IJCLab Irène Joliot-Curie
Laboratoire de Physique des 2 Infinis

Adam Falkowski

On-shell amplitude approach to gravitational radiation in general relativity and beyond

06 February 2025

Review talk at RPP'25 in Annecy

Plan:

1. Introduction

2. Quantum Amplitudes and Gravitational Waves

3. Scalar Radiation in Scalar-Tensor Theories

AA, Marinellis
2411.12909

4. Summary

Introduction: gravitational waves detection

Image Credit: EGO*

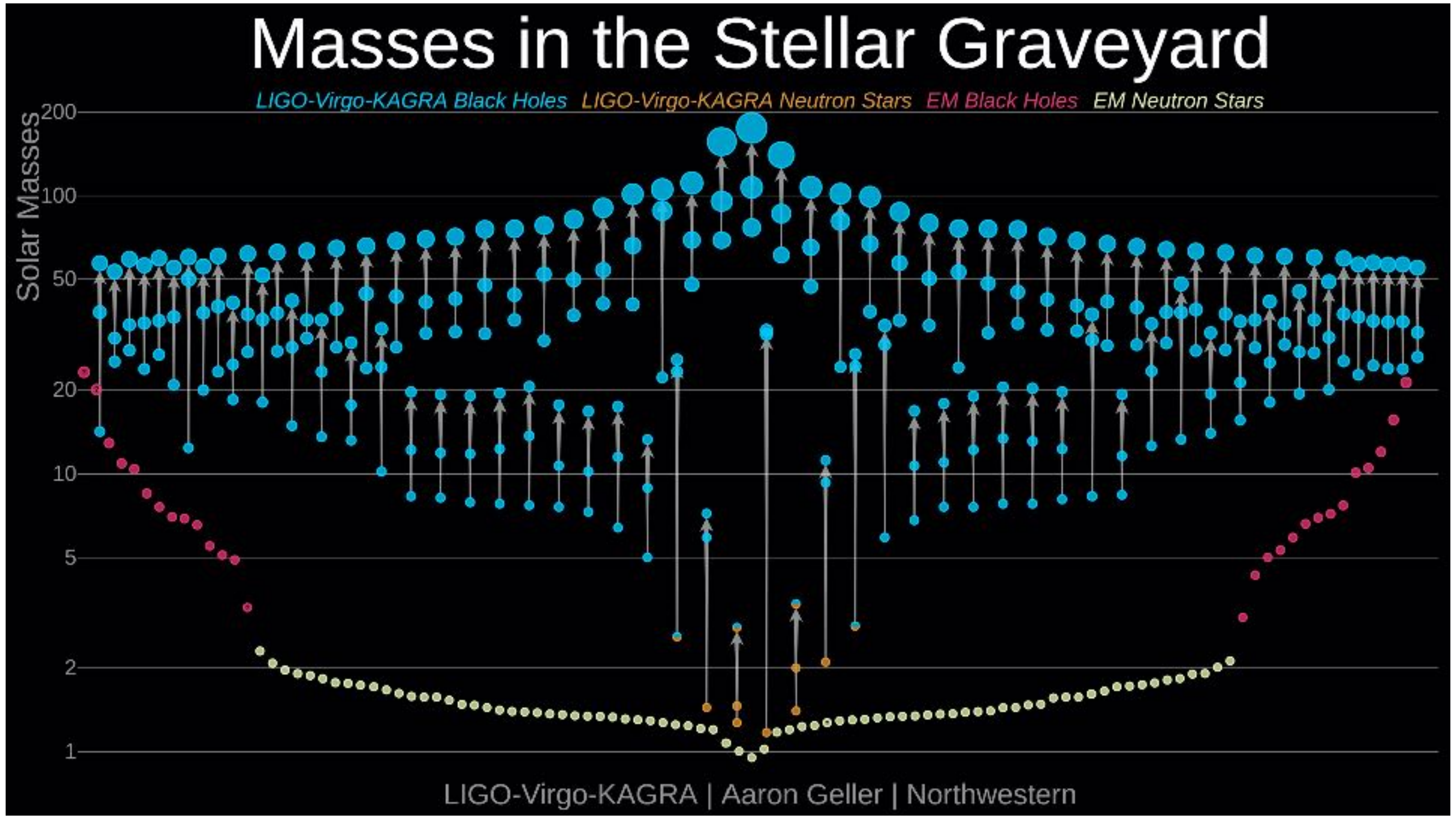
LIGO: First detection of **Gravitational Waves** (GWs) in **2015**

Many more black hole and neutron star mergers discovered by the LIGO-VIRGO-KAGRA collaboration

New era of precision measurements of GWs

Need for highly accurate GW templates, especially in view of the future upgrades of the LIGO/VIRGO/KAGRA network and future missions such as LISA, Einstein Telescope, Cosmic Explorer, Decigo, Tian-Qin, GEO-HF

New window to test physics beyond general relativity (GR)!



Credit:LIGO-Virgo/Aaron Geller/Northwestern University*

Introduction: gravitational waves detection

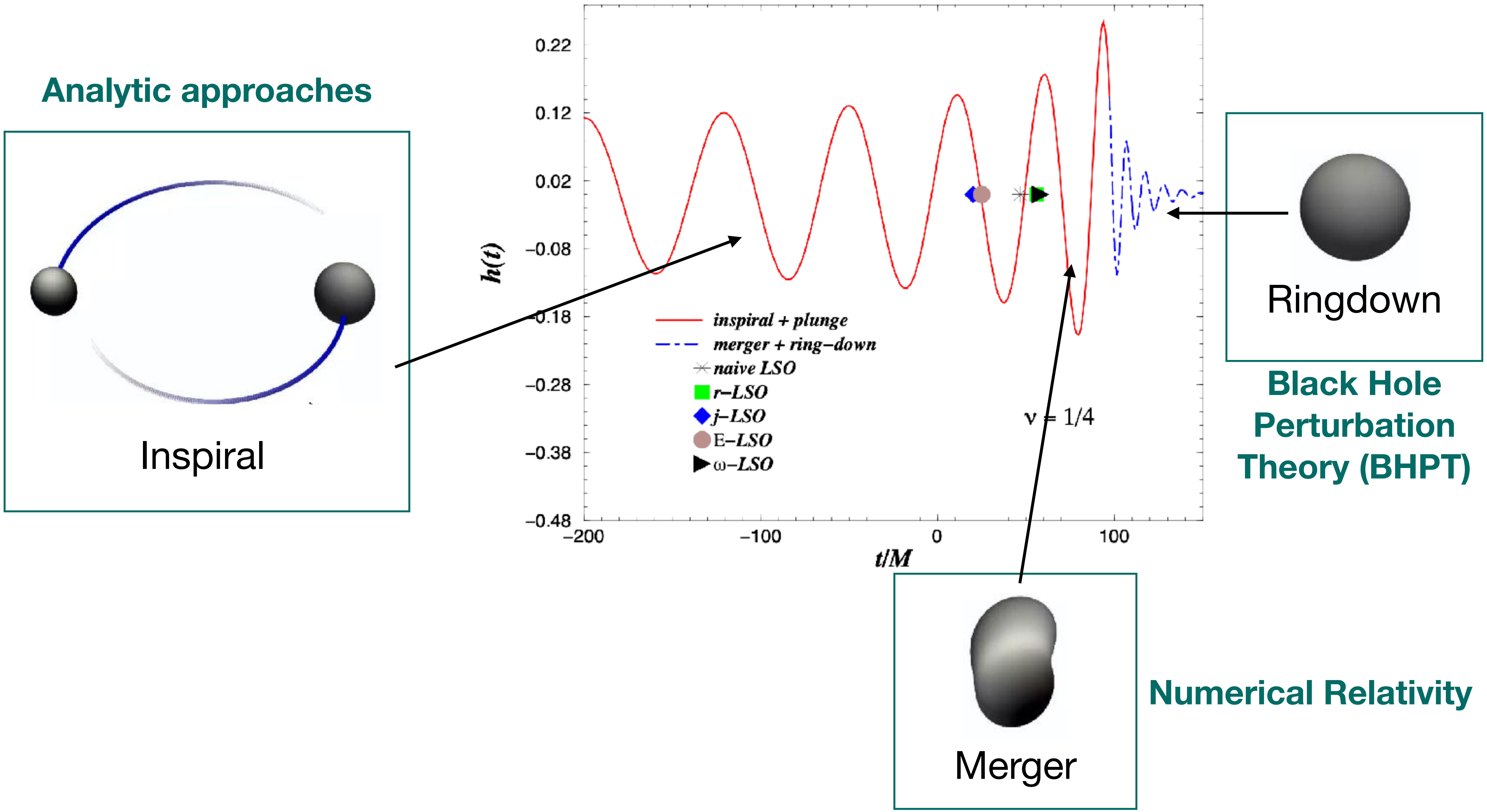
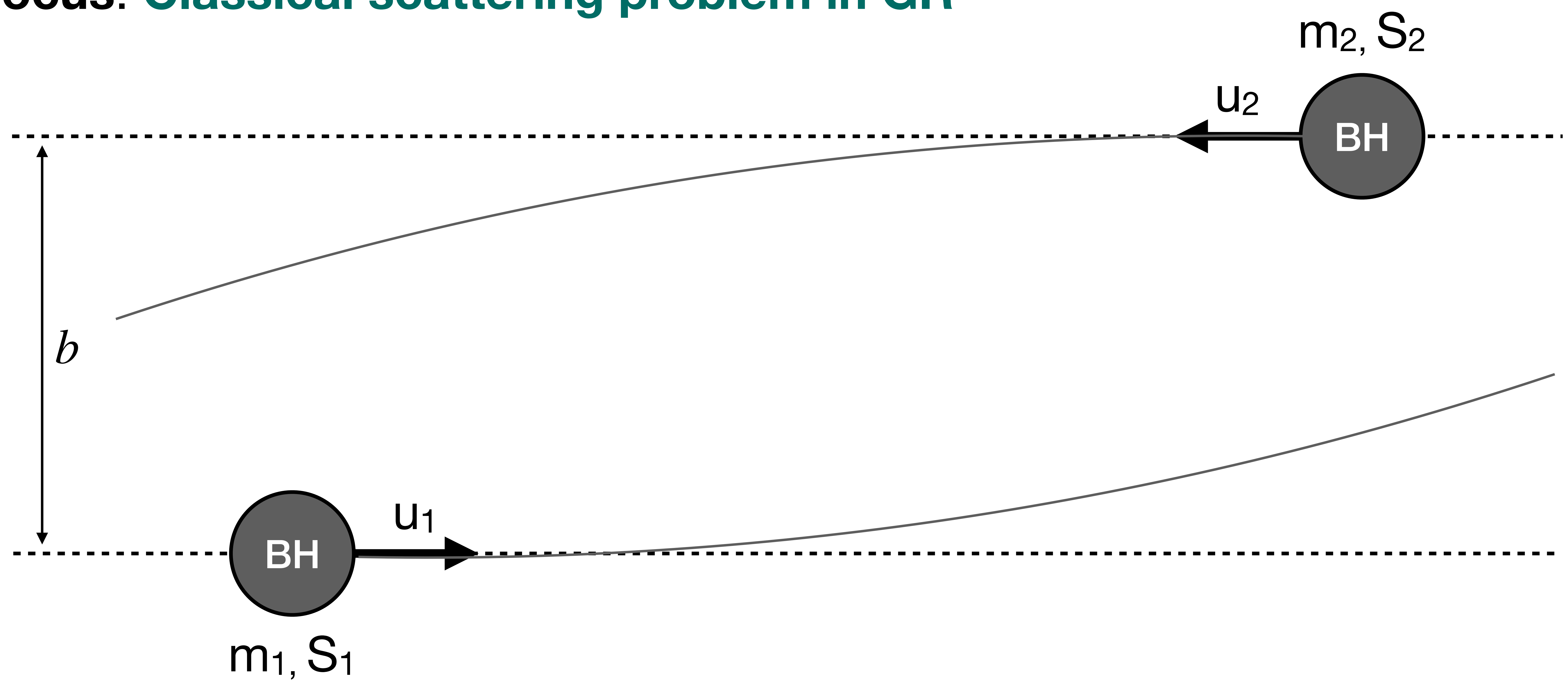


Illustration Credit: Max-Planck-Gesellschaft*

Quantum Amplitudes and Gravitational Radiation

Quantum Amplitudes and Classical Observables

- **Focus**: **Classical scattering problem in GR**



Flat background:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_{Pl}} h_{\mu\nu}$$

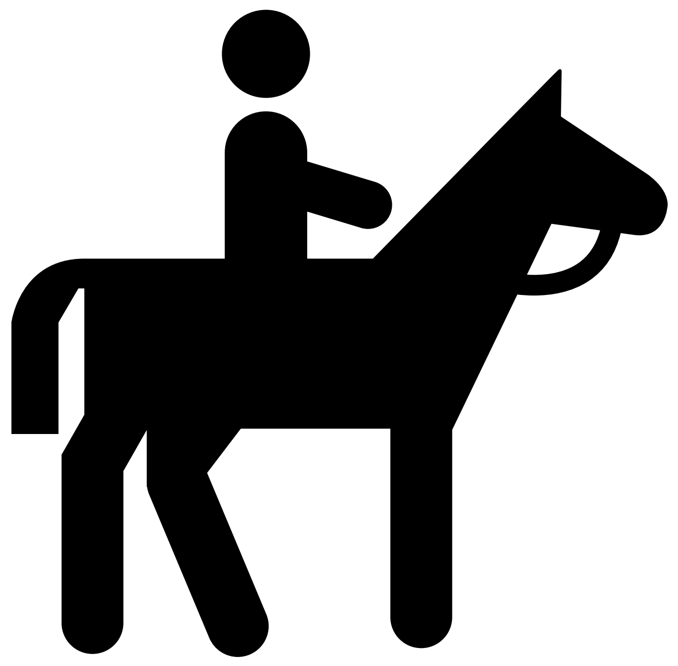
Large separation:

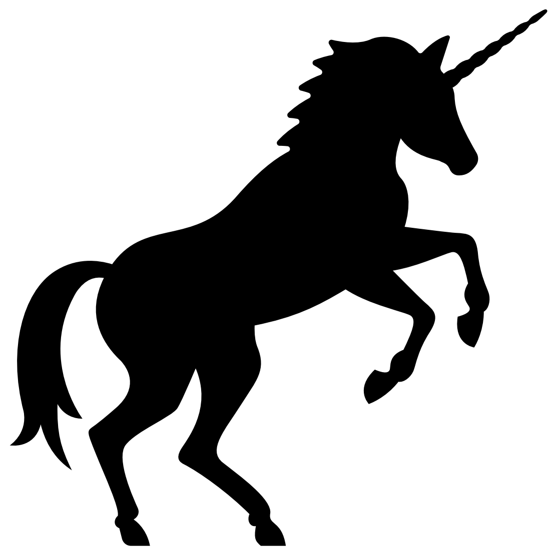
$$R_s \ll b$$

How can we describe this problem using Scattering Amplitudes computed in QFT?

