Integrable sigma-models and their quantisation

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Based on past and on-going works with Gleb Kotousov, Adrien Molines, Joerg Teschner, Benoît Vicedo, Charles Young

Introduction

Integrable σ-models

- Non-linear σ-models: 2-dimensional field theories
- Applications in string theory, condensed matter physics, ...
- Useful toy models to deepen our understanding of field theory

- Integrability: infinite number of symmetries, exact results
  $\rightarrow$ integrable σ-models are rare but valuable

- **Main topics of research:**
  - construction and classification of classical integrable σ-models
  - quantisation

Contents

Sigma-models

Free massless scalar fields

- Classical 2d field theory
- Free massless scalar fields $\phi^i(t,x)$:

$$S_{\text{free}}[\phi] = \frac{1}{2} \iint \mathrm{d}t\,\mathrm{d}x \, \sum_{i=1}^{n} \partial_+\phi^i \, \partial_-\phi^i \,, \qquad \partial_\pm = \partial_t \pm \partial_x$$

- Free equations of motion:

$$\partial_+\partial_-\phi^i = (\partial_t^2 - \partial_x^2)\phi^i = 0$$

Non-linear σ-model

- One possibility to add interactions: non-linear σ-model

$$S[\phi] = \frac{1}{2} \iint \mathrm{d}t \, \mathrm{d}x \; E_{ij}(\phi) \, \partial_+ \phi^i \, \partial_- \phi^j$$

- Non-linear coupled equations of motion:

$$\partial_+ \partial_- \phi^i + \Gamma^i{}_{jk}(\phi) \, \partial_+ \phi^j \, \partial_- \phi^k = 0$$

$$\Gamma^i{}_{jk} = (E_{il} + E_{li})^{-1} \left(\frac{\partial E_{lk}}{\partial \phi^j} + \frac{\partial E_{jl}}{\partial \phi^k} - \frac{\partial E_{jk}}{\partial \phi^l} \right)$$

- Non-constant $E_{ij}(\phi) \longrightarrow$ non-zero $\Gamma^i{}_{jk}(\phi) \longrightarrow$ interactions

Example: the $O(3)$-model

- The $O(3)$-model: two fields ϕ^1, ϕ^2

$$S[\phi] = \frac{1}{2} \iint \mathrm{d}t\,\mathrm{d}x \left(\partial_+\phi^1\,\partial_-\phi^1 + \sin^2(\phi^1)\,\partial_+\phi^2\,\partial_-\phi^2 \right)$$

- Non-linear coupled equations of motion:

$$\partial_+\partial_-\phi^1 - \sin(\phi^1)\cos(\phi^1)\,\partial_+\phi^2\,\partial_-\phi^2 = 0$$

$$\partial_+\partial_-\phi^2 + \frac{\cos(\phi^1)}{\sin(\phi^1)}\left(\partial_+\phi^1\,\partial_-\phi^2 + \partial_+\phi^2\,\partial_-\phi^1\right) = 0$$

- Applications in condensed matter (limit of spin chain) [Haldane '83]

Applications of non-linear sigma-models

$$S[\phi] = \frac{1}{2} \iint \mathrm{d}t \, \mathrm{d}x \; E_{ij}(\phi, g_a) \, \partial_+ \phi^i \, \partial_- \phi^j$$

- σ-model: interacting relativistic 2d field theory
- In general, E_{ij} depends on the fields ϕ^i and parameters g_a
  $\rightarrow$ g_a are coupling constants
- Applications in various domains:
  - condensed matter
  - string theory
  - AdS/CFT correspondence
  - toy models for field theory
  - ...

Integrable sigma-models

Integrable non-linear sigma-model

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  - infinite number of symmetries / conserved charges
  - exact results
  - example: $O(3)$-model [Pohlmeyer '76]
  - growing domain of research since the 70s

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- Recent activities: unifying approaches to classical integrable σ-models (affine Gaudin models / 4d Chern-Simons), towards classification

[Vicedo '17, Delduc SL Magro Vicedo '19 '20, Costello Yamazaki '19, Bassi SL '19, Schmidtt '19, Tian '20, Arutyunov Bassi SL '20, SL Vicedo '20, Derryberry '21, SL Wallberg '23, Cole Cullinan Hoare Liniado Thompson '23 '24, ...]

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Higher-spin local charges of integrable σ-models

[Evans Hassan MacKay Moutain '99, SL Magro Vicedo '17, ...]

