AN INTRODUCTION TO MULTILOOP AMPLITUDES

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Les Houches, PhysTev Workshop 2025

EVENTS AT HADRON COLLIDERS

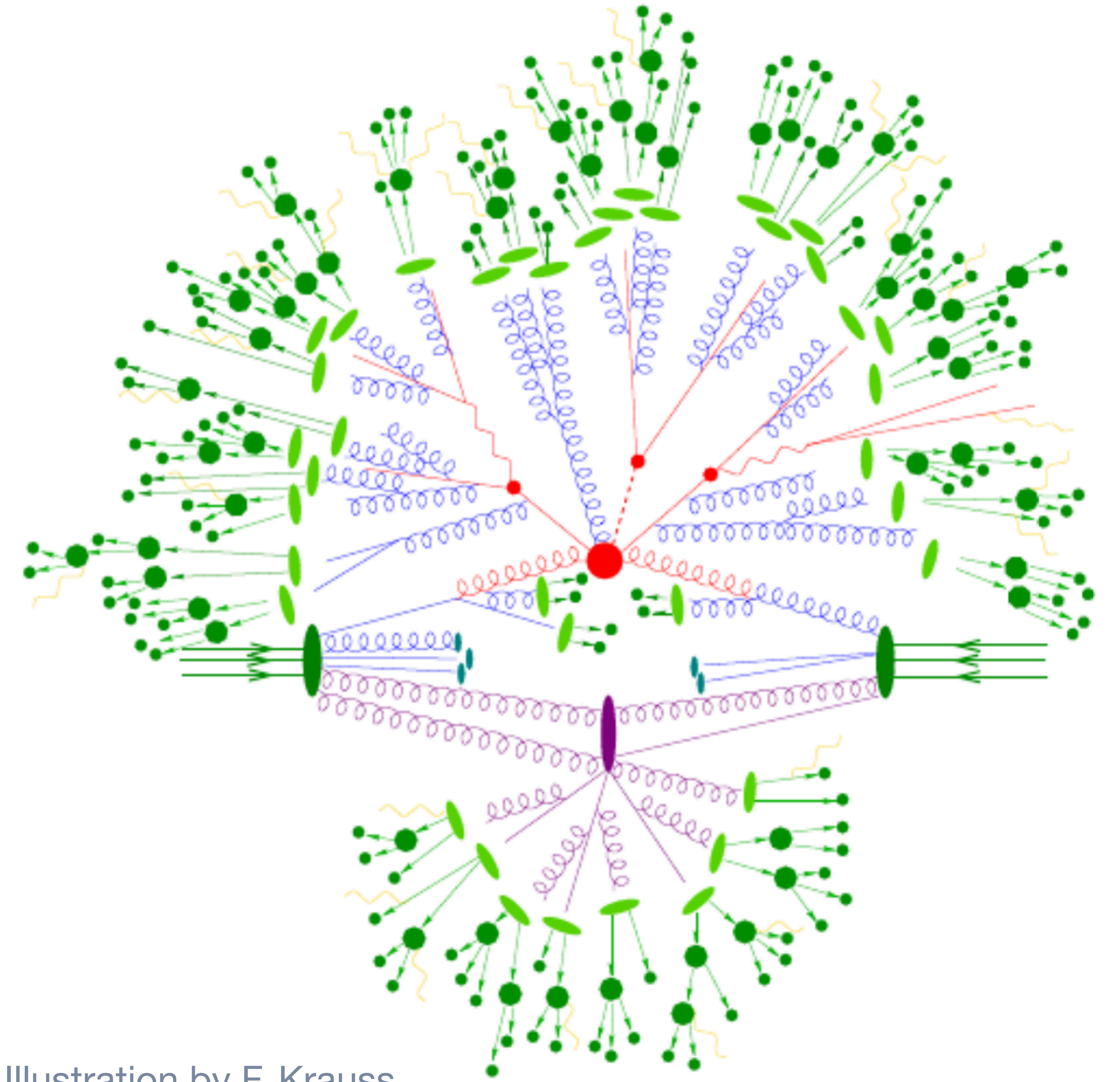


Illustration by F. Krauss

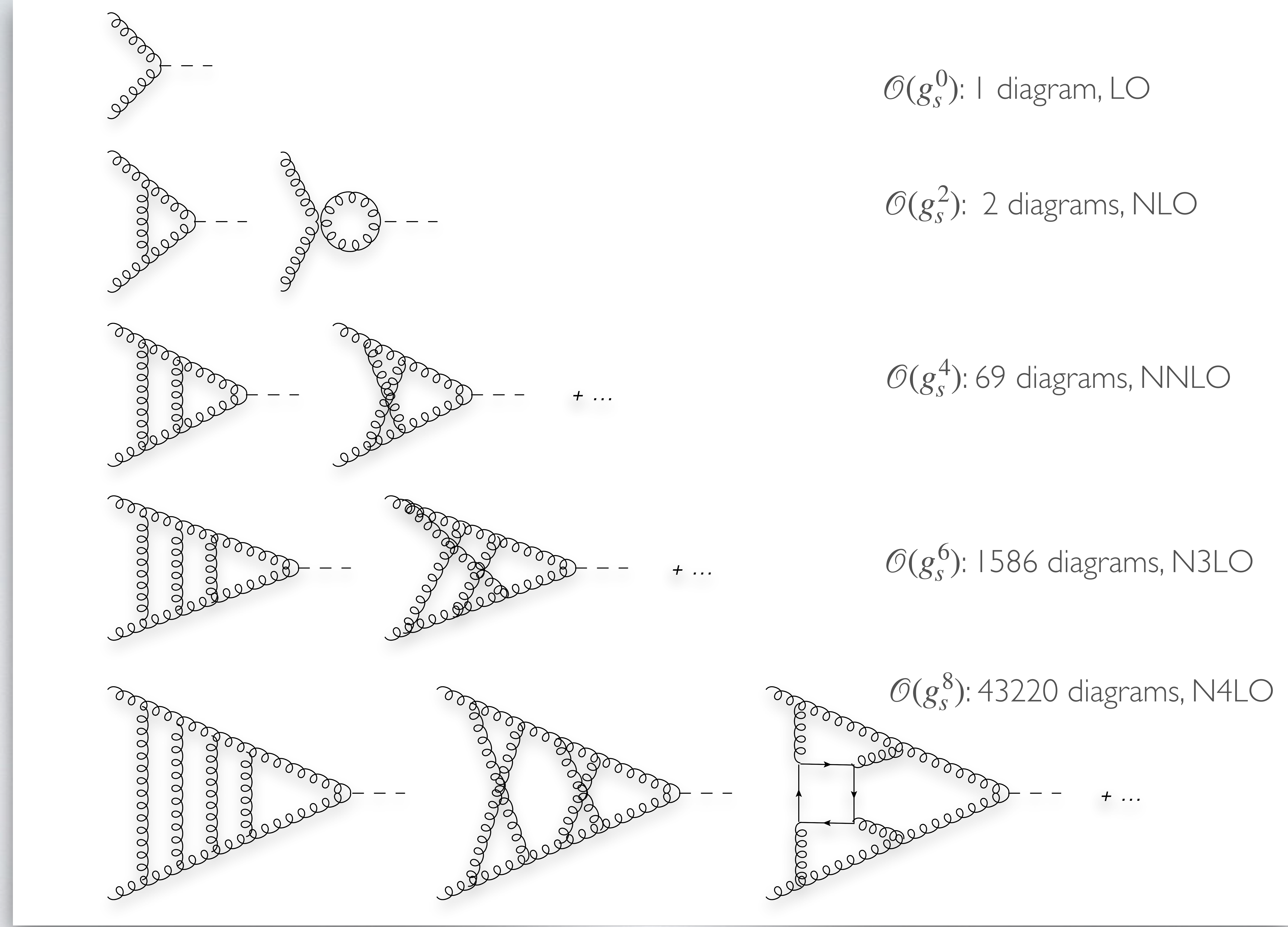
Hadronic scattering cross section $\sigma \propto \int f_a f_b \sigma_{ab}$

Universal parton distributions f_a

Partonic cross section $\sigma_{ab} \propto |\mathscr{M}|^2$

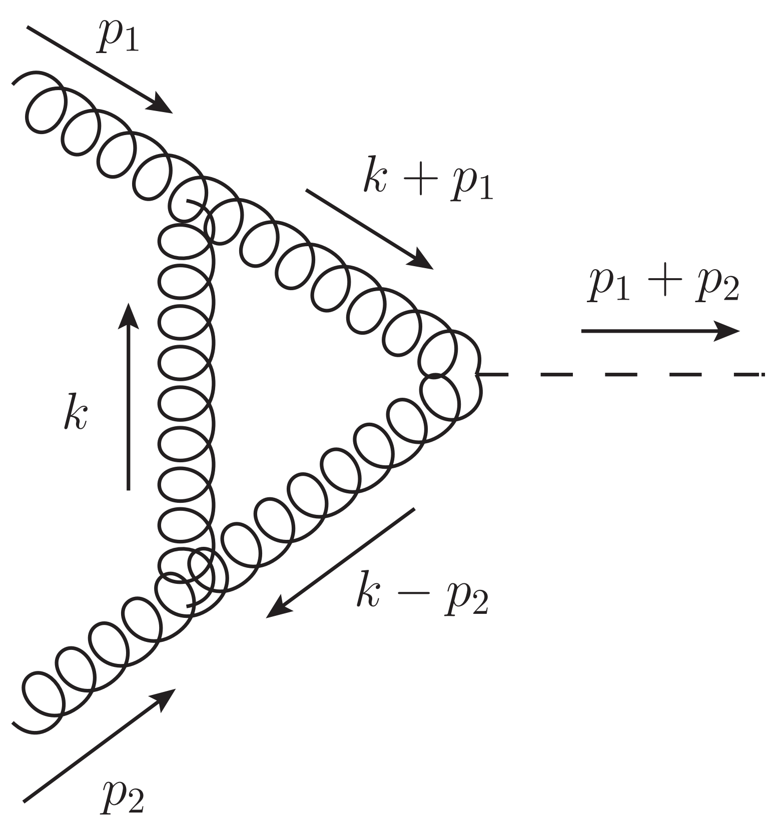
Partonic scattering amplitude $\mathscr{M}$ ⟵ this talk

PERTURBATIVE EXPANSION IN COUPLING



LOOP AMPLITUDES AND INTEGRALS

- ***Feynman diagrams*** describe ***scattering amplitudes***
- Quantum principle: ***sum coherently*** over ***unobserved*** possibilities



$\rightarrow$

"loop integral"

$$\int d^4k \frac{1}{k^2(k+p_1)^2(k-p_2)^2}$$

- ***Dimensional regularization*** of infrared and ultraviolet divergences

$$\int d^4k \frac{1}{k^2(k+p_1)^2(k-p_2)^2} \quad \rightarrow \quad \int d^dk \frac{1}{k^2(k+p_1)^2(k-p_2)^2}$$

- ***Observables*** in terms of observables finite for $d \rightarrow 4$

TYPICAL WORKFLOW FOR PERTURBATIVE CALCULATION

1. Generate Feynman diagrams to construct amplitude
2. Insert Feynman rules, perform Lorentz, Dirac, gauge algebra
3. Reduce loop integrals to a set of master integrals
4. Evaluate the basis integrals analytically or numerically
5. UV renormalization, IR subtractions
6. Monte-Carlo phase phase integration

INTEGRATION-BY-PART (IBP) IDENTITIES

- IBP identities in dimensional regularization since integrals over total derivatives vanish:

$$\int d^dk_1 \cdots d^dk_L \frac{\partial}{\partial k_i^\mu}\left(q^\mu \frac{1}{D_1^{\nu_1}\cdots D_N^{\nu_N}}\right) = 0 \quad \text{with } D_i = q_i^2 - m_i^2$$

[Chetyrkin, Tkachov ’81]

- **Linear relations between loop integrals**, allows for systematic **reduction** *[Laporta '00]*
- Finite number of integrals linearly independent: **basis** or **master integrals**
- In practice:
  - Reductions are **computational bottleneck**
  - Optimizations: finite field sampling + rational reconstruction, syzygies, choice of basis, …

Simple example

example: massive 1-loop tadpole

$$\int d^d k \frac{1}{(k^2 - m^2)^a} \qquad \text{with } a \in \mathbb{Z}$$

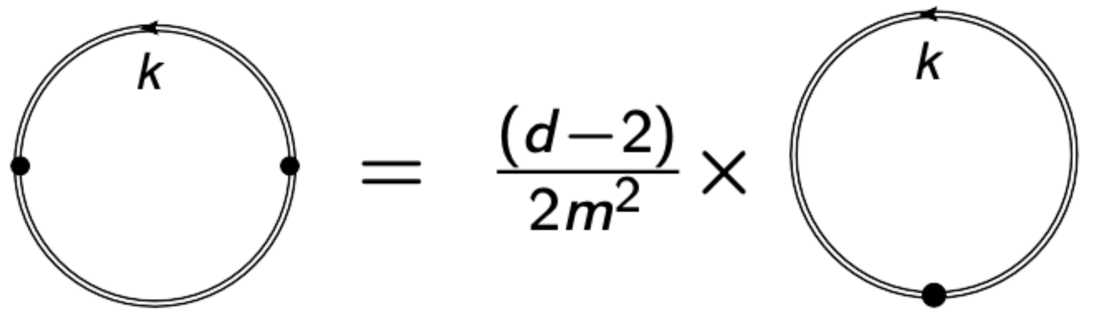
calculating IBP identity

$$\begin{aligned}
0 = & \int d^d k \frac{\partial}{\partial k_\mu} \left(k_\mu \frac{1}{(k^2 - m^2)^a} \right) \\
= & \int d^d k \left(\frac{d}{(k^2 - m^2)^a} - a \frac{2k^2}{(k^2 - m^2)^{a+1}} \right) \\
= & \int d^d k \left(\frac{d}{(k^2 - m^2)^a} - a \frac{2(k^2 - m^2 + m^2)}{(k^2 - m^2)^{a+1}} \right) \\
= & (d - 2a) \int d^d k \frac{1}{(k^2 - m^2)^a} - 2am^2 \int d^d k \frac{1}{(k^2 - m^2)^{a+1}}
\end{aligned}$$