Quantum Amplitudes and Radiation Observables

2 workhorses





KMOC formalism

Kosower, Maybee, O'Connell
[arXiv:1811.10950]

Cristofoli, R. Gonzo, D. A. Kosower, D. O'Connell
[arXiv:2107.10193]

On-shell methods

Arkani-Hamed, Huang, Huang
arXiv:1709.04891

Quantum Amplitudes and Classical Observables

Why employ quantum amplitudes for classical calculations?

- Computations organized in perturbative expansion with Lorentz covariance preserved at each step
- Often, **analytic results** in places where only numerical results previously available
- Can exploit many **modern techniques used in particle physics** to simplify calculation
- Easy to include spin of the colliding objects
- Can be easily applied to relevant theories such as QED or GR
- Can be straightforwardly **extended to beyond GR predictions**

One downside is that amplitudes naturally give scattering observables, while phenomenologically bound systems are more relevant.
The problem of bound-to-boundary continuation is not fully solved yet
Alternative formalism of WEFT where this particular problem is absent

KMOC formalism

One can define GR radiation observables as vacuum expectation values of metric field operators and its derivatives

GR

$$R_{\mu\nu} \equiv {}_{\text{out}}\langle \psi | h_{\mu\nu}(x) | \psi \rangle_{\text{out}}$$

or gauge invariant

$$R_{\mu\nu\alpha\beta} \equiv {}_{\text{out}}\langle \psi | R_{\mu\nu\alpha\beta}(x) | \psi \rangle_{\text{out}}$$

$$\boxed{R_h \equiv {}_{\text{out}}\langle \psi | h^{\mu\nu}(x) | \psi \rangle_{\text{out}} \epsilon^{-}_{\mu\nu}}$$

strain

Given radiation observable R_h one defines **waveform** W_h as

$$R_h(x) = \frac{W_h(t)}{|\boldsymbol{x}|} \qquad |\boldsymbol{x}| \to \infty \qquad t \equiv x^0 - |\boldsymbol{x}|$$

retarded time

Furthermore one defines **spectral waveform** or waveshape f_h as Fourier transform:

$$f_h(\omega) = \int dt e^{i\omega t} W_h(t)$$

KMOC formalism

Cristofoli, R. Gonzo, D. A. Kosower, D. O'Connell
[arXiv:2107.10193]

$$R_h \equiv {}_{\text{out}}\langle \psi | \, h^{\mu\nu}(x) \, | \psi \rangle_{\text{out}} \epsilon^{-}_{\mu\nu}$$

$$|\psi\rangle_{\text{out}} = S|\psi\rangle_{\text{in}} \quad \Rightarrow \quad R_h = {}_{\text{in}}\langle \psi | S^\dagger h^{\mu\nu}(x) S | \psi \rangle_{\text{in}} \epsilon^{-}_{\mu\nu}$$

"in-in observable"

$$|\psi\rangle_{\text{in}} = \Pi_{i=1,2}\left[\int d\Phi(p_i) f_i(p_i) e^{ip_i b_i}\right] |p_1 p_2\rangle_{\text{in}}$$

The radiation observables depends on an amplitude-like object

$$h_{\mu\nu} \sim \int d\Phi_k \, a_{\text{in}}(k) \, e^{-ikx} + \text{h.c} \quad \Rightarrow \quad R_h \supset {}_{\text{in}}\langle \psi | S^\dagger a_{\text{in}}(k) S | \psi \rangle_{\text{in}}$$

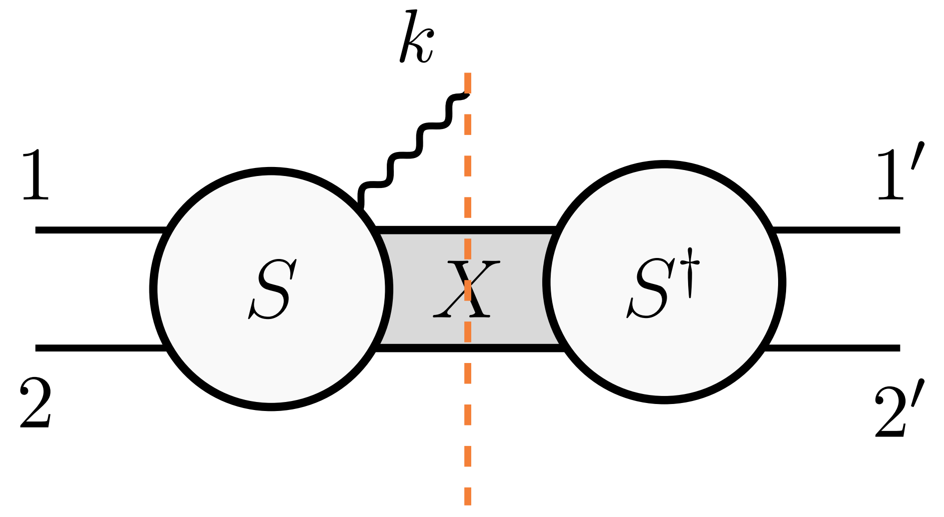
"generalized amplitude",
distinct from ordinary S matrix elements

$$ {}_{\text{out}}\langle \psi' | \psi \rangle_{\text{in}} = {}_{\text{in}}\langle \psi' | S^\dagger | \psi \rangle_{\text{in}}$$

Caron-Huot, Giroux, Hannesdottir, Mizera
arXiv:2310.12199

It can be related to usual S matrix elements via

$$ {}_{\text{in}}\langle \psi | S^\dagger a_{\text{in}}(k) S | \psi \rangle_{\text{in}} = \int d\Phi_X \;\; {}_{\text{in}}\langle \psi | S^\dagger | X \rangle_{\text{in}} \; {}_{\text{in}}\langle X | a_{\text{in}}(k) S | \psi \rangle_{\text{in}} = \int d\Phi_X \;\; {}_{\text{in}}\langle \psi | S^\dagger | X \rangle_{\text{in}} \; {}_{\text{in}}\langle X k | S | \psi \rangle_{\text{in}}$$

$\text{Exp}_k =$ 

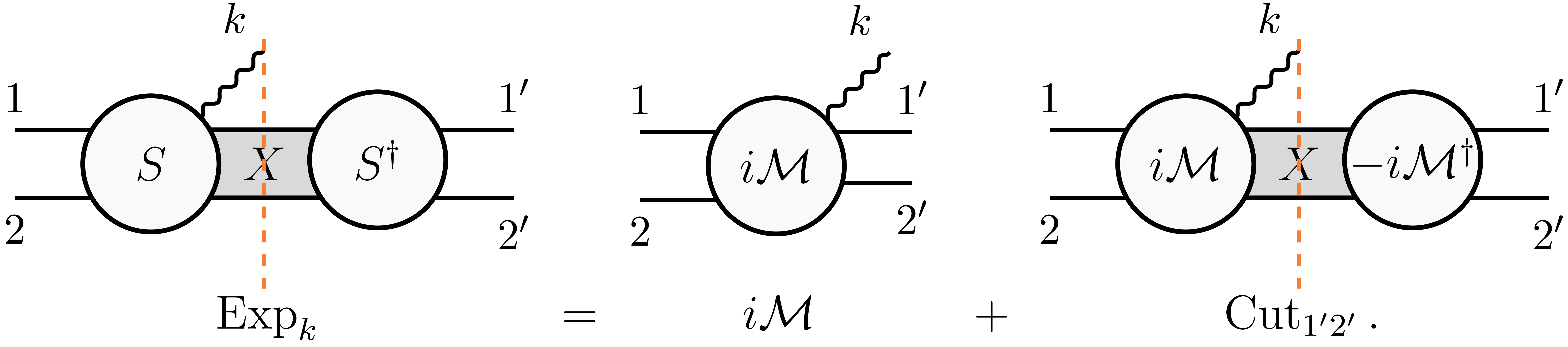
In the following the "in" label dropped to reduce clutter

Quantum Amplitudes and Radiation Observables

$$R_h \sim \int d\Phi_X \ \ {}_{\rm in}\langle\psi|S^\dagger|X\rangle_{\rm in}\ {}_{\rm in}\langle Xk|S|\psi\rangle_{\rm in}$$

In terms of usual amplitudes $S = 1 + i\delta^4(p)\mathscr{M}$

$$R_h \sim \langle \psi k|\mathscr{M}|\psi\rangle + \int d\Phi_X \ \ \langle\psi|\mathscr{M}^\dagger|X\rangle\langle Xk|\mathscr{M}|\psi\rangle$$



Well defined in classical limit

Contains superclassical terms $\mathscr{M} \sim e^{iS/\hbar}$

Cut terms cancel superclassical terms

Quantum Amplitudes and Radiation Observables

Classical limit of R_h is the leading term
under the classical (soft) scaling

momenta of matter particles representing classical object
$$p_i \to \hbar^0 p_i$$

masses of matter particles
$$m_i \to \hbar^0 m_i$$

momentum transfer for matter particles $q_i = p_i' - p_i$
$$q_i \to \hbar^1 q_i$$

momenta of radiation quanta
$$k_n \to \hbar^1 k_n$$

spins of matter particles
$$S_i \to \hbar^{-1} S_i$$

Quantum Amplitudes and Radiation Observables

Expansion in QFT vs in classical GR for 2-to-2 scattering

	0PN	1PN	2PN	3PN	4PN	
1PM	$\frac{1}{M_{\mathrm{Pl}}^2}$	$\frac{v^2}{M_{\mathrm{Pl}}^2}$	$\frac{v^4}{M_{\mathrm{Pl}}^2}$	$\frac{v^6}{M_{\mathrm{Pl}}^2}$	$\frac{v^8}{M_{\mathrm{Pl}}^2}$	tree level $\sim \frac{1}{bM_{\mathrm{Pl}}^2}$
2PM		$\frac{1}{M_{\mathrm{Pl}}^4}$	$\frac{v^2}{M_{\mathrm{Pl}}^4}$	$\frac{v^4}{M_{\mathrm{Pl}}^4}$	$\frac{v^6}{M_{\mathrm{Pl}}^4}$	1-loop $\sim \frac{1}{b^2M_{\mathrm{Pl}}^4} \sim \frac{R_s}{b}\frac{1}{bM_{\mathrm{Pl}}^2}$
3PM			$\frac{1}{M_{\mathrm{Pl}}^6}$	$\frac{v^2}{M_{\mathrm{Pl}}^6}$	$\frac{v^4}{M_{\mathrm{Pl}}^6}$	2-loop $\sim \frac{1}{b^3M_{\mathrm{Pl}}^6} \sim \left(\frac{R_s}{b}\right)^2\frac{1}{bM_{\mathrm{Pl}}^2}$
4PM				$\frac{1}{M_{\mathrm{Pl}}^8}$	$\frac{v^2}{M_{\mathrm{Pl}}^8}$	3-loop
5PM					$\frac{1}{M_{\mathrm{Pl}}^{10}}$	4-loop

Loop expansion

Velocity expansion

Kepler's law

$$v^2 \sim \frac{1}{bM_{\mathrm{Pl}}^2}$$

Schwarzschild radius

$$R_s \sim \frac{m}{M_{\mathrm{Pl}}^2}$$

Waveforms from amplitudes

The rest is an exercise in wave function integration and extracting the leading 1/|x| behaviour

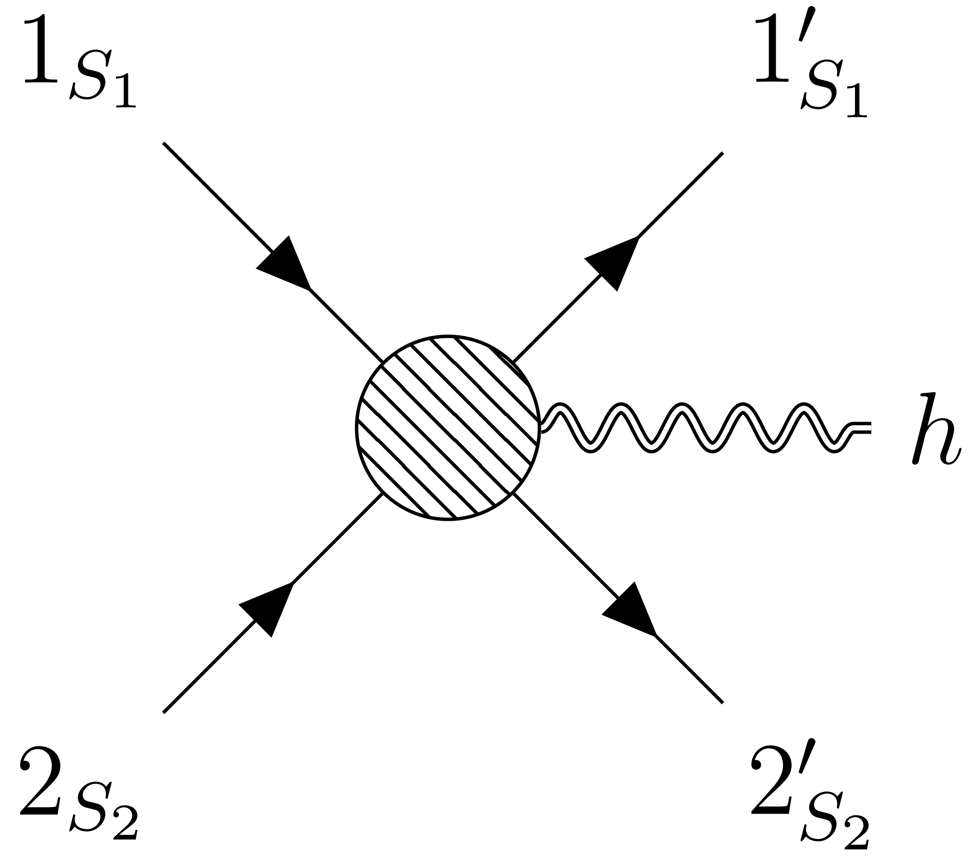
$$|\psi\rangle_{\text{in}} = \int \Pi_{i=1,2}[d\Phi(p_i) f_i(p_i) e^{ip_i b_i}]\,|p_1 p_2\rangle_{\text{in}}$$

At leading PM order KMOC formalism relates spectral waveform to integral of 5-point amplitude.

$$f_h(\omega) = \frac{1}{64\pi^3 m_1 m_2}\int d\mu\, \mathcal{M}^{\text{cl}}_{\text{tree}}[p_1 + w_1, p_2 + w_2 \to p_1, p_2, k]_{|k=\omega n}$$

$$d\mu \equiv \delta^4(w_1 + w_2 - k)\Pi_{i=1,2}[e^{ib_i w_i} d^4 w_i \delta(u_i w_i)]$$

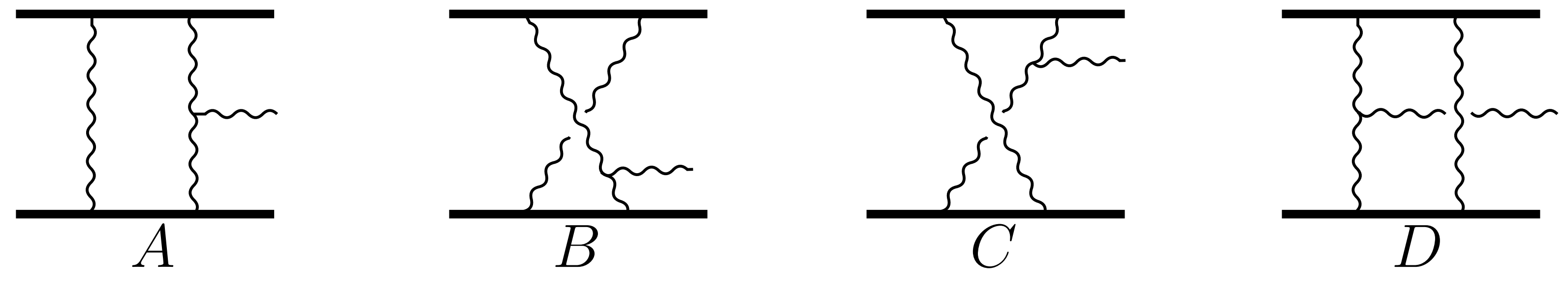
Integration measure



In fact, not full 5-point amplitude but just its certain residues are needed to calculate the integral!

