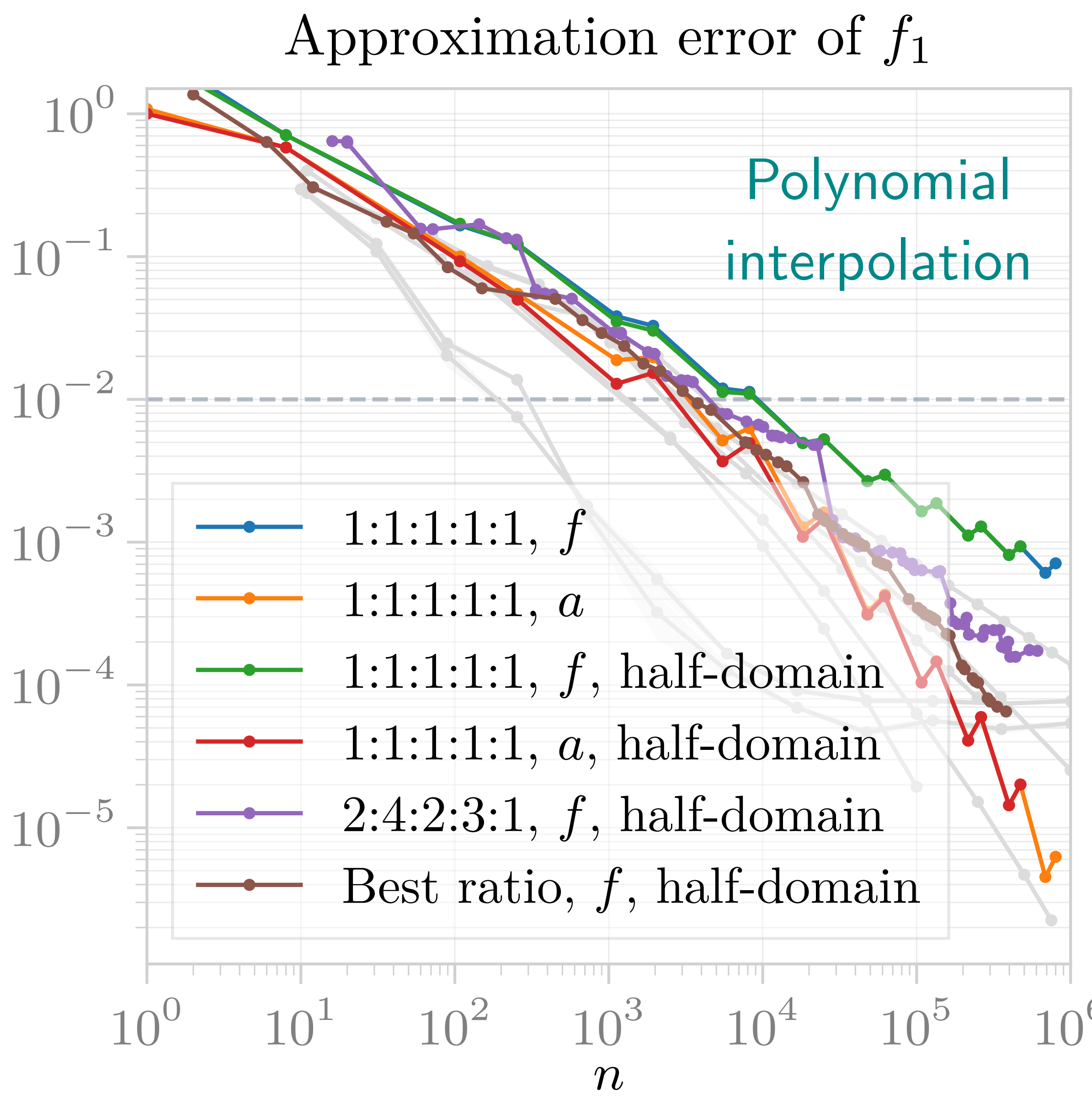
Chebyshev nodes for each dimension:

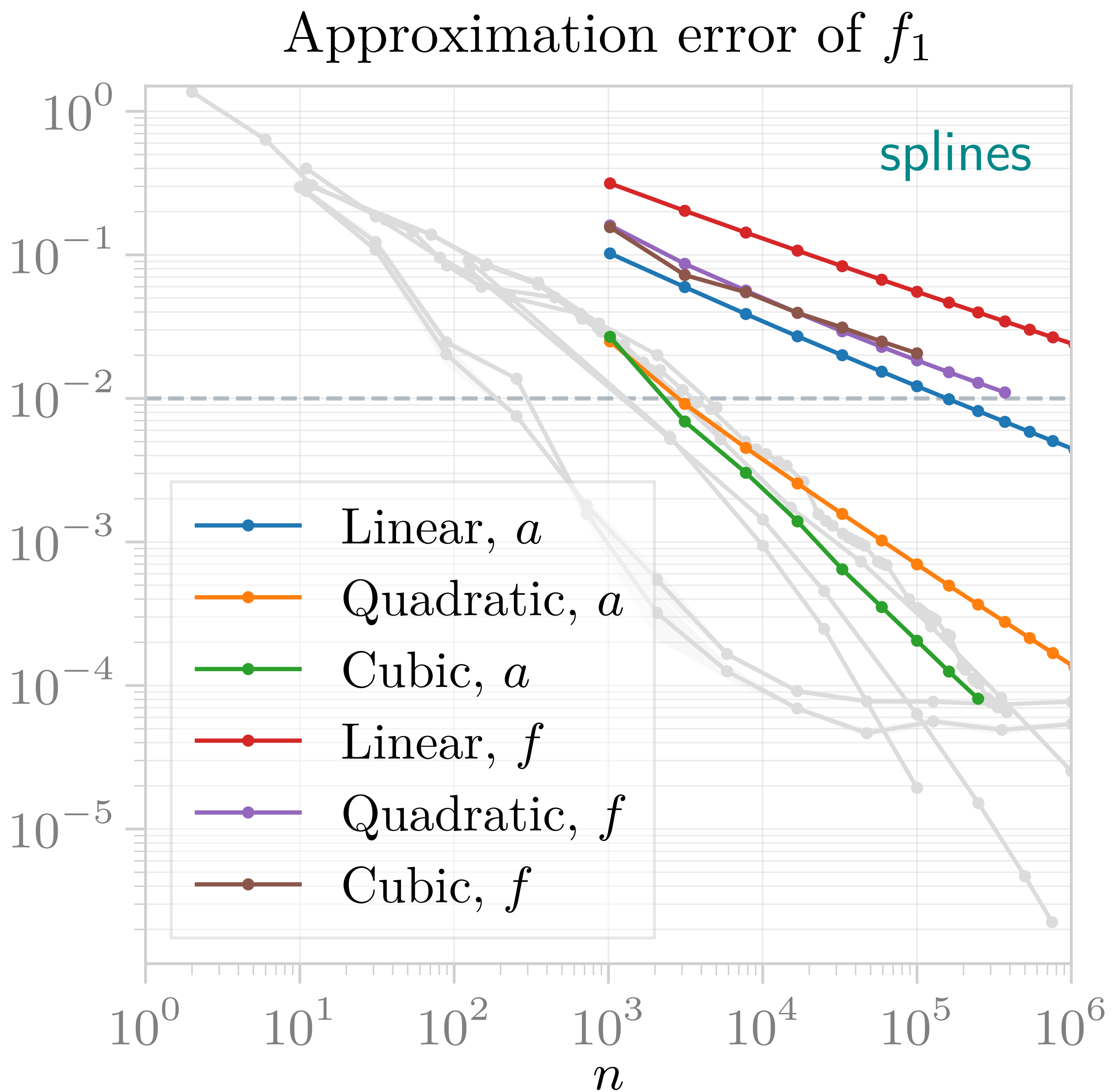
$$\{\vec{x}_i\} = \{x_{i_1}\} \otimes \cdots \otimes \{x_{i_d}\} \longrightarrow \tilde{f}(\vec{x}) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} c_{i_1\ldots i_d}\, T_{i_1}(x_1) \cdots T_{i_d}(x_d)$$