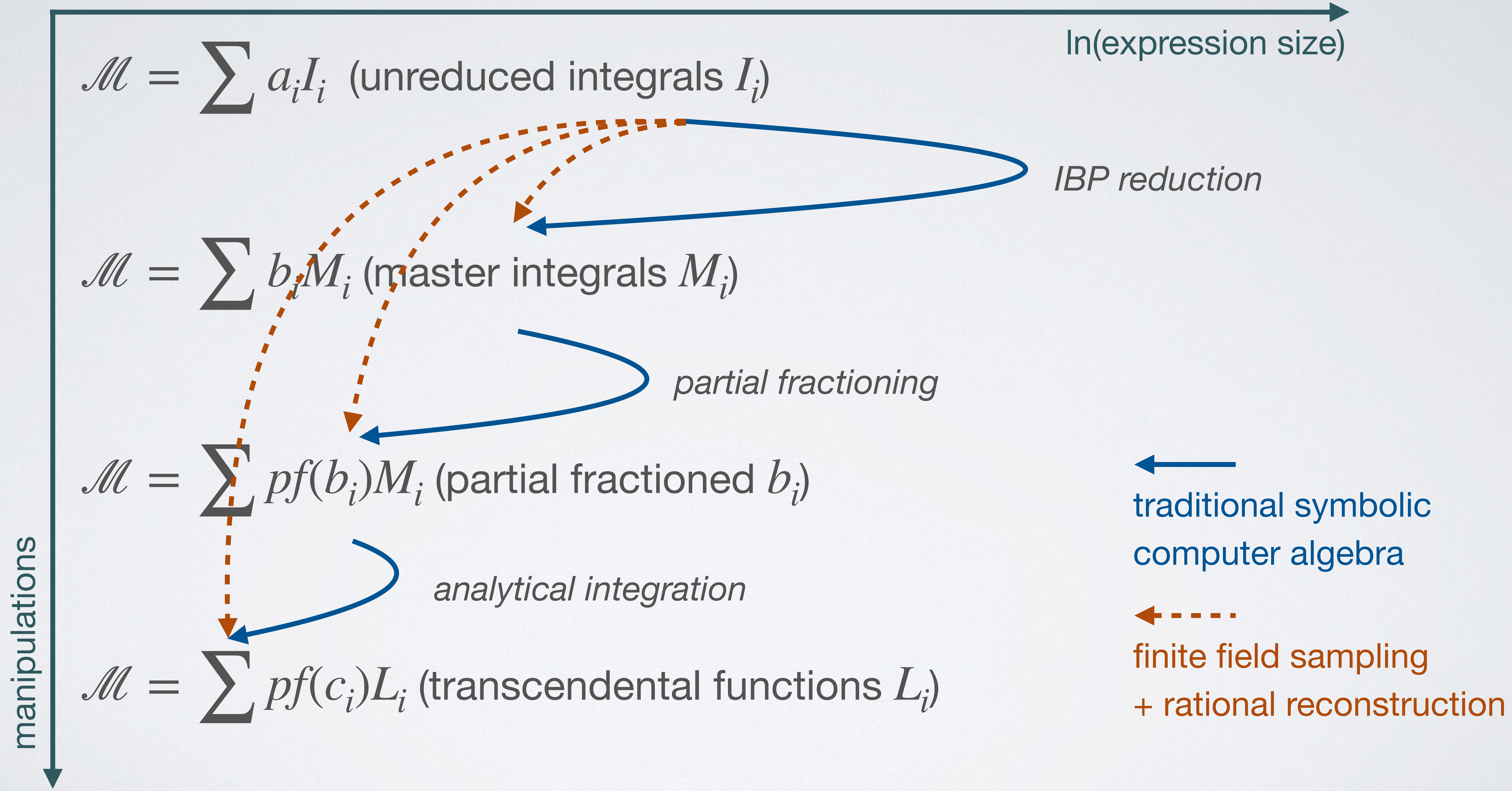
gives directly reduction of integral with additional numerator

$$\int d^d k \frac{1}{(k^2 - m^2)^{a+1}} = \frac{(d - 2a)}{2am^2} \int d^d k \frac{1}{(k^2 - m^2)^a}$$

diagrams for $a = 1$:



BY-PASSING COMPLEXITY



multivariateapart.nb

Univariate partial fractions separate terms with different poles:

In[1]:= $\text{Apart}\left[\frac{1}{x(1+x)}, x\right]$

Out[1]= $\frac{1}{x} - \frac{1}{1+x}$

Let' s consider a multivariate example :

In[2]:= $\text{multi} = \frac{2y - x}{y(x+y)(y-x)};$

Naive iteration of univariate partial fractioning introduced spurious poles (here 1/x) for multivariate case:

In[3]:= Apart[multi, y]

Out[3]= $\frac{1}{xy} + \frac{1}{2x(-x+y)} - \frac{3}{2x(x+y)}$

Solution: multivariate partial fraction decomposition, e.g. using methods from polynomial ideal theory

In[4]:= << MultivariateApart`

In[5]:= MultivariateApart[multi]

Out[5]= $-\frac{1}{2(x-y)y} + \frac{3}{2y(x+y)}$

Partial fraction decomposition of amplitudes:

- helps to reduce amplitude sizes potentially by orders of magnitude
- allows to reconstruct from fewer samples (with denominator guessing)
- helps to improve stabilty of numerical evaluations

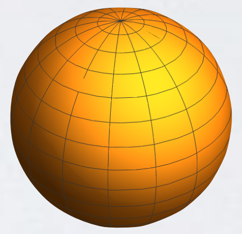
SOLVE INTEGRALS: DIFFERENTIAL EQUATIONS

- Integration of differential equations *[Kotikov '91, Remiddi '97]*:

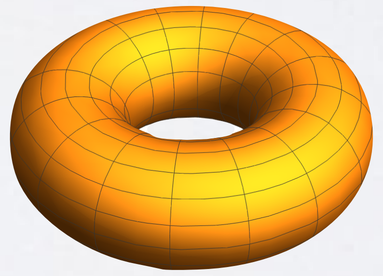
  $$\partial_x \vec{I}(x;\epsilon) = A(x;\epsilon)\vec{I}(x;\epsilon)$$

  where $\epsilon = (4-d)/2$ (analytical or through series expansions)

- Homogeneous solutions for $\epsilon = 0$ (leading singularities):
  - ***Rational number***, e.g. $1/2$



  - ***Rational functions***, e.g. $1/x$
  - ***Algebraic functions***, e.g. $\sqrt{x(x-4)}$
  - ***Elliptic integrals***, e.g. $K(x) = \int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-xz^2)}}, \ldots$

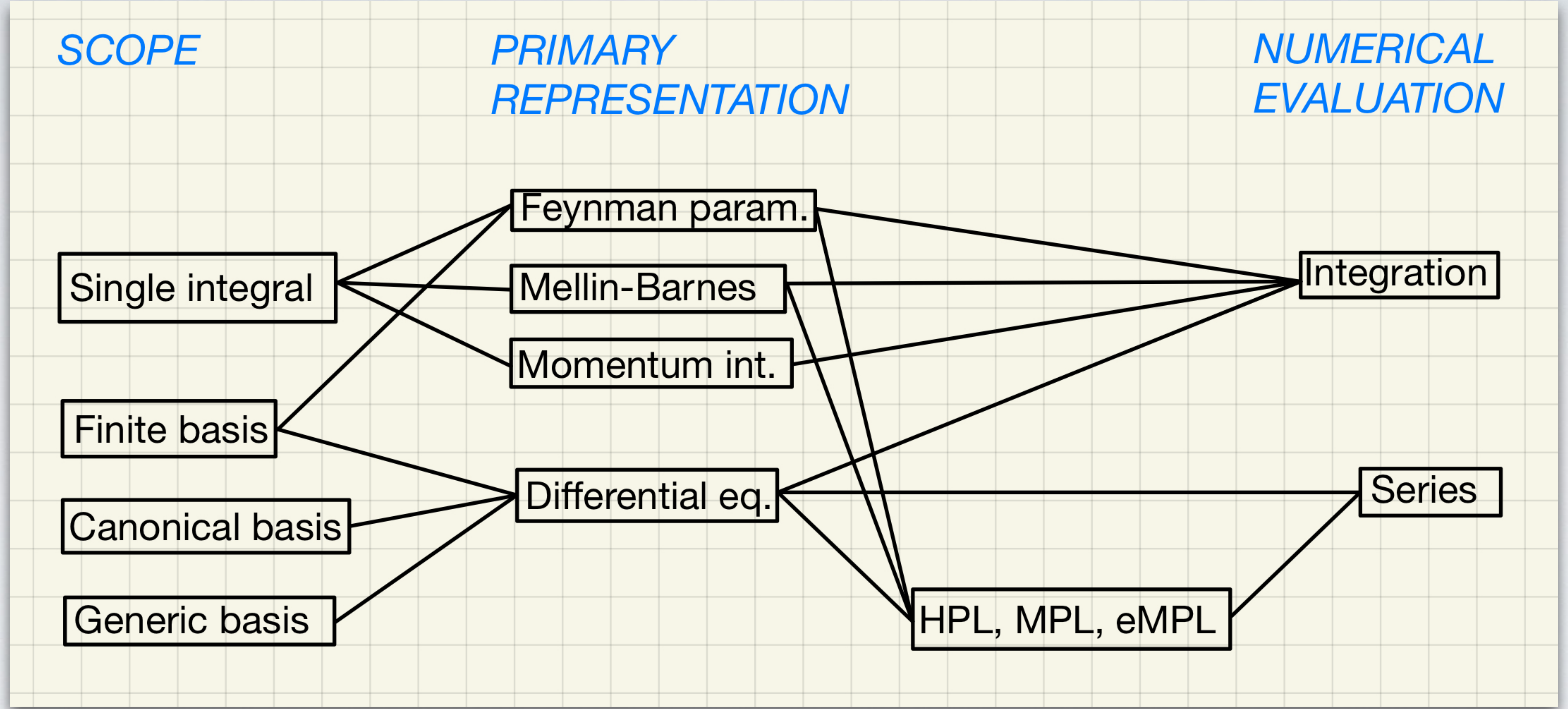


- Basis change involving homogenous solutions may allow to find ϵ***-form***:

  $$d\vec{m} = \epsilon\, \mathrm{d}A(x)\, \vec{m}$$

  [(Kotikov '10), Henn '13, (Adams, Weinzierl '18)]

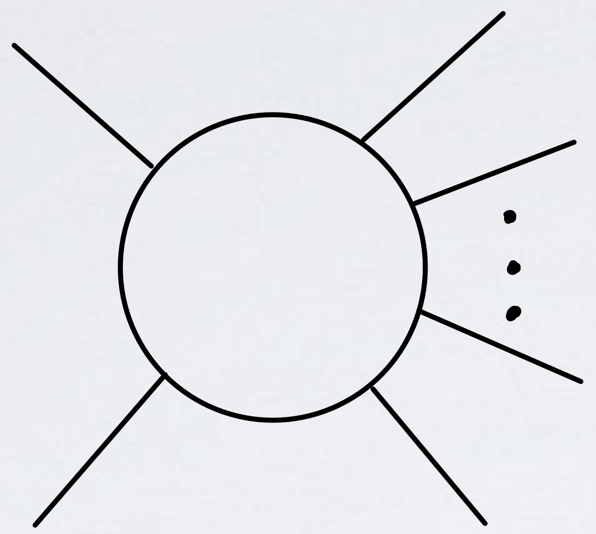
EVALUATION OF INTEGRALS

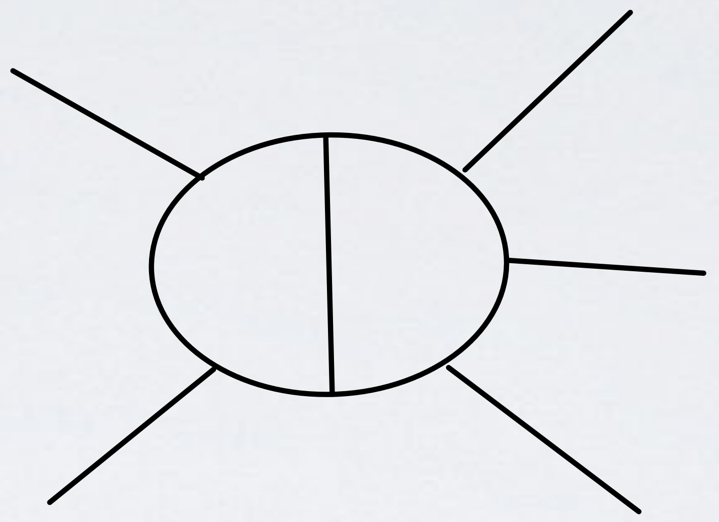


WHAT MAKES AMPLITUDES COMPLICATED ?

