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Symmetries of Black Hole Perturbations

based on arXiv: 2010.00593, 2105.01069, 2203.08832, 2312.05065, 2404.06544, 2410.03542, 2410.10952

with
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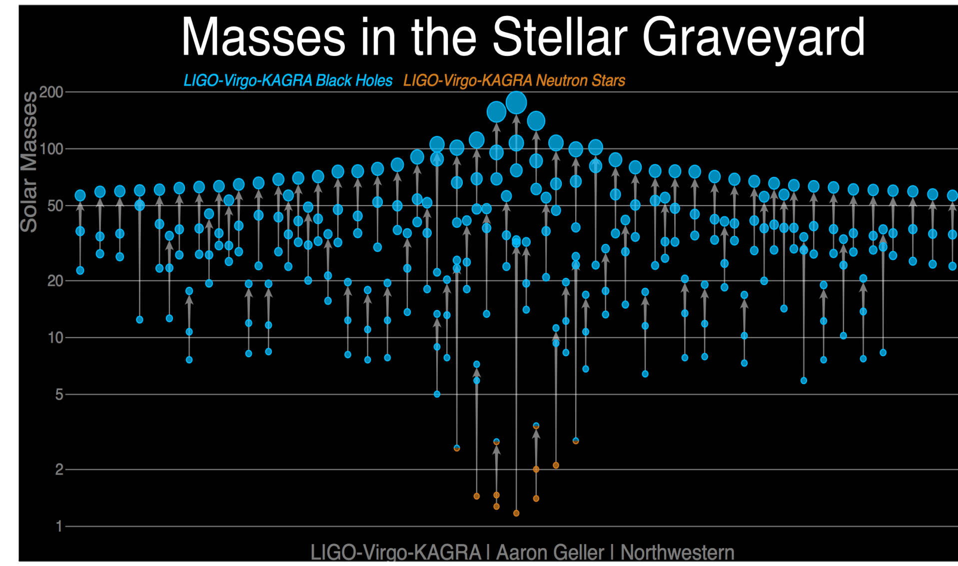
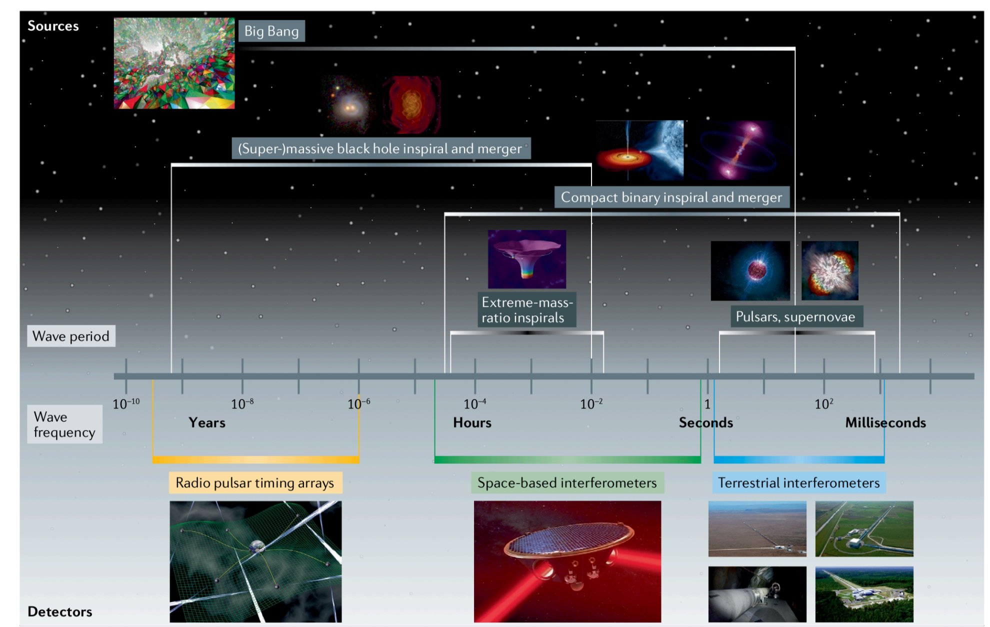
Gravitational-wave astronomy

- The direct detection of gravitational waves from merging binary systems sparked an explosive growth in the field of gravitational-wave astronomy.



- Unique opportunity to test general relativity in the strong-field regime, shed light on the fundamental aspects of gravity and black holes, probe the fundamental nature of astrophysical compact objects.



[Nature Reviews Physics, 3, 344–366 (2021)]

- Extraordinary scientific potential of upgraded detectors and future facilities.

- We are witnessing the dawn of the era of precision physics with gravitational waves.
  [Berti et al. '15], [Barack et al. '18], [Cardoso and Pani '19], [Baibhav et al. '19], [Barausse et al. '20], [Perkins, Yunes and Berti '20], [Bailes et al. '21], [Berti et al. '22]…

Symmetries of black holes

- Black hole perturbation theory has a long history starting from the work of Regge and Wheeler, Zerilli, Teukolsky, Chandrasekhar…
- Interestingly, recent investigations suggest the subject has depths yet to be plumbed.

Symmetries of black holes

- "The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time."
(S. Chandrasekhar, "*The mathematical theory of black holes*")

- Black holes are among the simplest and most robust objects in nature: uniquely determined by their mass and spin (and charge).

- This *simplicity* is inherited by the perturbations.

