lecture 4

Bose-statistics crossing

Recap. d.r. with 2 subtractions (fixed-t)

$\alpha^1 = \beta^0 = \beta^2 = 0, \quad m_\pi = 1$

Sum over I'

$$T^I(s,t) = C_{st}^{II'}\left(\alpha^{I'}(t) + (s-u)\,\beta^{I'}(t)\right) + \frac{1}{\pi^2}\int_4^\infty \frac{dx}{x^2}\left(\frac{x^2}{x-s} I^{II'} + \frac{x^2}{x-u} C_{su}^{II'}\right) A^{I'}(x,t)$$

If we consider $T^{(I)}(s,t)$ and project in t-channel we get Froissart-Gribov for $\ell \geq 2$

$\pi^0-\pi^0$ Bonnier & Vinhmau - eliminate subtraction function scattering length appears

Iso-spin amplitudes Roy 1971 (Basdevant, Lequillou, Navelet)

$$T^I(s,t,u) = g_1(s,t)^{II'} a_0^{I'} + \int_4^\infty dx\left[g_2(s,t,x)^{II'} A^{I'}(x,0) + g_3(s,t,x)^{II'} A^{I'}(x,t)\right]$$

Partial wave projection via dt int from $\left(\frac{4-s}{2}, 0\right)$

Roy equations obtained via projection and introduction of partial-wave expansion in RHS.
Valid in $4 \leq s \lesssim 60\,(68)$ in axiomatic (MR).

Because of this full t-u crossing symm implies the requirement

$$\int_4^\infty dx\Big[\{g_2(s,t,x) - C_{tu}\, g_2(s,u,x)\} A(x,0) + \{g_3(s,t,x)A(x,t) - C_{tu}\, g_3(s,u,x)\}A(x,t)\Big] = 0$$

Roy eqs.

$$t_\ell^I(s) = k_\ell^I(s) + \sum_{I'}\sum_{\ell'}\int_4^\infty dx\, K_{\ell\ell'}^{II'}(s,x)\, \mathrm{Im}\, t_{\ell'}^{I'}(x)$$

$$k_\ell^I(s) = a_0^I + \frac{s-4}{4}\left(2a_0^0 - 5a_0^2\right)\left(\frac{1}{3}\delta_0^I\delta_\ell^0 + \frac{1}{18}\delta_1^I\delta_\ell^1 - \frac{1}{6}\delta_2^I\delta_\ell^0\right)$$

Going beyond fixed-t

MRW Mahoux, Roy, Wanders 3-channel crossing symmetry

2 subtractions ⟹ S- and P-wave absorptive parts produce manifestly crossing symmetric amp (Lovelace)

Exercise: See eq. (9.2.10) of Martin, Morgan & Shaw.
show it is equivalent to decomp. in terms of W_0, W_1, W_2

MRW follow Roskies and produce crossing symmetric combinations out of T^I (π^0-π^0 $\equiv G^0$ already known)

$$G_1 \equiv \frac{F_1(s,t,u)}{t-u} + \frac{F_1(t,u,s)}{u-s} + \frac{F_1(u,s,t)}{s-t}$$

$$G_2 \equiv \left(\frac{F_1'(s,t,u)}{t-u} - \frac{F_1'(t,u,s)}{s-u} \right) \frac{1}{s-t} + \text{cyclic permutations}$$

Write d.r. on hyperbolae in Wanders' variables

$$x \equiv -\frac{1}{16}(st+tu+us) \qquad y \equiv \frac{1}{64} stu$$

Gives rise to new equations with extended holomorphy domains

Illustration of Roy consistency conditions on 2 resonances

Narrow width formula $\mathrm{Im}\, f_\ell^I(x) = \sqrt{\dfrac{x}{x-M_\pi^2}}\, \pi \Gamma_r M_r\, \delta(x-M_r^2)$