- Amplitudes become more difficult due to
  - Number of loops, number of legs
  - Number of scales
  - Massive propagators
  - Non-planar diagrams (subleading color)
- Complexity shows up in
  - Complexity of rational functions (combinatorics, higher degree denominators)
  - Functions space (multiple polylogs, elliptic, K3, more complicated Calabi-Yau)

FULL-COLOR MASSLESS QCD AMPLITUDES





$q\bar{q} \to \gamma\gamma j$

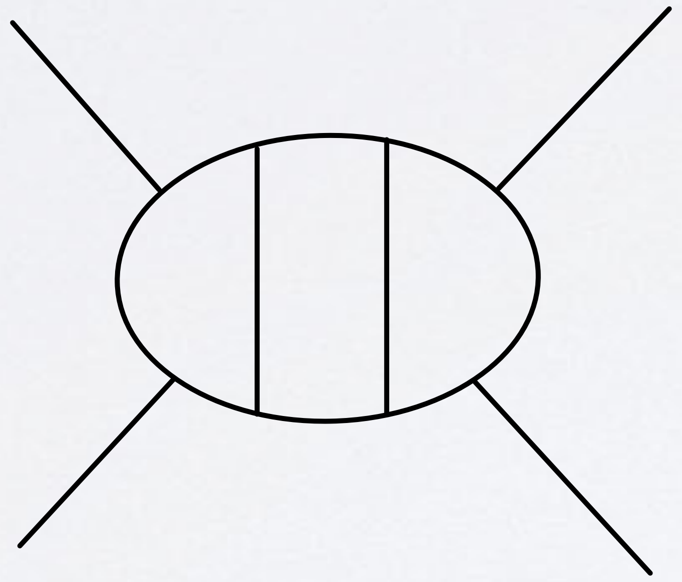
[Agarwal, Buccioni, AvM, Tancredi '21]

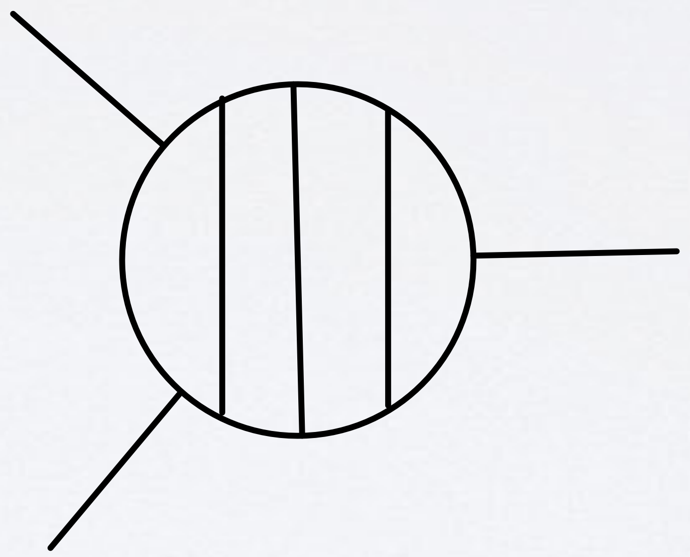
$gg \to \gamma\gamma j$

[Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]

$q\bar{q} \to \gamma\gamma\gamma$

[Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov '23]





$q\bar{q} \to \gamma\gamma$

[Caola, AvM, Tancredi '20]

$gg \to \gamma\gamma$

[Bargiela, Caola, AvM, Tancredi '21]

$q\bar{q} \to q'\bar{q}'$, $gg \to gg$, $q\bar{q} \to gg$:

[Caola, Chakraborty, Gambuti, AvM, Tancredi '21,'21,'22]

$q\bar{q} \to \gamma g$:

[Bargiela, Chakraborty, Gambuti '22]

$q\bar{q} \to \gamma^*$, $gg \to H$

[Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]

$b\bar{b} \to H$

[Chakraborty, Huber, Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '21]

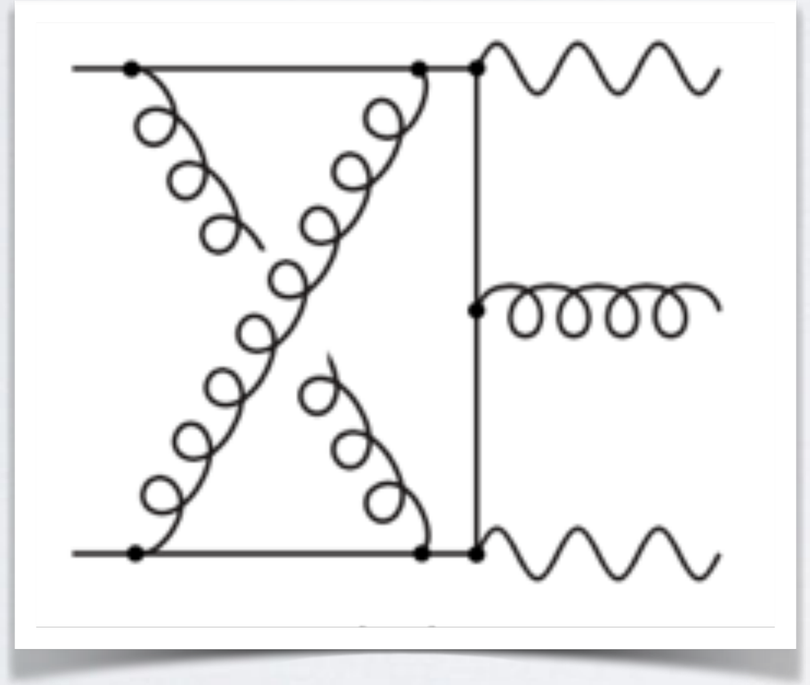
RELEVANCE OF SUBLEADING COLOR

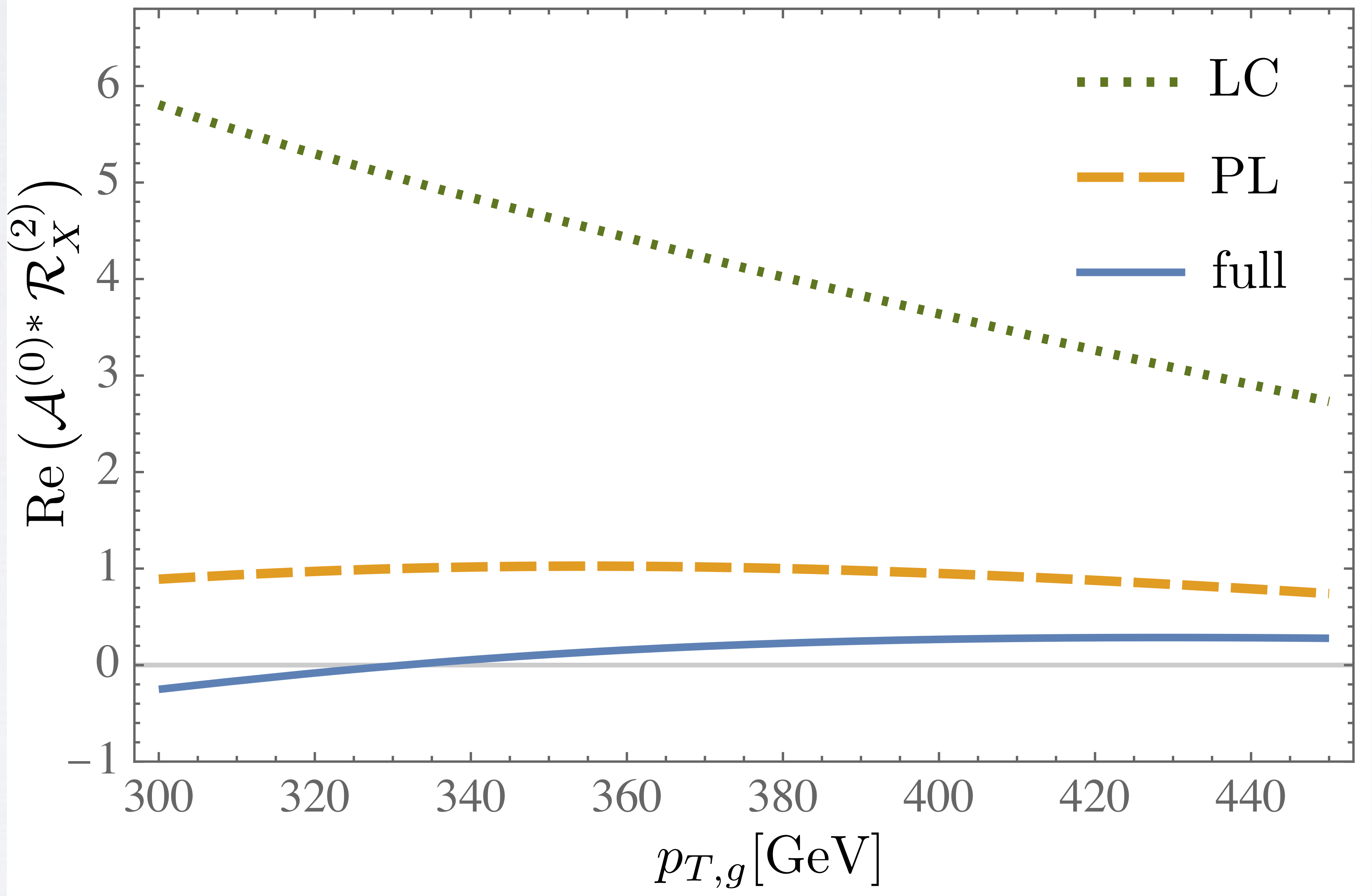
Ex.: $\gamma\gamma j$ @ NNLO: result w/ leading color virtual: *[Chawdhry, Czakon, Mitov, Poncelet 2021]*

Public library for master integrals: PentagonFunctions *[Chicherin, Sotnikov '20]*

Leading color easier to calculate, but not always a good approximation:

e.g. 2-loop finite remainder for $u\bar{u} \to g\gamma\gamma$ in Catani's scheme:



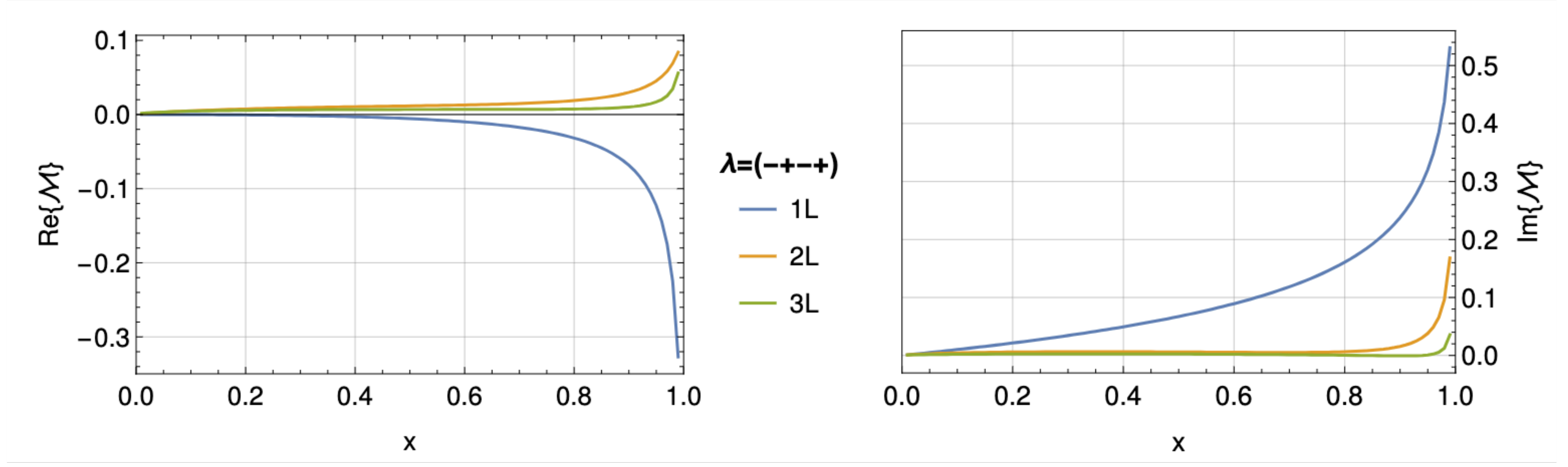


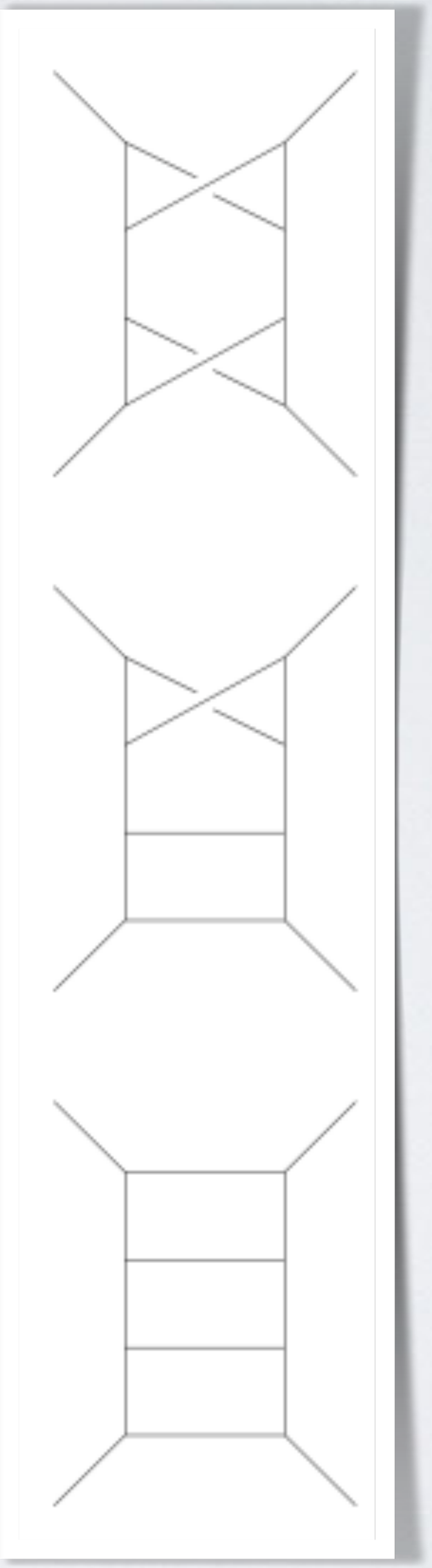
[Agarwal, Buccioni, AvM, Tancredi '21]