De Angelis, Novichkov, Gonzo
[arXiv:2309.17429]

Calculation can be extended to one-loop level



-

2303.06211
2303.06111
2303.06112
2303.07006

Caron-Huot, Giroux, Hannesdottir, Mizera
arXiv:2310.12199

On-shell calculation of gravitational amplitudes

Quantum Amplitudes and Classical Observables: On-shell methods

Quantum Mechanics + Poincaré invariance, locality and unitarity

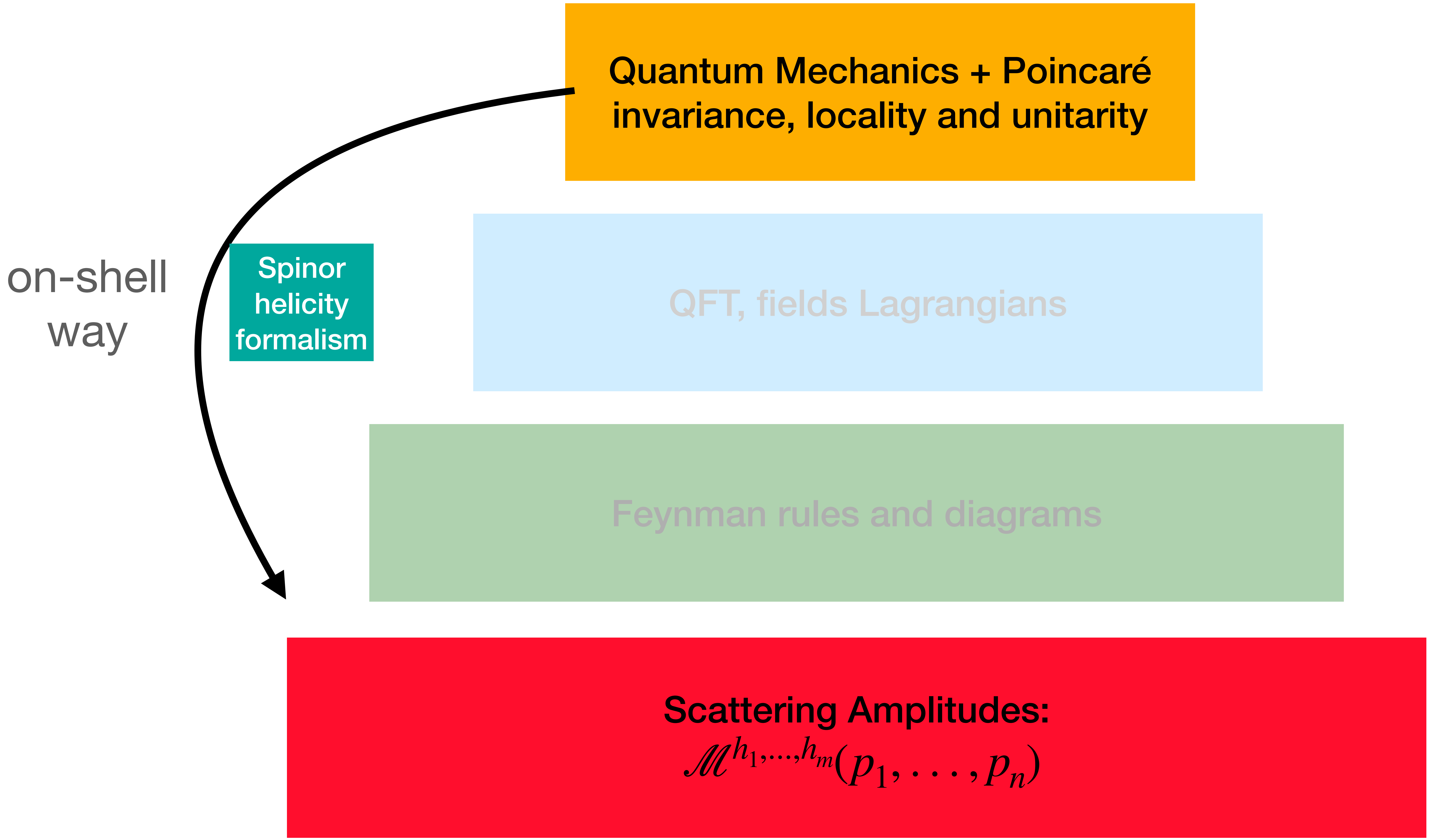
QFT, fields Lagrangians

Feynman rules and diagrams

Scattering Amplitudes:

$$\mathcal{M}^{h_1,\ldots,h_m}(p_1,\ldots,p_n)$$

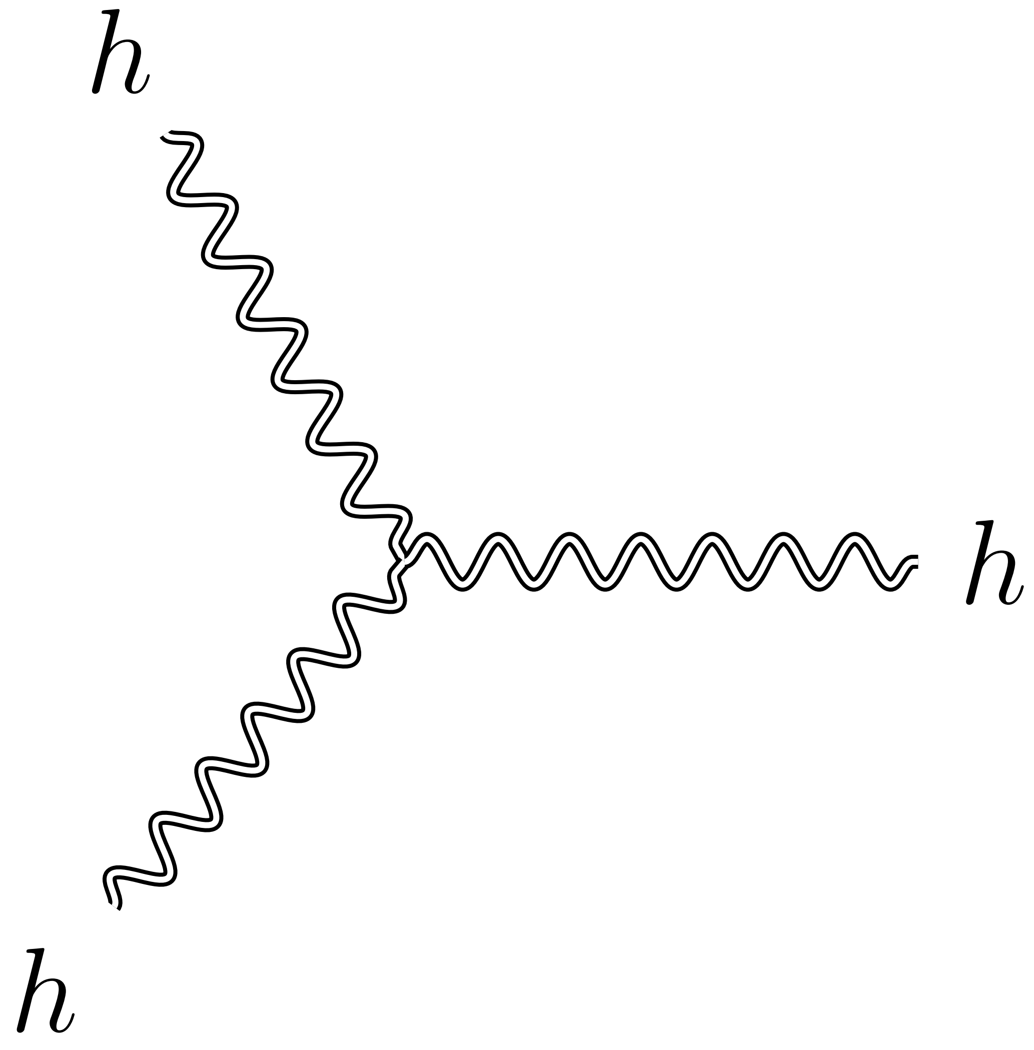
Quantum Amplitudes and Classical Observables: On-shell methods



Quantum Amplitudes and Classical Observables: On-shell methods

Basis building block are **on-shell 3-point amplitudes**

Pure general relativity



$$\mathscr{M}(1_h^-, 2_h^-, 3_h^+) = -\frac{1}{M_{Pl}} \frac{\langle 12\rangle^6}{\langle 13\rangle^2 \langle 23\rangle^2}$$

$$\mathscr{M}(1_h^+, 2_h^+, 3_h^-) = -\frac{1}{M_{Pl}} \frac{[12]^6}{[13]^2 [23]^2}\,.$$

$$|n\rangle \equiv \lambda_n$$

$$|n] \equiv \tilde{\lambda}_n$$

$$p_n \sigma = \lambda_n \tilde{\lambda}_n$$

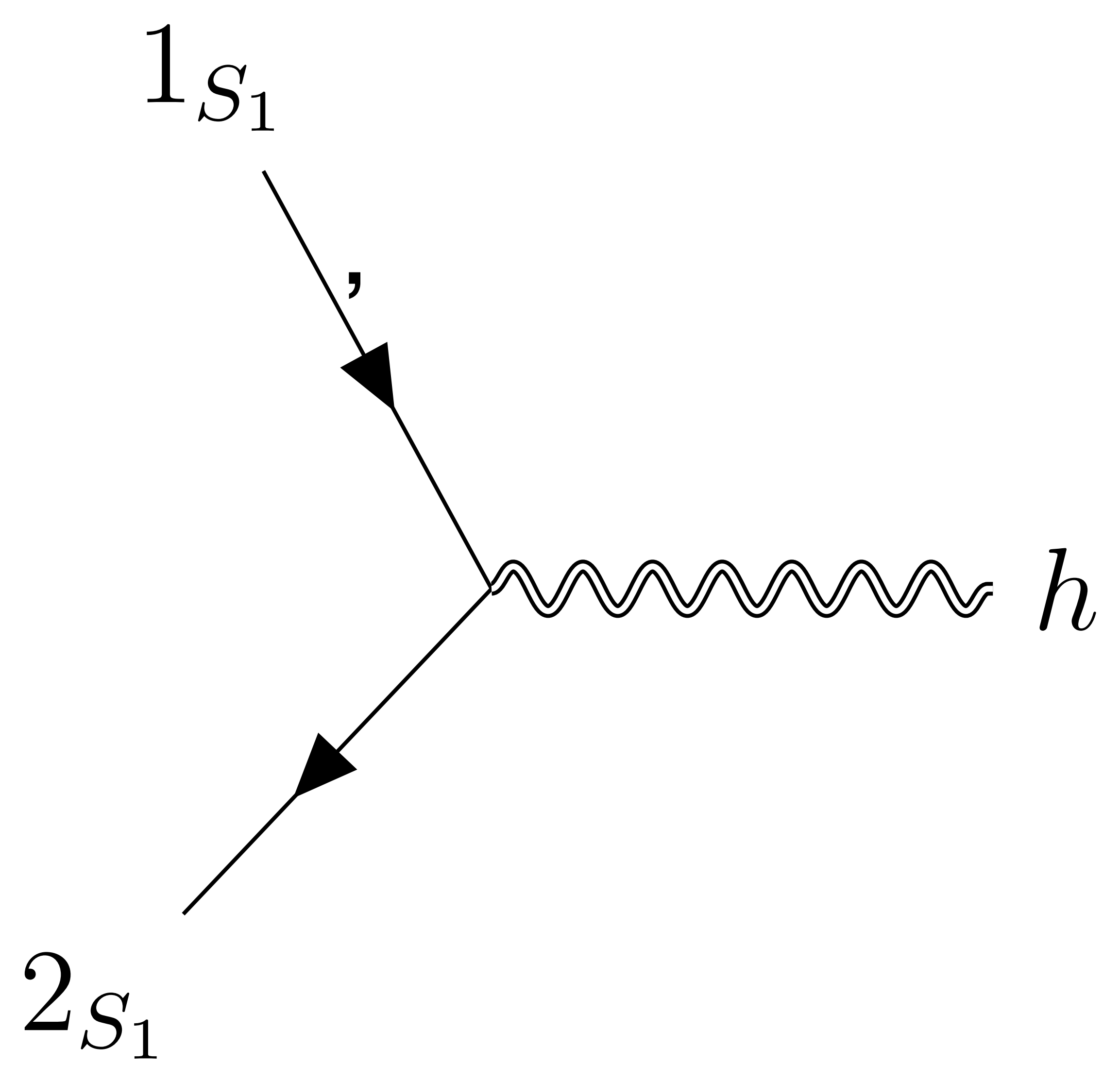
Quantum Amplitudes and Classical Observables: On-shell methods