We can consider $I=0,\ \ell=2$ f_2 resonance and
$I=1,\ \ell=3$ g_3 resonance

Wanders – holomorphy domain
Functions of several complex variables

Plugging f_2 into Roy representation s, t, u dimensionless

$$T^0(s,t,u)_f^{FT} = \frac{5}{12}\pi\frac{\Gamma_f}{M_f^7}\sqrt{\frac{M_f^2}{M_f^2 - M_\pi^2}}\, M_\pi^4$$

$$(-32 M_f^2 - 192 M_\pi^2 - 48 M_f^2 s - 560 M_\pi^2 s + 14 M_f^2 s^2$$
$$+ 204 M_\pi^2 s^2 + 2 M_f^2 (t-u)^2 + 12 M_\pi^2 (t-u)^2 - 13 M_\pi^2 s^3$$
$$+ 21 M_\pi^2 s (t-u)^2)$$

Similarly with g_3 resonance

T^1 has terms that violate crossing symmetry and Bose statistics

If we tune widths of f_2 and g_3 we can cancel.

. D_F_waves.nb

Use of crossing constraints leads to same tuning condition.

We have studied this in MRW representation

BA, PRD 58 (1998) 036002

Several notebooks to illustrate the Roy equations, their structure, nature of terms in the equations.

Crossing constraints

see G. Wanders, Springer Tracts Vol. 57, 1971

Also MMS, Ch. 7.

Partial wave amplitudes inside the Mandelstam triangle

BNR Balachandran - Nuyts - Roskies constraints

$f_2 - g_3$: same condition as before.

Note presence of 2 subtractions did not constrain $\mathrm{Im}\, f^0_0$, $\mathrm{Im}\, f^1_1$, $\mathrm{Im}\, f^2_0$ in physical region.

What about models of partial waves in EFTs, bootstrap ... ?

Example. $\int_0^4 ds\,(4-s)(3s-4)\,(f^0_0(s) + 2 f^2_0(s)) = 0$

$$\int_0^4 ds\,(4-s)\,s\,(2 f^0_0(s) - 5 f^2_0(s)) = -3 \int_0^4 ds\,(4-s)^2 f^1_1(s)$$

Families of constraints arise from moments of partial waves [crossing + positivity]

Uses F-G rep, properties of Jacobi polynomials

Could be of use in modern bootstrap EFTs.

Solutions of Roy equations [ACGL, Phys. Rep. 353, 2001]

separate out S- and P-waves

We solve $\sqrt{s_2} = 2\,\text{GeV}$

$s_2 = 205.8\, m_\pi^2$

$$t^I_\ell(s) = k^I_\ell(s) + \sum_{I'=0}^{2} \sum_{\ell'=0}^{1} \int_4^{s_2} ds'\, K^{II'}_{\ell\ell'}(s,s')\, \mathrm{Im}\, t^{I'}_{\ell'}(s')$$
$_{SP}$

Remainder – driving terms

$$d^I_\ell(s) = \sum_{I'=0}^{2} \sum_{\ell'=2}^{\infty} \int_4^{\infty} ds'\, K^{II'}_{\ell\ell'}(s,s')\, \mathrm{Im}\, t^{I'}_{\ell'}(s')$$
$$+ \sum_{I'=0}^{2} \sum_{\ell'=0}^{\infty} \int_{s_2}^{\infty} K^{II'}_{\ell\ell'}(s,s')\, \mathrm{Im}\, t^{I'}_{\ell'}(s')$$

Solve for $s_1 = 68\, M_\pi^2$, $\sqrt{s_1} = 1.15\,\text{GeV}$

Driving terms for partial waves from background amplitude

Solutions from physical ansatz

Enforce uniqueness by absence of cusp at matching point

Experimental information at higher energies, elasticity, phase shifts δ^I_ℓ $\ell = 0, 2,\ I = 0$
$\ell = 1,\ I = 1$

Many different experiments η^I_ℓ

In elastic region an effective range formula with 11 parameters for $\tan\delta^I_\ell$

Compare LHS & RHS and minimize

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Driving terms for partial waves from background amplitude