Results for f_1:

$(q\bar{q} \to t\bar{t}H \text{ @ } 0L)$

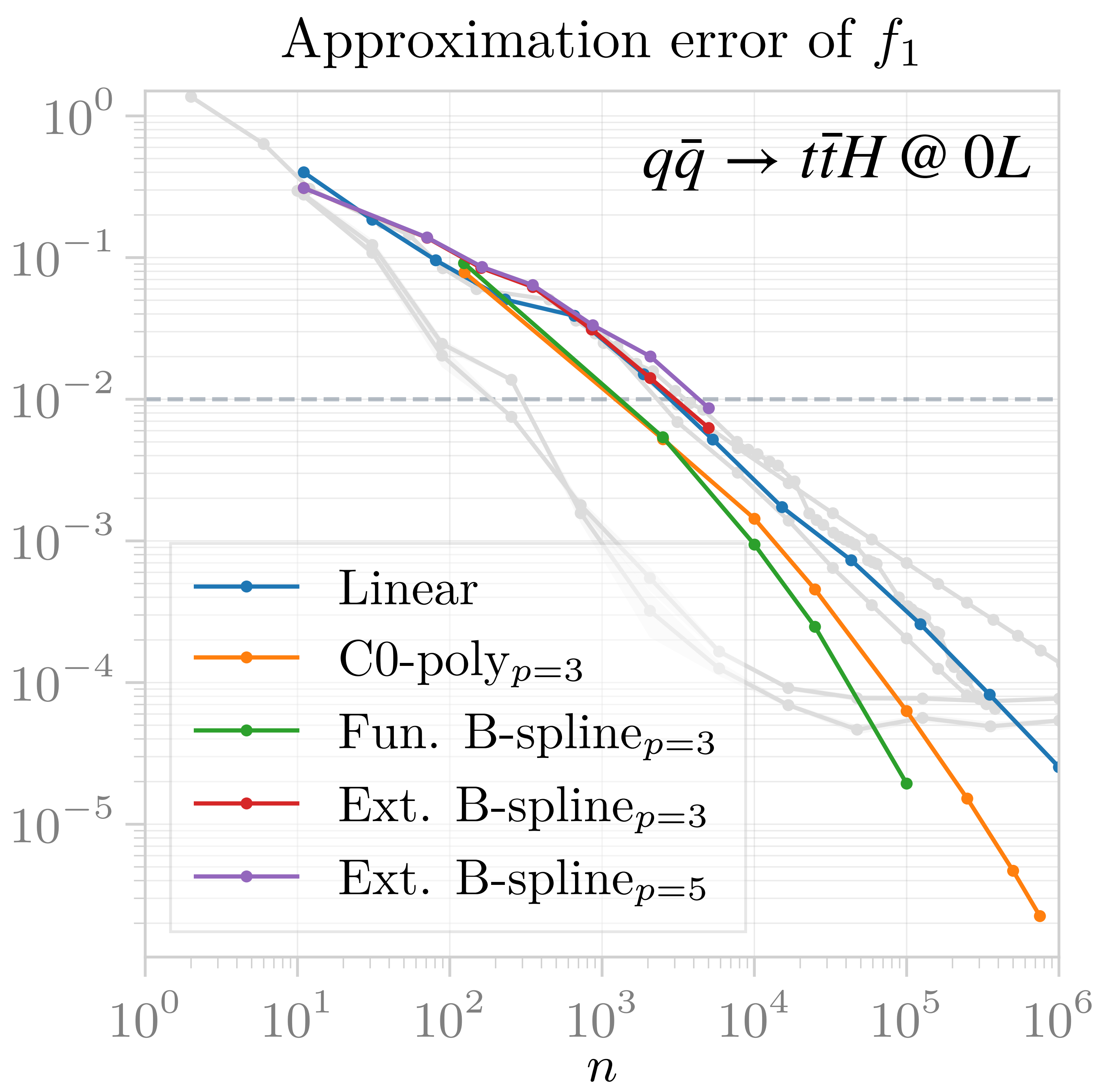


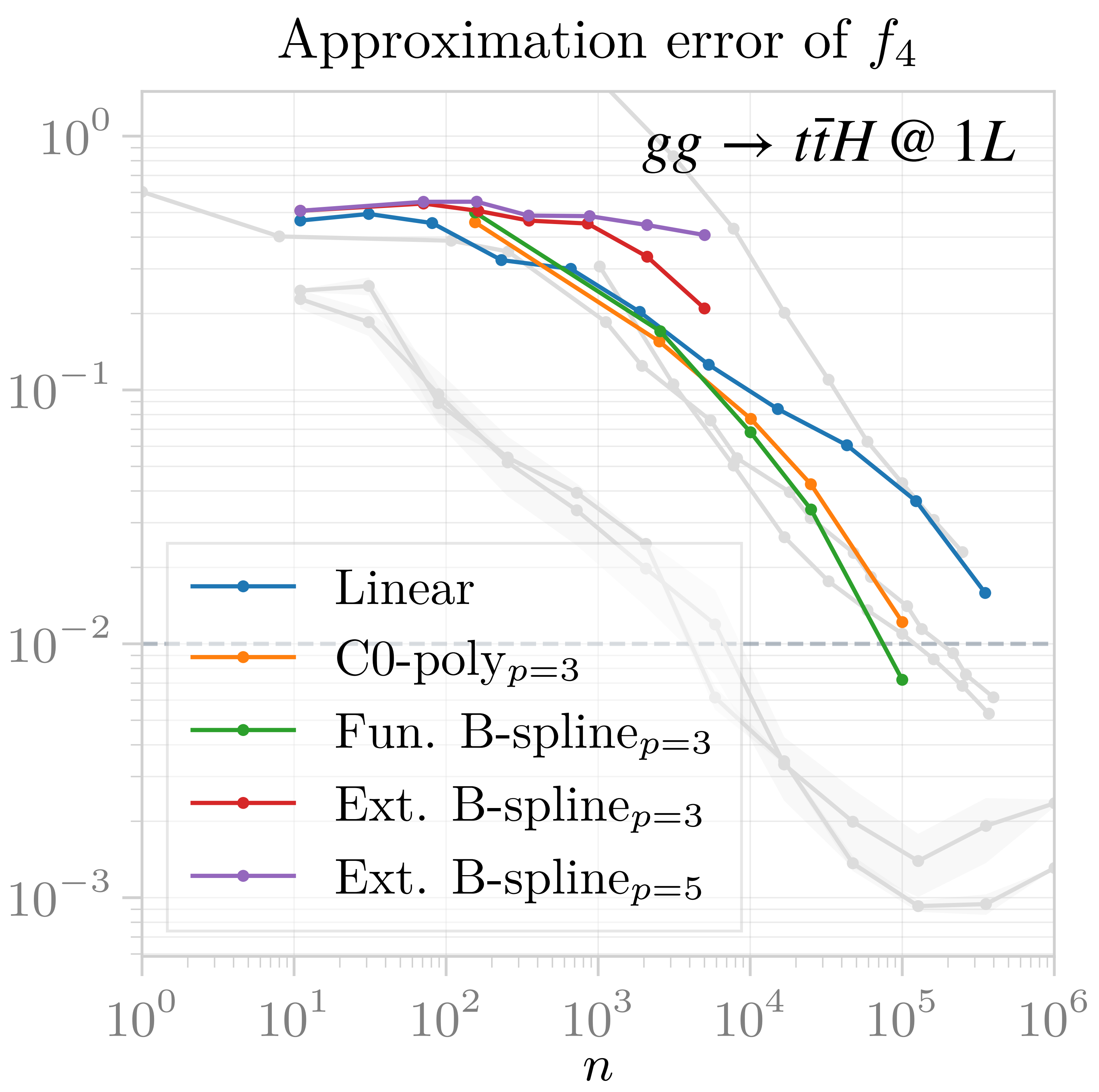