Basis building block are **on-shell 3-point amplitudes**

Spinning matter representing black holes

$$\mathscr{M}\left[1_\Phi 2_{\bar\Phi} 3_h^-\right] = -\frac{\langle 3|p_1|\tilde\zeta]^2}{M_{Pl}[3\tilde\zeta]^2}\frac{[\mathbf{21}]^{2S}}{m^{2S}},$$

$$\mathscr{M}\left[1_\Phi 2_{\bar\Phi} 3_h^+\right] = -\frac{\langle \zeta|p_1|3]^2}{M_{Pl}\langle 3\zeta\rangle^2}\frac{\langle\mathbf{21}\rangle^{2S}}{m^{2S}},$$



$$|n\rangle \equiv \lambda_n$$

$$|n] \equiv \tilde\lambda_n$$

$$p_n\sigma = \lambda_n\tilde\lambda_n$$

Quantum Amplitudes and Classical Observables: On-shell methods

Basis building block are **on-shell 3-point amplitudes**

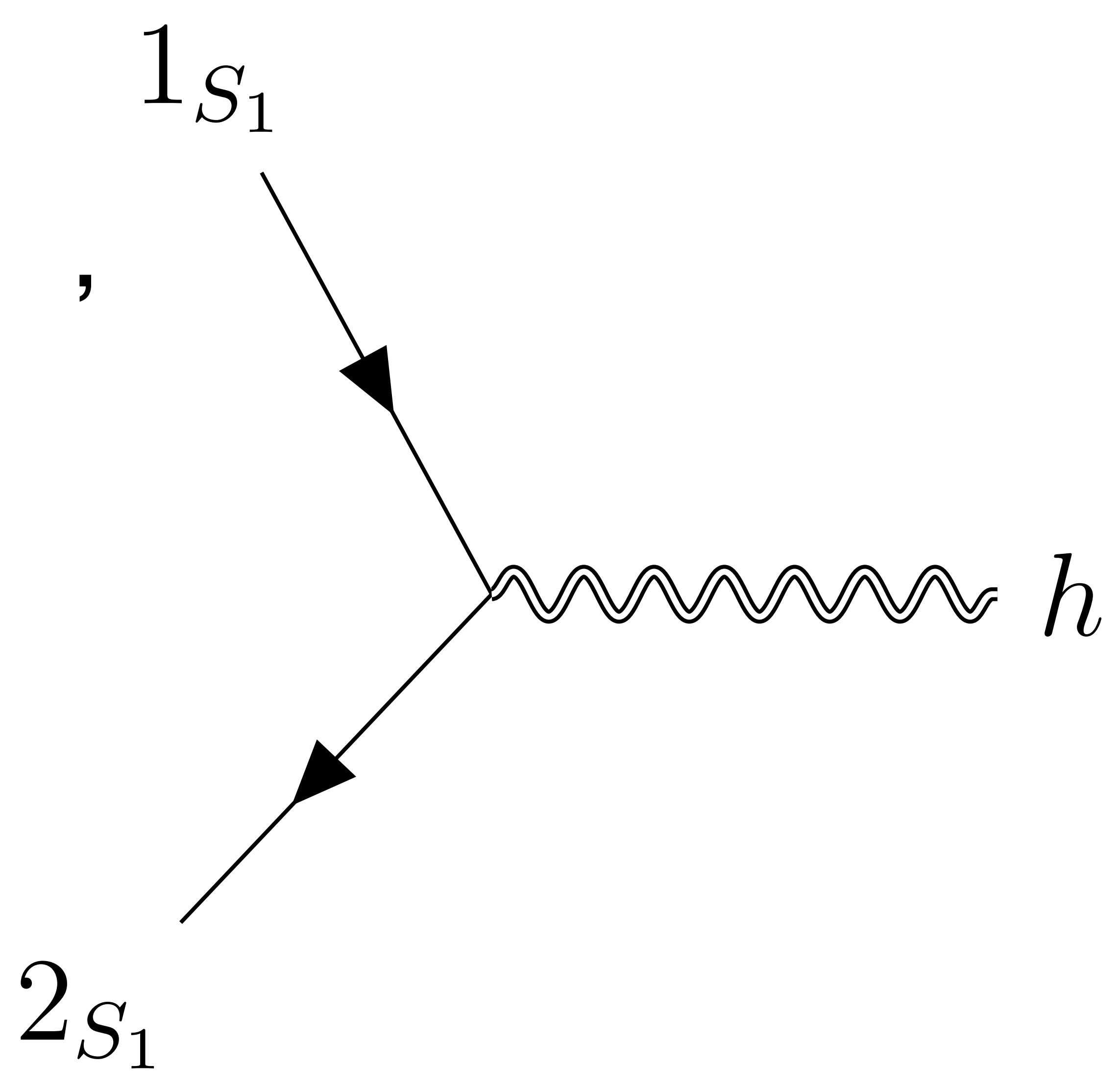
Spinning matter representing black holes

Classical limit

$$\mathcal{M}^{\rm cl}\left[1_\Phi 2_{\bar\Phi} 3_h^-\right] = -\frac{\langle 3|p_1|\tilde\zeta]^2}{M_{Pl}[3\tilde\zeta]^2}\exp(+p_3 a_1) \quad ,$$

$$\mathcal{M}^{\rm cl}\left[1_\Phi 2_{\bar\Phi} 3_h^+\right] = -\frac{\langle \zeta|p_1|3]^2}{M_{Pl}\langle 3\zeta\rangle^2}\exp(-p_3 a_1)$$

Spin vector



$|n\rangle \equiv \lambda_n$

$|n] \equiv \tilde\lambda_n$

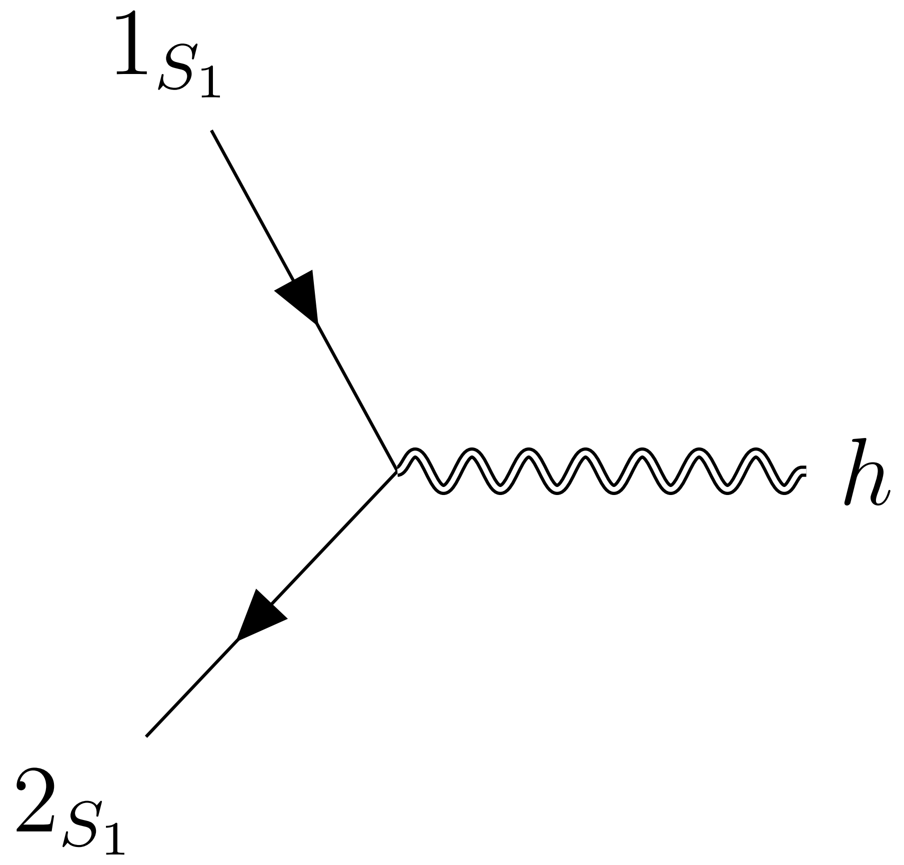
$p_n\sigma = \lambda_n\tilde\lambda_n$

Quantum Amplitudes and Classical Observables: On-shell methods

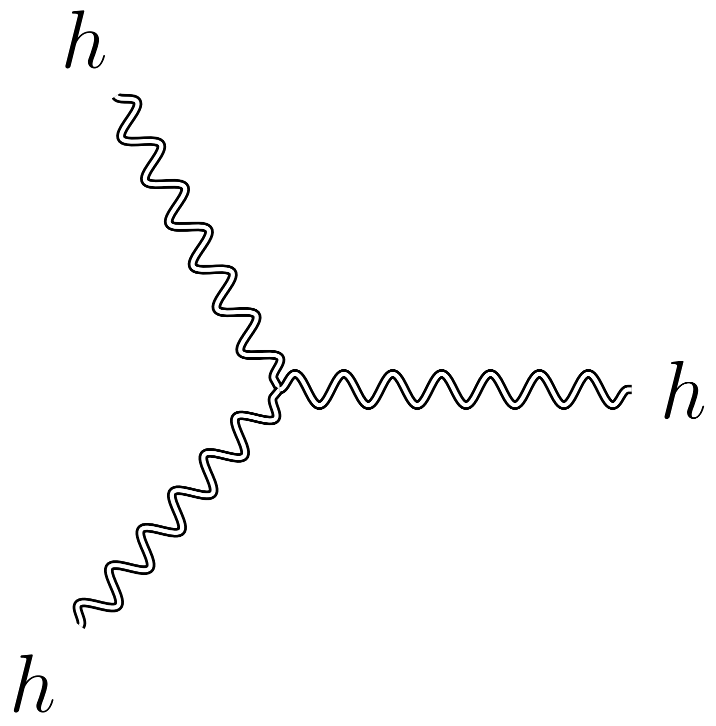
Basis building block are **on-shell 3-point amplitudes**

$$\mathscr{M}^{\rm cl}\left[1_\Phi 2_{\bar\Phi} 3_h^-\right] = -\frac{\langle 3|p_1|\tilde\zeta]^2}{M_{Pl}[3\tilde\zeta]^2}\exp(+p_3 a_1)$$

$$\mathscr{M}^{\rm cl}\left[1_\Phi 2_{\bar\Phi} 3_h^+\right] = -\frac{\langle \zeta|p_1|3]^2}{M_{Pl}\langle 3\zeta\rangle^2}\exp(-p_3 a_1)$$



,

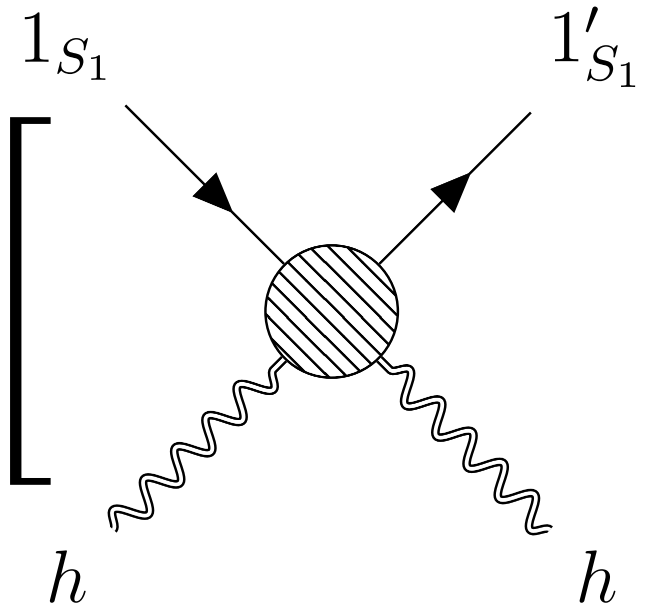


$$\mathscr{M}(1_h^-,2_h^-,3_h^+) = -\frac{1}{M_{Pl}}\frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2}$$

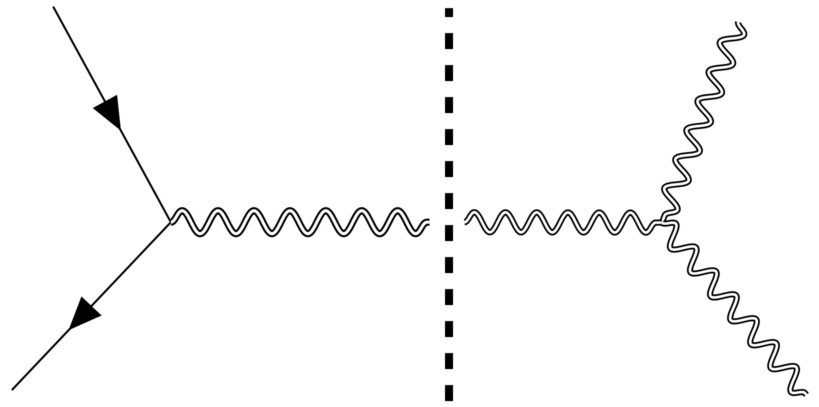
$$\mathscr{M}(1_h^+,2_h^+,3_h^-) = -\frac{1}{M_{Pl}}\frac{[12]^6}{[13]^2[23]^2}.$$

→ Build higher-point amplitudes (up to contact terms) from their **residues** at kinematic poles in the **complex** plane

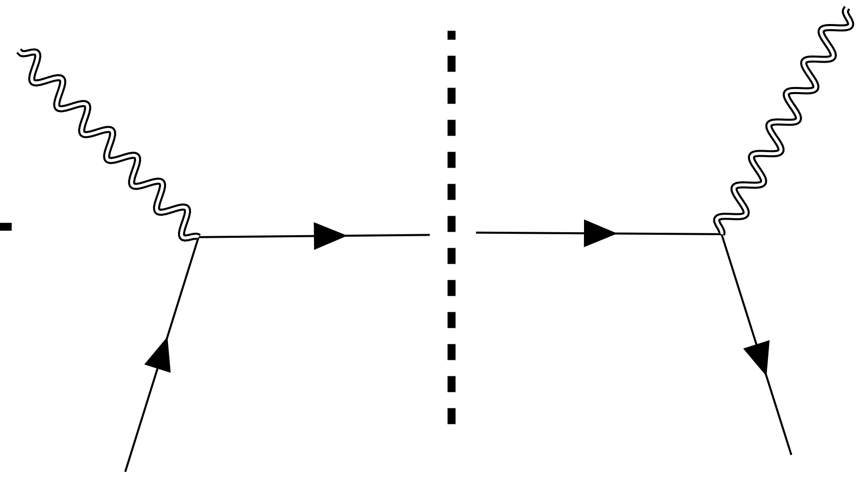
$$\sum_{i=2,3,4} \text{Res}_{(p_1+p_i)^2\to 0}\left[\right.$$



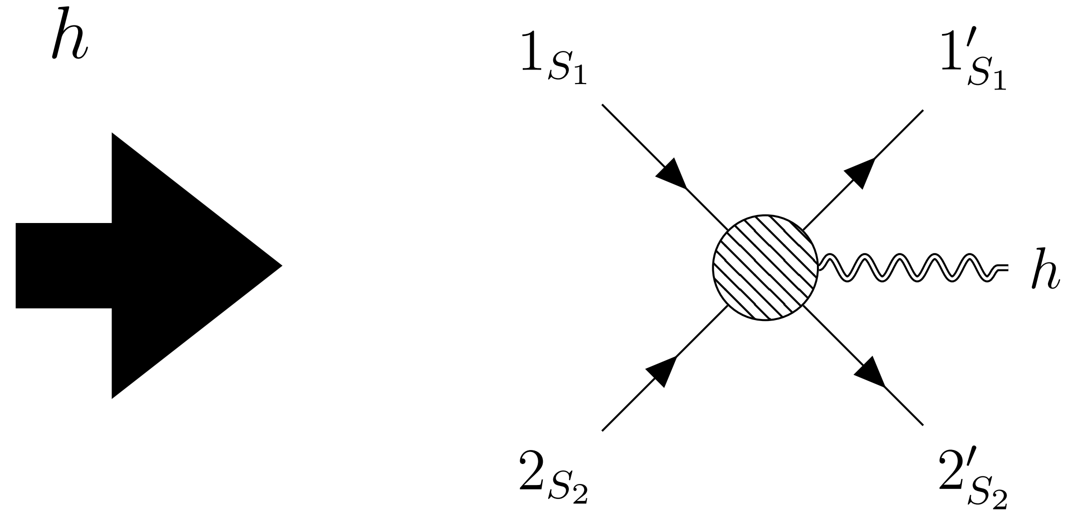
$$\left.\right]\Big|_{tree} = -\left[\right.$$