$f_2(1270)$ $I=0, l=2$

$g_3(1540)$ $I=1, l=3$

Asymptotic contributions | S-, P-waves at high energy

Solutions from physical ansatz

Ensure uniqueness at matching point

In elastic region introduce ansatze

$$\tan \delta^I_l = \sqrt{1-\frac{4}{s}}\, q^{2l} \{ A^I_l + B^I_l q^2 + C^I_l q^4 + D^I_l q^6 \} \left(\frac{4 - s^I_l}{s - s^I_l} \right)$$

$$a^I_l = A^I_l, \quad b^I_l = B^I_l + \frac{4}{s^I_l - 4} A^I_l - (A^I_l)^3 \delta_{l0}$$

11 free parameters fixed by minimizing difference of LHS & RHS

Appendix D
Table 7 of ACGL

Important properties of Roy equation solutions needed for phenomenology

Roy equations as integral equations

If Im parts are known, plug into RHS to give Re and Im parts on LHS.

Re part known from elastic unitarity.

Compare and minimize difference (a^0_0, a^2_0) goal

Comparison with K_{e4} measurements
BNL

Pionium life-time measurements
DIRAC expt.

Kaon decays: Cabibbo & Isidori

Lattice determinations
(Lüscher)

Anupam

Important properties of Roy equation solutions — use of and implications to phenomenology.

Solutions of coupled integral equations.

Re part produced by RHS, to be compared with value from elastic formula.

Goal is accurate range for a_0^0 and a_0^2

$2a_0^0 - 5a_0^2$ Olsson Sum Rule

$18a_1^1 - (2a_0^0 - 5a_0^2)$ Wanders Sum Rule

Many more

Uniqueness study.

Described in detail in ACGL.

G. Wanders, Role of the Input in Roy's eqs. for $\pi\pi$ scattering, EPJC 17 (2000) 323.

LSW model

$$\alpha_\rho(t) = \alpha_f(t) = \alpha_0 + t\,\alpha_1$$

Adler zero condition $\alpha(M_\pi^2) = \frac{1}{2}$

$$\alpha_1 = \frac{1}{2}\left(M_\rho^2 - M_\pi^2\right)^{-1}$$

$$\alpha_0 = \frac{1}{2} - \alpha_1 M_\pi^2$$

$$\beta_\rho(t) = \frac{2}{3}\beta_f(t) = \frac{\pi\,\lambda(\alpha_1)^{\alpha(t)}}{\Gamma\left[\alpha(t)\right]}$$

$$\lambda = 96\pi\Gamma_\rho\, M_\rho^2 \left(M_\rho^2 - 4M_\pi^2\right)^{-3/2}$$

ACGL – driving terms

Driving terms $f_2(1275)$

Asymptotic region: Pomeron 20 mb
$\rho - f$ trajectory from LSV model

App. B describes background amplitude

B.4 Asymptotic contributions

A trajectory with iso-spin I generates a contribution proportional to $s^{\alpha(t)}$ to the t-channel iso-spin component

$$\mathrm{Im}\, T^{(1)}(s,t) \propto \beta_\rho(t)\, s^{\alpha_\rho(t)}$$
$$s \to \infty,\ t\text{-fixed}$$

LSV model

$\rho - f$ exchange degenerate

$\alpha_\rho(t) = \alpha_f(t)$
$= \alpha_0 + \alpha_1 t$
intercept, slope

Pomeron dominates $I_t = 0$

$$\mathrm{Im}\, T^0(s,t) = P(s,t) + \frac{1}{3}\beta_f s^{\alpha_f(t)} + \beta_\rho s^{\alpha_\rho(t)} + (t \leftrightarrow u)$$

$T^1(s,t)$
$T^2(s,t)$

Special traj

$P(s,t) = \sigma_s e^{\frac{1}{2}bt}$
$b = 8\ \mathrm{GeV}^{-2}$
$\sigma = (16 \pm 5)\,\mathrm{mb}$

At fixed-t $P(s,t)$
$\beta(t) s^{\alpha(t)}$
generate peak in forward

$t \to u$ backward

Other facts noted in ACGL

at fixed-t discard cross-term.

Regge Poles

Motivation – possible conflict between asymptotic behavior in s for different t.