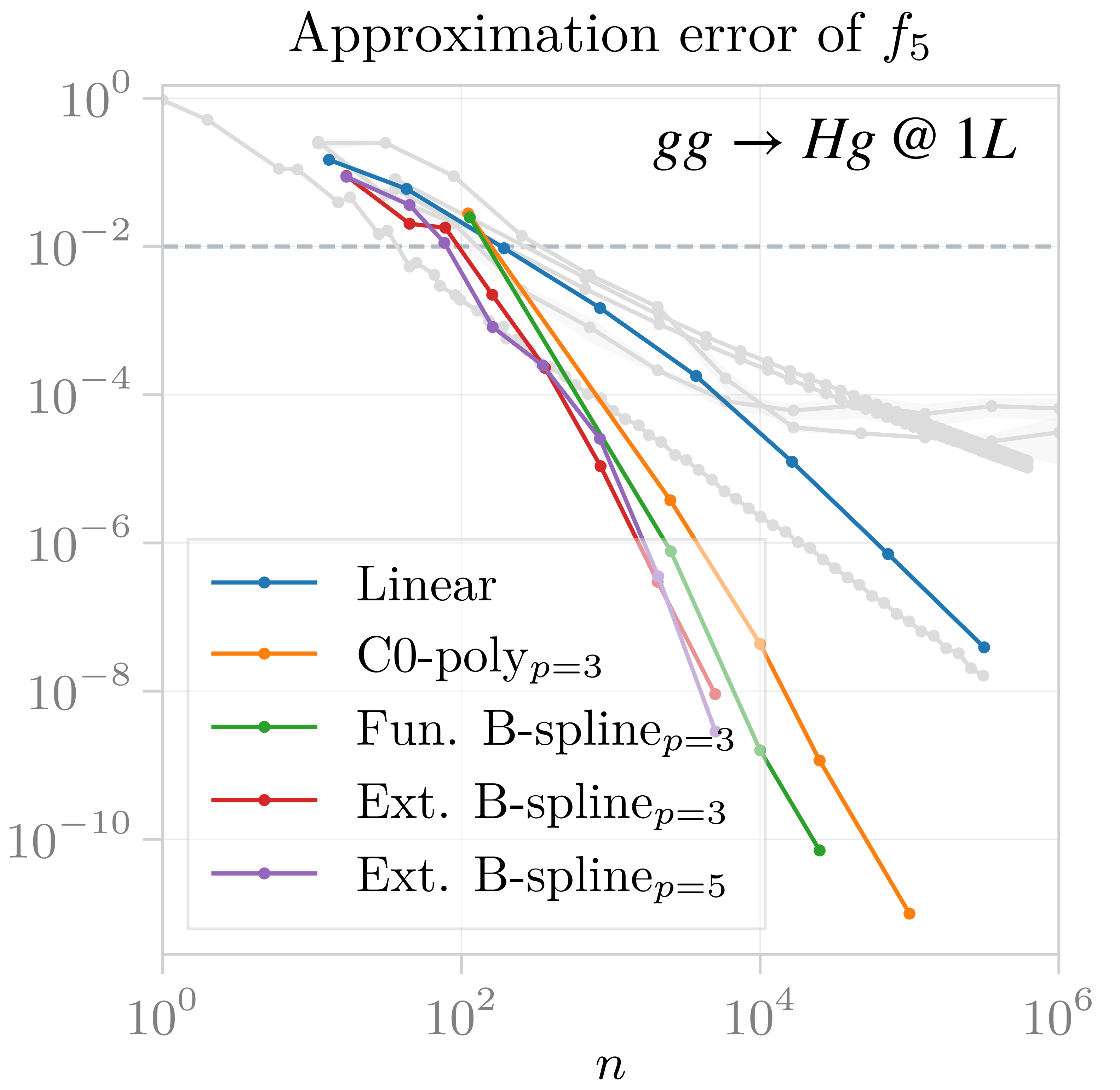
Interpolating Amplitudes [Bresó, Heinrich, Margery, Olsson 2412.09534]

