# Quantum cosmology as automorphic dynamics

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## Motivation

We are in a golden era of observational cosmology with a wealth of incoming experimental data. On the theory side, we have good effective models that have been widely successful: $\Lambda$-CDM and slow-roll inflation (Lucas' talk).

However, we don't have a fundamental understanding of the theory:

- String theory provides a UV completion of gravity but is hard to reconcile with cosmology. (Severin's talk).
- Even the semi-classical theory is not understood: we don't understand quantum gravity in cosmological spacetimes.

This is due to a variety of technical and conceptual issues:

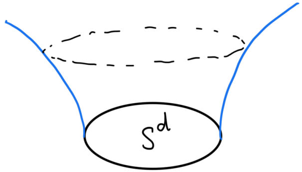
- What is the Hilbert space?
- There are no local observables so how to define gauge-invariant observables?
- Time evolution is "pure gauge" so how to understand dynamics?
- The observer is part of the system.

## Semi-classical cosmology

The theory we consider is Einstein gravity with matter in $d+1$ dimensions:

$$S = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}(R-2\Lambda) + S_{\text{matter}} \ . \qquad (1)$$

We take $\Lambda > 0$ and focus on expanding spacetimes with closed spatial slices.




$$ds^2 = -dt^2 + a(t)^2 d\Omega_d^2$$

de Sitter: $a(t) = \cosh t$

Two approaches for quantization of a classical system:

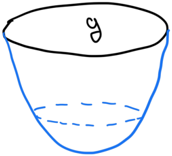
- Lagrangian approach: gravitational path integral
- Hamiltonian approach: canonical gravity

# The Hartle-Hawking state

The path integral defines objects in the quantum theory by summing over all classical configurations. We obtain a functional of a spatial metric $g$ obtained by summing over all smooth geometries ending on it.

[Hartle-Hawking]

$$\Psi_{HH}[g] =$$




In cosmology, this defines a physical quantum state: the Hartle-Hawking state.

Hartle-Hawking proposed that this is the state describing our universe. However, this appears to be incompatible with observations. [Maldacena, ...]

In any case the gravity path integral is a "blackbox" that is not really understood and cannot be used as a starting point to define the theory.

## Canonical quantum gravity

Canonical gravity provides an approach for quantization from first principles: we apply to gravity the canonical method to quantize constrained systems.

[Dirac]

The spacetime is written in a Hamiltonian decomposition

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

and the canonical Hamiltonian is

$$H_{\rm can} = \int d^d x \left( N\mathcal{H} + N^i \mathcal{H}_i \right) \ .$$

The lapse and shift behave as Lagrange multipliers. They enforce the Hamiltonian and momentum constraints that are due to the diffeomorphism gauge invariance of gravity. [Arnowitt-Deser-Misner]

The Hamiltonian constraint takes explicitly the form

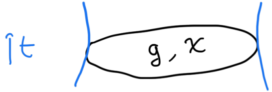
$$\mathcal{H} = \frac{16\pi G}{\sqrt{g}} \left( \pi_{ij}\pi^{ij} - \frac{1}{d-1}\left(g_{ij}\pi^{ij}\right)^2 \right) - \frac{1}{16\pi G}\sqrt{g}(R - 2\Lambda) + \mathcal{H}_{\rm m}$$

where $\pi^{ij}$ is the conjugate momentum of the spatial metric $g_{ij}$.

# Wheeler-DeWitt equation

In the quantum theory, states are wavefunctionals of spatial configurations

$$\Psi[g, \chi]$$




of the metric $g$ and matter fields $\chi$. The Hamiltonian constraint becomes an operator equation

$$\widehat{\mathcal{H}}\,\Psi = 0$$

obtained with $\widehat{\pi}^{ij} = -i\frac{\delta}{\delta g_{ij}}$. This leads to a second-order functional equation known as the Wheeler-DeWitt equation:

$$\left[\frac{16\pi G}{\sqrt{g}}\left(\frac{\delta}{\delta g_{ij}}\frac{\delta}{\delta g^{ij}} - \frac{1}{d-1}\left(g_{ij}\frac{\delta}{\delta g_{ij}}\right)^2\right) + \frac{1}{16\pi G}\sqrt{g}(R - 2\Lambda) + \widehat{\mathcal{H}}_{\mathrm{m}}\right]\Psi = 0\ ,$$

which determines the physical states. The Hilbert space of quantum cosmology is the space of solutions of this equation. [Wheeler, DeWitt]

The difficulty in solving this equation has been one of the main stumbling block in developing the canonical approach.