1 Froissart $\quad |T(s, t=0)| \underset{s\to\infty}{<} s(\log s)^2$

2 if $\exists$ t-channel bound state of mass M and spin j

$$T(s, t \approx M^2) \approx g^2 \frac{P_j(\cos\theta_t)}{M^2 - t} \underset{s\to\infty}{\sim} s^j$$

Elastic scattering of Spin-0 $\Big\}$ $j > 1$ will violate Froissart bound.

Let us try to resolve the different behaviour for $t = 0$ and $t \approx M^2$

Variable asymptotic behavior $T(s,t) \sim \beta(t)\, s^{\alpha(t)}$

$\alpha(t)$ is some fn. of t $\quad \leq 1$ for $t < 0$
(s - physical region)
and j for $t = M^2$

Such behaviour found by Regge in NR scattering
Chew - Frautschi proposed in rel. scatt.

Consider $\alpha(t)$. For $t > 0$ family of bound states
$t < 0$ asymptotic behavior of s-channel scatt. amp.

Complex Angular Momentum - Spin 0 scattering.

elastic, spin-0. P.W.E $T(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(s) P_\ell(z)$

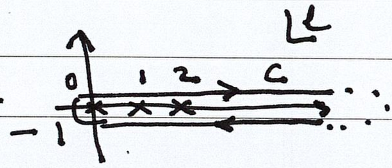
$z = \cos\theta$, θ : c.m. scatt. angle - limited range of validity is $\cos\theta$ hence t

ℓ - integer - physical ; now make general $\ell \in \mathbb{C}$ so that P.W.E. valid everywhere in t

large $t \in \mathbb{R}$

1.1 The Sommerfeld - Watson transform.

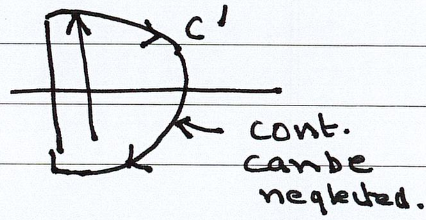
Replace $\sum_{\ell=0}^{\infty}$ by contour integral



$$T(s,t) = -\frac{1}{2i} \int_C \frac{(2\ell+1) f(\ell,s) P_\ell(-z)}{\sin \pi \ell} d\ell$$

See fn in MUS p. 415 on convention } equivalence due to Cauchy.

Deform contour



Definition $f(\ell,s)$? Uniqueness ?

$f(\ell,s) + g(s) \sin \pi \ell$?

Impose some conditions to enforce requirements of Carlson's theorem.

Usual partial wave projection fails.

Exercise : see why not. 2 reasons - Carleson and failure to complete S-W transform

However, before we can complete the Sommerfeld-Watson transform, we must examine the function $f(l, s)$: to see whether a definition exists with the required asymptotic properties in l; to ascertain whether the definition is unique; to study its analytic properties as a function of l. We begin with the question of uniqueness. If $f(l, s)$ is a continuation to complex l of $f_l(s)$ then clearly another possible continuation is

$$f(l, s)+g(s) \sin \pi l.$$

However if we impose a limitation on the behaviour of $f(l, s)$ as $l \to \infty$ then the continuation can be shown to be unique. The result follows from Carlson's Theorem† which states that given a function $f(z)$ such that

(i) $f(z)$ is regular in $\operatorname{Re} z > C$ where C is some finite real constant,

(ii) $f(z) < \exp(k|z|)$, where $k < \pi$, in $\operatorname{Re} z > C$,

(iii) $f(z) = 0$ at an infinite sequence of positive integers $z = N+1, N+2, \ldots$,

then $f(z)$ is identically zero.
This means that if $f(l, s)$ as a function of l satisfies conditions (i) and (ii) then it will be uniquely defined when its values at the positive integral points beyond some starting point N are specified.