COMPLEXITY FOR INCREASING #LOOPS

- $gg \to \gamma\gamma$ helicity amplitudes: *[Bargiela, Caola, AvM, Tancredi '21]*
  - Symbolic intermediate expressions sizable but allow for easy crossings, simple workflow
  - Compact analytical results for amplitudes

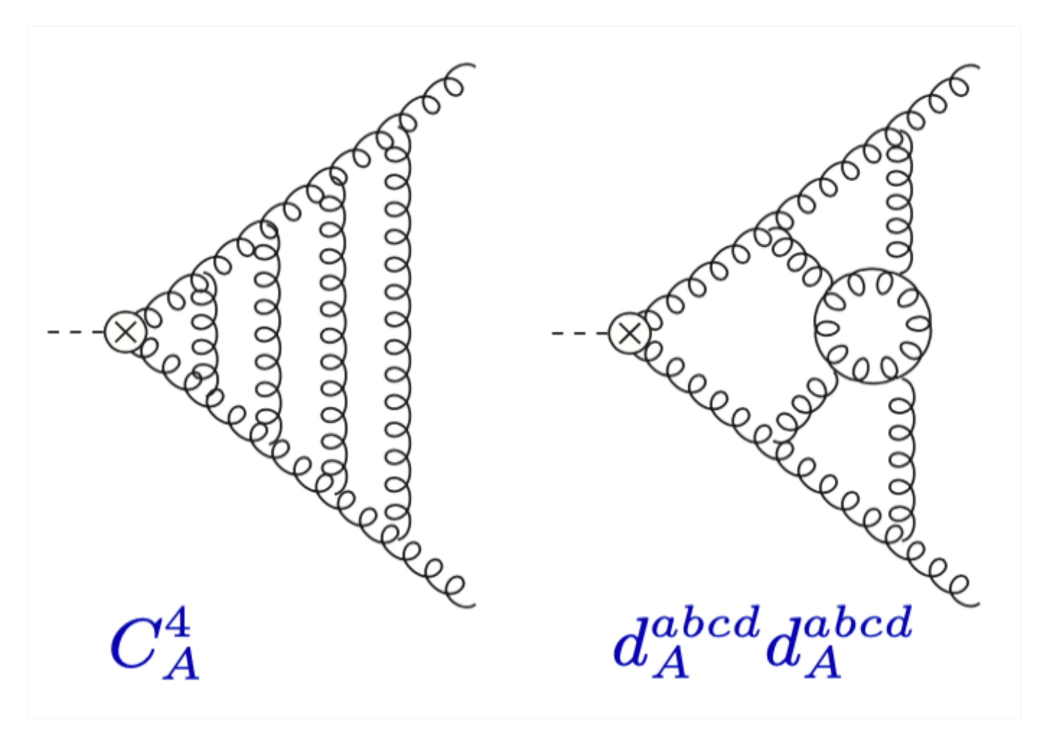
	1L	2L	3L
Number of diagrams	6	138	3299
Number of inequivalent integral families	1	2	3
Number of integrals before IBPs and symmetries	209	20935	4370070
Number of master integrals	6	39	486
Size of the `Qgraf` result [kB]	4	90	2820
Size of the `Form` result before IBPs and symmetries [kB]	276	54364	19734644
Size of helicity amplitudes written in terms of MIs [kB]	12	562	304409
Size of helicity amplitudes written in terms of HPLs [kB]	136	380	1195