- Some aspects of this *simplicity* are well understood in terms of (hidden) symmetries of general relativity.

Outline

I will focus on static perturbations and tidal Love numbers of black holes in general relativity.

Static response and tidal deformability

- The Love numbers are the coefficients encoding the (conservative) tidal deformability of a compact object (analogous to the electric and magnetic susceptibilities in EM).

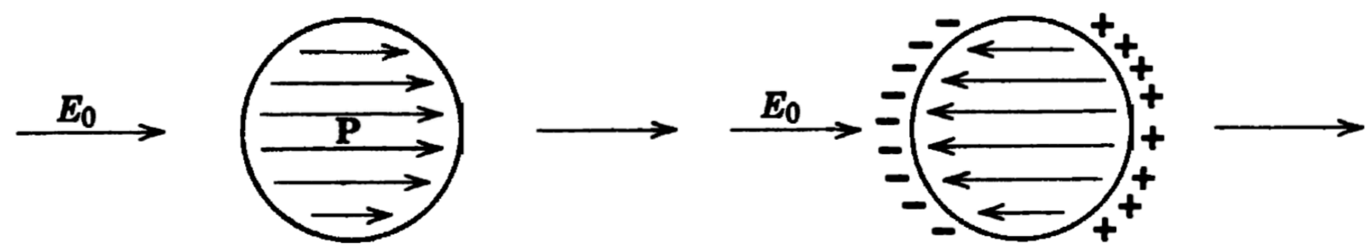


Figure 4.7 Dielectric sphere in a uniform field E_0, showing the polarization on the left and the polarization charge with its associated, opposing, electric field on the right.

[J. D. Jackson, "*Classical Electrodynamics*"]

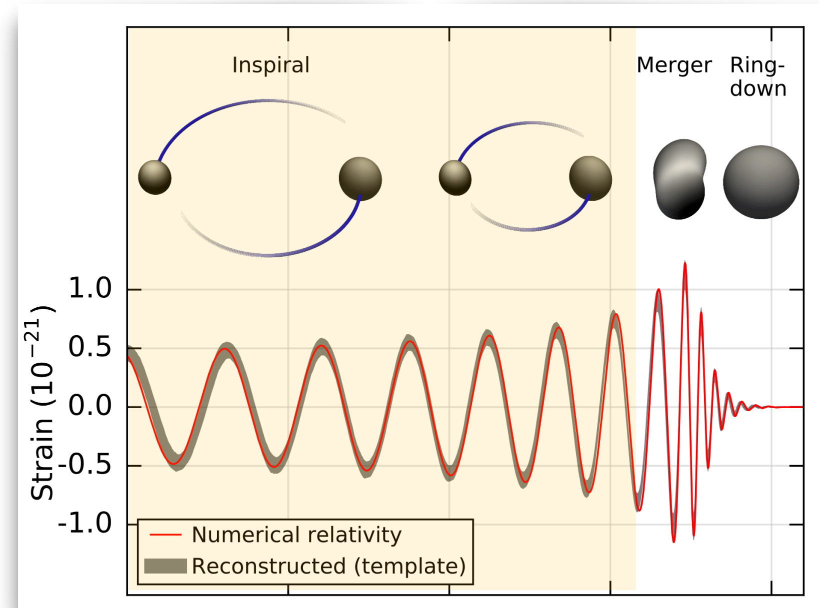
- In EM we solve $\vec{\nabla}^2\Phi = 0$:

$$\Phi_{\text{ext}} = \sum_\ell A_\ell \left[r^\ell + k_\ell r^{-\ell-1}\right] P_\ell(\cos\theta)\,, \qquad \Phi_{\text{int}} = \sum_\ell B_\ell r^\ell P_\ell(\cos\theta)\,.$$

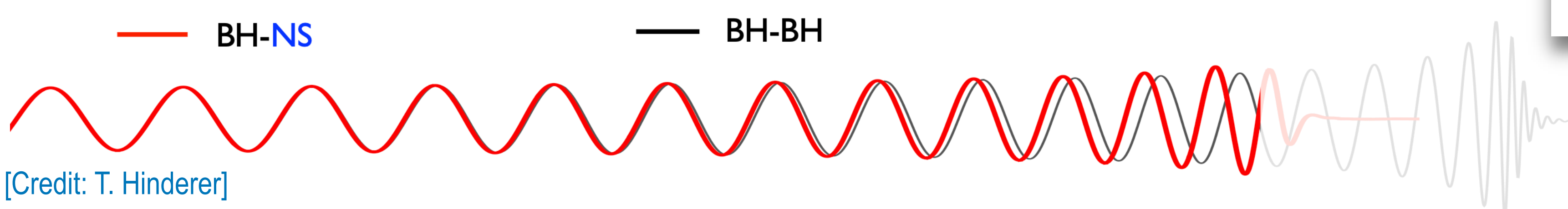
- The boundary condition at $r = +\infty$ fixes A_ℓ, while k_ℓ and B_ℓ are determined by regularity conditions across the surface (continuity of $\vec{E}_{\|}$ and $\vec{D}_\perp$).
- For instance, if $\vec{E}_0 = A_1\hat{z}$, one finds $k_{\ell=1} = -\dfrac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} r_0^3$ (ϵ_0 and ϵ are the vacuum and dielectric permittivities).
- k_ℓ are the coefficients of the induced response.

