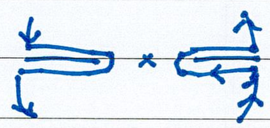
Lecture 3

Recap

Writing a d.r. for an amplitude (fixed-t)

$$T(s,t) = \frac{1}{2\pi i}\oint \frac{T(s',t)}{s'-s}\, ds' \quad \text{(Cauchy)}$$



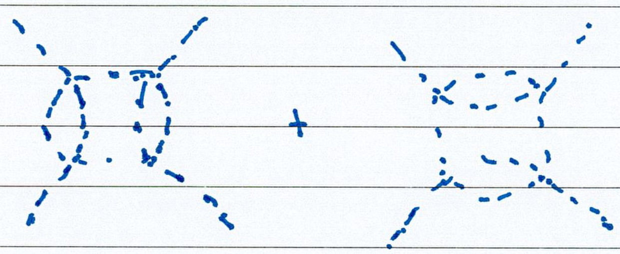
$$= \frac{1}{\pi}\int_{-\infty}^{s_L} \frac{A_u(s'+i\varepsilon,t)}{s'-s}\,ds' + \frac{1}{\pi}\int_{s_R}^{\infty}\frac{A_s(s'+i\varepsilon,t)}{s'-s}\,ds'$$

I. Caprini book

Martin, Morgan and Shaw:

Start with Mandelstam Rep

$$T^I(s,t,u) = \frac{1}{\pi^2}\int_4^\infty ds' \int_4^\infty dt'\, \frac{\rho^I_{st}(s',t')}{(s'-s)(t'-t)} + us + tu$$



Motivated by this 'double spectral function'

Boundary given by Kibble function

Boundary of support given by $(s-4)(t-16)-64=0$, $(s-16)(t-4)-64=0$

us, tu

M.R + Schwartz reflection

$$T^I(s,t,u) = \frac{1}{\pi}\sum_{I'}\int ds'\, \mathrm{Im}\,T^{I'}(s',t)\left(\frac{\delta_{II'}}{s'-s} + \frac{C^{su}_{II'}}{s'-u}\right)$$

$-32 < t < 4$

Crossed channel absorptive part explicit

Mandelstam Representation diagrams and reduction to single variable dispersion relations. Finding the result $-32 < t < 4$

See detailed discussion in Ch. 2 Sec 1(e) of Martin, Morgan and Shaw.

Fig 2.2 in Mandelstam plane
Fig 2.4 Cartesian co-ordinates

Defining $$\mathcal{D}_s^I(s,t) = \frac{1}{\pi}\int_4^\infty dt'\, \rho_{st}^I(s,t')\left(\frac{1}{t'-t} + \frac{(-1)^I}{t'-u}\right)$$

has no singularities for $-32 < t < 4$

Exercise: obtain the boundary curves from Fig 6.8 in Martin and Spearman and eq. (6.85) of the Kibble function, see problem 7 of chapter 6.

The importance of the forward dispersion relations (6.60) and (6.61) is that, apart from the πNN coupling constant, they contain only physically measurable quantities. Re $F(\omega_L)$ can be obtained up to a sign by extrapolation of the elastic cross section to the forward direction

$$\mathrm{Re}\, F = \sqrt{\left.\frac{\mathrm{d}\sigma(0)}{\mathrm{d}\Omega}\right|_{\mathrm{lab}} - \left(\frac{k_L \sigma_{\mathrm{tot}}}{4\pi}\right)^2}. \tag{6.64}$$

The sign of Re F is determined by the interference with Coulomb scattering. The use of these relations therefore allows a determination of the πNN coupling constant and also provides an important check on the validity of dispersion relations. Using the available experimental data we find that the relations are satisfied to a high degree of accuracy.

§ 2. Double Variable Dispersion Relations — The Mandelstam Representation

So far we have only examined the analytic properties of the elastic scattering amplitude in terms of one of its variables at a time while keeping the other variable fixed. Thus we can write down a dispersion relation for $T(s, t, u)$ in s for a fixed value of t: equally well we could write down a single variable dispersion relation in t or u keeping in each case the other independent variable fixed. The next step is to examine the analytic properties of the amplitude as a function of its two independent variables simultaneously. When we have determined these analytic properties in terms of two variables we will be able to construct a representation for the amplitude in the form of a double variable dispersion relation. This representation, originally formulated by MANDELSTAM [1958], provides a concise and complete statement of the postulated analyticity properties of the elastic scattering amplitude.