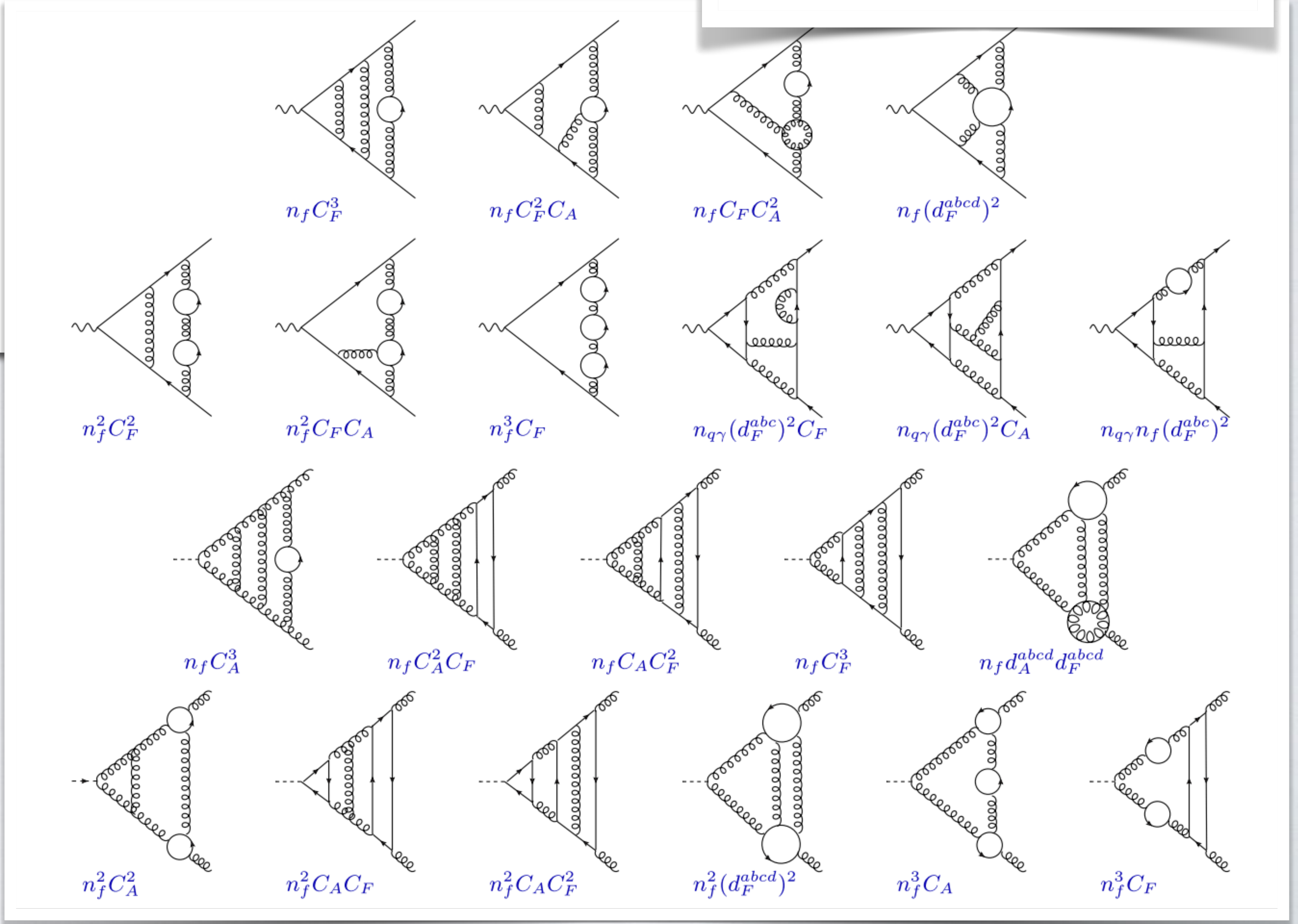




QCD 4 LOOP FORM FACTORS

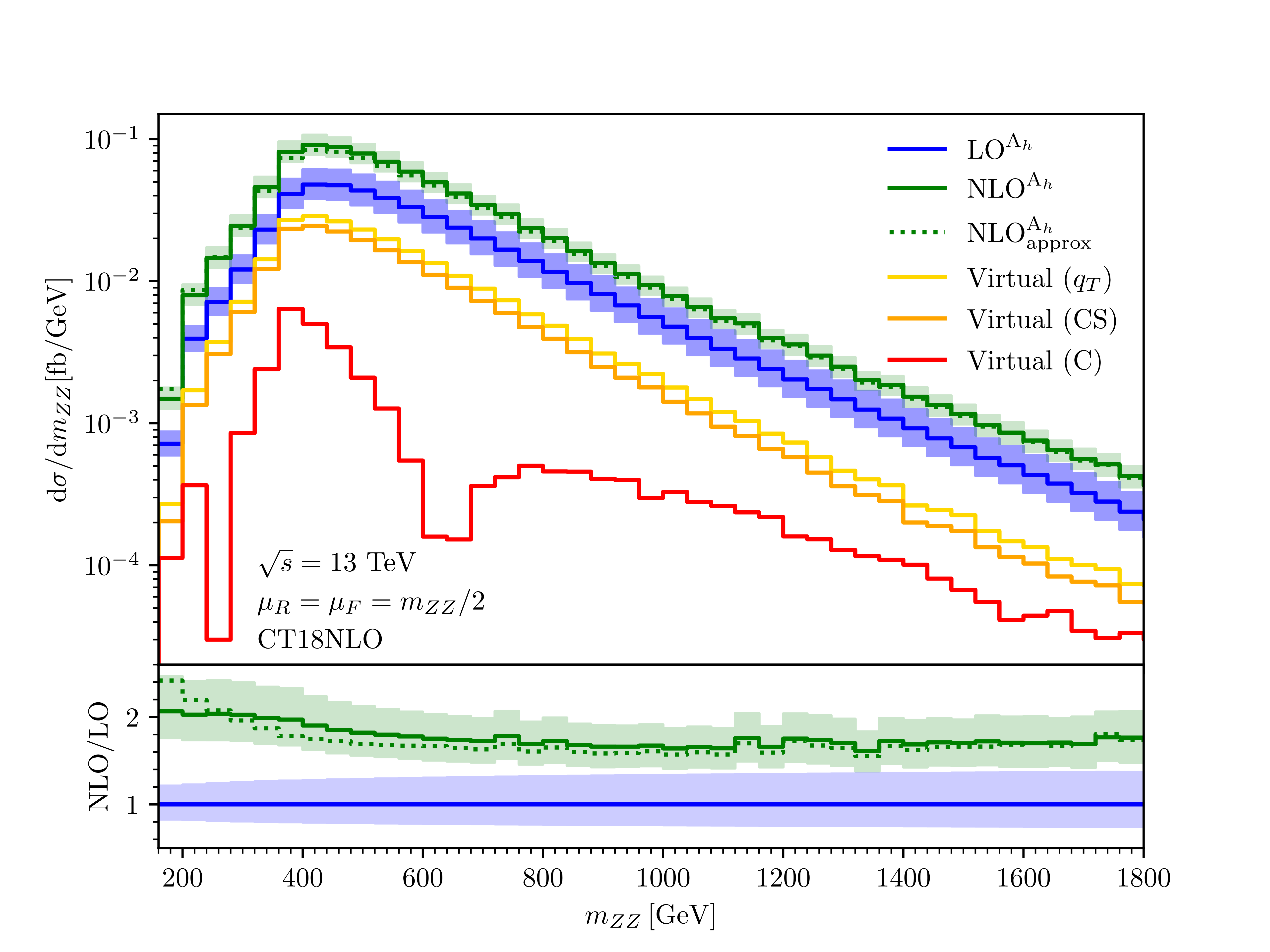
$$
\begin{aligned}
F^{\text{fin}}_{g,4} = {} & C_A^4\Big(-\frac{181}{30}\zeta_{5,3}+\frac{2377}{6}\zeta_5\zeta_3+\frac{271}{9}\zeta_3^2\zeta_2+\frac{4583689}{27000}\zeta_2^4-\frac{224939}{72}\zeta_7+\frac{5423}{6}\zeta_5\zeta_2+\frac{18931}{90}\zeta_3\zeta_2^2+\frac{418801}{162}\zeta_3^2+\frac{353093}{1620}\zeta_2^3+\frac{1203647}{135}\zeta_5-\frac{1806605}{486}\zeta_3\zeta_2-\frac{778313}{5832}\zeta_2^2-\frac{47586469}{1944}\zeta_3+\frac{32379341}{104976}\zeta_2+\frac{5165679667}{139968}\Big) \\
& +\frac{d_A^{abcd}d_A^{abcd}}{N_A}\Big(260\,\zeta_{5,3}-5092\,\zeta_5\zeta_3-16\,\zeta_3^2\zeta_2-\frac{496766}{525}\zeta_2^4-\frac{6776}{3}\zeta_7-5016\,\zeta_5\zeta_2+\frac{2992}{3}\zeta_3\zeta_2^2+\frac{31588}{3}\zeta_3^2+\frac{1073972}{945}\zeta_2^3-6460\,\zeta_5+\frac{6752}{9}\zeta_3\zeta_2+\frac{24616}{45}\zeta_2^2+\frac{68410}{9}\zeta_3-\frac{4682}{27}\zeta_2-\frac{1310}{9}\Big) \\
& +n_fC_A^3\Big(-\frac{8390}{9}\zeta_7+\frac{991}{9}\zeta_5\zeta_2-\frac{2129}{45}\zeta_3\zeta_2^2-\frac{32425}{324}\zeta_3^2-\frac{702253}{5670}\zeta_2^3+\frac{566977}{540}\zeta_5+\frac{67831}{162}\zeta_3\zeta_2-\frac{2333729}{29160}\zeta_2^2+\frac{9686917}{1944}\zeta_3+\frac{113944685}{104976}\zeta_2-\frac{20463665839}{839808}\Big) \\
& +n_fC_A^2C_F\Big(\frac{16003}{12}\zeta_7+\frac{230}{9}\zeta_5\zeta_2-\frac{44}{15}\zeta_3\zeta_2^2-\frac{1787}{3}\zeta_3^2+\frac{32254}{945}\zeta_2^3+\frac{143197}{36}\zeta_5+\frac{78590}{81}\zeta_3\zeta_2-\frac{44839}{540}\zeta_2^2+\frac{8317937}{1944}\zeta_3-\frac{293267}{3888}\zeta_2-\frac{573672965}{46656}\Big) \\
& +n_fC_AC_F^2\Big(-\frac{9580}{3}\zeta_7-300\,\zeta_5\zeta_2+12\,\zeta_3\zeta_2^2-368\,\zeta_3^2-\frac{39328}{945}\zeta_2^3-\frac{92317}{18}\zeta_5+\frac{193}{3}\zeta_3\zeta_2-5\,\zeta_2^2+\frac{700879}{108}\zeta_3-\frac{217}{36}\zeta_2+\frac{1156175}{1296}\Big) \\
& +n_fC_F^3\Big(3360\,\zeta_7-2940\,\zeta_5-156\,\zeta_3+\frac{169}{2}\Big) \\
& +n_f\frac{d_A^{abcd}d_F^{abcd}}{N_A}\Big(\frac{2464}{3}\zeta_7+1824\,\zeta_5\zeta_2-\frac{1088}{3}\zeta_3\zeta_2^2-\frac{15700}{3}\zeta_3^2-\frac{245536}{945}\zeta_2^3+\frac{108692}{9}\zeta_5+\frac{1544}{9}\zeta_3\zeta_2-\frac{35108}{45}\zeta_2^2-\frac{89932}{9}\zeta_3+\frac{9580}{27}\zeta_2+\frac{6944}{9}\Big) \\
& +n_f^2C_A^2\Big(\frac{9452}{81}\zeta_3^2+\frac{15044}{945}\zeta_2^3-\frac{38071}{135}\zeta_5+\frac{3113}{486}\zeta_3\zeta_2+\frac{78953}{3240}\zeta_2^2+\frac{1103621}{1944}\zeta_3-\frac{25105537}{104976}\zeta_2+\frac{3255482741}{839808}\Big) \\
& +n_f^2C_AC_F\Big(-270\,\zeta_3^2-\frac{10084}{945}\zeta_2^3-\frac{23572}{27}\zeta_5-\frac{944}{9}\zeta_3\zeta_2-\frac{764}{135}\zeta_2^2-\frac{724883}{486}\zeta_3-\frac{4790}{27}\zeta_2+\frac{48037931}{11664}\Big) \\
& +n_f^2C_F^2\Big(\frac{800}{3}\zeta_3^2+\frac{13696}{945}\zeta_2^3+\frac{3920}{3}\zeta_5+\frac{32}{3}\zeta_3\zeta_2-\frac{212}{15}\zeta_2^2-1592\,\zeta_3+\frac{58}{9}\zeta_2+\frac{32137}{216}\Big) \\
& +n_f^2\frac{d_F^{abcd}d_F^{abcd}}{N_A}\Big(512\,\zeta_3^2-960\,\zeta_5+\frac{384}{5}\zeta_2^2+1520\,\zeta_3-\frac{9008}{9}\Big) \\
& +n_f^3C_A\Big(-\frac{194}{15}\zeta_5+\frac{124}{27}\zeta_3\zeta_2-\frac{944}{405}\zeta_2^2-\frac{17818}{243}\zeta_3+\frac{9430}{729}\zeta_2-\frac{8399887}{52488}\Big) \\
& +n_f^3C_F\Big(\frac{640}{27}\zeta_5-\frac{64}{9}\zeta_3\zeta_2+\frac{112}{45}\zeta_2^2+\frac{4060}{27}\zeta_3+\frac{64}{3}\zeta_2-\frac{233953}{972}\Big)
\end{aligned}
$$

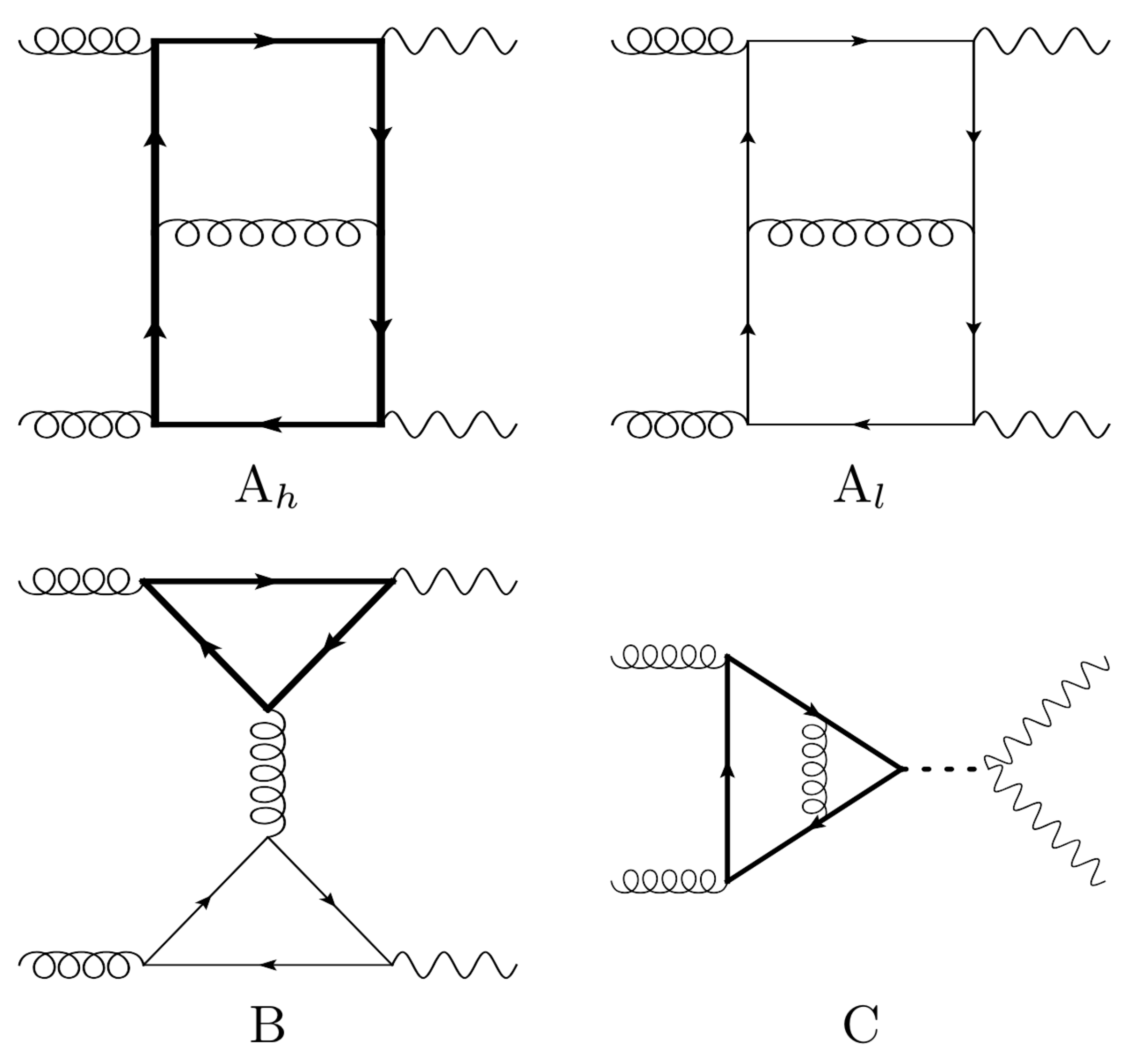


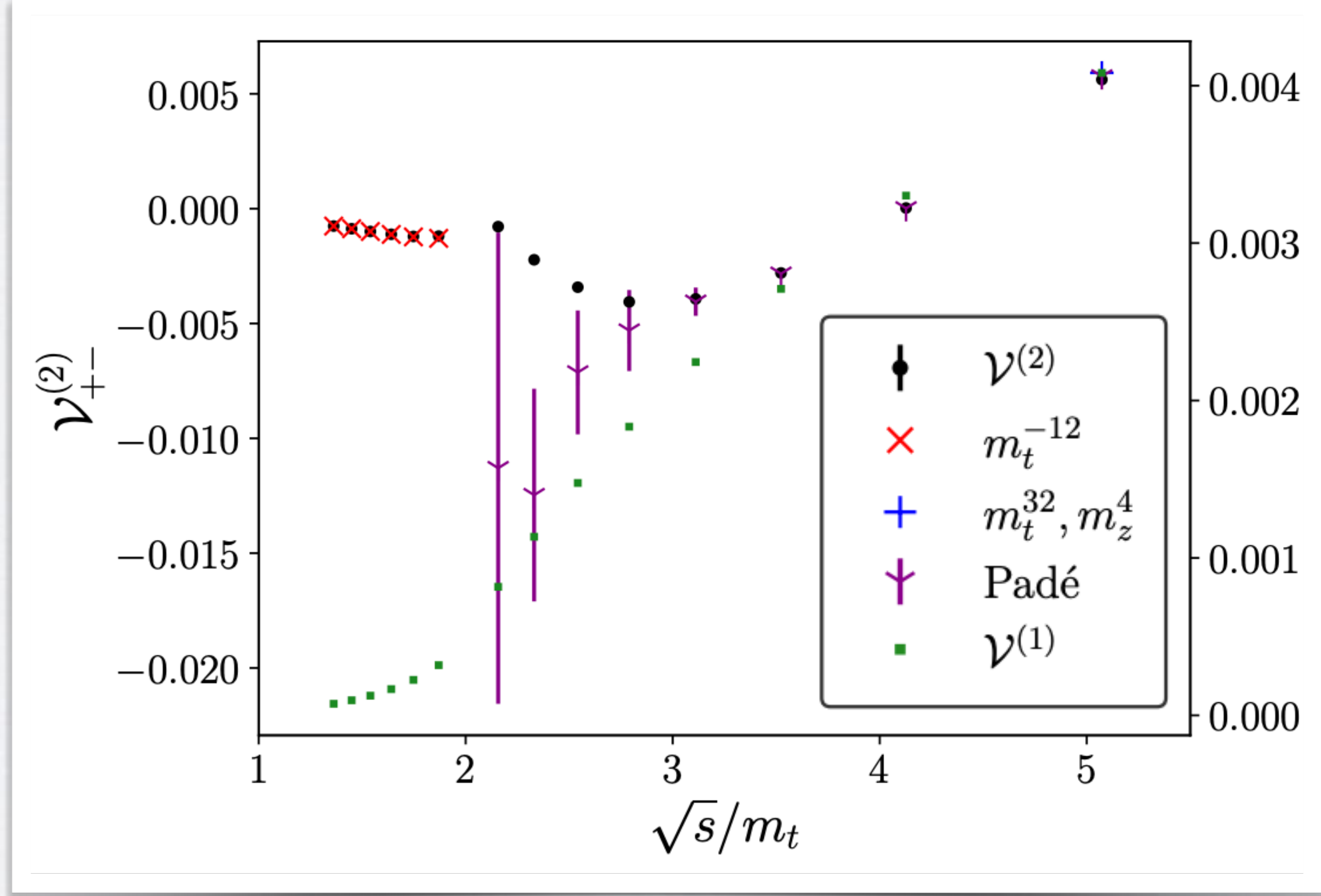


- $ggH, q\bar{q}\gamma^*$ [Lee, AvM, Schabinger, Steinhauser, Smirnov, Smirnov ’21, ’22]
- $b\bar{b}H$ [Chakraborty, Lee, Huber, AvM, Schabinger, Steinhauser, Smirnov, Smirnov ’22]