Spatially adaptive sparse grids

$\rightarrow$ reduce number of grid nodes





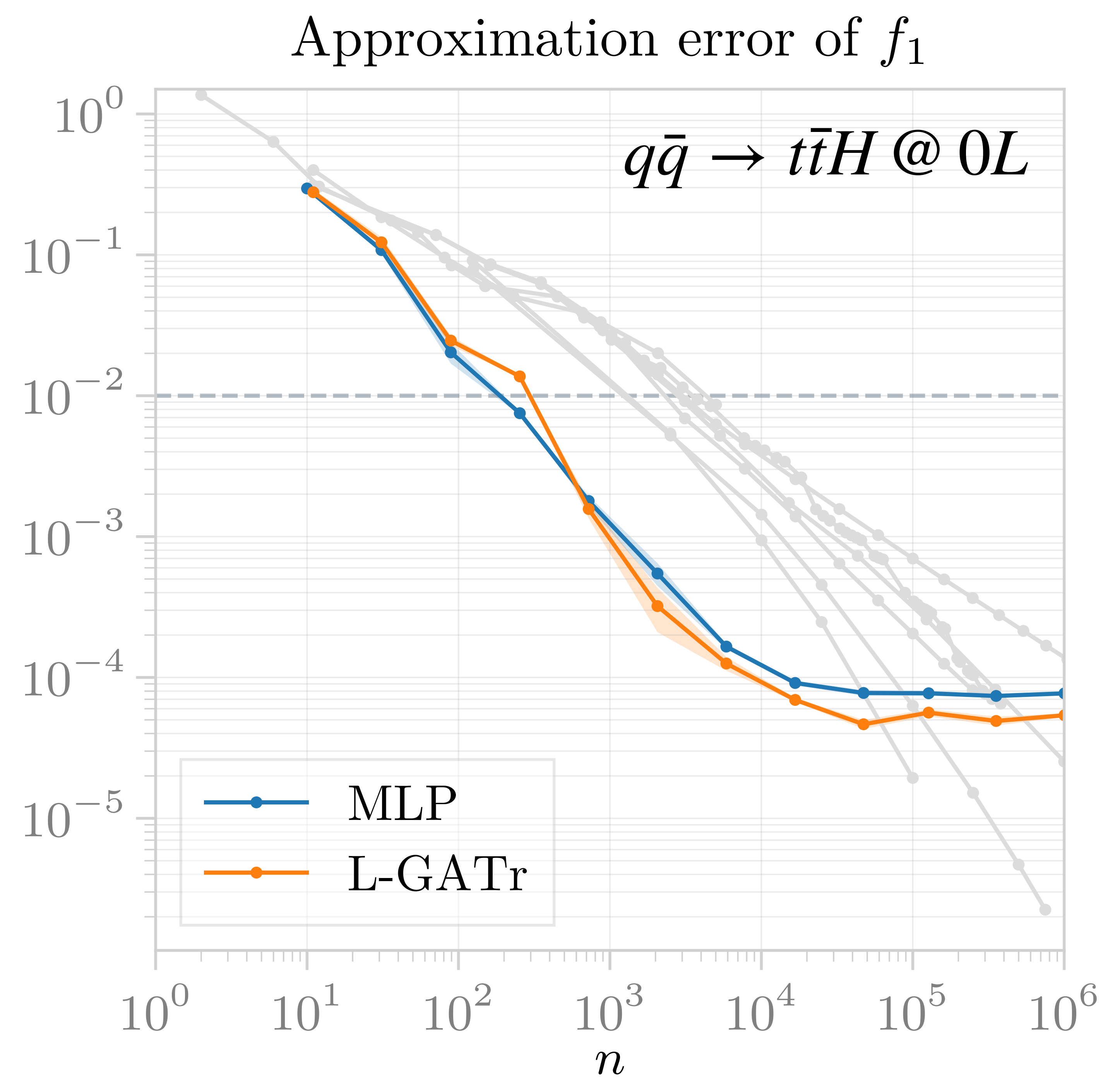


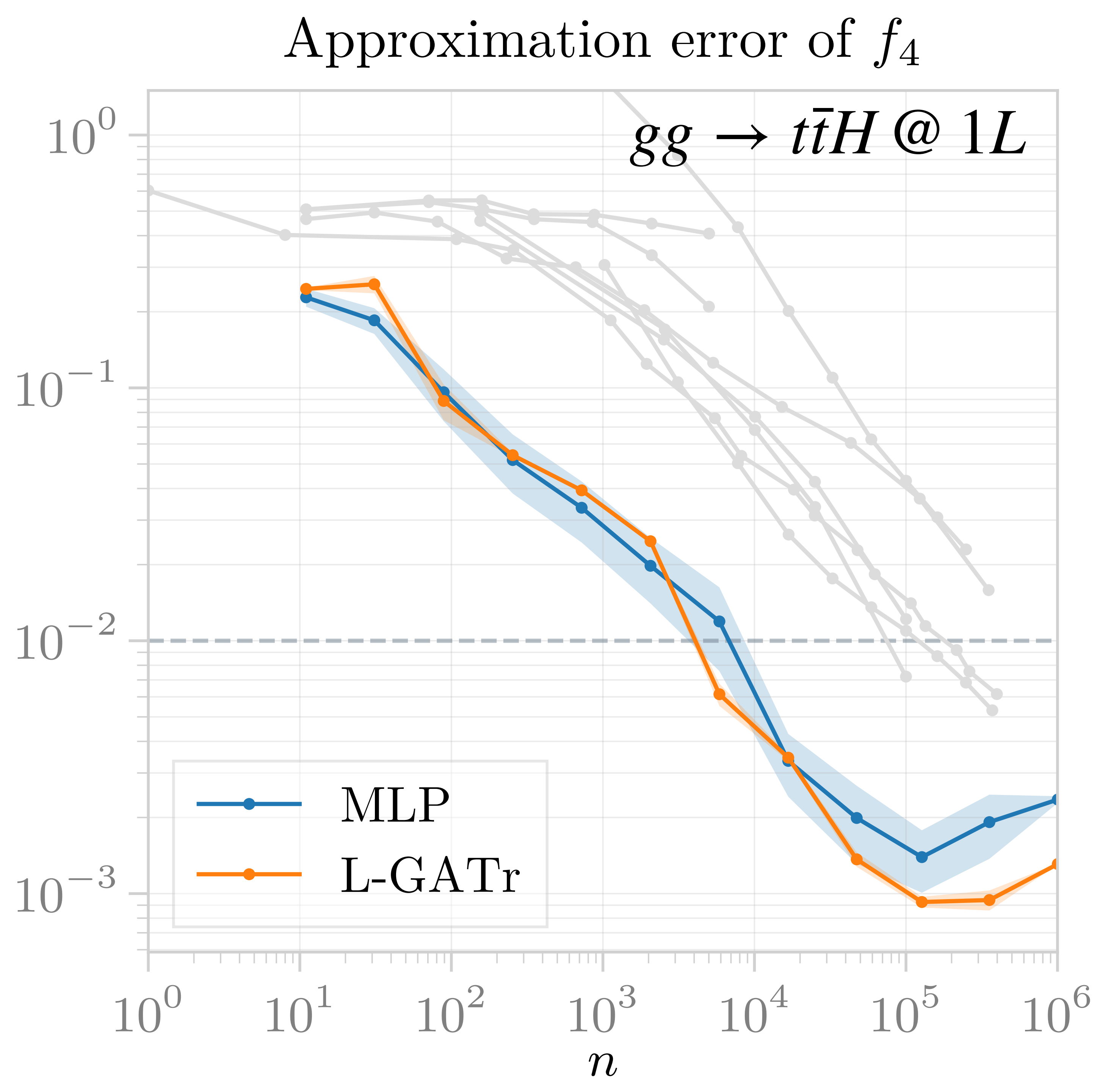
$\rightarrow$ Adaptive grids are preferred method @ large n