Tidal deformability of compact objects

- One can distinguish between conservative response (Love numbers) and dissipative response (*tidal heating*).
  [Fang and Lovelace '05], [Damour and Nagar '09], [Binnington and Poisson '05]



- Tidal effects change the dynamics during the inspiral.



[Credit: T. Hinderer]

- Love numbers and dissipative numbers carry relevant information about the object's structure and interior dynamics:

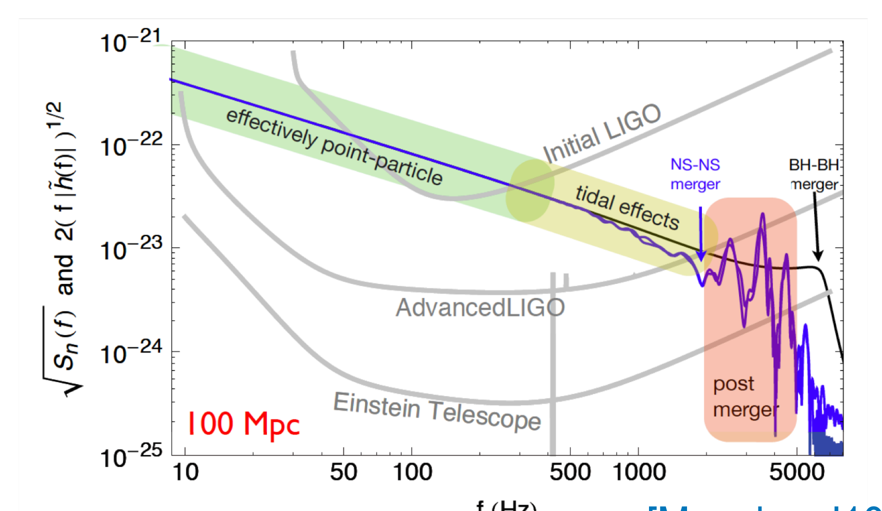
  _ equation of state of neutron stars; [Flanagan and Hinderer '07], [Vines, Flanagan and Hinderer '11], [Bini, Damour and Faye '12], [Baiotti and Rezzolla '17], […]

  _ physics at the horizon of black holes and fundamental aspects of gravity in strong-field regime; [Hui, Joyce, Penco, LS and Solomon '21, '22], [Charalambous, Dubovsky and Ivanov '21], […]

  _ new physics and existence of new types of compact objects.
  [Franzin et al. '17], [Cardoso et al. '17], [Cardoso and Pani '19], […]

Tidal deformability of compact objects

- The measurement of tidal deformation is challenging with current detectors.
- In the Post-Newtonian (PN) regime of the inspiral, leading-order tidal effects are associated with tidal heating: start from 2.5PN order for spinning objects and 4PN order for non-rotating ones. [Poisson and Sasaki '94], [Tagoshi et al. '97], [Blanchet '13], […]
  Conservative effects start from 5PN. [Damour '83], [Porto '16], […]
- Example of constraints on tidal heating from LKV O1-O3 data: [Chia, Zhou and Ivanov '24].
  Constraints on black hole dissipative coefficients still 2 orders of magnitudes larger than theoretical value.
- We can search for exotic compact objects (e.g., boson stars, DM stars…) with large λ.
  Example of matched-filtering search for binaries with compact objects with $10^2 \lesssim \lambda \lesssim 10^6$: [Chia et al. '23].
- Precise measurements of tidal coefficients possible in the future.
  [Piovano, Maselli, Pani '22], [Maggiore et al '19], [Iacovelli et al '23], […]



[Maggiore '18]

Luca Santoni

Tidal Love numbers of black holes

- In linear perturbation theory, an explicit calculation in general relativity shows that Love numbers of black holes vanish, as opposed to other types of compact objects:

  _ Schwarzschild BHs: [Fang and Lovelace ’05], [Binnington and Poisson ‘09], [Damour and Nagar ’09], […]
  _ Kerr BHs: [Le Tiec, Casals ’20], [Le Tiec, Casals, Franzin ’20], [Chia ’20], [Charalambous, Dubovsky, Ivanov ’21], […]
  _ Reissner–Nordström BHs: [Cardoso et al. ’17], [Rai and LS ’24]

- This is special of 4D general relativity: Love numbers are nonzero for different objects, modified gravity theories, BHs in higher spacetime dimensions.

- As opposed to the EM example, the calculation of the induced response in general relativity is affected by ambiguities in the choice of coordinate system, source/response split, and nonlinearities.
  [Kol and Smolkin ’11], [Hui, Joyce, Penco, LS, Solomon ’20], [Le Tiec, Casals ’20], [Charalambous, Dubovsky, Ivanov ’21], […]

Point-particle effective theory

- There is a completely unambiguous way of defining the tidal response coefficients based on an effective field theory. [Goldberger and Rothstein '04, '05, ...], [Kol and Smolkin '11], [Porto '16]

- Let us consider e.g. a scalar field around a black hole:

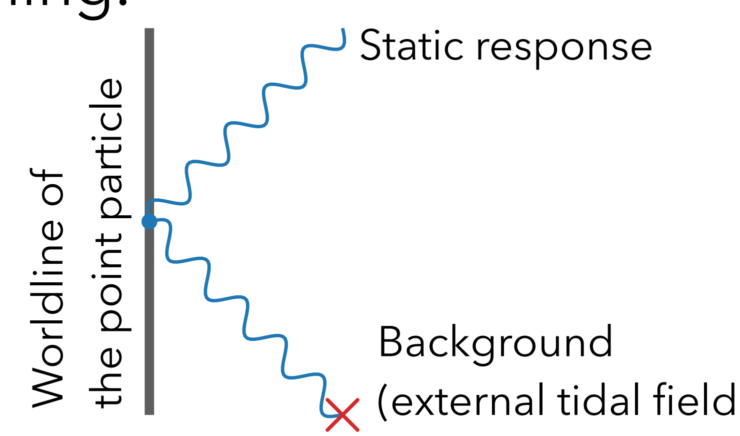
$$S = -\frac{1}{2}\int \mathrm{d}^4x\,(\partial\phi)^2 - M\int \mathrm{d}\tau + \int \mathrm{d}\tau\left[-g\phi + \sum_{\ell=0}^{\infty}\frac{\lambda_\ell}{2\ell\,!}\left(\partial_{(a_1}\cdots\partial_{a_\ell)_T}\phi\right)^2\right].$$

- λ_ℓ are the Love number coefficients.

- One generically expects: $\lambda_\ell \sim \mathcal{O}(1) r_s^{2\ell-1}$ and to find (classical) RG running.

- After matching with the UV result: $\lambda_\ell = 0$ in D=4 and no running.

- Generically non-zero in D>4.
[Kol and Smolkin '11], [Hui, Joyce, Penco, LS and Solomon '21], [Charalambous and Ivanov '23], [Rodriguez, LS, Solomon and Temoche '23]



Point-particle effective theory

- Following 't Hooft's naturalness principle, the vanishing of the Love numbers in D=4 is a naturalness puzzle from an EFT perspective. [Rothstein '14], [Porto '16]

$$S = -\frac{1}{2}\int \mathrm{d}^4x\,(\partial\phi)^2 - M\int \mathrm{d}\tau + \int \mathrm{d}\tau\left[-g\phi + \sum_{\ell=0}^{\infty}\frac{\lambda_\ell}{2\ell\,!}\left(\partial_{(a_1}\cdots\partial_{a_\ell)_T}\phi\right)^2\right]$$

- Looks like something that can very likely follow from a symmetry in the theory.

Symmetries of vanishing Love numbers

- In [2105.01069] we showed that the vanishing of Love numbers is the consequence of linearly realized symmetries governing static perturbations around black holes.

- Take the Klein–Gordon equation with $\omega = 0$ (static limit):

$$\partial_r \left(\Delta \partial_r \phi_\ell\right) - \ell(\ell+1)\phi_\ell = 0\,, \qquad \Delta = r(r - r_s)\,.$$

- The raising and lowering operators

$$D_\ell^+ \equiv -\Delta\partial_r + \frac{\ell+1}{2}(r_s - 2r), \qquad D_\ell^- \equiv \Delta\partial_r + \frac{\ell}{2}(r_s - 2r)$$

are a symmetry of the static KG equation – they "commute" with the "Hamiltonian"

$$H_\ell \equiv -\Delta\partial_r\left(\Delta\partial_r\right) + \ell(\ell+1)\Delta\,, \qquad H_\ell\phi_\ell = 0\,,$$

$$H_{\ell+1}D_\ell^+ = D_\ell^+ H_\ell\,, \qquad H_{\ell-1}D_\ell^- = D_\ell^- H_\ell\,.$$

- This representation has a "ground state", $\ell = 0$, which satisfies $\partial_r\left(\Delta\partial_r\phi_0\right) = 0$. The good ground state (regular at the horizon) is $\phi_0 =$ const., s.t. $D_0^-\phi_0 = 0$.

- From this one, apply a string of raising operators to construct the solution at level ℓ, $\phi_\ell = D_{\ell-1}^+ D_{\ell-2}^+ \cdots D_0^+\phi_0$. This is a *polynomial* in *r*.

Symmetries of vanishing Love numbers

[Hui, Joyce, Penco, LS and Solomon '21]

$$\partial_r \left(\Delta \partial_r \phi_0\right) = 0\,, \qquad\qquad \Delta = r(r - r_s)\,.$$

- One could have chosen an $\ell = 0$ state such that it goes as $\frac{1}{r}$ as $r \to \infty$.
  This is not a good ground state: $\sim \log(r - r_s)$ as $r \to r_s$.

- The vanishing of the Love numbers (absence of $r^{-\ell-1}$ falloff) follows from two facts:

  (a) the existence of a ladder structure (*generalized special conformal symmetry*) allowing one to connect any level-ℓ solution to the level-0 solution;

  (b) the good level-0 solution obeys a first-order differential equation, thus connecting a single (regular) asymptotic behavior at the horizon with a single (r^ℓ) asymptotic behavior at large r.