- Local conserved charges $\mathcal{Q}_n^\pm$ of higher-spin $\pm s$:

$$\mathcal{Q}_s^\pm = \int A_{i_1 \cdots i_{s+1}}(\phi, g_a)\, \partial_\pm \phi^{i_1} \cdots \partial_\pm \phi^{i_{s+1}}\, \mathrm{d}x$$

- For $s = 1$: Hamiltonian $\mathcal{H}$ and momentum $\mathcal{P}$

$$\mathcal{Q}_1^\pm = \mathcal{H} \pm \mathcal{P}$$

- Allowed spins s are model-dependent (for $O(3)$-model, all odd numbers)

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- Poisson-commuting:

$$\left\{ \mathcal{Q}_s^\pm, \mathcal{Q}_{s'}^\pm \right\} = \left\{ \mathcal{Q}_s^+, \mathcal{Q}_{s'}^- \right\} = 0$$

Quantisation

Quantising σ-models: renormalisation

$$S[\phi] = \frac{1}{2} \iint \mathrm{d}t \, \mathrm{d}x \; E_{ij}(\phi, g_a) \, \partial_+ \phi^i \, \partial_- \phi^j$$

- Short-distance quantum divergences
- Renormalisable: absorb divergences into coupling constants g_a
- Dependence on energy scale μ

  $\rightarrow$ Renormalisation Group (RG) flow

$$\mu \frac{\partial}{\partial \mu} g_a = \beta_a(g_b)$$

Renormalisation of integrable σ-models

- **Conjecture:** classically integrable σ-models are renormalisable [Fateev Onofri Zamolodchikov '93, Fateev '96]
- Progresses towards a proof at 1-loop using unifying frameworks
- Recent activities and future perspectives for higher-loops

[**1-loop:** Delduc SL Sfetsos Siampos '20, Hassler '20, Costello, Derryberry '21, Levine '22, Hassler SL Vicedo '23, SL Wallberg '23, SL Levine Wallberg *in progress*, ...]

[**Higher-loop:** Hoare Levine Tseytlin '19 '19 '21, Georgiou Sagkrioti Sfetsos Siampos '19, Hassler '20, Alfimov Litvinov '22, Gamayun Losev Shifman '23, ...]

UV and IR behaviours

- Ultraviolet (UV) limit $\mu \to \infty$: high-energy
- Well-behaved for many (but not all) integrable σ-models:
  - asymptotic freedom: massless bosons as UV fixed point
  - asymptotic safety: interacting CFT as UV fixed point

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- Often (but not always): strong-coupling regime in the infrared $\mu \to 0$
- Conjecture: dynamically generated mass-gap M

  $\to$ large distance physics gives scattering of massive particles

- Properties analogous to QCD

  $\to$ toy models with exact results

First-principle quantum integrability

- Integrable bootstrap: suppose quantum integrability and derive exact results on the emerging IR-physics (S-matrix)
  - possible only for simplest models
  - relies on strong assumptions, in particular quantum integrability
  - difficult to connect to microscopic theory

First-principle quantum integrability

- Integrable bootstrap: suppose quantum integrability and derive exact results on the emerging IR-physics (S-matrix)
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- **My main goal:** first-principle quantum integrability
  - start from micoscropic Lagrangian/Hamiltonian
  - define the states and the algebra of quantum observables
  - construct commuting higher-spin local charges $\widehat{Q}_s^{\pm}$
  - simultaneously diagonalise them
- All steps are hard!
- **Strategy** [Zamolodchikov '89, Bazhanov Lukyanov Zamolodchikov '94]: start at conformal UV fixed-point, described by chiral algebras

Chiral algebra of the free bosons

- Free scalar bosons:

$$S_{\text{free}}[\phi] = \frac{1}{2} \iint \mathrm{d}t \, \mathrm{d}x \, \sum_{i=1}^{n} \partial_+ \phi^i \, \partial_- \phi^i \,, \qquad \partial_-(\partial_+ \phi^i) = 0$$

- Left-moving fields $\partial_+ \phi^i$, functions of $x^+ = t + x$ only
- Chiral algebra: all left-moving fields, normal ordered products

$$:\partial_+^{k_1} \phi^{i_1} \cdots \partial_+^{k_p} \phi^{i_p}: \qquad \text{with } k_i \in \mathbb{Z}_{\geq 1}$$

- Quantum commutation relation:

$$[\partial_+ \phi^i(x), \partial_+ \phi^j(y)] = 2\mathrm{i}\pi\hbar \, \delta^{ij} \, \delta'(x - y)$$