$+$



$$+(t\leftrightarrow u)\left.\right]$$



Quantum Amplitudes and Classical Observables: gravitational waves

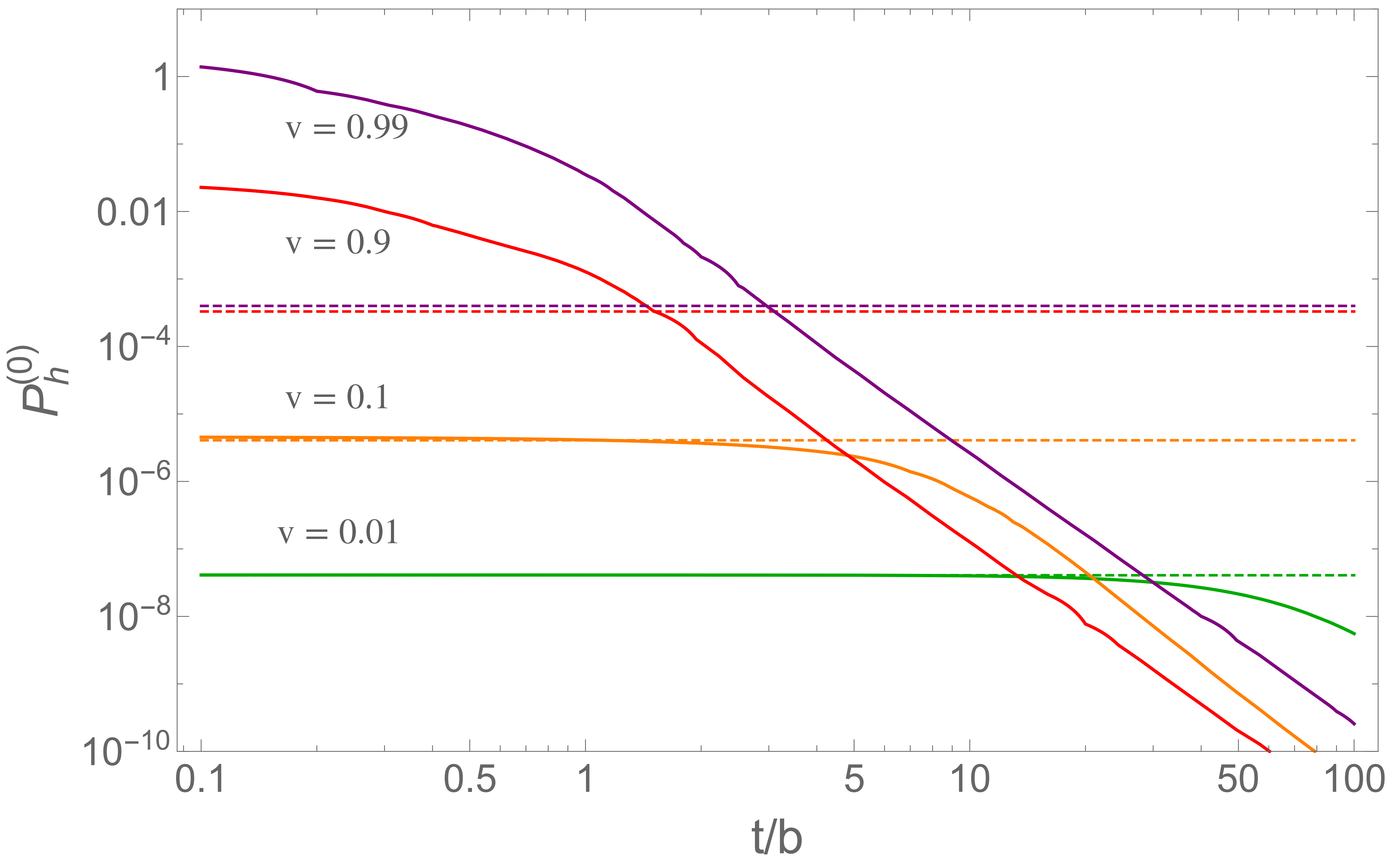
Waveform in GR calculated as expansion in spin
$$W_h = W_h^{(0)} + W_h^{(1)} + \ldots$$

$$T_i \equiv \frac{t - b_i n}{(\hat{u}_i n) b} \qquad \hat{u}_i = \frac{u_i}{\sqrt{\gamma^2 - 1}} \qquad \gamma = u_1 u_2 = \frac{1}{\sqrt{1 - \mathrm{v}^2}}$$

At leading order
$$W_h^{(0)} = -\frac{m_1 m_2}{512\pi^2 M_{\mathrm{Pl}}^3 b (\hat{u}_1 n)^2 \sqrt{\gamma^2 - 1}} \frac{1}{\sqrt{z^2+1}} \mathscr{R}\left\{ \frac{\left(\mathscr{F}_1^-(z)\right)^2 + \left(\mathscr{F}_2^-(z)\right)^2}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + z(\tilde{b} n) + i\sqrt{z^2+1}(\tilde{v} n)} \right\}_{|z=T_1} + (1 \leftrightarrow 2)$$

$$\mathscr{F}_1^-(z) = \langle n | \left(\hat{u}_1 \hat{u}_2 + 2\gamma z \hat{u}_1 \tilde{b} + z \tilde{b} \hat{u}_2 + 2i\gamma\sqrt{z^2+1}\, \hat{u}_1 \tilde{v} + i\sqrt{z^2+1}\, \tilde{v} \hat{u}_2 \right) | n \rangle$$
$$\mathscr{F}_2^-(z) = \langle n | \left(\hat{u}_1 - z\tilde{b} - i\sqrt{z^2+1}\, \tilde{v} \right) \hat{u}_2 | n \rangle$$

Emitted power in gravitational waves $\quad \dfrac{dP_h}{d\Omega} = 2\,|\partial_t W_h|^2$



For $|t| \lesssim b/\mathrm{v}$ in the rest frame of particle 2
$$\partial_t W_h^{(0)} = -\, e^{-2i\phi} \frac{m_1 m_2}{16\pi^2 M_{\mathrm{Pl}}^3 b^2} \left\{ (\hat{\boldsymbol{v}}\hat{\boldsymbol{n}})(\hat{\boldsymbol{b}}\hat{\boldsymbol{n}}) + i(\hat{\boldsymbol{v}} \times \hat{\boldsymbol{b}}) \cdot \hat{\boldsymbol{n}} \right\} \mathrm{v} + O(\mathrm{v}^2)$$

velocity suppression

Emitted power
$$\frac{dP_h^{(0)}}{d\Omega} = \frac{m_1^2 m_2^2}{128\pi^4 M_{\mathrm{Pl}}^6 b^4} \left[(\hat{\boldsymbol{v}}\hat{\boldsymbol{n}})^2 (\hat{\boldsymbol{b}}\hat{\boldsymbol{n}})^2 + (\hat{\boldsymbol{v}} \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}})^2 \right] \mathrm{v}^2 + \mathcal{O}(\mathrm{v}^3)$$

Total emitted power
$$P_h^{(0)} = \frac{m_1^2 m_2^2}{80\pi^3 M_{\mathrm{Pl}}^6 b^4} \mathrm{v}^2$$

Quantum Amplitudes and Classical Observables: gravitational waves

Waveform in GR calculated as expansion in spin

$$W_h = W_h^{(0)} + W_h^{(1)} + \dots$$

$$T_i \equiv \frac{t - b_i n}{(\hat{u}_i n) b} \qquad \hat{u}_i = \frac{u_i}{\sqrt{\gamma^2 - 1}} \qquad \gamma = u_1 u_2 = \frac{1}{\sqrt{1 - \mathrm{v}^2}}$$

$$\Lambda[a,b] \equiv (\lambda_n a \sigma b \bar{\sigma} \lambda_n)$$

$$\begin{aligned}
W_h^{(1)}(t) = {} & \frac{i m_1 m_2}{512\pi^2 b^2 M_{\mathrm{Pl}}^3 \sqrt{\gamma^2-1}(\hat{u}_1 n)^2} \frac{d}{dz}\left(\frac{1}{\sqrt{z^2+1}} \mathscr{R}\left\{ \frac{1}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + z(\tilde{b} n) + i\sqrt{z^2+1}(\tilde{v} n)} \right[\right. \\
& \left(\frac{(a_1 n)}{(\hat{u}_1 n)} + (a_1 + a_2)^\mu (\gamma \hat{u}_2 - \hat{u}_1 + z\tilde{b} + i\sqrt{z^2+1}\,\tilde{v})^\mu \right) \left(\Lambda[\hat{u}_1, \hat{u}_2] - \Lambda[z\tilde{b} + i\sqrt{z^2+1}\,\tilde{v}, 2\gamma \hat{u}_1 - \hat{u}_2] \right)^2 \\
& + \left(\frac{(a_1 n)}{(\hat{u}_1 n)} - (a_1 + a_2)^\mu (\gamma \hat{u}_2 - \hat{u}_1 + z\tilde{b} + i\sqrt{z^2+1}\,\tilde{v})^\mu \right) \left(\Lambda[\hat{u}_1, \hat{u}_2] - \Lambda[z\tilde{b} + i\sqrt{z^2+1}\,\tilde{v}, \hat{u}_2] \right)^2 \\
& - 2\left(\Lambda[\hat{u}_1, \hat{u}_2] - \Lambda[z\tilde{b} + i\sqrt{z^2+1}\,\tilde{v}, 2\gamma \hat{u}_1 - \hat{u}_2] \right) \\
& \times \left(\gamma \Lambda[\hat{u}_1, a_1] - \Lambda[\hat{u}_2, a_1] - (\gamma^2 - 1)\Lambda[\hat{u}_1, \hat{u}_2, z\tilde{b} + i\sqrt{z^2+1}\,\tilde{v}, a_1] \right) \Big] \Big\} \Big)_{|z=T_1} + (1 \leftrightarrow 2)\,.
\end{aligned}$$

For $|t| \lesssim b/\mathrm{v}$ in the rest frame of particle 2

$$\begin{aligned}
\partial_t W_h^{(1)} = -e^{-2i\phi} \frac{m_1 m_2}{64\pi^2 b^3 M_{\mathrm{Pl}}^3} \Big\{ & i(\hat{\boldsymbol{v}}\boldsymbol{a_2})\left[1 - (\hat{\boldsymbol{v}}\boldsymbol{n})^2\right] + (\hat{\boldsymbol{b}}\boldsymbol{a_2})\left[(\hat{\boldsymbol{v}} \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}}) - i(\hat{\boldsymbol{b}}\hat{\boldsymbol{n}})(\hat{\boldsymbol{v}}\hat{\boldsymbol{n}})\right] \\
& - (\hat{\boldsymbol{v}} \times \hat{\boldsymbol{b}} \cdot \boldsymbol{a_2})\left[(\hat{\boldsymbol{b}}\hat{\boldsymbol{n}}) + i(\hat{\boldsymbol{v}}\hat{\boldsymbol{n}})(\hat{\boldsymbol{v}} \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}})\right] \Big\} \mathrm{v} + \mathcal{O}(\mathrm{v}^2)
\end{aligned}$$

Emitted power

$$\frac{dP_h^{(1)}}{d\Omega} = \frac{m_1^2 m_2^2}{256\pi^4 M_{\mathrm{Pl}}^6 b^5}\left[1 - (\hat{\boldsymbol{v}}\boldsymbol{n})^2\right]\left\{ (\hat{\boldsymbol{v}}\boldsymbol{a_2})(\hat{\boldsymbol{v}} \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}}) - (\hat{\boldsymbol{v}} \times \hat{\boldsymbol{b}} \cdot \boldsymbol{a_2})(\hat{\boldsymbol{v}}\hat{\boldsymbol{n}}) \right\} \mathrm{v}^2 + \mathcal{O}(\mathrm{v}^3)$$

Total emitted power

$$P_h^{(1)} = \mathcal{O}(\mathrm{v}^3)$$

Scalar radiation in scalar-tensor theories

Scalar-Tensor Theories

→ **Scalar-tensor theories** have long stood as popular direction to study **extensions of GR**

→ They consist of gravity theories with the introduction of an additional **massless scalar** degree of freedom

$$\boxed{S_{GR}[g_{\mu\nu}] \to S_{ST}[g_{\mu\nu}, \phi]}$$

Example: Scalar Gauss-Bonnet and Dynamical Chern Simons gravity

$$S = \int d^4x \frac{M_{Pl}^2}{2}\sqrt{-g}R + S_{SGB,DCS}[\phi, g_{\mu\nu}] + S_m[\Psi_m, \mathcal{A}(\phi)g_{\mu\nu}]$$

Gauss Bonnet invariant

$$\boxed{R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2}$$

Chern Simons invariant

$$\boxed{R^{\mu\nu\rho\sigma}\tilde{R}_{\mu\nu\rho\sigma} \quad , \quad \tilde{R}^{\mu}{}_{\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\rho\sigma}{}^{\alpha\beta}R^{\mu}{}_{\nu\alpha\beta}}$$

GW observations constrain them!