[Hui, Joyce, Penco, LS and Solomon '21, '22], [Achour, Livine, Mukohyama, Uzan '22]

Symmetries of vanishing Love numbers

[Hui, Joyce, Penco, LS and Solomon '21]

- The symmetry has a geometric origin: it arises from the (E)AdS isometries of a dimensionally reduced black hole spacetime.
  Static scalar action on a Schwarzschild background:

$$S = \frac{1}{2}\int \mathrm{d}\theta \mathrm{d}\varphi \mathrm{d}r \sqrt{g}\, \phi \,\square\, \phi\,, \qquad \mathrm{d}s^2 = \mathrm{d}r^2 + \Delta\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right).$$

After a Weyl rescaling, the metric becomes EAdS_3 with

$$\tilde{g}_{ij} = \Omega^2 g_{ij}\,, \qquad \tilde{\phi} = \Omega^{-\frac{1}{2}}\phi\,, \qquad \text{where} \qquad \Omega \equiv L^2/\Delta\,,$$

$$S = \frac{1}{2}\int \mathrm{d}^3x\sqrt{\tilde{g}}\left(\tilde{\phi}\,\tilde{\square}\,\tilde{\phi} + \frac{r_s^2}{4L^4}\tilde{\phi}^2\right), \qquad \mathrm{d}\tilde{s}^2 = \mathrm{d}r_\star^2 + \frac{4L^4}{r_s^2}\sinh^2\!\left(\frac{r_\star r_s}{2L^2}\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

where $\mathrm{d}r_\star = (L^2/\Delta)\mathrm{d}r$. The space has 6 Killing vectors: 3 rotations and 3 translations (or "boosts"). The translation that mixes $r_\star$ and θ acts on the original ϕ as

$$\delta\phi = -\,2\Delta\cos\theta\partial_r\phi + (r_s - 2r)\partial_\theta(\sin\theta\,\phi)$$

or, equivalently,

$$\delta\phi_\ell = c_{\ell+1}D^-_{\ell+1}\phi_{\ell+1} - c_\ell D^+_{\ell-1}\phi_{\ell-1}\,.$$

Symmetries of vanishing Love numbers

[Hui, Joyce, Penco, LS and Solomon '21]

- At large *r*, $\delta\phi$ reduces to a SCT, $\delta\phi = c_i\left(x^i - \vec{x}^2\partial^i + 2x^i\vec{x}\cdot\vec{\partial}\,\right)\phi$.

- We claim that this is the infrared symmetry that forbids Love number couplings in the point-particle effective action.

- Straightforward the generalization to Kerr and higher spins. [Hui, Joyce, Penco, LS and Solomon '21]
Similar conclusions for Reissner–Nordström (although slightly more involved). [Rai and LS '24]

Outlook and conclusions

Nonlinear tidal effects

- Nonlinearities in the Einstein field equations affect the tidal response of a compact object.

- In electromagnetism, the nonlinear polarization of an optical medium can be studied in nonlinear polarization theory.

  - *What is the nonlinear static response of a black hole?*
  - *Are the symmetries of* [Hui, Joyce, Penco, LS and Solomon '21, '22] *an 'accident' of linear response theory, or do they admit an extension to nonlinear order?*

Nonlinear tidal effects

- The worldline EFT naturally provides a framework for describing nonlinear deformation of compact objects.
[Bern et al '20], [Riva, LS, Savić, Vernizzi '23], [Iteanu, Riva, LS, Savić, Vernizzi '24]

- In the scalar field example, one shall just add operators with more powers of ϕ:

$$S = -\frac{1}{2}\int \mathrm{d}^4x\,(\partial\phi)^2 - M\int \mathrm{d}\tau + \int \mathrm{d}\tau\,\lambda_{\ell,n}\partial^{\ell}\phi^{n}$$

For gravitational waves, one shall replace ϕ with metric perturbations.

- In [Riva, LS, Savić, Vernizzi '23], [Iteanu, Riva, LS, Savić, Vernizzi '24], we showed that quadratic Love numbers of Schwarzschild black holes are zero in general relativity.
(See also [Gürlebeck '15], [Poisson '20, '21], [De Luca, Khoury, Wong '23])

- In [Combaluzier-Szteinsznaider, Hui, LS, Solomon, Wong '24], we showed that the ladder symmetries admit a fully nonlinear extension to all orders in perturbation theory (for axisymmetric and static spacetimes) and form an $\mathfrak{sl}(2,\mathbb{R})$ algebra.
(See also [Kehagias, Riotto '24])

Conclusions and open directions

- What is the nonlinear tidal deformability of a Kerr black hole?
- Black holes with charge? Higher spacetime dimensions?
- Quantify importance of nonlinearities in late inspiral of compact objects with future interferometers.

Backup

Nonlinear tidal effects

- The worldline EFT naturally provides a framework for describing nonlinear deformation of compact objects.
[Bern et al '20], [Riva, LS, Savić, Vernizzi '23], [Iteanu, Riva, LS, Savić, Vernizzi '24]

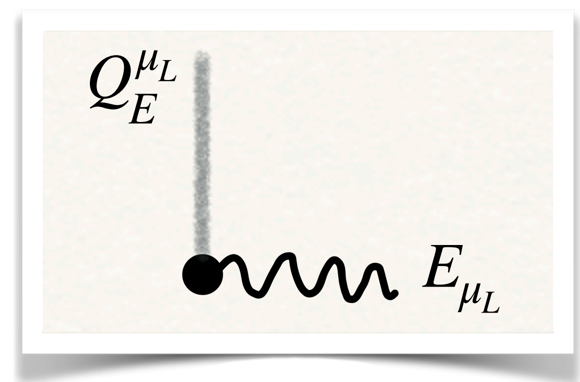
$$S = S_{\rm EH} + S_{\rm pp} + S_{\rm int}$$

$$S_{\rm EH} = \frac{M_{\rm Pl}^2}{2}\int {\rm d}^4x\sqrt{-g}\,R\,, \qquad S_{\rm pp} = -M\int {\rm d}\tau\,, \qquad S_{\rm int} = \sum_{\ell=2}^{\infty}\int {\rm d}\tau\left(Q_E^{\mu_L}E_{\mu_L} + Q_B^{\mu_L}B_{\mu_L}\right) + \dots\,,$$