Chiral algebra of $SU(2)$ WZW-model

- $SU(2)$ WZW-model: conformal σ-model

$$S_{\mathrm{WZW}}[\phi] = \frac{K}{2} \iint \mathrm{d}t\, \mathrm{d}x \left(\sum_{i=1}^{3} \partial_+ \phi^i \, \partial_- \phi^i + 2\cos(\phi^1)\, \partial_+ \phi^2 \, \partial_- \phi^3 \right)$$

- Left-moving current $\widehat{\mathrm{J}}^1(x^+)$, $\widehat{\mathrm{J}}^2(x^+)$, $\widehat{\mathrm{J}}^3(x^+)$: $\qquad \partial_- \widehat{\mathrm{J}}^a = 0$

$$\widehat{\mathrm{J}}^1 = \partial_+ \phi^3 + \cos(\phi^1)\, \partial_+ \phi^2 \,, \qquad \ldots$$

Non-linear analogues of left-moving $\partial_+ \phi^i$ for free fields

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$$\widehat{\mathrm{J}}^1 = \partial_+\phi^3 + \cos(\phi^1)\, \partial_+\phi^2\,, \qquad \dots$$

Non-linear analogues of left-moving $\partial_+\phi^i$ for free fields

- Chiral algebra: normal-ordered products $:\partial_+^{k_1}\widehat{\mathrm{J}}^{a_1} \cdots \partial_+^{k_p}\widehat{\mathrm{J}}^{a_p}:$ with

$$\left[\widehat{\mathrm{J}}^a(x), \widehat{\mathrm{J}}^b(y)\right] = 2\pi\hbar \left(\epsilon^{abc}\, \widehat{\mathrm{J}}^c(x)\, \delta(x-y) + \mathrm{i}\, K\, \delta^{ab}\, \delta'(x-y) \right)$$

Quantum local charges for the $SU(2)$ WZW-model

$$\left[\widehat{\mathrm{J}}^a(x), \widehat{\mathrm{J}}^b(y)\right] = 2\mathrm{i}\pi\hbar \left(\epsilon^{abc}\,\widehat{\mathrm{J}}^c(x)\,\delta(x-y) + K\,\delta^{ab}\,\delta'(x-y)\right)$$

- Commuting local charges of odd spins $2n-1$ [SL Molines *in progress*]:

$$\widehat{\mathrm{Q}}^+_{2n-1} = \int :\widehat{\mathrm{J}}^{a_1}\widehat{\mathrm{J}}^{a_1}\dots\widehat{\mathrm{J}}^{a_n}\widehat{\mathrm{J}}^{a_n}: \,+\, \mathrm{corr}\,, \qquad \left[\widehat{\mathrm{Q}}^+_{2n-1}, \widehat{\mathrm{Q}}^+_{2m-1}\right] = 0$$

- Explicit charges up to spin 7:

$$\widehat{\mathrm{Q}}^+_1 = \int :\widehat{\mathrm{J}}^a\widehat{\mathrm{J}}^a:\,, \qquad \widehat{\mathrm{Q}}^+_3 = \int \left(:\widehat{\mathrm{J}}^a\widehat{\mathrm{J}}^a\widehat{\mathrm{J}}^b\widehat{\mathrm{J}}^b: - \frac{10\hbar(K+\hbar)}{3} :\partial\widehat{\mathrm{J}}^a\,\partial\widehat{\mathrm{J}}^a: \right)$$

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- Spectrum: diagonalisation of $\widehat{\mathrm{Q}}^+_{2n-1}$ on Hilbert space (first results)
- Charges $\widehat{\mathrm{Q}}^-_{2n-1}$ of spin $-2n+1$ built from right-moving current

Generalisation: chiral affine Gaudin models

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  - $\mathfrak{su}(2)$ replaced by Lie algebra $\mathfrak{g}$
  - N left-moving currents $\widehat{\mathrm{J}}^a_1, \dots, \widehat{\mathrm{J}}^a_N$
  - non-local charges
  - coset algebras, Takiff currents, ...

- Various preliminary results and conjectures
- Relations to (affinisation of) Geometric Langlands Correspondence
- Still a lot to explore and many interesting future perspectives

[Feigin Frenkel '07, Frenkel Hernandez '16, SL Vicedo Young '18 '18, Gaiotto Lee Wu '19, Gaiotto Lee Vicedo Wu '20, Kotousov Lukyanov '21, Kotousov SL Teschner '22, Franzini Young '22, SL Molines *in progress*]

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- **Important question:** massive perturbations?

Thank you for your attention!