## Late time expansion

We focus on expanding universe so the large volume regime is also the late time regime. We take the "canonical time" to be the spatial volume

$$T = \int d^d x \sqrt{g} \ . \tag{2}$$

The expansion can be achieved by introducing intermediate variables $\Omega, \gamma_{ij}$ such that

$$g_{ij} = \Omega^2 \gamma_{ij}, \qquad \det(\gamma_{ij}) = 1, \tag{3}$$

and taking the limit $\Omega \to +\infty$.

Different terms in the Wheler-DeWitt equation have different scalings with $\Omega$. This allows to solve the equation order by order in $\Omega$.

The solution can be expanded in the form

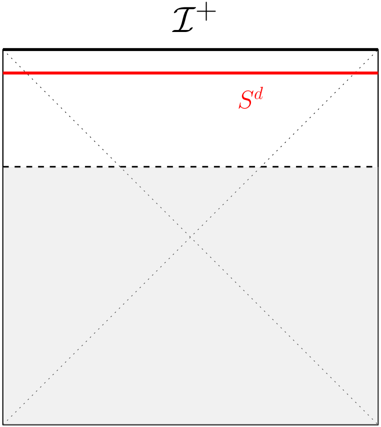
$$\Psi[g, \chi] = \exp\left[i(X_d + \cdots + X_0) + O(\Omega^{-1})\right] \tag{4}$$

where $X_k$ scales as $\Omega^k$.

## Late time expansion

The Wheeler-DeWitt equation can be solved at large volume, *i.e.* in the late time expansion of an asymptotically de Sitter spacetime.

[Chakraborty-Chakravarty-VG-Paul-Raju]



## The Hilbert space as theory space

The late time solutions then take the form

$$\Psi[g,\chi] = e^{iS[g,\chi]} Z[g,\chi] \tag{5}$$

where $S[g,\chi]$ is a universal divergent phase viewed as a counterterm.

The Wheeler-DeWitt equation is equivalent to the Weyl anomaly equation:

$$\Omega \frac{\delta}{\delta\Omega} Z = \mathcal{A}_d Z \ . \tag{6}$$

Thus the Hamiltonian and momentum constraints become at late time the Ward identities for the Weyl $\times$ diff symmetry of a CFT partition function.

As a result, we can view the wavefunction $Z[g,\chi]$ as the partition function of a CFT$_d$ on the metric $g$ and with sources $\chi$:

$$Z[g,\chi] \sim Z_{\rm CFT}[g,\chi] = \langle e^{-\int d^dx\, \chi\phi} \rangle_{\rm CFT}^{g_{ij}} \tag{7}$$

This gives a description of the Hilbert space at future infinity: it is the space of all CFT partition functions.

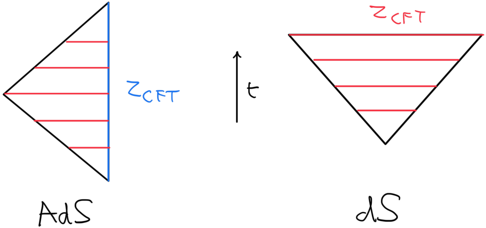
[Chakraborty-Chakravarty-VG-Paul-Raju]

## Relation to dS/CFT

The analytic continuation $\ell_{\rm AdS} = i\ell_{\rm dS}$ gives a way to relate AdS/CFT results to dS. This is the dS/CFT philosophy.

[Strominger, Maldacena, ...]

However, the role played by the CFT is quite different in cosmology.