where $\mu_L \equiv \mu_1 \cdots \mu_\ell$ and, schematically,

$$E_{\mu_1\cdots\mu_\ell} \sim \nabla_{\mu_1}\cdots\nabla_{\mu_{\ell-2}}E_{\mu_{\ell-1}\mu_\ell}\,, \qquad B_{\mu_1\cdots\mu_\ell} \sim \nabla_{\mu_1}\cdots\nabla_{\mu_{\ell-2}}B_{\mu_{\ell-1}\mu_\ell}\,,$$

$$E_{\mu\nu} \equiv C_{\mu\rho\nu\sigma}u^\rho u^\sigma, \qquad B_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\gamma\mu}{}^{\alpha\beta}C_{\nu\delta\alpha\beta}u^\delta u^\gamma\,.$$



- $Q_{E,B}^{\mu_L}$ carry information about the object's finite-size properties and microscopic physics: absorption across the horizon, dissipation, tidal response…

Nonlinear tidal Love numbers

- To define nonlinear response, we shall proceed as in EM. Focusing e.g. on E-sector:

$$\langle Q_E^{i_L}(\tau)\rangle = \sum_{n=1}^{\infty}\int \mathrm{d}\tau_1\cdots\int \mathrm{d}\tau_n\, {}^{(n)}\mathscr{R}^{i_L|i_{L_1}\cdots i_{L_n}}(\tau-\tau_1,\ldots,\tau-\tau_n)E_{i_{L_1}}(\tau_1)\cdots E_{i_{L_n}}(\tau_n) \ ,$$

where ${}^{(n)}\mathscr{R}$ is the n$^{\text{th}}$-order response function.

- To leading-order in the derivative expansion (quadrupole), considering non-rotating objects and conservative sector,

$${}^{(n)}\mathscr{R}^{ij|i_1j_1\cdots i_nj_n} \supset \delta^i_{j_n}\delta^j_{i_1}\delta^{i_2}_{j_1}\delta^{j_2}_{i_3}\cdots\delta^{j_{n-1}}_{i_n}\delta(\tau-\tau_1)\cdots\delta(\tau-\tau_n) + \text{perms.}$$

and $S_{\rm int}$ boils down to *local* contact operators:

$$S_{\rm int} = \int \mathrm{d}\tau \sum_{n=1}^{\infty}\left[\lambda_n^{(E)}E_{\mu_1}{}^{\mu_2}\cdots E_{\mu_{n+1}}{}^{\mu_1} + \text{higher multipoles}\right] \sim \sum_{n,l}\int \mathrm{d}\tau\, \lambda^{(E)}_{l,n}\partial^l E^{n+1} \ .$$

Nonlinear tidal Love numbers

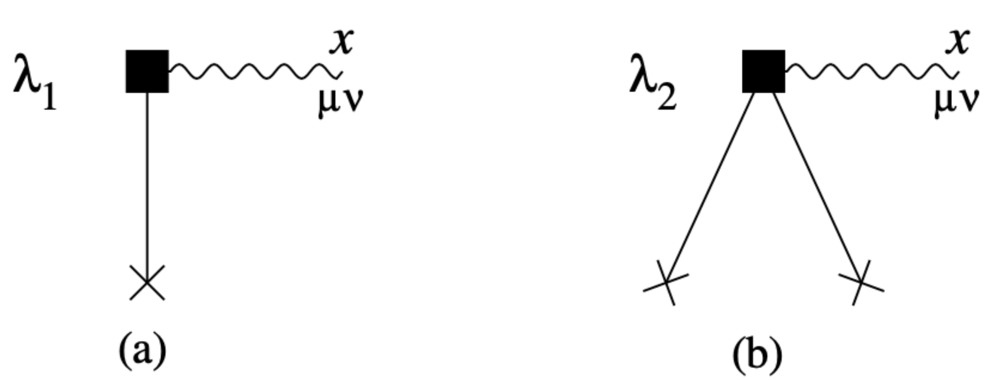
$$S_{\rm int} = \int \mathrm{d}\tau \sum_{n=1}^{\infty} \left[\lambda_n^{(E)} E_{\mu_1}{}^{\mu_2} \cdots E_{\mu_{n+1}}{}^{\mu_1} + \text{higher multipoles}\right] \sim \sum_{n,l} \int \mathrm{d}\tau\, \lambda_{l,n}^{(E)} \partial^l E^{n+1}$$

- $\lambda_{l,n}^{(E)}$ are the (nonlinear) Love numbers.

- As in any EFT, $\lambda_{l,n}^{(E)}$ can be either constrained experimentally or determined via matching to some explicit UV model.

- In [Riva, LS, Savić, Vernizzi '23], [Iteanu, Riva, LS, Savić, Vernizzi '24] we computed the quadratic Love numbers of BHs by matching with GR:

$$S_{\rm int} = \int \mathrm{d}\tau \left[\lambda_1^{(E)} E_{ij}E^{ij} + \lambda_1^{(B)} B_{ij}B^{ij} + \lambda_2^{(E)} E^i{}_j E^j{}_k E^k{}_i + \lambda_2^{(EB)} E^i{}_j B^j{}_k B^k{}_i + \text{higher multipoles}\right]$$



Feynman diagrams for the (a) linear and (b) nonlinear tidal deformation. These vertices scale as $\sim 1/r^{\ell+1}$.