The structure in s of the amplitude $T(s, t)$ is given by the single variable dispersion relation

$$T(s, t) = \frac{\rho}{s - M^2} - \frac{\rho}{s - 4m^2 + M^2 + t} + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\mathrm{Im}\, T(s', t)}{s' - s}\,\mathrm{d}s' +$$

$$+ \frac{1}{\pi}\int_{-\infty}^{-t} \frac{\mathrm{Im}\, T(s', t)}{s' - s}\,\mathrm{d}s' \tag{6.15}$$

which refers to some real fixed value of the variable t. Recall that for simplicity we are considering the elastic scattering of spin zero particles of equal mass m. Further we shall assume that unsubtracted dispersion relations are meaningful. To display the symmetry between the s- and u-channels we

To attempt to make an analytic continuation of $D_s(s, t)$ in t, from its known values in the s-channel physical region, we fall back on the s-channel unitarity relation. This may be regarded as the defining relation for $D_s(s, t)$ and it will form the basis for the continuation procedure. Before writing down the unitarity relation we recall that for the s-channel we use the familiar variables

$$s = 4(m^2+q^2) = W^2$$

$$t = -2q^2(1-\cos\theta)$$

where $\boldsymbol{q}$ and θ are the c.m. 3-momentum and scattering angle respectively and $q=|\boldsymbol{q}|$. Remember that s lies in the physical range, it is real and greater than $4m^2$. Initially we may suppose that the value of t corresponds to a point in the s-channel region, eq. (6.66): this permits us to write down the s-channel unitarity relation for $D_s(s, t)$, but we will then allow t to assume arbitrary values and define $D_s(s, t)$ by the analytically continued unitarity relation. For values of s below the inelastic threshold the s-channel unitarity relation is (cf. eqs. (6.6) and (4.35))

$$D_s(s, t) = \frac{2W}{q}\int \mathrm{d}\Omega' \langle\theta\phi\,|T_{\mathscr{P}}^{\dagger}|\,\theta'\phi'\rangle\,\langle\theta'\phi'\,|T_{\mathscr{P}}|\,00\rangle \tag{6.69}$$

where the initial and final states are specified by polar angles (0, 0) and (θ, ϕ) respectively. The angular integration $\mathrm{d}\Omega' = \mathrm{d}(\cos\theta')\,\mathrm{d}\phi'$ is over all possible configurations of the two-particle intermediate state. We choose the axes so that $\phi=0$, that is the zx-plane is the plane of scattering. Now since the T-matrix is invariant under rotations we can write

$$\langle\theta 0\,|T_{\mathscr{P}}^{\dagger}|\,\theta'\phi'\rangle = \langle 00\,|T_{\mathscr{P}}^{\dagger}|\,\theta''0\rangle \tag{6.70}$$

where θ'' is the angle between the directions specified by the two sets of polar angles $(\theta, 0)$ and (θ', ϕ'), see Fig. 6.3. These angles are related as follows

$$\cos\theta'' = \cos\theta\,\cos\theta' + \cos\phi'\,\sin\theta\,\sin\theta'. \tag{6.71}$$

Now recall that

$$\begin{aligned}\langle\theta'\phi'\,|T_{\mathscr{P}}|\,00\rangle &= \frac{q}{4W}\,T(s, t')\\ \langle 00\,|T_{\mathscr{P}}^{\dagger}|\,\theta''0\rangle &= \frac{q}{4W}\,T^*(s, t''),\end{aligned} \tag{6.3}$$

with $s+t'+u' = 4m^2$, and a similar relation for $T(s, t'')$. When these dispersion relations are substituted into eq. (6.73) it becomes

$$D_s(s,t) = \frac{q}{8W}\int \mathrm{d}\Omega' \left[\frac{g^2}{M^2-t'} + \frac{g^2}{M^2-u'} + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{D_t(s,t_1)}{t_1-t'}\,\mathrm{d}t_1 + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{D_u(s,u_1)}{u_1-u'}\,\mathrm{d}u_1\right] \times \left[\frac{g^2}{M^2-t''} + \frac{g^2}{M^2-u''} + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{D_t(s,t_2)}{t_2-t''}\,\mathrm{d}t_2 + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{D_u(s,u_2)}{u_2-u''}\,\mathrm{d}u_2\right]^* \tag{6.75}$$