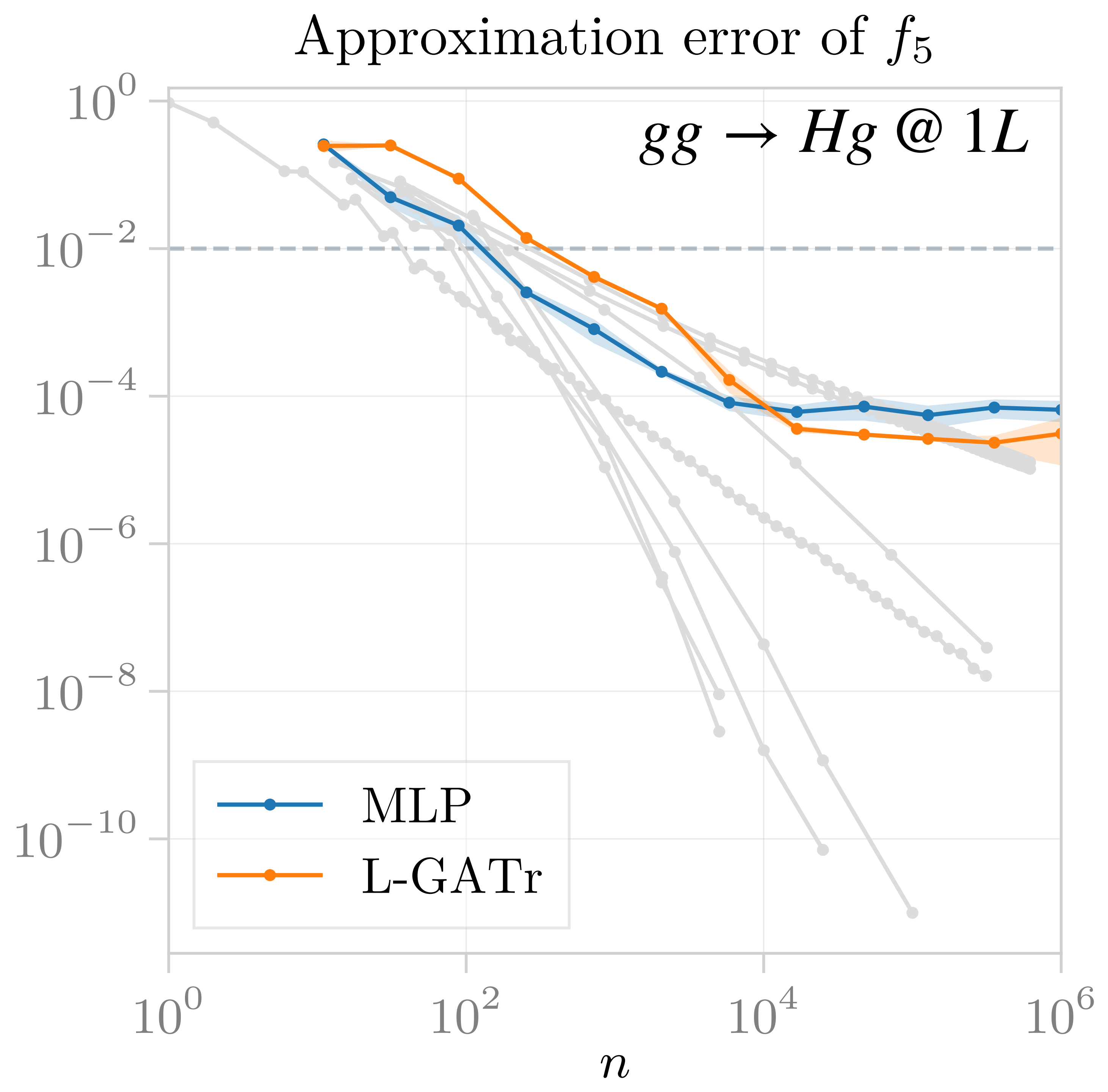
Interpolating Amplitudes [Bresó, Heinrich, Margery, Olsson 2412.09534]

Machine Learning Techniques:

- MPL (Multilayer perceptron, 5 layers, 512 channels each)
- L-GATr (Lorentz-Equivariant Geometric Algebra Transformer) [Brehmer, Bresó, de Haan, Plehn, Qu, Spinner, Thaler 2405.14806, 2412.09534] respects space-time symmetry: $f\left(\Lambda(x)\right) = \Lambda\left(f(x)\right)$







$\to$ Error does not improve beyond $\mathcal{O}(10^{-4})$ level, even for HJ amplitude

Optimising simulations for diphoton production at hadron colliders using amplitude neural networks

Aylett-Bullock, Badger, Moodie 2106.09474

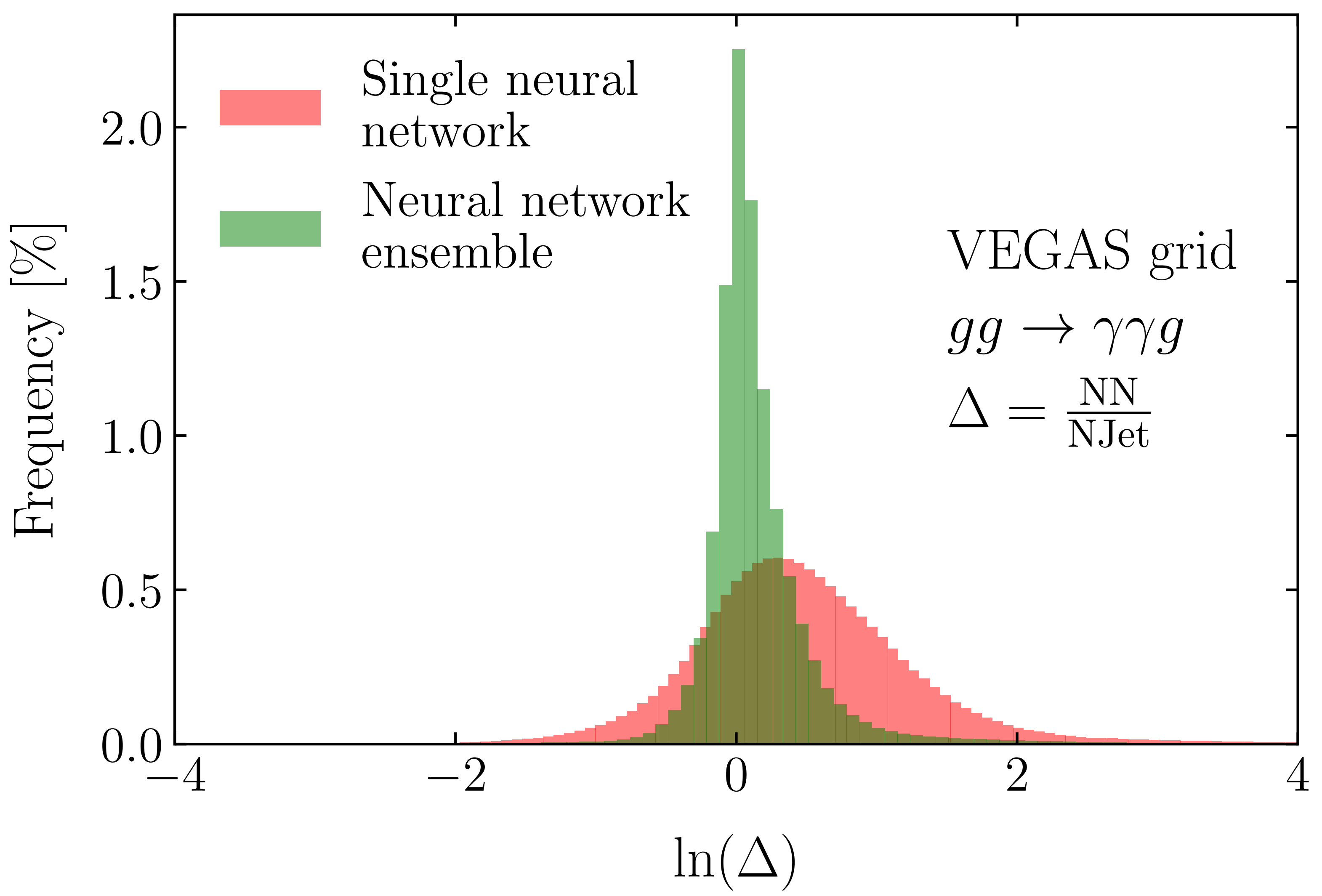
partition phase-space into IRC regions according to FKS and train one NN for each region → ensemble of neural networks