For integral values of l the partial wave amplitude is usually defined by

$$f_l(s) = \tfrac{1}{2} \int_{-1}^{1} P_l(z)\, T(s, t(z))\, \mathrm{d}z\,. \tag{9.4}$$

This definition will certainly permit an analytic continuation to complex values of l. However the function so defined will not satisfy condition (ii) of Carlson's Theorem as is seen when we look at the asymptotic behaviour in l of $P_l(z)$ (see ERDÉLYI [1953], p. 162),

$$P_l(\cos\theta) \approx l^{-\frac{1}{2}}(\text{const. } e^{il\theta} + \text{const. } e^{-il\theta})$$

where

$$-1 < \cos\theta < 1.$$

So if we were to use eq. (9.4) to define $f(l, s)$ for general l we would not be able to use Carlson's Theorem to prove uniqueness. Further we would be unable to complete the Sommerfeld-Watson transform, as the only direction in which

† For a proof of Carlson's Theorem see TITCHMARSH [1939], p. 185.

Froissart - Gribov projection

Write a fixed-s d.r. (we saw this earlier) with N subtractions

$$T(s,t) = \sum_{n=0}^{\infty} r_n z^n + \frac{z^N}{\pi}\left\{ \int_{z_R}^{\infty} \frac{D_t(s,z')}{z'^n (z'-z)} + \int_{-z_L}^{-\infty} \frac{D_u(s,z')}{z'^n (z'-z)} dz' \right\}$$

D_t, D_u discontinuities of T across t- and u-channel cuts. $z_R, z_L > 1$. Insert into partial wave expansion.

For $l \in \mathbb{Z}^+$ we can perform z integration

Partial wave

$$f_l(s) = \frac{1}{\pi} \int_{z_R}^{\infty} D_t(s,z') Q_l(z') dz' + \frac{1}{\pi} \int_{-z_L}^{-\infty} D_u(s,z') Q_l(z') dz'$$

Neumann formula $Q_l(z') = -\frac{1}{2} \int_{-1}^{1} dz \frac{P_l(z)}{z - z'}$

has been used. For $l \in \mathbb{Z}'$ $Q_l(-z') = (-1)^{l+1} Q_l(z')$

$$f_l(s) = \frac{1}{\pi} \int_{z_0}^{\infty} \{ D_t(s,z') + (-1)^l D_u(s,-z') \} Q_l(z') dz'$$

z_0 smaller of z_R & z_L

Now continue to complex-l : $Q_l(z')$

$\sim \frac{1}{\sqrt{l}} \exp\{-(l+1)\xi\}$ as $|l| \to \infty$ $\quad \xi = \cosh^{-1} z'$

causes no problem for $\mathrm{Re}\, l > 1/2$. However $(-1)^l$ does: $(-1)^l = \exp(i\pi l)$. Problem with Carlson reappears

Problem needs to be resolved.

Separate even and odd l values of $f_l(s)$ before making the continuation in l. (f^+ and f^-)

$$f^{\pm}(l,s) = \frac{1}{\pi}\int_{z_0}^{\infty} \{D_t(s,z') \pm D_u(s,z')\} Q_l(z')\,dz'$$

Froissart-Gribov projection.

Integrals converge for $l > N$

$$f_l(s) = f^+(l,s) \text{ for even } l$$
$$= f^-(l,s) \text{ for odd } l$$

f^+, f^- even and odd signatures.

Back to deforming the contour. Using $f^{\pm}$ make S-W transformation

Procedure: define $T^{\pm}(s,t) = T^{\pm}(s,z)$

$$= \sum (2l+1) f^{\pm}(l,s) P_l(z)$$

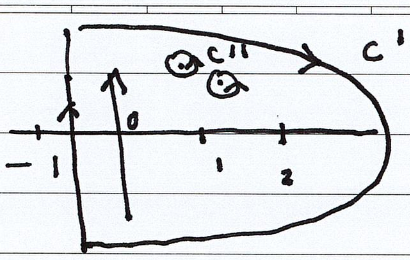
and perform S-W separately on each

$T^{\pm}$ are not physical.

physical $T(s,z) = \frac{1}{2}\{T^+(s,z) + T^+(s,-z) + T^-(s,z) - T^-(s,-z)\}$

Now replace sum over l by contour integral. So far analytic only for $l > N$.