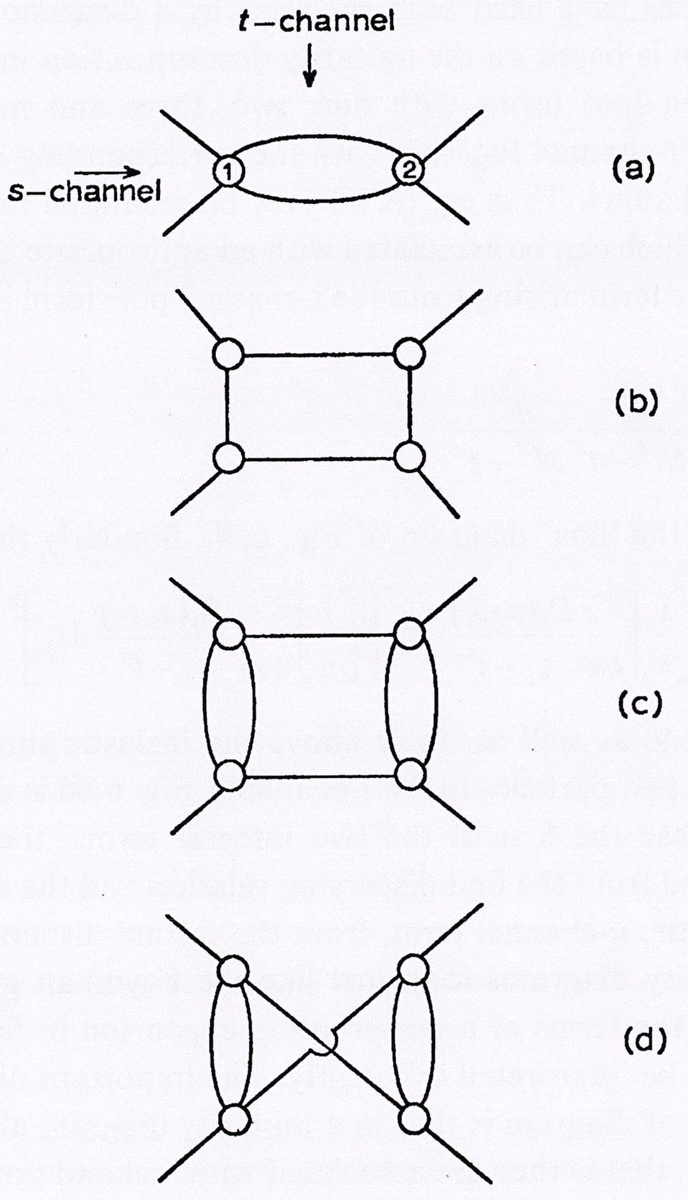


Fig. 6.4. Some low order unitarity diagrams.

immediately deduce that each of these diagrams will only contribute a singularity to the discontinuity in some quite restricted region of the s, t, u plane. Thus the box diagram of Fig. 6.4b describes two physical intermediate particles in both the s and the t-channels which means that it can only contribute to $D_s(s, t)$ in the region

$$s \geqslant 4m^2, \qquad t \geqslant 4M^2 . \tag{6.79}$$

Similarly the diagram of Fig. 6.4c can only contribute a singularity in the region

$$s \geqslant 4m^2, \qquad t \geqslant 16m^2 . \tag{6.80}$$

It is clear from this discussion that the simplest unitarity diagrams with small numbers of internal lines can contribute singularities to the discontinuities for small values of s, t and u ($\sim 4m^2$) and in regions of the s, t, u-plane which approach quite close to the physical regions. The diagrams containing larger numbers of internal lines only contribute for correspondingly larger values of s, t or u and in general only in regions further removed from the physical region. Because of this the diagrams with a small number of internal lines, which we shall refer to as the *lowest* unitarity diagrams, are expected to be particularly important. These diagrams will determine the singularities of the scattering amplitude lying closest to the low energy physical region.