- $$S_{SGB,DCS} = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{\Lambda^2}\left(f(\phi)\mathcal{G} + \tilde{f}(\phi)R\tilde{R}\right) + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi\right)\right]$$

$$f(\phi) = \text{const} + \alpha\frac{\phi}{M_{\rm Pl}} + O(\phi^2) \qquad \tilde{f}(\phi) = \text{const} + \tilde{\alpha}\frac{\phi}{M_{\rm Pl}} + O(\phi^2)$$

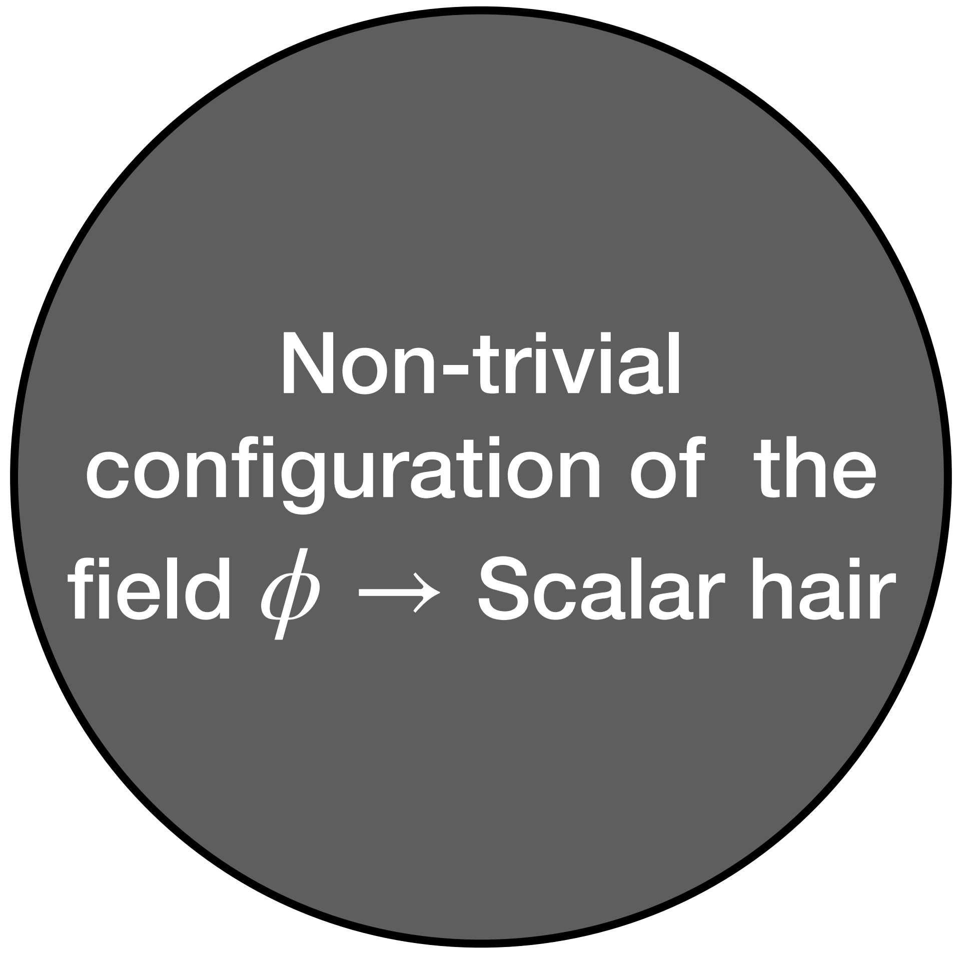
Phys.Rev.Lett. 126 (2021) 18, 181101 [Silva, Holgado, Cárdenas-Avendaño, Yunes]
Phys.Rev.D 107 (2023) 4, 044030 [Silva, Ghosh, Buonan
arXiv: 2406.13654 [Julié, Pompili, Buonanno]

<u>Experimental window:</u>

$$\frac{\sqrt{\alpha}}{\Lambda} \lesssim 0.22km \quad , \quad \frac{\sqrt{\tilde{\alpha}}}{\Lambda} \lesssim 9.5km$$

Scalar-Tensor Theories: scalar hair

Compact objects can acquire scalar hair in scalar-tensor theories
This is the case for black holes in SGB and DCS



BH solution in ST theory

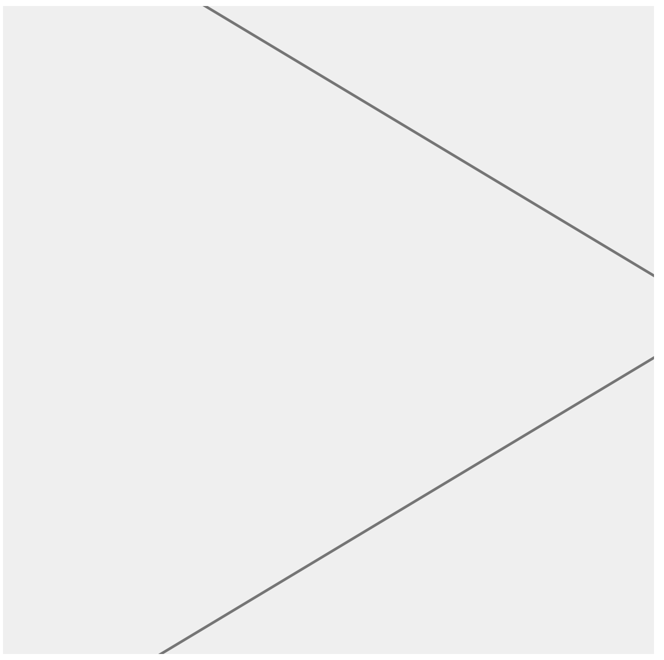
Far zone

$x \to \infty$

"Monopole hair" "Dipole hair"

$$\phi = \frac{c_i}{r} + \frac{d_i}{r^2} + \dots$$

$$c_n \sim \alpha \frac{M_{\mathrm{Pl}}^4}{\Lambda^2 m_n^2}$$



How can we model this behaviour with amplitudes?

Scalar-Tensor Theories: scalar hair on shell

AA, Marinellis
2411.12909

We model the black hole as a point-particle interacting with the scalar field via an effective metric (scalar conformal coupling)

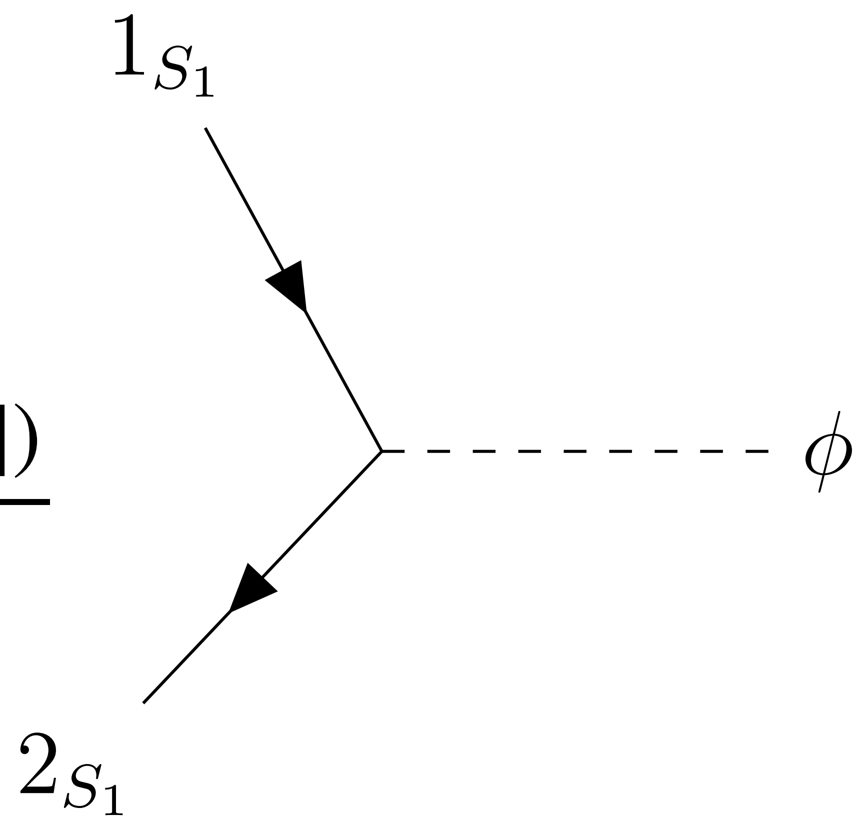
$$\tilde{g}_{\mu\nu} = \exp\left[C\left(\frac{\phi}{M_{Pl}}\right)\right] g_{\mu\nu}$$

3-point amplitudes for arbitrary spinning black holes:

$$\boxed{\exp\left[C\left(\frac{\phi}{M_{Pl}}\right)\right] \approx 1 + c\frac{\phi}{M_{Pl}}}$$

$$\mathscr{M}_{3,bos.}[1_{\Phi_n}, 2_{\bar{\Phi}_n} 3_\phi] = -\frac{c_n}{M_{Pl}} \frac{\langle \mathbf{21} \rangle^{S_n} [\mathbf{21}]^{S_n}}{m_n^{2S_n - 2}}$$

$$\mathscr{M}_{3,ferm.}[1_{\Psi_n}, 2_{\bar{\Psi}_n} 3_\phi] = -\frac{c_n}{M_{Pl}} \frac{\langle \mathbf{21} \rangle^{S_n - 1/2} [\mathbf{21}]^{S_n - 1/2} (\langle \mathbf{21} \rangle + [\mathbf{21}])}{m_n^{2S-2}}$$



In the classical limit
conformal coupling is spin-independent!

$$\mathscr{M}^{\text{cl}}[1_{\Phi_n}, 2_{\bar{\Phi}_n} 3_\phi] = -\frac{c_n m_n^2}{M_{Pl}}$$

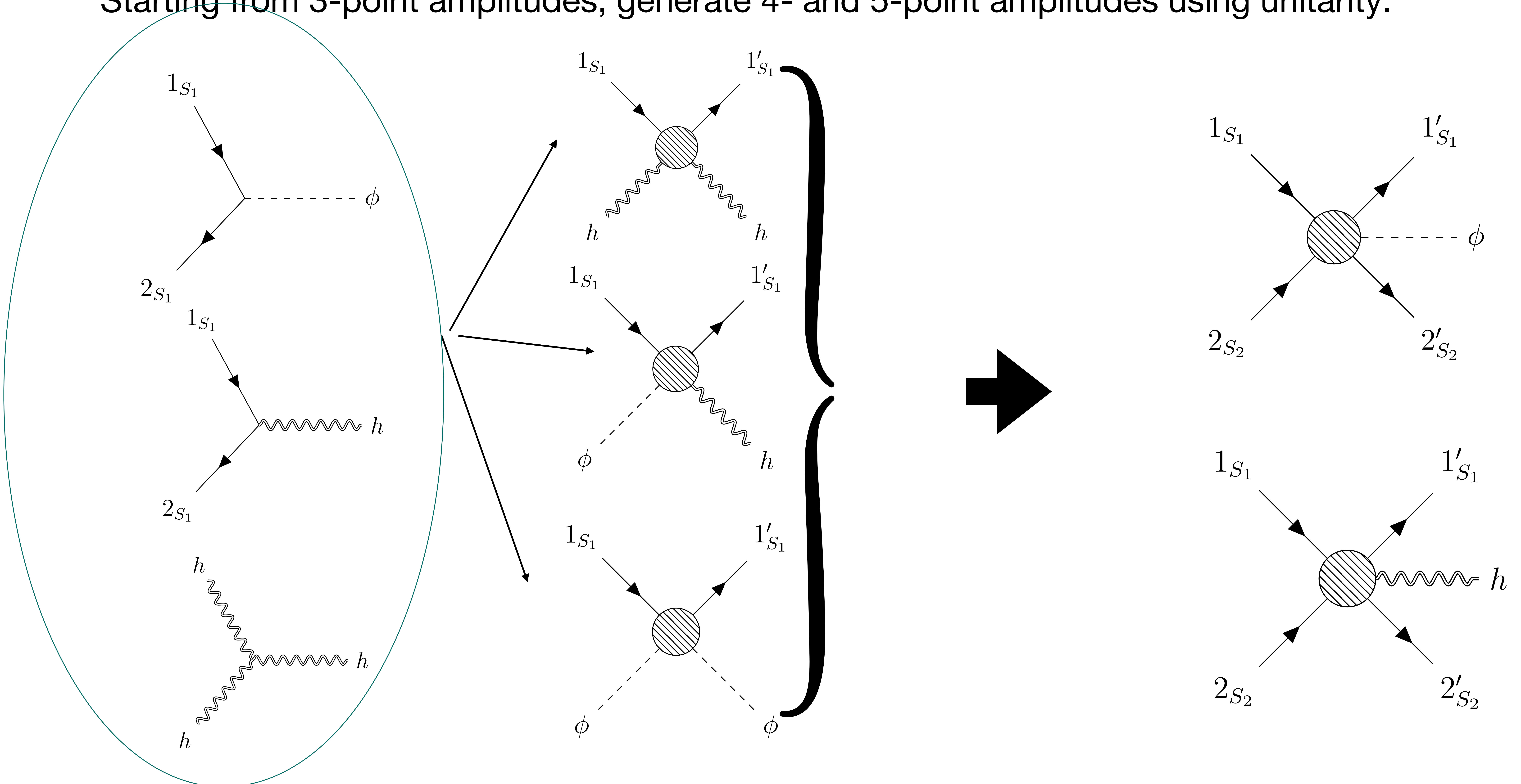
At the lagrangian level for any spin coupling can be obtained by mass redefinition:

$$m \to e^{C/2} m$$

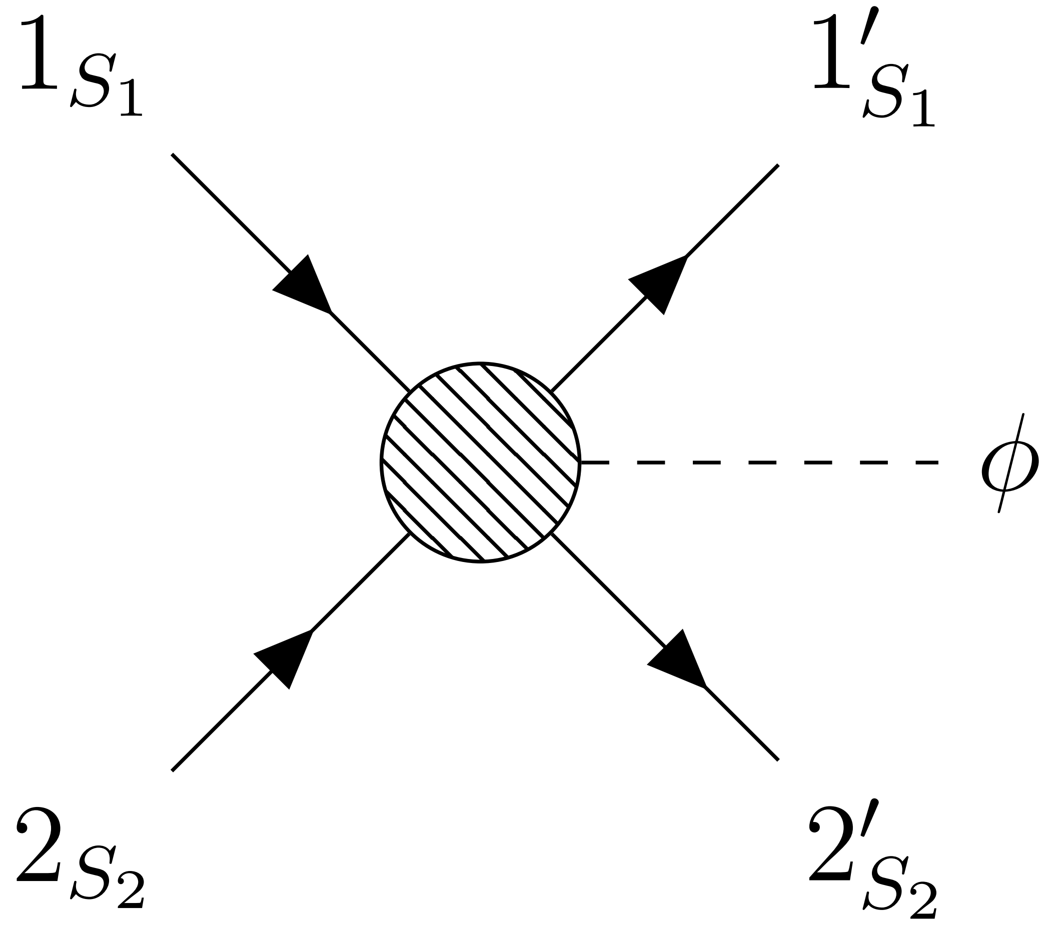
Classical conformal coupling
maps to monopole charge
in scalar tensor theory

Scalar-Tensor Theories: amplitudes

Starting from 3-point amplitudes, generate 4- and 5-point amplitudes using unitarity:



Scalar-Tensor Theories: amplitudes



$$R_X = -\sum_{i=h,\phi} \mathscr{M}^{\mathrm{cl}}_X[(p_1+w_1)_{\Phi_1}(-p_1)_{\bar{\Phi}_1}(-k)_{\phi}(w_2)_i]\,\mathscr{M}^{\mathrm{cl}}[(p_2+w_2)_{\Phi_2}(-p_2)_{\bar{\Phi}_2}(-w_2)_i]$$

One finds the pole part of the residue to all orders in spin, plus contact terms in systematic expansion in spin vector

Part originating from pole terms in 4-point amplitude

$$R_U = \frac{2}{M_{\mathrm{Pl}}^3}\Bigg\{ \frac{(kw_2)[(p_1p_2)^2 - \frac{1}{2}m_1^2m_2^2] + m_2^2(p_1k)^2 - 2(p_1p_2)(p_1k)(p_2k)}{(p_1k)^2} c_1 m_1^2 \cosh(w_2a_2)$$
$$+ i\frac{c_1m_1^2(p_1p_2)}{(p_1k)^2} p_1^{\mu}p_2^{\nu}k^{\rho}w_2^{\sigma}\varepsilon_{\mu\nu\rho\sigma}\sinh(w_2a_2) - 2\frac{c_2m_2^2(p_1k)^2\cosh(w_1a_1) + c_1m_1^2(p_2k)^2\cosh(w_2a_2)}{w_1^2}$$
$$+ \frac{2i\varepsilon_{\mu\nu\rho\sigma}p_1^{\mu}k^{\nu}w_2^{\rho}}{w_1^2}\left[a_1^{\sigma}(p_1k)\frac{c_2m_2^2\sinh(w_1a_1)}{w_1a_1} + p_2^{\sigma}\frac{c_1m_1^2(p_2k)}{p_1k}\sinh(w_2a_2)\right]\Bigg\}$$

Part originating from contact terms in 4-point amplitude

$$R_C^{(0)} = \frac{C_1^{(0)}c_2m_1^2m_2^2}{M_{\mathrm{Pl}}^3} \qquad R_C^{(1)} = -i\frac{C_1^{(1)}c_2m_1^2m_2^2}{M_{\mathrm{Pl}}^3}(w_1a_1) \qquad R_C^{(2)} = \ldots$$

Scalar-Tensor Theories: scalar waveforms

Scalar radiation observable $$R_\phi \equiv {}_{\text{out}}\langle \psi | \phi(x) | \psi \rangle_{\text{out}}$$

Scalar waveform $$R_X(x) = \frac{W_X(t)}{|\boldsymbol{x}|} \qquad |\boldsymbol{x}| \to \infty \qquad t \equiv x^0 - |\boldsymbol{x}|$$

Using KMOC one can relate the scalar waveshape to the residues of the 5-point scalar emission amplitude

$$f_\phi(\omega) = -\frac{1}{64\pi^2 m_1 m_2 \sqrt{\gamma^2 - 1}} \int_{-\infty}^{\infty} \frac{dz e^{ib_1 k + iz(\hat{u}_1 k) b}}{\sqrt{z^2+1}} \frac{1}{2} \Big\{ R\big(w_2 \to (\hat{u}_1 k)\big[\gamma \hat{u}_2 - \hat{u}_1 + z\tilde{b} + i\sqrt{z^2+1}\tilde{v}\big]\big) + R\big(w_2 \to (\hat{u}_1 k)\big[\gamma \hat{u}_2 - \hat{u}_1 + z\tilde{b} - i\sqrt{z^2+1}\tilde{v}\big]\big) \Big\} + (1 \leftrightarrow 2) \qquad k = \omega n$$

From this, waveform is calculated via inverse Fourier transform

$$W_\phi(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} f_\phi(\omega)$$

Scalar-Tensor Theories: scalar waveforms

$$W_\phi = W_\phi^{(0)} + W_\phi^{(1)} + \dots$$

$$T_i \equiv \frac{t - b_i n}{(\hat{u}_i n) b} \qquad \hat{u}_i = \frac{u_i}{\sqrt{\gamma^2 - 1}} \qquad \gamma = \frac{1}{\sqrt{1 - \mathrm{v}^2}}$$

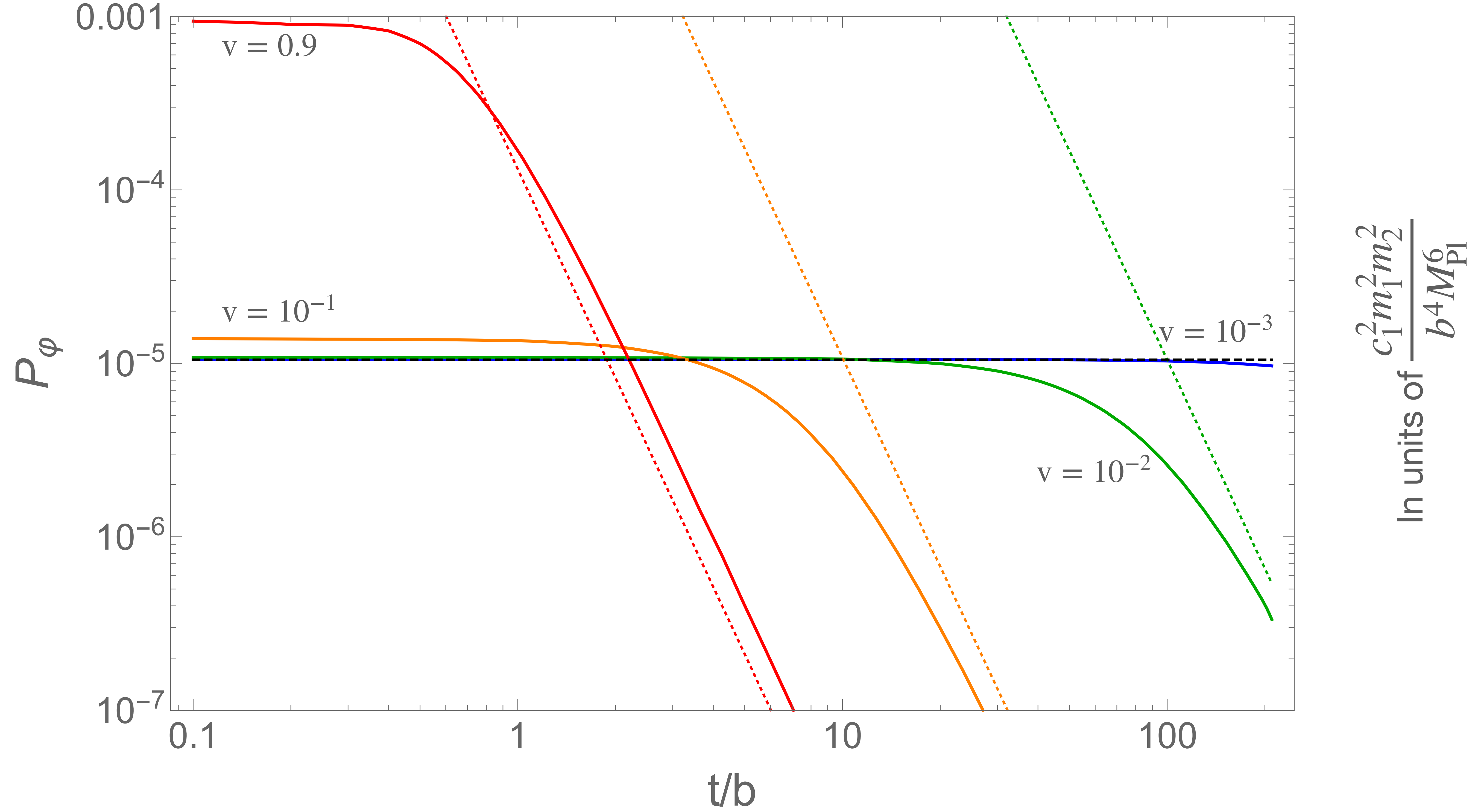
LO Scalar Waveforms-Spinless part:

$$W_\phi^{(0)} = -\frac{m_1 m_2}{32\pi^2 M_{Pl}^3 \sqrt{\gamma^2-1}(\hat{u}_1 n)^2 b} \frac{1}{\sqrt{1+T_1^2}} \left\{ \frac{(\gamma^2-1)[c_1(\hat{u}_2 n)^2 + c_2(\hat{u}_1 n)^2][\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + (\tilde{b} n) T_1]}{[-(\hat{u}_1 n) + \gamma(\hat{u}_2 n) + (\tilde{b} n) T_1]^2 + (\tilde{v} n)^2 (1 + T_1^2)} - \frac{c_1}{2} \frac{(\hat{u}_1 n) + (2\gamma^2 - 3)\gamma(\hat{u}_2 n) - (2\gamma^2 - 1)(\tilde{b} n) T_1}{\gamma^2 - 1} + \frac{C_1^{(0)}}{2} c_2 (\hat{u}_1 n) \right\} + (1 \leftrightarrow 2)$$

Contact term in 4-point matter-scalar scattering amplitude

Emitted power

$$\frac{dP_\phi}{d\Omega} = (\partial_t W_\phi)^2$$



Scalar-Tensor Theories: scalar waveforms

$$W_\phi = W_\phi^{(0)} + W_\phi^{(1)} + \dots$$

$$T_i \equiv \frac{t - b_i n}{(\hat{u}_i n) b} \qquad \hat{u}_i = \frac{u_i}{\sqrt{\gamma^2 - 1}} \qquad \gamma = u_1 u_2 = \frac{1}{\sqrt{1 - \mathrm{v}^2}}$$

LO Scalar Waveforms-Spinless part

$$W_\phi^{(0)} = -\frac{m_1 m_2}{32\pi^2 M_{Pl}^3 \sqrt{\gamma^2 - 1}(\hat{u}_1 n)^2 b} \frac{1}{\sqrt{1 + T_1^2}} \left\{ \frac{(\gamma^2 - 1)[c_1(\hat{u}_2 n)^2 + c_2(\hat{u}_1 n)^2][\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + (\tilde{b} n) T_1]}{[-(\hat{u}_1 n) + \gamma(\hat{u}_2 n) + (\tilde{b} n) T_1]^2 + (\tilde{v} n)^2 (1 + T_1^2)} \right.$$

$$\left. - \frac{c_1}{2} \frac{(\hat{u}_1 n) + (2\gamma^2 - 3)\gamma(\hat{u}_2 n) - (2\gamma^2 - 1)(\tilde{b} n) T_1}{\gamma^2 - 1} + \frac{C_1^{(0)}}{2} c_2(\hat{u}_1 n) \right\} + (1 \leftrightarrow 2)$$

Center-of-mass frame, non-relativistic limit of small relative velocity v.
PN expansion of the waveform independent of contact terms at this order

$$W_\phi^{(0)} = \frac{m_1 m_2 (c_1 - c_2)}{64\pi^2 M_{\rm Pl}^3 b} \left\{ -\frac{(\hat{\mathbf{v}}\hat{\boldsymbol{n}})}{\mathrm{v}} + (\hat{\boldsymbol{b}}\hat{\boldsymbol{n}}) \frac{t}{b} \right\} + O(\mathrm{v})$$

Phys.Rev.D 85 (2012) 064022, Phys.Rev.D 93 (2016) 2, 029902 [Yagi, Stein, Yunes, Tanaka]
Class.Quant.Grav. 39 (2022) 3, 035002 [Shiralilou, Hinderer, Nissanke, Ortiz, Witek]

Emitted power in the limit of of small velocity

$$P_\phi^{(0)} = \frac{m_1^2 m_2^2}{3072\pi^3 M_{\rm Pl}^6 b^4} (c_1 - c_2)^2 + O(\mathrm{v}) \xrightarrow{\text{Kepler's law}} P_\phi^{(0)} = \frac{4\pi m_1^2 m_2^2 M_{\rm Pl}^2 (c_1 - c_2)^2}{3(m_1 + m_2)^4} \mathrm{v}^8 + O(\mathrm{v}^9)$$

$$b^{-1} \to \frac{8\pi M_{\rm Pl}^2 \mathrm{v}^2}{m_1 + m_2}$$

Agreement with existing classical results for SGB!

In particular, $\dfrac{P_\phi^{(0)}}{P_h(0)} \sim \dfrac{1}{\mathrm{v}^2}$

Scalar-Tensor Theories: scalar waveforms

Comparison beyond leading PN order for quasi-circular orders

Amplitudes

$$P_\phi^{(0)} = \frac{m_1^2 m_2^2}{3072\pi^3 M_{\rm Pl}^6 b^4}(c_1-c_2)^2 + \frac{m_1^2 m_2^2}{\pi^3 M_{\rm Pl}^6 b^4}\left[\frac{(c_1+c_2)^2}{3840} + \frac{(c_1+c_2)(c_1-c_2)}{3840}\frac{m_1-m_2}{m_1+m_2} + (6-\eta)\frac{(c_1-c_2)^2}{7680} + \frac{(c_1-c_2)\left(m_2 c_2 C_1^{(0)} - m_1 c_1 C_2^{(0)}\right)}{1536(m_1+m_2)} - \frac{(c_1-c_2)^2}{1536}\frac{t^2}{b^2}\right]\mathrm{v}^2,$$