VIRTUAL CORRECTIONS ARE SCHEME DEPENDENT

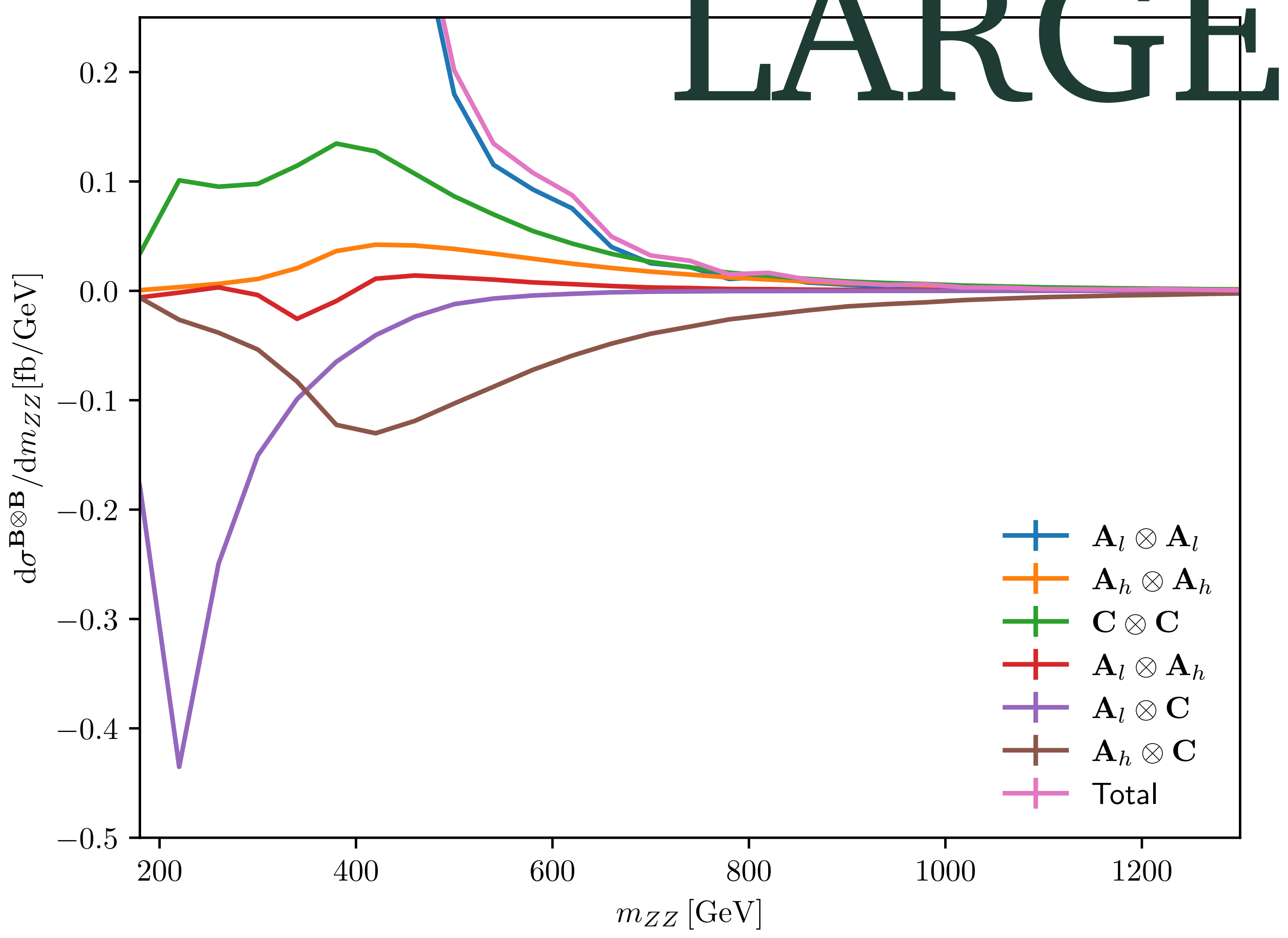


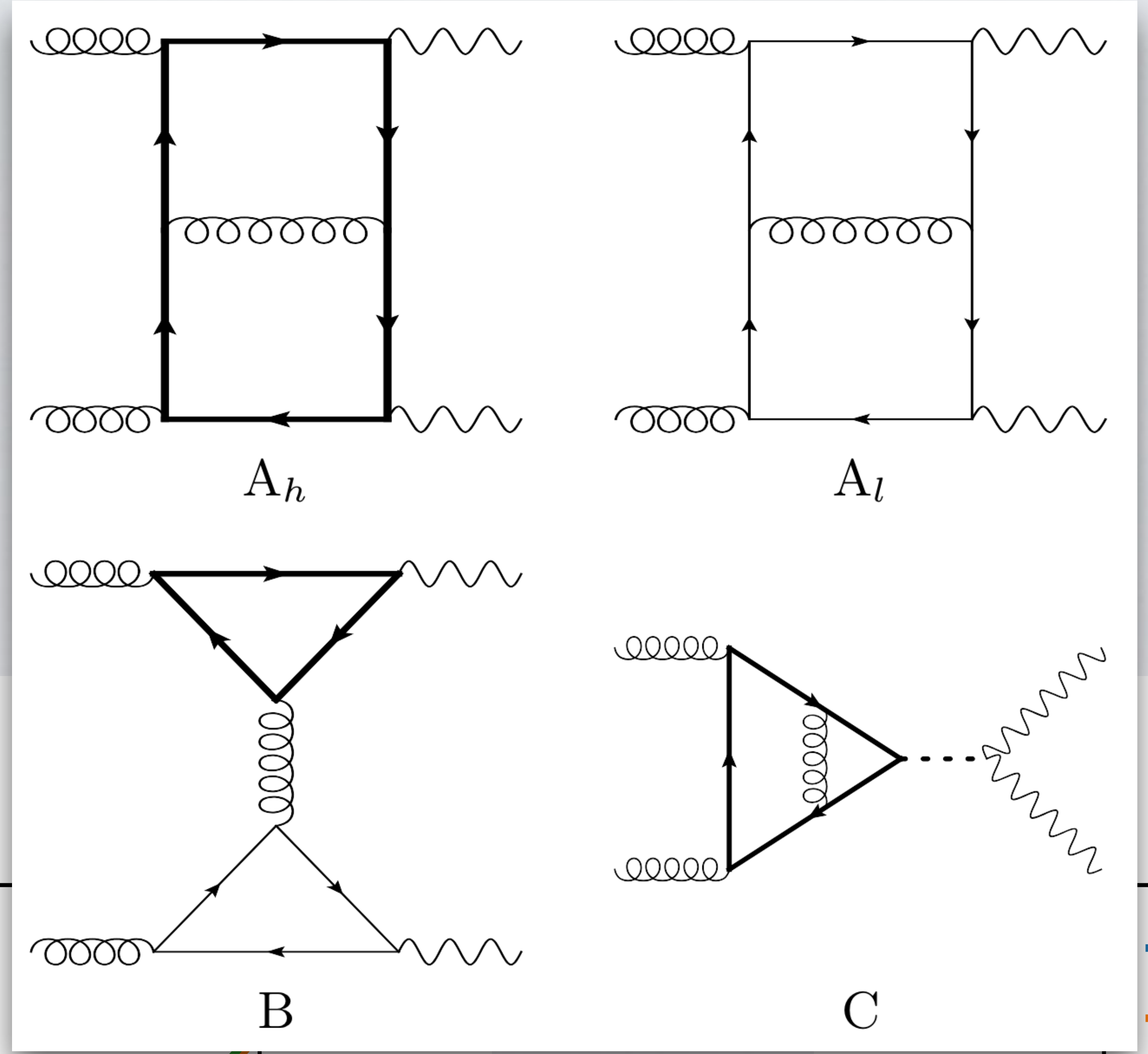


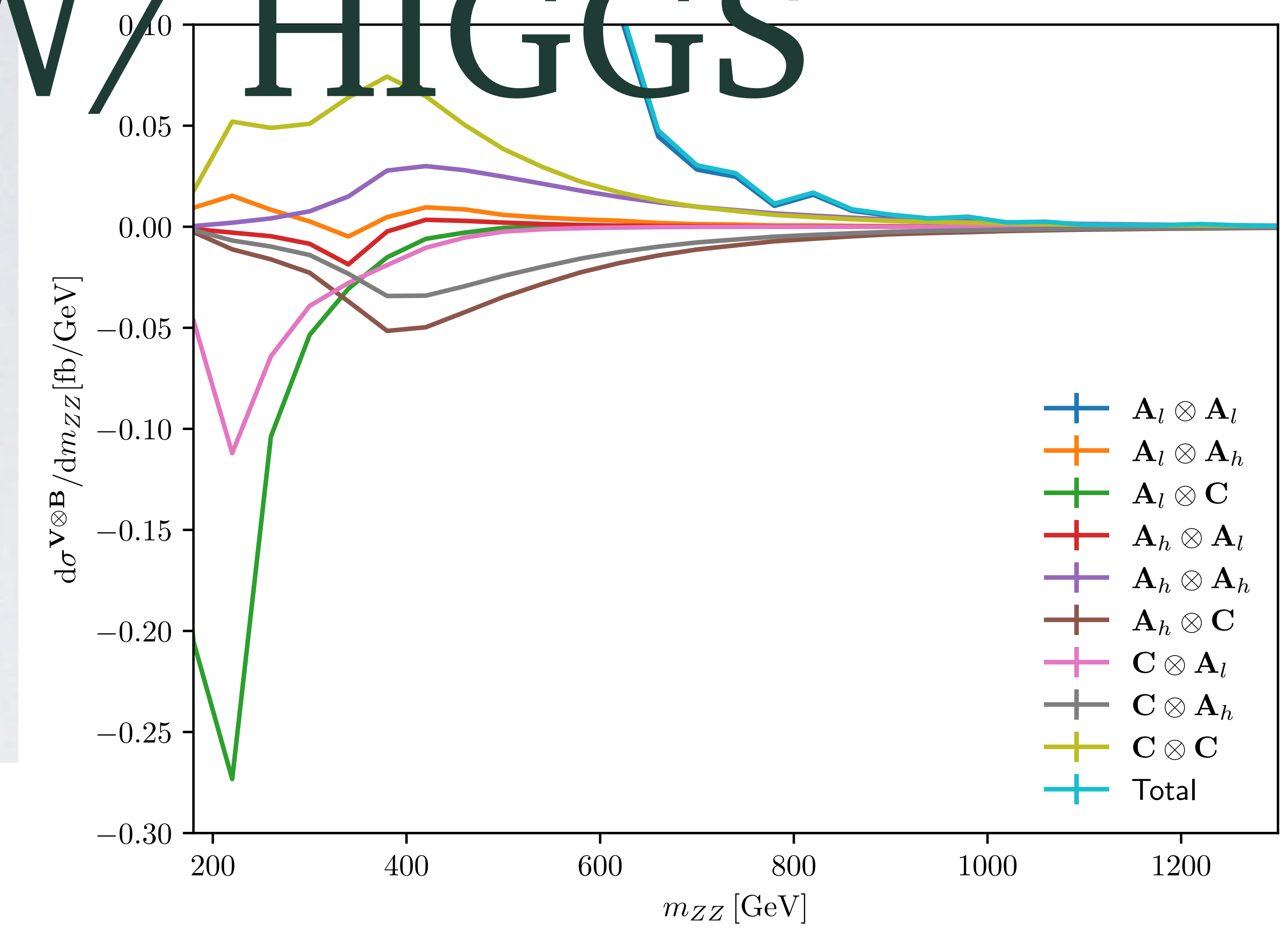


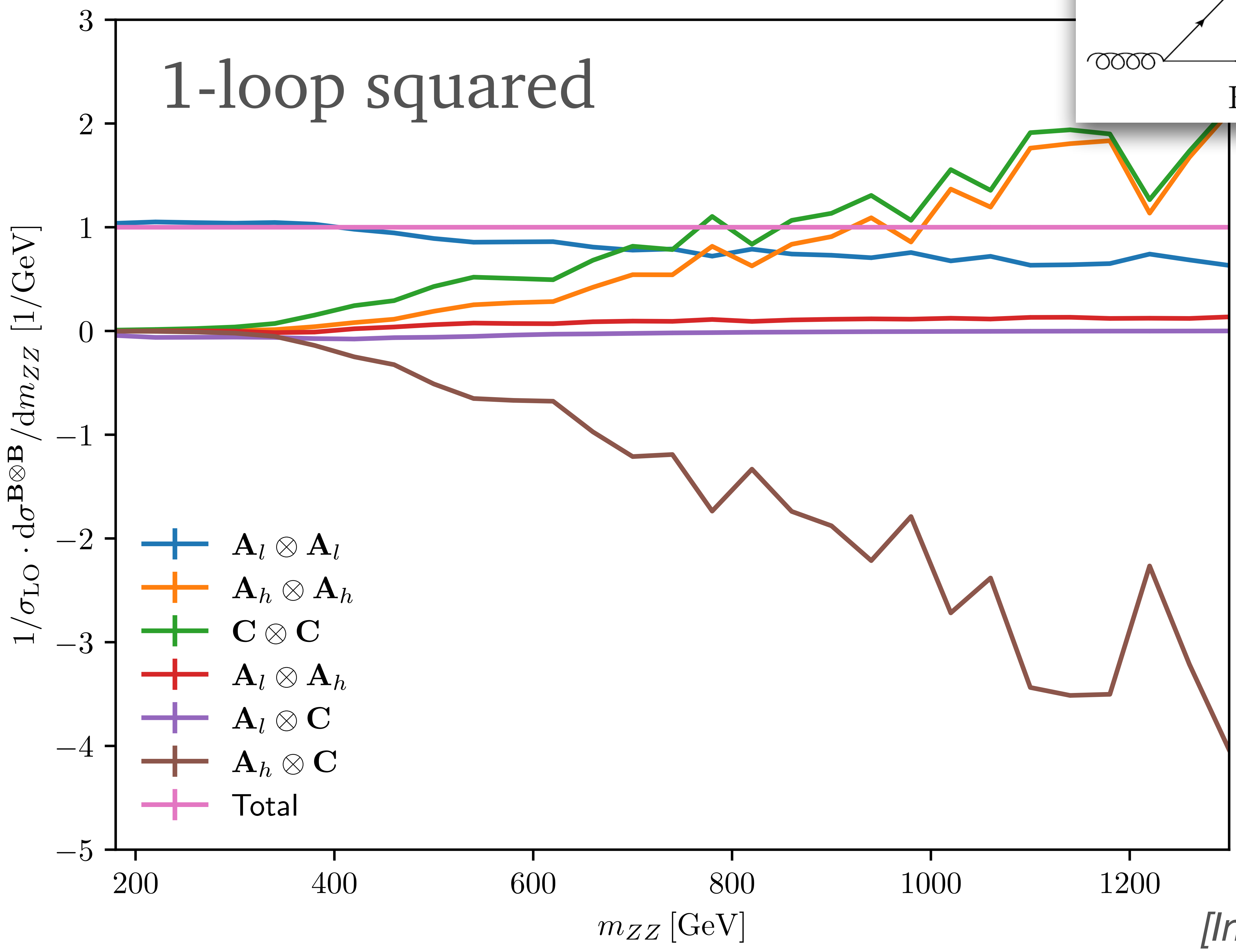
$gg \to ZZ$ top quark corrections *[Agarwal, Jones, Kerner, AvM, 2024]*