An interesting and physically important situation arises when there is a selection rule which prohibits a three-particle vertex. We saw, for example, in Chapter 5 that a three-pion vertex is prohibited by parity conservation. This means that there is no box diagram (with internal pion lines) for $\pi\pi$ scattering. The lowest unitarity diagrams for this process are of the type shown in Figs. 6.4c and 6.4d. We shall see that this means that there are no 'singularities' of the scattering amplitude in the region

$$4m^2 \leqslant s \leqslant 16m^2, \qquad 4m^2 \leqslant t \leqslant 16m^2 .$$

A similar situation occurs for πN scattering, again because the three-pion vertex is forbidden.

We study the analytic continuation of $D_s(s, t)$ of eq. (6.75) for $\pi\pi$ scattering. In this case there are no bound state pole terms and therefore, as we have just remarked, no box diagram. For the moment let us ignore the u-channel discontinuity D_u. Then eq. (6.75) becomes, after interchanging the integrations

$$D_s(s, t) = \frac{q}{8W\pi^2} \int_{4m^2}^{\infty} \mathrm{d}t_1 \int_{4m^2}^{\infty} \mathrm{d}t_2\, D_t(s, t_1)\, D_t^*(s, t_2) \int \mathrm{d}\Omega' \frac{1}{(t_1 - t')(t_2 - t'')} . \tag{6.81}$$

and that t_L increases as either t_1 or t_2 increases. Therefore the minimum value of t_L, say $t_L = b(s)$, is obtained by taking the lowest values of t_1 and t_2 occurring in the integration of eq. (6.81), that is $t_1 = t_2 = 4m^2$. Now

$$K(s, t; 4m^2, 4m^2) = t\left(t - 16m^2 - \frac{16m^4}{q^2}\right)$$

and so, taking the larger root, we have

$$(t_L)_{\min} \equiv b(s) = 16m^2 + \frac{16m^4}{q^2}. \tag{6.86}$$

The boundary curve of the double spectral function $\rho_{st}(s, t)$ is therefore given by $t = b(s)$. Note that this curve, which is shown in Fig. 6.5, is asymptotic to the lines $s = 4m^2$ and $t = 16m^2$. From eqs. (6.81), (6.82) and (6.84) we find for $t > b(s)$ that

$$\rho_{st}(s, t) = \frac{1}{4Wq} \int_{4m^2}^{K(s,t;t_1,t_2)=0} \mathrm{d}t_1 \int_{4m^2} \mathrm{d}t_2 \, \frac{D_t(s, t_1)\, D_t^*(s, t_2)}{[K(s, t; t_1, t_2)]^{\frac{1}{2}}} \tag{6.87}$$