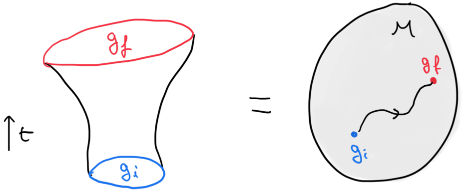

In AdS/CFT, $Z_{\rm CFT} = Z_{\rm AdS}$ computes the AdS partition function. So AdS quantum gravity is described by a single CFT.

In cosmology, $Z_{\rm CFT} = \Psi_{\rm dS}$ describes a physical state at late time. The cosmological Hilbert space is the space of all CFT partition functions.

## Time evolution and automorphic dynamics

The state at late time can be viewed as a CFT partition function. Then time evolution corresponds to running the renormalization group flow.

Another perspective: cosmological spacetimes correspond to particle trajectories on the moduli space $\mathcal{M}$ of spatial metrics. This suggests that quantum cosmology is equivalent to automorphic dynamics: the theory of a particle on moduli space.




The Einstein-Hilbert action is the worldline action of the particle and the Wheeler-DeWitt equation is the corresponding Klein-Gordon equation. The sum over particle trajectories is the sum over geometries in the gravity path integral.

## Pure 3d cosmology

Pure quantum cosmology in three dimensions is described by the action

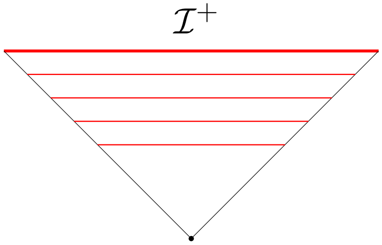
$$S = \frac{1}{16\pi G}\int_M d^3x\sqrt{-g}\,(R-2) - \frac{1}{8\pi G}\int_{\partial M}\sqrt{h}K \ . \tag{8}$$

We focus on spacetimes of the form $\mathbb{R}\times\Sigma$ where $\Sigma$ is a torus:

$$ds^2_\Sigma = T\frac{|du+\tau dv|^2}{y}, \qquad \tau = x+iy \in \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z}) \ . \tag{9}$$

The classical solutions are inflationary universes with torus slices starting from a singularity. They are analytic continuations of the BTZ geometries in $\mathrm{AdS}_3$.

[Castro-Maloney]



## Pure 3d cosmology

- The wavefunctions are modular form which can be viewed as $\mathrm{CFT}_2$ partition functions

$$\Psi(T, \tau = x + iy), \qquad \tau = x + iy \in \mathcal{F} = \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z}) \qquad (10)$$

- The Wheeler-DeWitt equation reduces to a Klein-Gordon equation on the auxiliary spacetime

$$ds^2_{\mathsf{Aux}} = dT^2 - T^2 \frac{dx^2 + dy^2}{y^2}, \qquad (\Box + M^2)\Psi = 0, \qquad M = \frac{\pi}{2G},$$

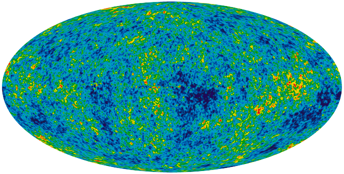
which determines the evolution in $T$ of the wavefunction.

- The resulting dynamics is equivalent to the $T\overline{T}$ deformation of the $\mathrm{CFT}_2$ partition function: a solvable irrelevant deformation. [Zamolodchikov, ...]

The Hilbert space is the space $L^2(\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z}))$ of modular cusp forms which plays an important role in number theory. In particular, we obtain a number-theoretic interpretation of the Hartle-Hawking state in this setting.

[VG]

# Takeaways



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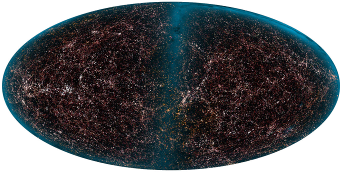

- We should understand better quantum cosmology. It is our best hope to connect theoretical quantum gravity ideas to experiments.
- Canonical quantum gravity provides an approach from first principles. The Wheeler-DeWitt equation can be solved in a late time expansion and the Hilbert space is the space of CFT partition functions.
- The dynamics can be view as a renormalization group flow.
- The three-dimensional case gives a solvable toy model with interesting connections to number theory.

Thank you!