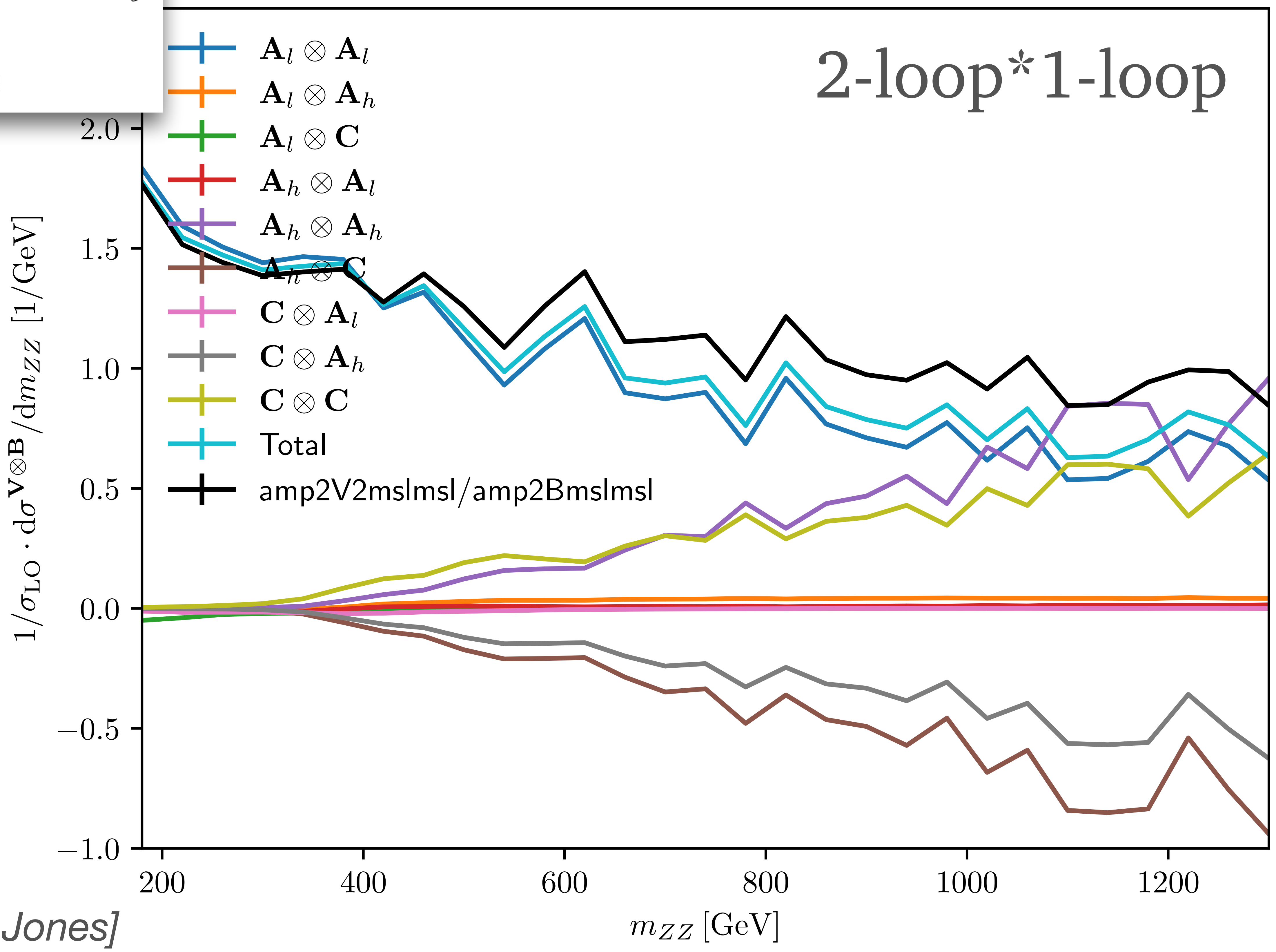
LARGE CANCELLATIONS W/ HIGGS





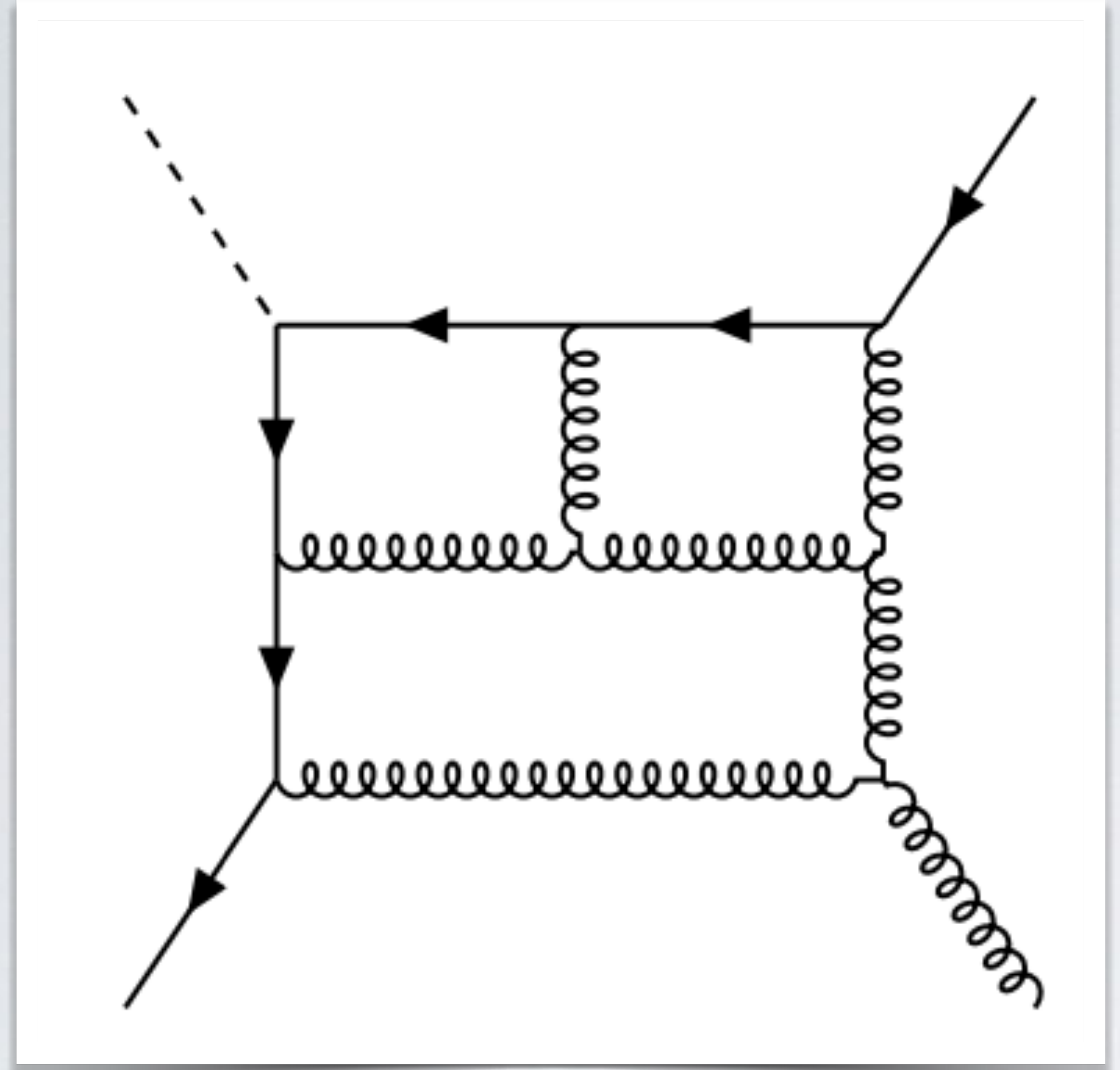






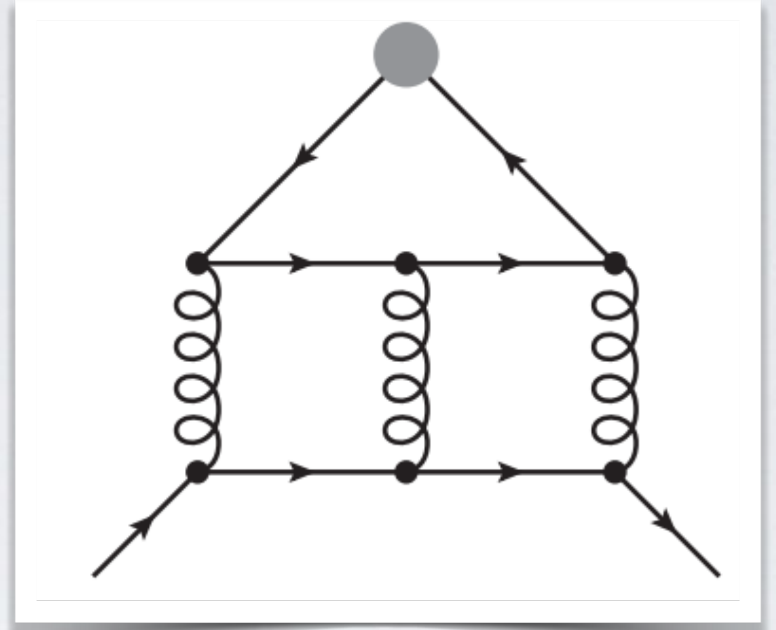
[Image credit: Stephen Jones]

AMPLITUDES, INTEGRALS: EXAMPLES W/ MORE LOOPS, MORE LEGS

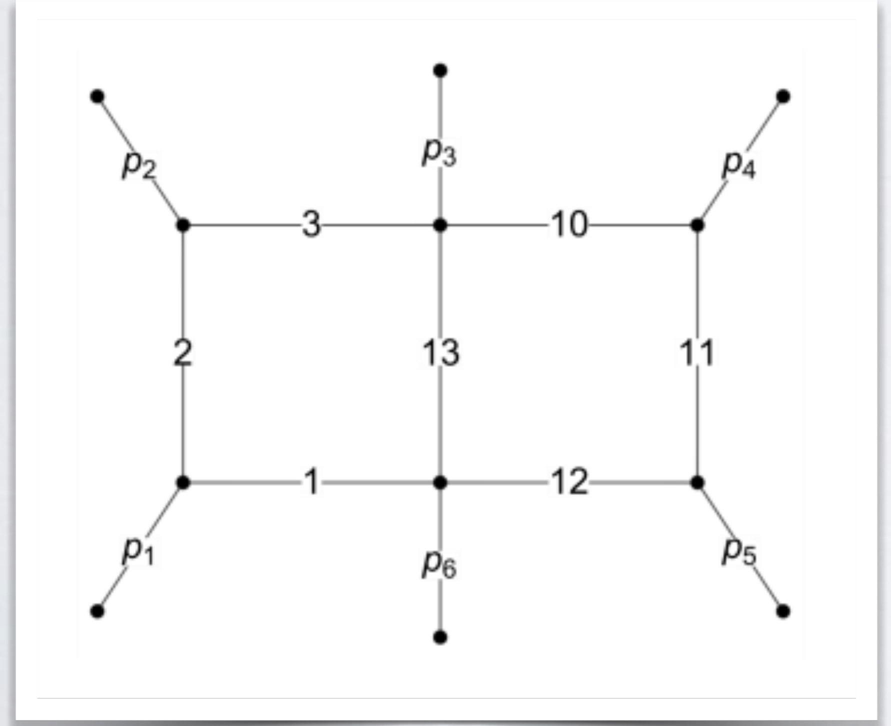


Vj: *[Vita, Mastrolia, Schubert, Yundin, Syrrakos '14], [Canko, Syrrakos '21], [Gehrmann, Jakubzik, Mella, Syrrakos, Tancredi '23]*