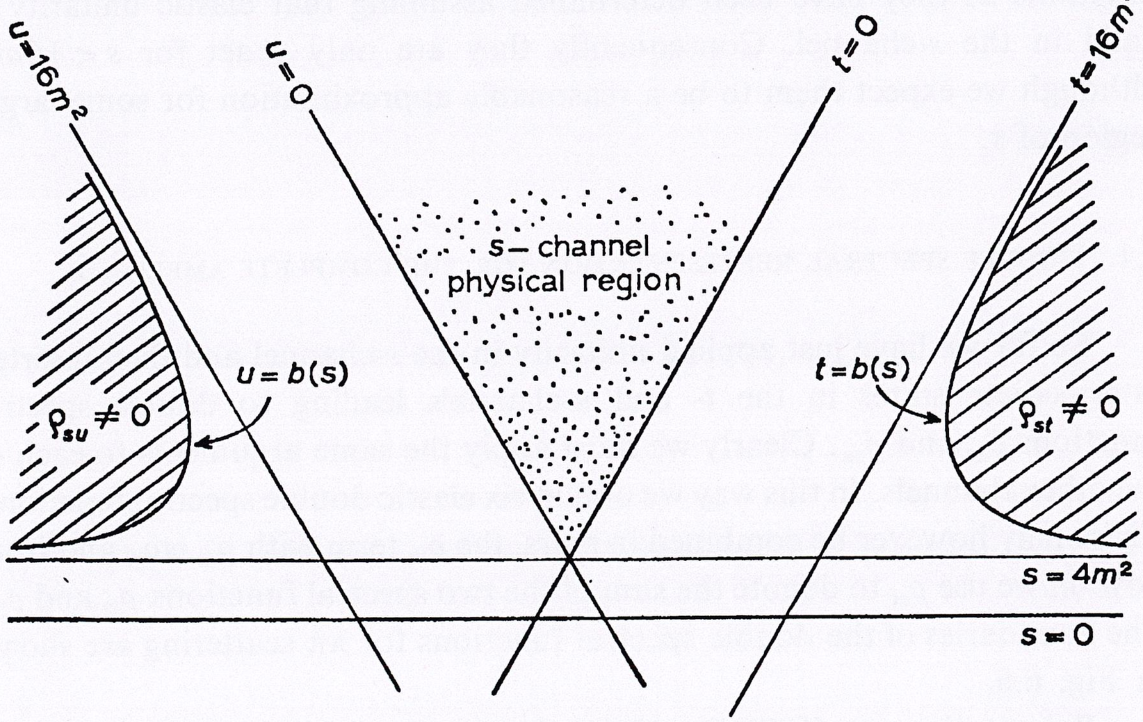


Fig. 6.5. The Mandelstam diagram for $\pi\pi$ scattering, showing the elastic s-channel double spectral functions.

amplitude may be written as

$$
\begin{aligned}
T(s,t) = {} & \frac{1}{\pi^2}\int_{4m^2}^{\infty} \mathrm{d}s' \int_{b(s')}^{\infty} \mathrm{d}t' \, \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} \\
& + \frac{1}{\pi^2}\int_{4m^2}^{\infty} \mathrm{d}s' \int_{b(s')}^{\infty} \mathrm{d}u' \, \frac{\rho_{su}(s',u')}{(s'-s)(u'-\bar{u})} \\
& + \frac{1}{\pi^2}\int_{4m^2}^{\infty} \mathrm{d}u' \int_{b(u')}^{\infty} \mathrm{d}t' \, \frac{\rho_{tu}(t',u')}{(u'-u)(t'-t)} \\
& + \frac{1}{\pi^2}\int_{4m^2}^{\infty} \mathrm{d}u' \int_{b(u')}^{\infty} \mathrm{d}s' \, \frac{\rho_{su}(s',u')}{(u'-u)(s'-\bar{s})}
\end{aligned}
$$

where

$$\bar{u} = 4m^2 - s' - t, \qquad \bar{s} = 4m^2 - t - u'.$$

Combining the second and fourth terms, this expression simplifies to

$$
\begin{aligned}
T(s,t) = {} & \frac{1}{\pi^2}\int_{4m^2}^{\infty} \mathrm{d}s' \int_{b(s')}^{\infty} \mathrm{d}t' \, \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} \\
& + \frac{1}{\pi^2}\int_{4m^2}^{\infty} \mathrm{d}s' \int_{b(s')}^{\infty} \mathrm{d}u' \, \frac{\rho_{su}(s',u')}{(s'-s)(u'-u)} \\
& + \frac{1}{\pi^2}\int_{4m^2}^{\infty} \mathrm{d}t' \int_{b(t')}^{\infty} \mathrm{d}u' \, \frac{\rho_{tu}(t',u')}{(t'-t)(u'-u)} .
\end{aligned}
\tag{6.89}
$$

This representation has been obtained by simply using elastic unitarity, and all the Cutkosky diagrams considered have been elastic in at least one channel. Clearly above the inelastic thresholds we will have additional contributions to the double spectral functions. However since the contributions corresponding to higher unitarity diagrams have boundaries which lie inside those arising from the lowest diagrams the form of the representation for $T(s, t)$ will be unchanged. Notice that eq. (6.89) involves the assumption that the amplitude has only those singularities that are required by

Subtractions (Roy starts with this)

$$T^{I}(s,t) = C_{st}^{II'}\left(\alpha^{I'}(t) + \beta^{I'}(t)(s-u) \right)$$

$$+ \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dx}{x^2} \left[\frac{\delta^{II'} s^2}{x-s} + \frac{C_{su}^{II'} u^2}{x-u} \right] \times \operatorname{Im} T^{I'}(x,t)$$

Exercise for tutorial: $\alpha^1 = 0$ $\beta^0 = \beta^2 = 0$ $\rightarrow$ show from properties of crossing matrices.