Nonlinear equations for static perturbations

- There are two expansion parameters:
  _ $\kappa \equiv 1/M_{\rm Pl}$ which controls nonlinearities of gravity;
  _ $\mathscr{E}$: the amplitude of the external tidal field.

- At quadratic order, the (static) equations are schematically: $\mathscr{D}\delta g \sim \kappa\, \delta g^2$.
  Solving perturbatively as $\delta g_{\mu\nu} = \delta g^{(1)}_{\mu\nu} + \kappa \delta g^{(2)}_{\mu\nu}$ and imposing regularity at the horizon, one can find the nonlinear solution for $\delta g_{\mu\nu}$.

- $\delta g_{\mu\nu} = \delta g^{\rm even}_{\mu\nu} + \delta g^{\rm odd}_{\mu\nu}$. As an example, let us focus on $\delta g^{\rm even}_{\mu\nu}$.
  In the Regge–Wheeler gauge, the most general parametrization of $\delta g^{\rm even}_{\mu\nu}$ is:

$$\delta g^{\rm even}_{\mu\nu}(r,\theta,\phi) = \sum_{\ell m} {\rm diag}\left[\left(1-\frac{r_s}{r}\right) H_0(r), H_2(r), K(r),\ \sin^2\theta\, K(r)\right] Y^m_\ell(\theta,\phi)\ .$$

- After some algebra, the Einstein equations boil down to

$$H_0'' + \frac{2r-r_s}{r(r-r_s)} H_0' - \frac{\ell(\ell+1)r(r-r_s)+r_s^2}{r^2(r-r_s)^2} H_0 = S_{H_0}\,, \qquad \text{where } S_{H_0} \sim O(\delta g^2).$$

Nonlinear equations for static perturbations

- Let us assume that the external tidal field is a quadrupole: asymptotically, $H_0^{(\ell=2,m)} \sim \mathscr{E} r^2$.

- The solution to the linearized equation is $H_0^{(\ell=2,m)} = \mathscr{E}\left(r^2 - r_s r\right)$, which is notoriously a polynomial (i.e., no induced linear static response).

- $S_{H_0} \sim O(\delta g^2)$ contains a product of 3 spherical harmonics:
$$\mathscr{G}^{\ell,\ell_1,\ell_2}_{m,m_1,m_2} \equiv \int Y_\ell^{m*}(\theta,\phi) Y_{\ell_1}^{m_1}(\theta,\phi) Y_{\ell_2}^{m_2}(\theta,\phi) \sin\theta \mathrm{d}\phi \mathrm{d}\theta \,,$$
which enforces the standard angular momentum selection rule $\ell = \ell_1 \otimes \ell_2$, i.e. $|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2$ and $m = m_1 + m_2$.

- If $\ell_1 = \ell_2 = 2 \Rightarrow \ell = 0, 2, 4$. At the next order:
$$H_0^{(\ell=2,m')} = \mathscr{E}\left(r^2 - r_s r\right)\left[\delta_m^{m'} - \frac{\mathscr{E}}{4r_s^2}\mathscr{G}^{2,2,2}_{m',m,m} r(2r + 3r_s)\right] .$$

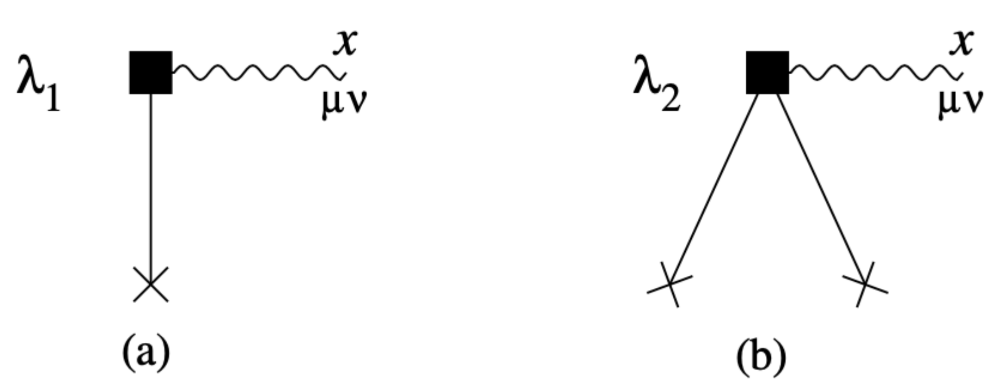
- Note that the quadratic terms in $\mathscr{E}$ are small corrections as long as $\mathscr{E} r^2 \ll r_s^2$.

Matching with point-particle EFT

- The full quadratic static solution

$$H_0^{(\ell=2,m')} = \mathcal{E}\left(r^2 - r_s r\right)\left[\delta_m^{m'} - \frac{\mathcal{E}}{4r_s^2}\mathcal{G}^{2,2,2}_{m',m,m} r(2r+3r_s)\right]$$

should then be compared (after a suitable gauge transformation) with



$$\sim \frac{1}{r^{\ell+1}} \, .$$

- The result of the matching (in the region $r_s \ll r \ll r_s/\sqrt{\mathcal{E}}$) is:

$$\lambda_1 = \lambda_2 = 0 \, .$$

[Riva, LS, Savic, Vernizzi '23]

Vanishing of quadratic Love numbers

[Iteanu, Riva, LS, Savic, Vernizzi '24]

- The result can be extended to the odd sector and to all multipoles.

$$H_0'' + \frac{2r - r_s}{r(r - r_s)} H_0' - \frac{\ell(\ell+1)r(r-r_s) + r_s^2}{r^2(r-r_s)^2} H_0 = S_{H_0} ,$$

$$h_0'' - \frac{\ell(\ell+1)r - 2r_s}{r^2(r-r_s)} h_0 = S_{h_0} ,$$

where $S_{H_0}, S_{h_0} \sim O(\delta g^2)$.