From Froissart and stronger bounds for s physical in t-channel $N(s) \leq 1$ and we can go to $\mathrm{Re}\, l = -\frac{1}{2}$ (but with singularities $-\frac{1}{2} < \mathrm{Re}\, l < N$)



Deform C to C'

$C \rightarrow C' + C''$

In N-R scattering Regge for many potentials $f^{\pm}(\ell, s)$ meromorphic in ℓ for $\mathrm{Re}\,\ell > -1/2$

Singularities are only poles (Regge poles)
All in UHP in ℓ-plane

Relativistic also ? (Mandelstam - branch points - to be ignored)

Pole
Positions $\alpha^{\pm}(s)$
Residues $\beta^{\pm}(s)$
. In general they are functions of s

$\alpha^{\pm}(s)$ for real s in complex-ℓ plane
- Regge trajectory.

Contributions from C'' are simply the pole parameters. So the result is.

$$T^{\pm}(s, z) = -\frac{1}{2i}\int_{C'} \frac{(2\ell+1)\, f^{\pm}(\ell, s)\, P_\ell(-z)}{\sin \pi \ell}\, d\ell$$

$$- \sum \frac{\pi\,(2\alpha_i^{\pm}(s)+1)\,\beta_i^{\pm}(s)\, P_{\alpha_i^{\pm}}(-z)}{\sin \pi \alpha_i^{\pm}(s)}$$

Cont. from large circle $\rightarrow 0$

Physical amplitude

$$T(s,t) = -\frac{1}{2i}\int_{-1/2-i\infty}^{-1/2+i\infty} \frac{2\ell+1}{2\sin\pi\ell}\Big\{ f^{+}(\ell,s)\,(P_\ell(-z)+P_\ell(z)) + f^{-}(\ell,s)\,(P_\ell(-z)-P_\ell(z))\Big\}\,d\ell$$

$$-\sum \frac{\pi\,(2\alpha_i^{\pm}(s)+1)}{2\sin\pi\alpha_i^{\pm}}\,\beta_\ell^{\pm}(s)\left(P_{\alpha_i^{\pm}}(-z) \pm P_{\alpha_i^{\pm}}(z)\right)$$

To be interpreted on where pole appears.

Importance due to behaviour of contributions to $T(s,z)$ as $z \to \infty$. z-dependence comes only from Legendre polynomials

$$\sqrt{\pi}\;P_\alpha(z) \sim \frac{(\alpha-1/2)!}{\alpha!}\,(2z)^{\alpha} + \frac{(-\alpha-3/2)!}{(-\alpha-1)!}\,(2z)^{-\alpha-1}$$

dominates $\text{Re}\,\alpha > -1/2$ $\text{Re}\,\alpha < -1/2$

$\text{Re}\,\ell = -1/2$ integral : background integral

vanishes as $z^{-1/2}$ for $z \to \infty$

For large z only largest value of $\text{Re}\,\alpha$ will dominate

Leading Regge trajectory

$z \propto t$ (energy^2 in t-channel)

Regge poles relevant to high-energy scattering

So far s lies in physical energy^2 region

Can be analytically continued to $-$ve s.

(t-channel physical region)

t-channel amplitude behaves as $T(s,t) \sim \beta(s)\, t^{\alpha(s)}$
$t \to \infty$

Froissart: $\mathrm{Re}\,\alpha(s) \leq 1$ in the t-channel physical region

$$-4q_t^2 \leq s \leq 0$$

Relavance of Regge poles in t-channel

Physical importance in s-channel.

Each Regge trajectory associated with family of bound states and resonances

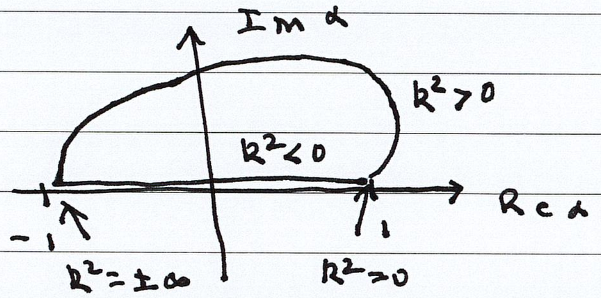
First - push the background integral further

Mandelstam - Sommerfeld - Watson transform.