Exercise: start with unsubtracted and carry out 2 subtractions and show this has to be the structure. give sum rules for $\alpha^0, \alpha^2, \beta^1$ in terms of moments of $\operatorname{Im} T^{I'}(x,t)$.

Another common way of writing this is for t-channel amplitudes

very useful

$$T_t^{(I)}(s,t) = \alpha^I(t) + \beta^I(t)(s-u) + \frac{1}{\pi}\int_4^{\infty} \frac{dx}{x^2}\left[\frac{s^2}{x-s} + (-1)^I \frac{u^2}{x-u}\right] \times \operatorname{Im} T^{(I)}(x,t)$$

All follow from properties of crossing matrices

Note $C_{st} \cdot C_{su} = ((C_{st} \cdot C_{su}) \cdot C_{st}) \cdot C_{st}$
$= \operatorname{diag}(1,-1,1)\, C_{st}$

Wanders "superconvergent sum Rules..."

Exercise

$$T^{(1)}(s,t) = \frac{1}{\pi}\int dx \left[\frac{1}{x-s} - \frac{1}{x-u}\right] T^{(1)}(x,t)$$

Pomeranchuk theorem.

Looks unsubtracted, but only because of symmetry.

Prove Olsson sum rule $2a_0^0 - 5a_0^2 = \frac{M_\pi^2}{4\pi}\int_4^{\infty} ds \, \frac{2\operatorname{Im}T^0(s,0) + 3\operatorname{Im}T^1(s,0) - 5\operatorname{Im}T^2(s,0)}{s\,(s-4m^2)}$

Sum rule $\rightarrow$ RHS has integral over cross-sections.

common to project onto partial waves via
$P_\ell(\cos\theta_s)$
$\cos\theta_s = 1 + \frac{2t}{s-4}$ to get $f_\ell^I(s)$

fixed-s $d\cos\theta_s \rightarrow dt$ integral
t-variable Limits -1 $t = s-4$
$+1$ $t = 0$
} Note t is variable of integration

Sometimes $\cos\theta_s$ from 0 to 1 $t = \frac{4-s}{2}$ } Symmetry of P_ℓ.

Another way is to project w.r.t $\cos\theta_t$ at fixed-t and s-variable

New kind of projection

In this case $\underline{\text{Im}\, T^{(I)}(x,t) \text{ stays intact in d.r.}}$
For $\ell \geq 2$ the first 2 terms ($P_0(\cos\theta_t)$ and $P_1(\cos\theta_t)$ being orthogonal to $P_\ell(\cos\theta_t)$) vanish upon integration.

Gives rise to the Froissart - Gribov representation.

$$f_\ell^I(t) = \left[\frac{1+(-1)^{I+\ell}}{\pi}\right] \frac{4}{t-4} \int_4^\infty dx \, \text{Im}\, T^{(I)}(x,t) \, Q_\ell\left(1 + \frac{2x}{t-4}\right)$$

Froissart-Gribov

$\ell \geq 2$, $-28 < t < 4$ and $|t| < 4$

$$\frac{1}{2}\int_{-1}^{1} dx \, \frac{x^n}{y^n} \frac{P_\ell(x)}{y-x} = Q_\ell(y) = (-1)^{\ell+1} Q_\ell(-y)$$
$\ell \geq n$

Legendre function of the 2nd kind.

Useful in Regge theory.

See book of Martin & Cheung

Work of A. Martin primarily

Also possible to project on to s-wave (Froissart-Gribov)

$$T(s,t) = c(t) + \int_4^\infty ds' \frac{A(s',t)}{s'^2}\left(\frac{s^2}{s'-s} + \frac{u^2}{s'-u}\right)$$

$\pi^0-\pi^0$: $\frac{1}{3}(T^0 + 2T^2)$ A - absorptive part

$$f_0(t) = c(t) + \int_4^\infty \frac{ds'}{s'^2}\left\{(t-4-2s') + \frac{2s'^2}{(t-4)} \ln\left(\frac{s'+t-4}{s'}\right)\right\} A(s',t)$$

Survives because $P_0(z) = 1$.