$$\mathrm{d}\sigma = \sum_{i,j} S_{i,j}\,\mathrm{d}\sigma \qquad \text{with} \qquad \mathcal{S}_{i,j} = \frac{1}{D_1 s_{ij}}, \quad D_1 = \sum_{i,j\in\mathcal{P}_{\mathrm{FKS}}} \frac{1}{s_{ij}}$$

3 hidden layers: 30-40-30

100k training points

$gg \to \gamma\gamma g$:



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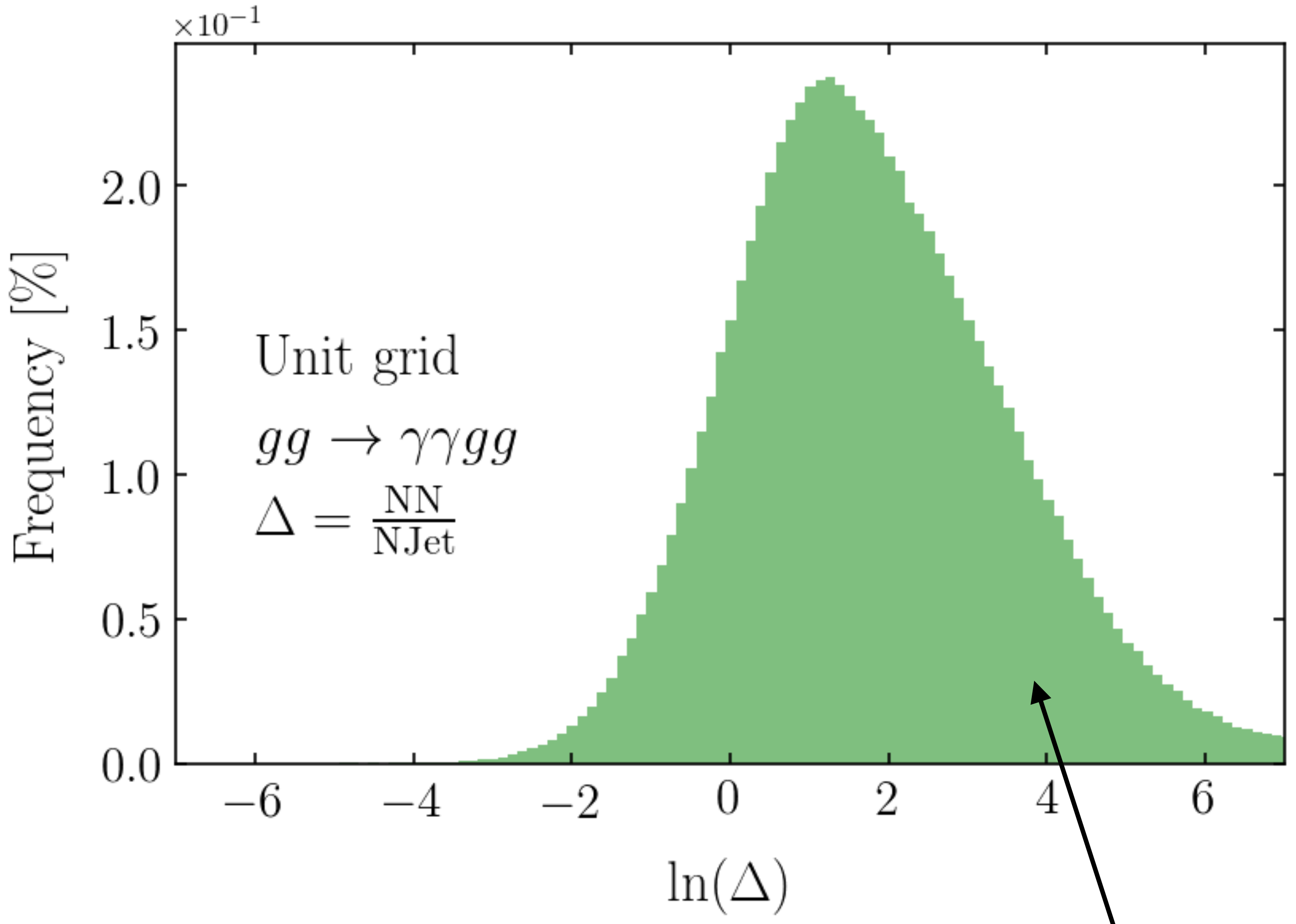
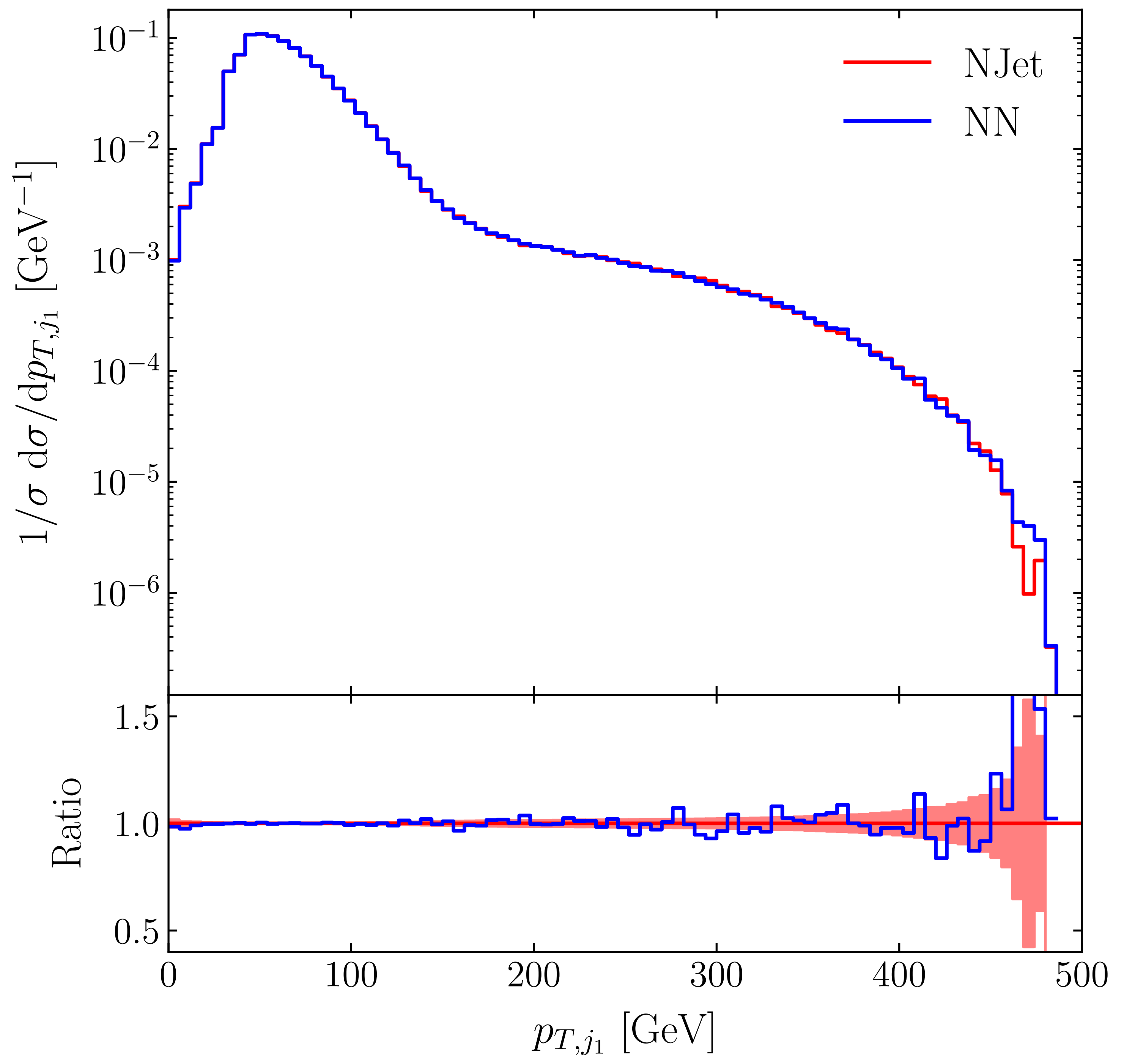