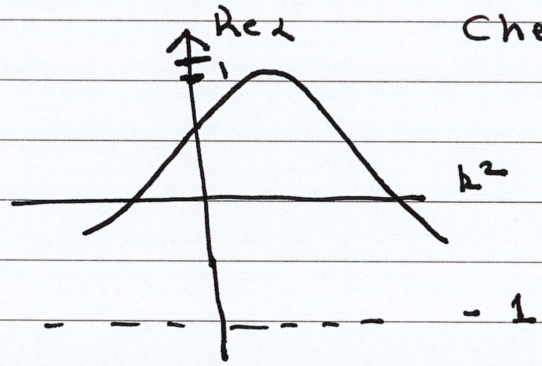
Relationship between Regge poles and resonances and bound states

Projecting and using properties of Legendre polynomials

N-R scattering Yukawa Regge traj

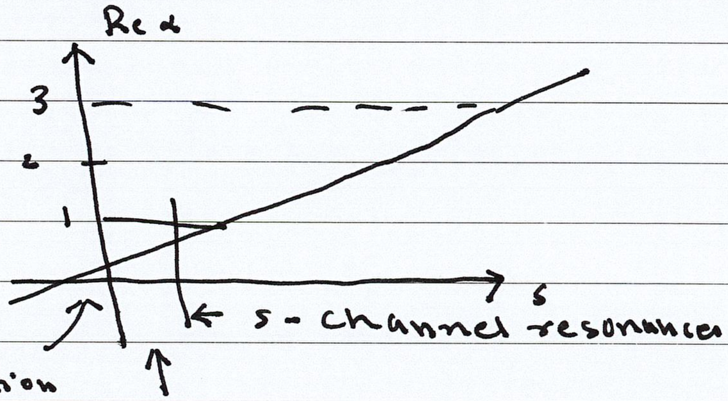


Chew-Frautschi plot



Chew - Frautschi in relativistic scattering.

odd signature
passing through
spin 1 & 3.



Connection
with t-channel s-channel bound state region
high energy scattering.

Properties of the Regge-Pole functions $\alpha(s)$ and $\beta(s)$

Information on analyticity properties of $\alpha(s)$, $\beta(s)$ from Mandelstam rep.

$\alpha(s)$ $\qquad \dfrac{\beta(s)}{q^{2\alpha}}$ real analytic fns. of s in cut s-plane

Unitarity relations

For amplitudes of definite signature.

Predictions for high energy scattering

(9.28)

(9.29)

(9.30) $$T(s,t)_{pole} = \phi(t)\, s^{\alpha(t)}$$

Asymptotic value of $P_\alpha(z_t)$ and replace z_t by s.

Optical theorem $\sigma_{tot} \sim \frac{1}{s} \operatorname{Im} T(s, t=0)$

$\sim s^{\alpha(0)-1}$

There is a trajectory $\alpha(t)$ which passes through maximum allowed value at $t=0$

This leading pole is called the Pomeranchuk trajectory [$I=0$, $S=0$, vacuum q.n. apart from ang. mom.]
- even signature

For even signature, forward amplitude becomes pure imaginary at high energy (diffraction scattering)

Factorization of residues

Super convergence relations and FESR

Duality

Too advanced

Venezinno model shows all desired properties.

From Martin and Spearman

Simplicity consider s-channel $\pi^+\pi^- \to \pi^+\pi^-$

t-channel is the same

$$T(s,t) = -\beta \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}$$

$\alpha(t)<0$

$\alpha(s) = \alpha(0)+\alpha' s \qquad \alpha(0) \approx 1/2$

$$T(s,t) = -\beta \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha(t))}{\Gamma(n)\,\Gamma(\alpha(t))} \frac{1}{\alpha(s)-n}$$

n^{th} s-channel pole is a polynomial in t of degree n

can be expanded in Legendre series

$\to$ leading trajectory with an infinite series of equally spaced parallel daughter trajectories

$\Gamma(z+m)/\Gamma(z+n) \to z^{m-n}$

$-\pi < \arg z < \pi$

Asymptotic :

$T(s,t) \sim \beta\, \Gamma(1-\alpha(t))\,(\alpha' s)^{\alpha(t)}$

Regge behaviour

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