Classical

Class.Quant.Grav. 39 (2022) 3, 035002 [Shiralilou, Hinderer, Nissanke, Ortiz, Witek]

higher PM

$$\dot{E}_S = \frac{\eta^2}{G\bar{\alpha}c^3}\left(\frac{G\bar{\alpha}m}{r}\right)^4 \Bigg\{\frac{4}{3}\mathcal{S}_-^2 + \frac{8}{15\,c^2}\left(\frac{G\bar{\alpha}m}{r}\right)\Bigg[\left(-23+\eta-10\bar{\gamma}-10\beta_+ + 10\frac{\Delta m}{m}\beta_-\right)\mathcal{S}_-^2 - 2\frac{\Delta m}{m}\mathcal{S}_+\mathcal{S}_-\Bigg] + v^2\Bigg[2\mathcal{S}_+^2 + 2\frac{\Delta m}{m}\mathcal{S}_+\mathcal{S}_- + (6-\eta+5\bar{\gamma})\mathcal{S}_-^2 - \frac{10}{\bar{\gamma}}\frac{\Delta m}{m}\mathcal{S}_-\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) + \frac{10}{\bar{\gamma}}\mathcal{S}_-\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right)\Bigg] + \dot{r}^2\Bigg[\frac{23}{2}\mathcal{S}_+^2 - 8\frac{\Delta m}{m}\mathcal{S}_+\mathcal{S}_- + \left(9\eta - \frac{37}{2} - 10\bar{\gamma}\right)\mathcal{S}_-^2 - \frac{80}{\bar{\gamma}}\mathcal{S}_+\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) + \frac{30}{\bar{\gamma}}\frac{\Delta m}{m}\mathcal{S}_-\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) - \frac{10}{\bar{\gamma}}\mathcal{S}_-\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right) + \frac{120}{\bar{\gamma}^2}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right)^2\Bigg]\Bigg) - \frac{\Delta m}{m}\frac{\eta}{6\,c^2}\left(\frac{\alpha f'(\phi_0)\mathcal{S}_-\mathcal{S}_+}{\sqrt{\bar{\alpha}}r^2}\right)\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right)\left[-9\dot{r}^2 + 3v^2 - \frac{2G\bar{\alpha}m}{r}\right] + \mathcal{O}(c^{-3})\Bigg\}.$$

irrelevant for quasi-circular orbits

suppressed by more powers of distance

Scalar-Tensor Theories: scalar waveforms

Using amplitudes it is straightforward to continue beyond linear order in spin

$$W_\phi = W_\phi^{(0)} + W_\phi^{(1)} + \dots$$

$$a_i \equiv \frac{S_i}{m_i}$$

one more power of impact parameter

New contact term

$$\begin{aligned}
W_\phi^{(1)} = {} & \frac{m_1 m_2}{32\pi^2 M_{\mathrm{Pl}}^3 b^2 (\hat{u}_1 n)^2} \frac{\partial}{\partial z}\Bigg(\frac{1}{\sqrt{z^2+1}} \mathrm{Re}\Big\{ c_1 \big[z(\tilde{v}n) - i\sqrt{z^2+1}(\tilde{b}n)\big]\big[-(\hat{u}_1 a_2) + z(\tilde{b}a_2) + i\sqrt{z^2+1}(\tilde{v}a_2)\big] \\
& \times \left[\frac{\gamma}{\gamma^2-1} - \frac{(\hat{u}_2 n)}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + z(\tilde{b}n) + i\sqrt{z^2+1}(\tilde{v}n)}\right] - \frac{c_2 (\hat{u}_1 n)}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + z(\tilde{b}n) + i\sqrt{z^2+1}(\tilde{v}n)} \\
& \times \Big[\big[i\sqrt{z^2+1}(\tilde{b}n) - z(\tilde{v}n)\big](\hat{u}_2 a_1) + \big[\gamma(\tilde{v}n) + i\sqrt{z^2+1}\big(\gamma(\hat{u}_1 n) - \hat{u}_2 n\big)\big](\tilde{b}a_1) + \big[z\big(\hat{u}_2 n - \gamma(\hat{u}_1 n)\big) - \gamma(\tilde{b}n)\big](\tilde{v}a_1) \Big] \\
& + \frac{C_1^{(1)} c_2}{2\sqrt{\gamma^2-1}} \Big[(\hat{u}_1 n)\big[\gamma(\hat{u}_2 a_1) + z(\tilde{b}a_1)\big] - (a_1 n) \Big] \Big\} \Bigg) \Big|_{z=T_1} + (1 \leftrightarrow 2)
\end{aligned}$$

$$\begin{aligned}
W_\phi^{(1)} = {} & \frac{m_1 m_2}{32\pi^2 M_{\mathrm{Pl}}^3 b^2} \Bigg\{ \big[c_2 \boldsymbol{a_1} - c_1 \boldsymbol{a_2}\big] \cdot \hat{\mathbf{v}}\big(\hat{\mathbf{v}} \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}}\big) + \frac{1}{2}\big[c_1 C_2^{(1)} \boldsymbol{a}_2 - c_2 C_1^{(1)} \boldsymbol{a}_1\big] \cdot \hat{\boldsymbol{b}} \\
& + \mathrm{v} \left[\big[c_1 \boldsymbol{a_2} - c_2 \boldsymbol{a_1}\big] \cdot \big[2\hat{\boldsymbol{b}}\big(\hat{\mathbf{v}} \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}}\big) + \hat{\mathbf{v}} \times \hat{\boldsymbol{b}}\big(\hat{\boldsymbol{b}}\hat{\boldsymbol{n}}\big)\big] + \frac{1}{2}\big[c_1 C_2^{(1)} \boldsymbol{a}_2 - c_2 C_1^{(1)} \boldsymbol{a}_1\big] \cdot \hat{\mathbf{v}} \right] \frac{t}{b} \Bigg\} \\
& + O(\mathrm{v}^2)\,.
\end{aligned}$$

Linear-in-spin correction to emitted power

$$P_\phi^{(1)} = \frac{m_1^2 m_2^2 (c_1 - c_2)}{768\pi^3 M_{\mathrm{Pl}}^6 b^5} (\hat{\mathbf{v}} \times \hat{\boldsymbol{b}}) \cdot \big[c_1 \boldsymbol{a_2} - c_2 \boldsymbol{a_1}\big] \mathrm{v}$$

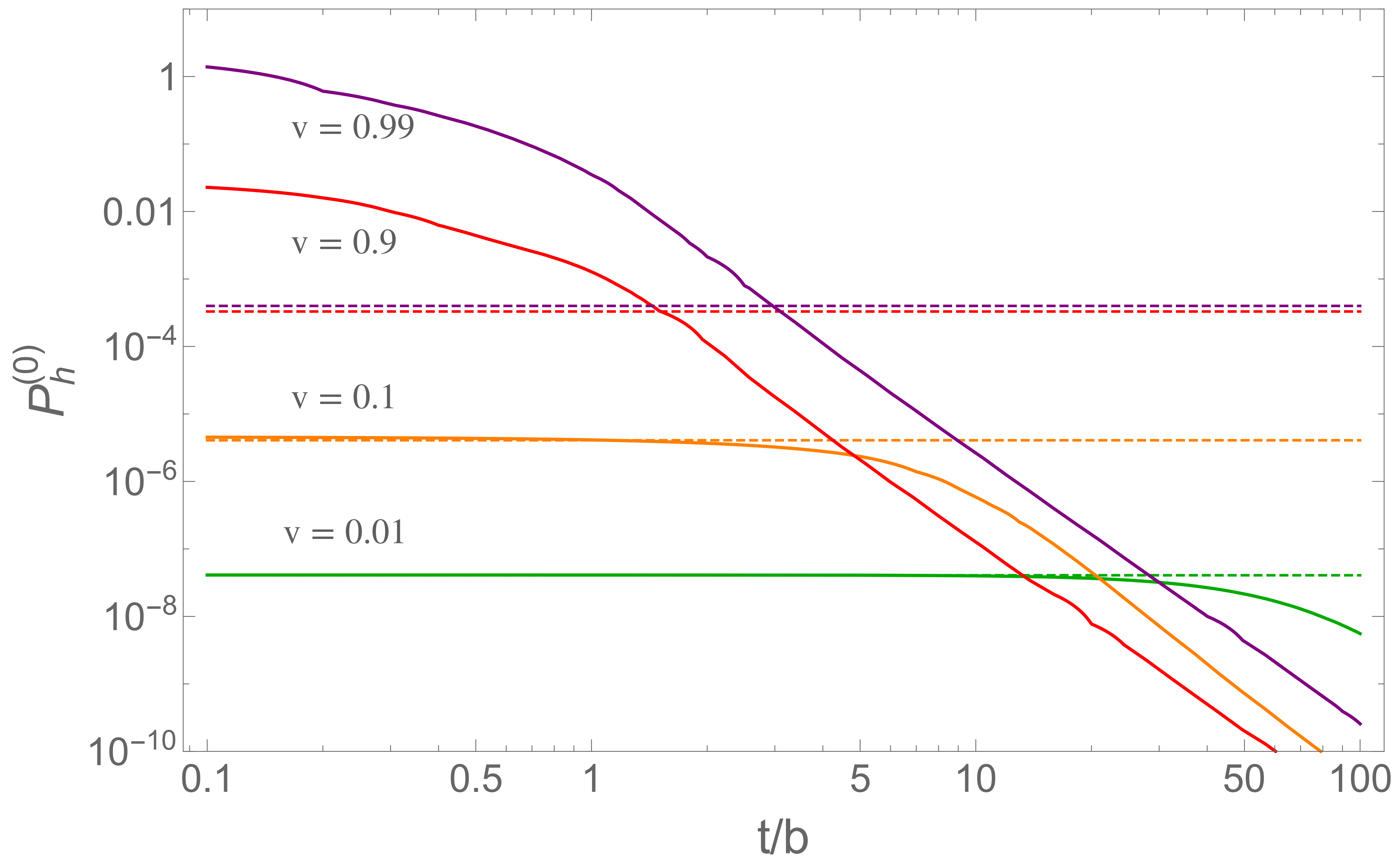
Scalar-Tensor Theories: gravitational waveforms

$$W_h = W_h^{(0)} + W_h^{(1)} + \dots$$

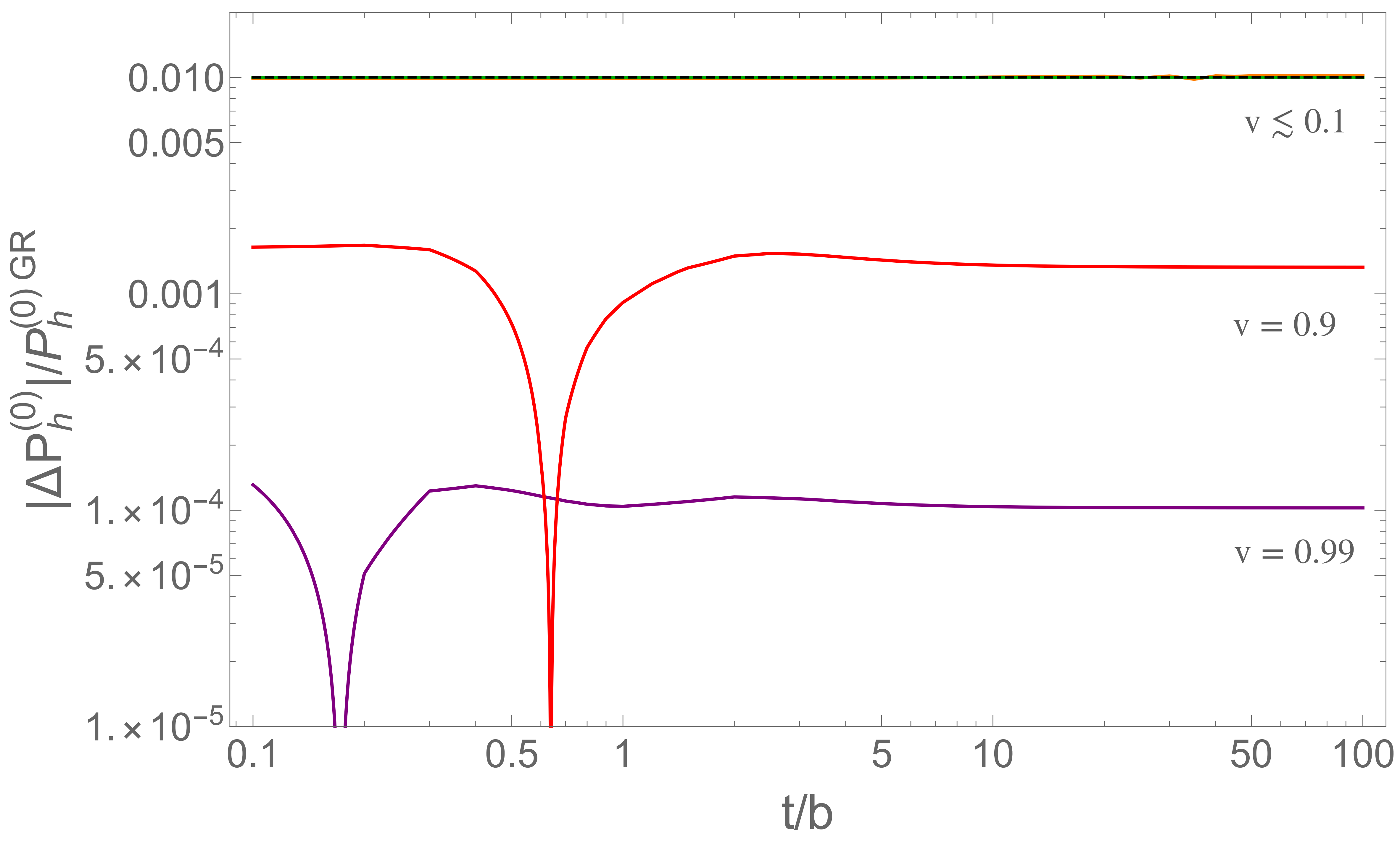
LO Gravitational Waveforms-Spinless part:

$$\Delta W_h^{(0)} = -\frac{c_1 c_2 m_1 m_2}{512\pi^2 M_{Pl}^3 \sqrt{\gamma^2-1}(\hat{u}_1 n)^2 b}\frac{1}{\sqrt{1+T_1^2}}\mathrm{Re}\left\{\frac{(\lambda_n[\gamma \hat{u}_2\sigma + T_1\tilde{b}\sigma + i\sqrt{T_1^2+1}\tilde{v}\sigma]\hat{u}_1\bar{\sigma}\lambda_n)^2}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + T_1(\tilde{b}n) + i\sqrt{T_1^2+1}(\tilde{v}n)}\right\} + (1 \leftrightarrow 2)\,.$$

Emitted power in gravitational waves $\frac{dP_h}{d\Omega} = 2|\partial_t W_h|^2$



Correction of emitted power due to scalar exchange



Scalar-Tensor Theories: gravitational waveforms

$$W_h = W_h^{(0)} + W_h^{(1)} + \ldots$$

LO Gravitational Waveforms-Spinless part:

$$\Delta W_h^{(0)} = -\frac{c_1 c_2 m_1 m_2}{512\pi^2 M_{Pl}^3 \sqrt{\gamma^2 - 1}(\hat{u}_1 n)^2 b} \frac{1}{\sqrt{1+T_1^2}} \mathrm{Re}\left\{ \frac{(\lambda_n[\gamma \hat{u}_2 \sigma + T_1 \tilde{b}\sigma + i\sqrt{T_1^2+1}\tilde{v}\sigma]\hat{u}_1 \bar{\sigma}\lambda_n)^2}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + T_1(\tilde{b}n) + i\sqrt{T_1^2+1}(\tilde{v}n)} \right\} + (1 \leftrightarrow 2)\,.$$

$$\left.\frac{dP_h}{d\Omega}\right|_{\mathcal{O}(a^0)} \sim \frac{\mathrm{v}^2}{b^4} \xrightarrow{\text{For closed orbits}} \left.\frac{dP_h}{d\Omega}\right|_{\mathcal{O}(a^0)} \sim \mathrm{v}^{10} \longrightarrow$$

Suppression compared to scalar radiation in agreement with classical literature

Phys.Rev.D 85 (2012) 064022, Phys.Rev.D 93 (2016) 2, 029902 (erratum) [Yagi, Stein, Yunes, Tanaka]
Class.Quant.Grav. 39 (2022) 3, 035002 [Shiralilou, Hinderer, Nissanke, Ortiz, Witek]

In fact, at leading PN order emitted power rescaled compared to GR by factor $1 + c_1 c_2$

Summary

- Classical observables often can be efficiently calculated using quantum amplitudes in the framework of the KMOC formalism
- One very fruitful application of this formalism is to calculate corrections to the gravitational potential and gravitational waveforms from systems of compact objects in general relativity
- The formalism can be readily extended to scalar-tensor theories of gravity, where both gravitational and scalar radiation is present
- Results for emitted power in gravitational and scalar waves found in the classical literature are reproduced in the KMOC approach. We also derive novel results at higher PN and spin orders