$$\ln(\Delta) = \ln\left(1 + \frac{\mathrm{NN} - \mathrm{NJet}}{\mathrm{NJet}}\right)$$

$$= \frac{\mathrm{NN} - \mathrm{NJet}}{\mathrm{NJet}} + \mathcal{O}((\mathrm{NN} - \mathrm{NJet})^2)$$

points with large deviations typically phase-space suppressed
→ still results in good predictions of cross sections

A factorisation-aware Matrix element emulator

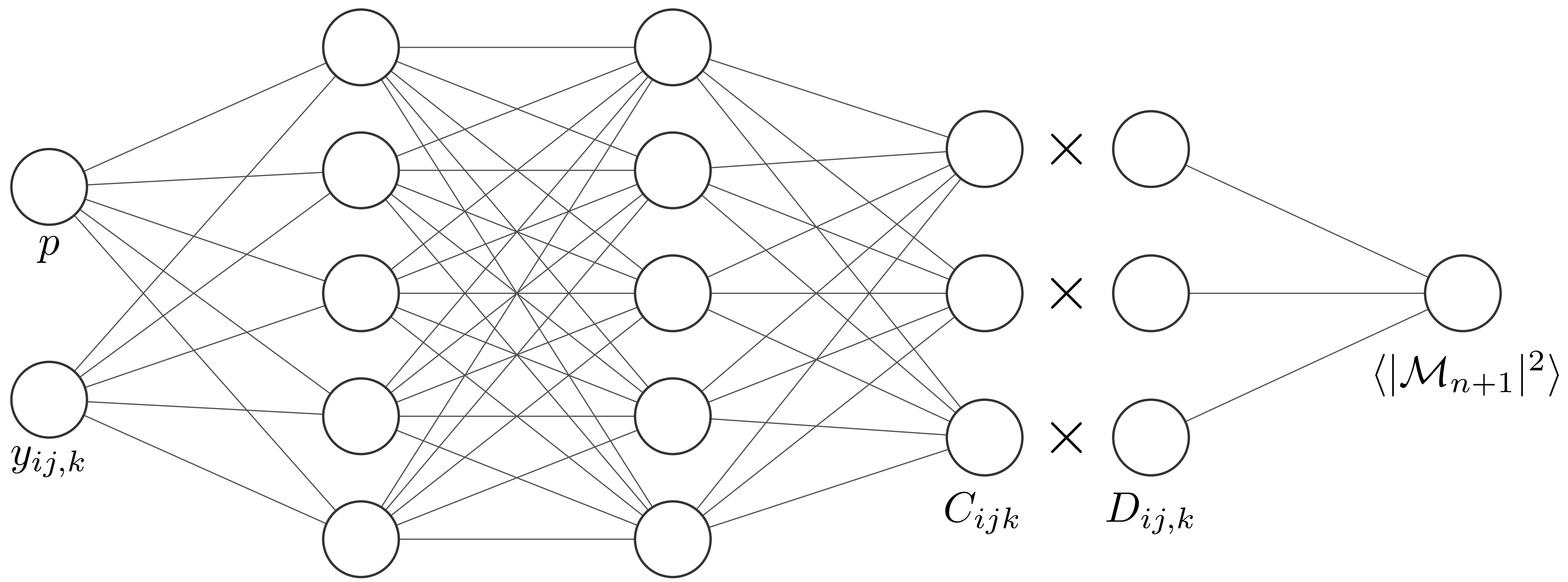
Maître, Truong 2107.06625

Ansatz for fit using NN:

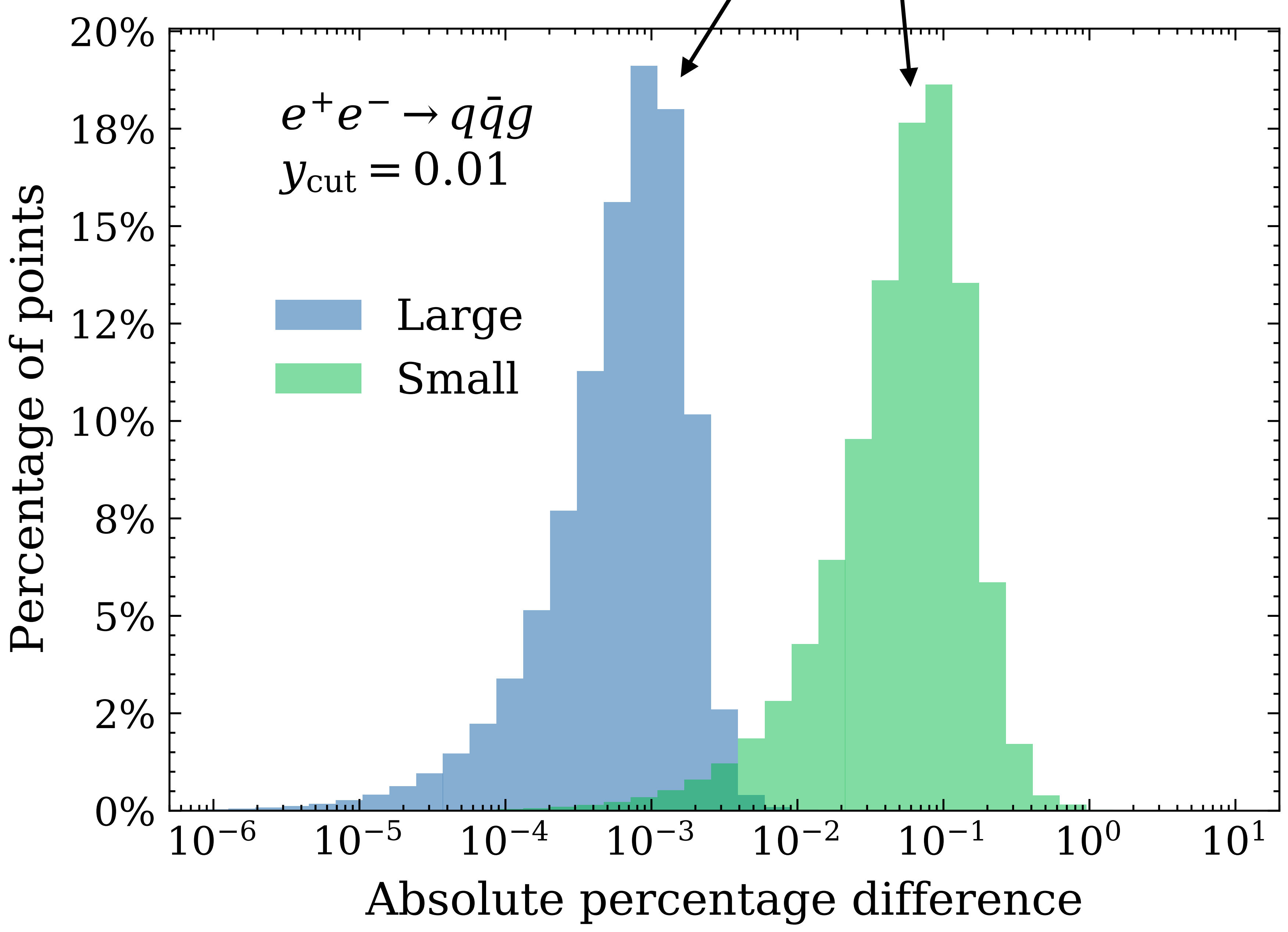
$$\langle|\mathcal{M}_{n+1}|^2\rangle = \sum_{\{ijk\}} C_{ijk} D_{ij,k}$$

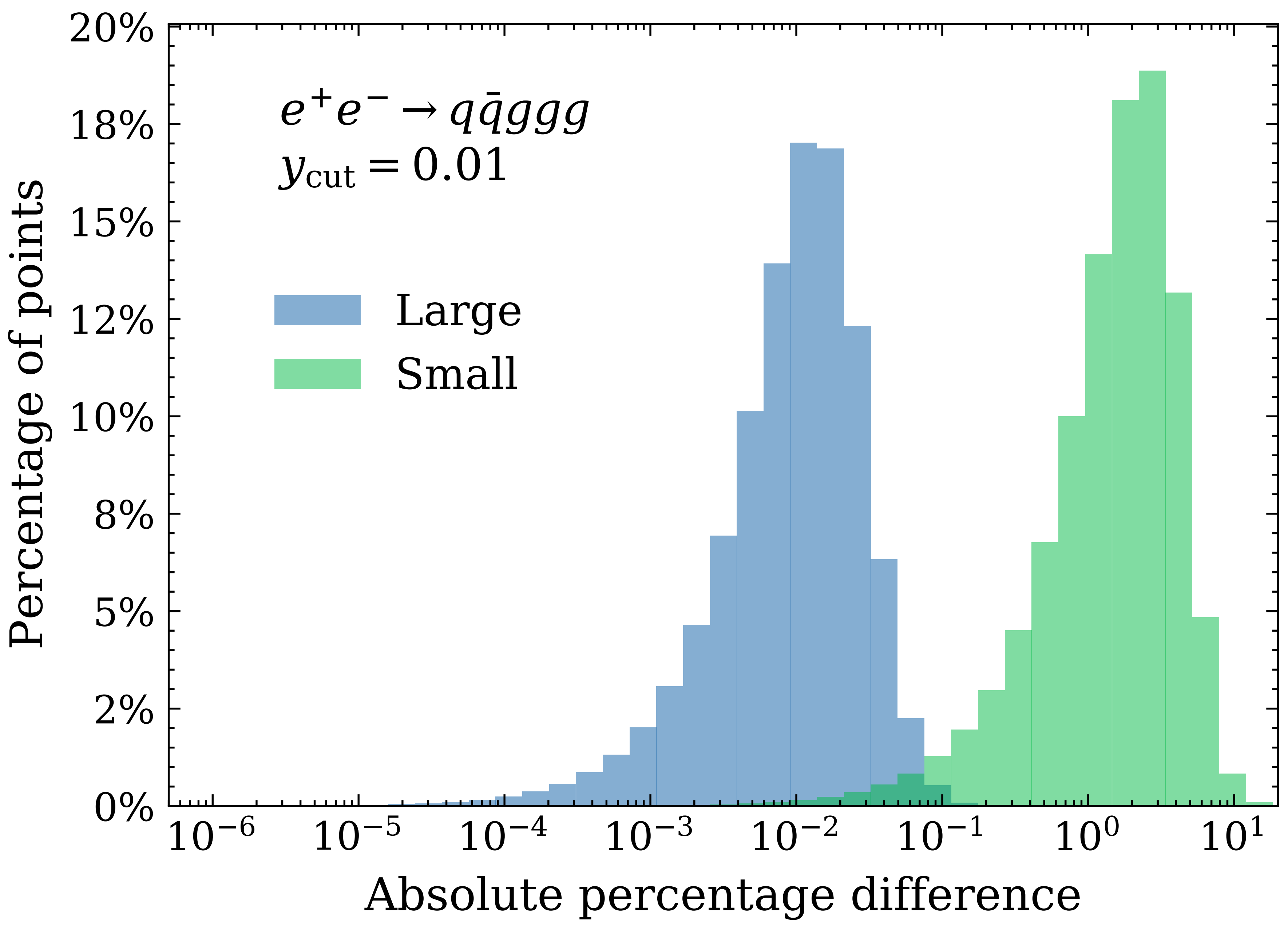
fit coefficients (C_{ijk}) Catani Seymour dipoles ($D_{ij,k}$)

$\rightarrow$ reliable results also in IRC limits



40M 500k training samples





Summary

- Multiple Method:
    - classical methods
      (polynomial interpolation, splines, sparse grids)
    - Machine learning

- Different strategies to deal with singularities:
    - subtract singular terms before fit
    - fit coefficients of singular terms
    - ensembles of neural networks
    - avoid by multiplication with appropriate factors

Summary & Open Questions

- Multiple Method:
  - classical methods
    (polynomial interpolation, splines, sparse grids)
  - Machine learning
- Different strategies to deal with singularities:
  - subtract singular terms before fit
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- Which method works best for a given use case?
- Which precision is required?
  (on amplitude / cross section)
- How to deal with uncertainties?
  (of input data; uncertainties due to interpolation)
- How can we incorporate parametric dependencies?
  (anomalous couplings, scale dependence, ...)
- ...