- We show that, for all ℓ's, the structure of the eqs in GR is so special that, up to second order in PT, the solutions H_0 and h_0 are simple **polynomials** in r.

- This follows from: (a) the simple form of the source, and (b) the general properties of the homogeneous (hypergeometric) solutions:

$$h_0(r) = h_{0,1}(r) \int_1^{r/r_s} \mathrm{d}y\, h_{0,2}(y) S_{h_0}(y) - h_{0,2}(r) \int^{r/r_s} \mathrm{d}y\, h_{0,1}(y) S_{h_0}(y)$$

Vanishing of quadratic Love numbers

[Iteanu, Riva, LS, Savic, Vernizzi '24]

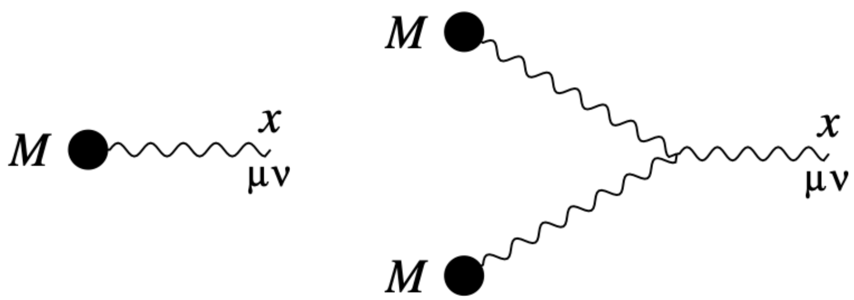
- In the EFT, all linear and quadratic Love number couplings vanish:

$$S_{\rm int} \sim \int {\rm d}\tau \left[\lambda_{1,n}^{(E)}(\partial^n E)^2 + \lambda_{1,n}^{(B)}(\partial^n B)^2 + \lambda_{2,nml}^{(E)}\partial^n E\,\partial^m E\,\partial^l E + \lambda_{2,nml}^{(EB)}\partial^n E\,\partial^m B\,\partial^l B + \dots\right]$$

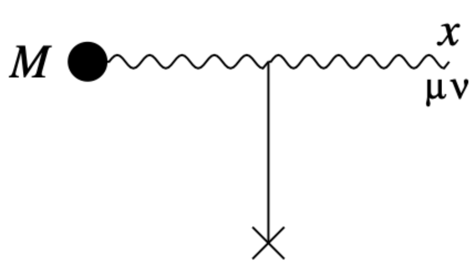
$$\boxed{\lambda_{1,n}^{(E)} = \lambda_{1,n}^{(B)} = \lambda_{2,nml}^{(E)} = \lambda_{2,nml}^{(EB)} = 0\,.}$$

Vanishing of quadratic Love numbers

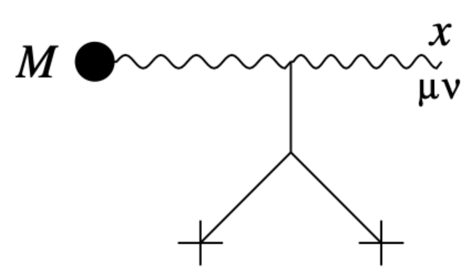
- The point-particle EFT can be matched to the full solution without turning on Love number couplings.
  Nonlinear corrections to the static solution in GR can be reconstructed from the EFT, to all orders in r_s, via just graviton bulk nonlinearities.



Feynman diagrams that reconstruct the Schwarzschild metric up to $O(r_s^2)$.

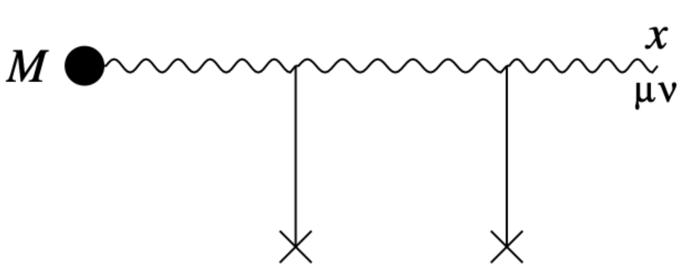


(a)

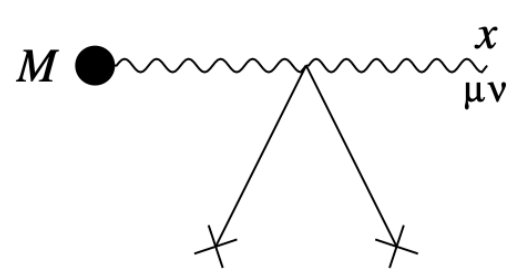


(b)

Diagram (a) yields the order-r_s correction to the linear tidal field solution.



(c)



(d)

Diagrams (b), (c) and (d) represent order-r_s corrections to the tidal source at second order in the external field amplitude.