Hj: *[Bobadilla, Henn, Lim '23], [Canko, Syrrakos '23], [Bobadilla, Gehrmann, Henn, Jakubcik, Lim, Mella, Syrrakos, Tancredi]*

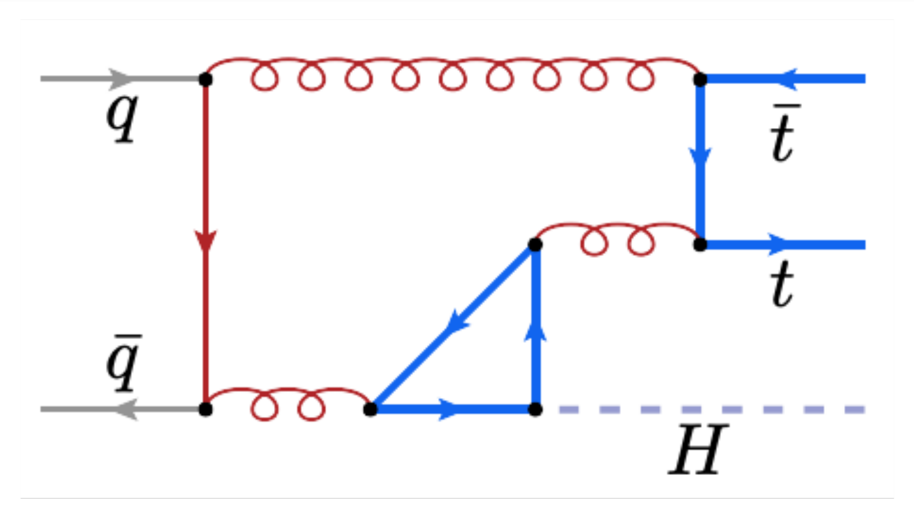


H with finite top mass: *[Fael, Lange, Schönwald, Steinhauser], [Czakon, Eschment, Niggetiedt, Poncelet, Schellenberger], [Niggetiedt, Usovitsch]*

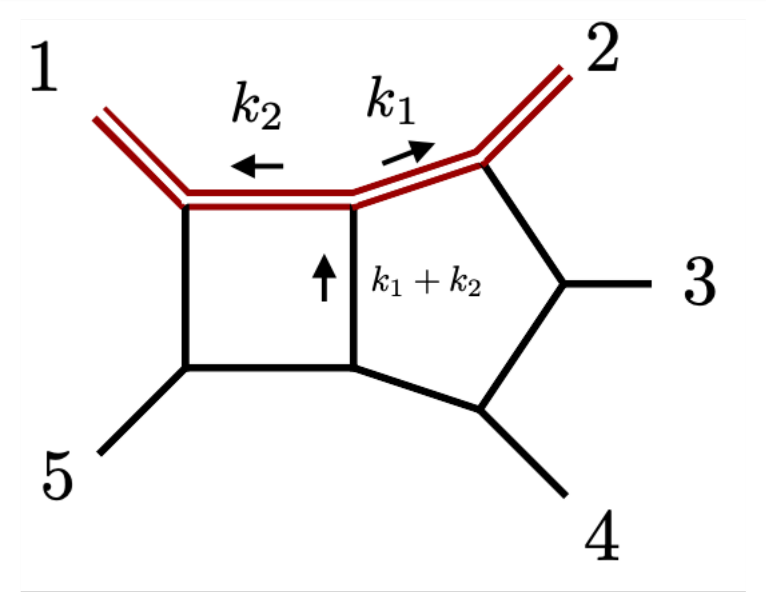


6pt integrals: *[Henn, Matijasic, Miczajka, Peraro, Xu, Zhang '24]*

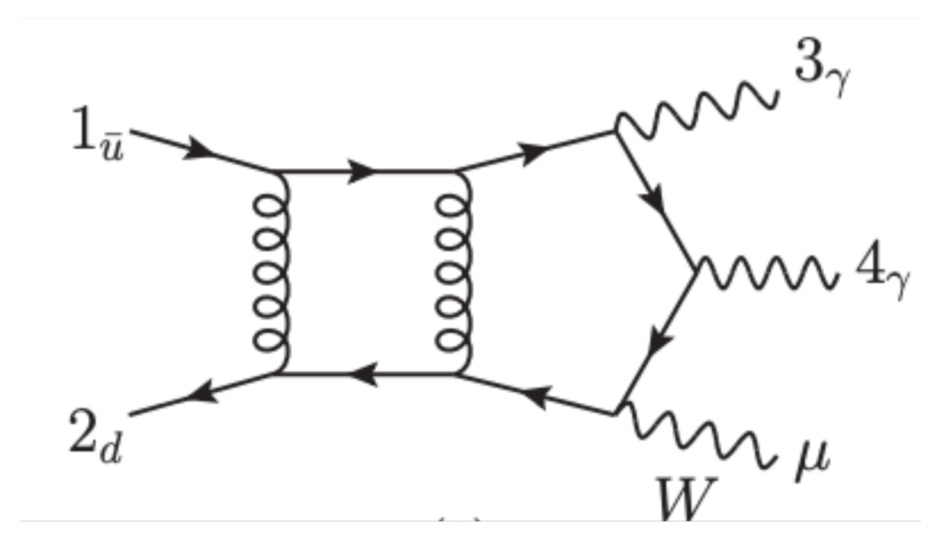
AMPLITUDES: EXAMPLES W/ MORE MASSES



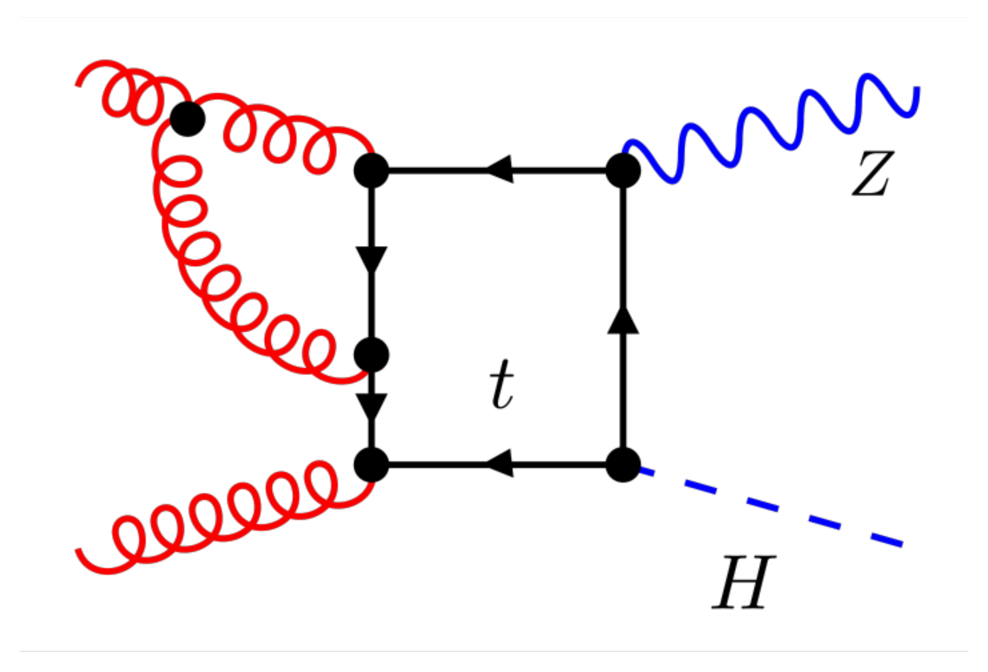
ttH: *[Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson '24], [Febres-Cordero, Figueiredo, Kraus, Page, Reina '23]*



ttj: *[Badger, Becchetti, Giraudo, Zoia '23]*



Vjj,Hjj,…: *[Badger, Hartanto, Krys, Zoia], [Abreu, Chicherin, Febres-Cordero, Ita, Klinkert, Page, Sotnikov, Tschernow, Zoia], [Mazzitelli, Sotnikov, Wiesemann]*



HZ: *[Hasselhuhn, Luthe Steinhauser], [Wang, Zu, Zu, Yang], [Chen, Davies, Jones, Kerner], [Degrassi, Gröber, Vitti, Zhao]*

BEYOND QCD: CHALLENGES WITH γ_5

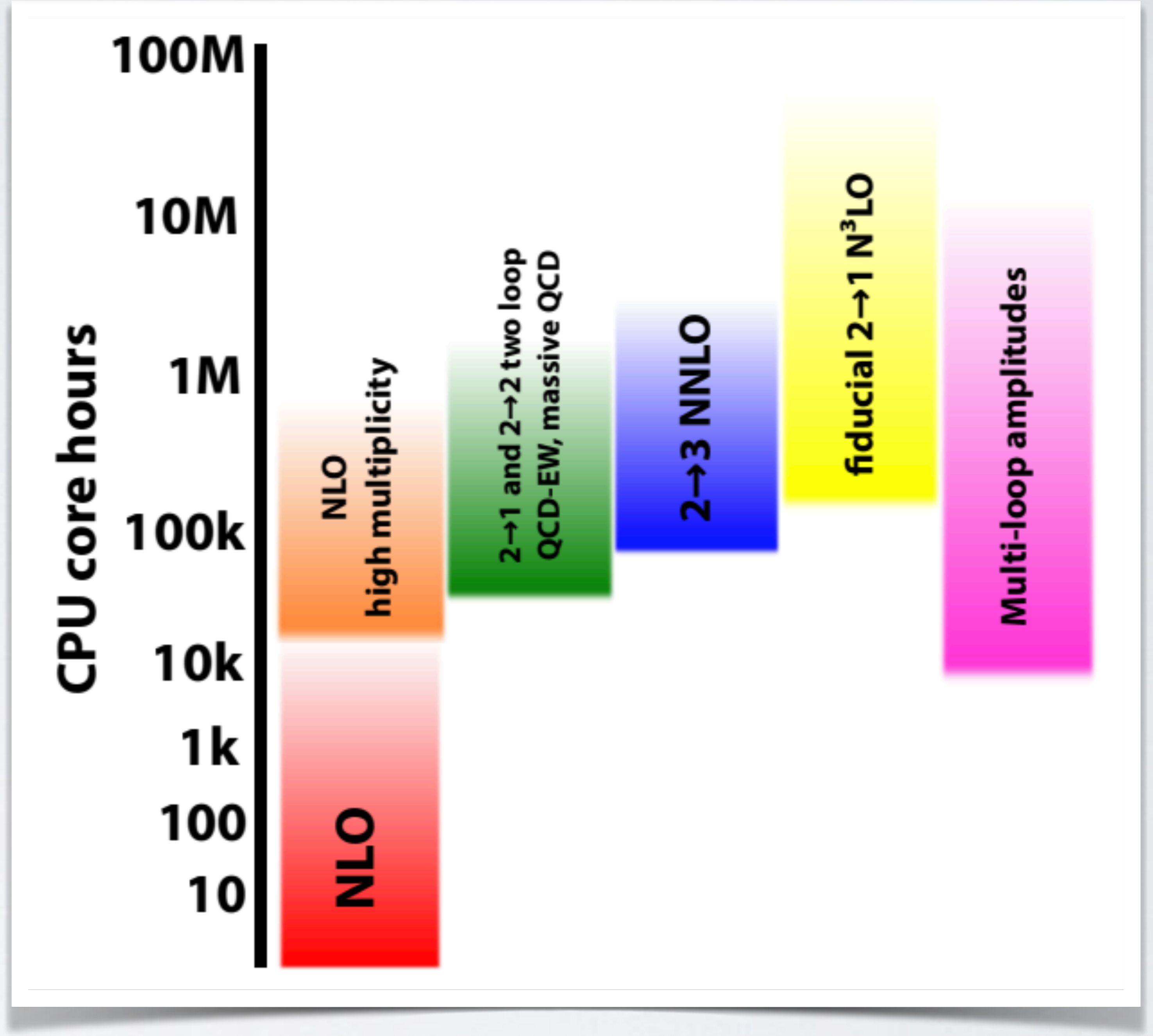
- In four space-time dimensions:
  - $\{\gamma_5, \gamma_\mu\} = 0$
  - $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$
  - $\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda) = -4i\epsilon^{\mu\nu\kappa\lambda}$
- But **anticommuting** γ_5 w/ **cyclic trace** in dimensional regularization:

  $\Rightarrow$ $(d-4)\,\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda\gamma^5) = 0$, i.e. inconsistent !

  $\Rightarrow$ need to give up some property
- **'t Hooft-Veltman,Breitenlohner-Maison** scheme:
  - give up anti-commutativity
  - violation of gauge symmetry, but can be fixed systematically
  - only known consistent scheme
- Always **technical implications** also for loop integrals ("μ" terms/tensor red.)

AMPLITUDES: STATUS AND OUTLOOK

- Amplitudes are bottleneck of fixed-order calculations
- Improvements due to better methods, better codes
- Want fast and reliable numerical evaluations
- (Semi-)numerical methods easier to automate, avoid expression swell
- Analytical insights can improve numerical performance
- Systematic treatment of γ_5



From: *Snowmass survey of 53 perturbative calculations* *[Febres-Cordero, AvM, Neumann '22]*