Eliminate $c(t)$

$$T(s,t,u) = f_0(t) + \frac{1}{\pi}\int_4^\infty ds' \left[\frac{1}{s'-s} + \frac{1}{s'+t+s-4} - \frac{2}{t-4}\ln\left(\frac{s'+t-4}{s'}\right)\right] \times A(s',t)$$

New d.r.

$$= f_0(t) + R_2(s,t)$$

Use crossing symmetry $f_0(s) - f_0(t) = R_2(t,s) - R_2(s,t)$
$|s| < 4,\ |t| < 4$

Positivity! Used to constrain 1-loop ChPT parameters.

Inside the Mandelstam triangle $A(s',t)$ is also +ve.

because $0 \le t \le 4$, $0 \le s \le 4$

$s'-4 > 0$ $1 + \frac{2t}{s'-4} \geq 1$ $P_\ell(z) > 0$ for $z > 1$

B.A., D. Toublan, G. Wanders [originally by A. Martin]

$f_0(0) > f_0(3.155)$ Series of arguments to prove these.

$f_0(0) > f_0(3.189)$

Extremely useful review.

Reviewed also in S.M. Roy, Helv. Physica Acta Vol. 63 (1990)

Other constraints $f_0(s) < f_0(4) \qquad 0 < s < 4$

$$\frac{d f_0(s)}{ds} > 0 \qquad 2 < s < 4$$

$$\frac{d f_0(s)}{ds} < 0 \qquad 0 < s < 1.127$$
$$< 1.217$$

$$\frac{d^2 f_0(s)}{ds^2} > 0 \qquad 0 < s < 1.7$$

$\Rightarrow$ unique minimum between 1.217 and 1.7

Other inequalities involving S- and D-waves using Froissart-Gribov

These methods can be updated.

Absolute bounds Lukaszuk & Martin
Lopez & Menessier

Write a simple d.r. $z \equiv \frac{(s-u)^2}{4} = (s-2+t/2)^2$

$$F(s,t) = G(z,t) = A(t) + \frac{z}{\pi}\int_{z_0(t)}^{\infty} \frac{\operatorname{Im} G(z',t)}{z'(z'-z)}\, dz'$$

$$z_0(t) = (2+t/2)^2$$

$F(s,0) < F(4,0) \quad 0 < s < 4$

Absolute bounds in Mandelstam triangle
$F(4/3, 4/3, 4/3)$ minimum
Table 1 gives 2 sided inequalities

Substituting (II.38) in

$$0 = [f(s) - f(s_1)] + [f(s_1) - f(t)] + [f(t) - f(s)]$$

we get the following relation between physical absorptive parts (R6)

$$0 = [R_2(s_1, s) - R_2(s, s_1)] + [R_2(t, s_1) - R_2(s_1, t)] + [R_2(s, t) - R_2(t, s)]\,, \text{ for } |s|, |s_1|, |t| < 4. \tag{II.41}$$

Similar relations have also been derived by Wanders (W3), Roskies (R3), and Auberson and Khuri (A6). Grassberger (G1) has shown, for example in deriving (II.28), that results following from positivity of the absorptive part can be improved by a judicious use of (II.41). Common and Pidcok (C3) have derived very useful inequalities on the D-wave below threshold using (II.41). The relation (II.41) can be regarded as a crossing relation between physical absorptive parts. Such relations would be discussed further in Sec. III.

(b) **Constraints involving a few low partial waves.** We quote for illustration a few results for $\pi^0\pi^0$ scattering (M8), and a few for other iso-spin combinations (A4). For $\pi^0\pi^0$ scattering,

$$4.067 f_2(0.0341) < f_0(3.839) - f_0(0.0341) \tag{II.42}$$

$$3.061 f_2(0.0730) > f_0(3.654) - f_0(0.0730) \tag{II.43}$$

$$\begin{aligned} &1.494 f_2(0.537) - 1.623 f_2(2.363) \\ &< f_0(0.537) - f_0(2.363) < 1.510 f_2(0.537) - 1.622 f_2(2.363) \end{aligned} \tag{II.44}$$

and for other iso-spin combinations

$$\begin{aligned} &1.844 f_1^1(0.2937) + 3.765 f_1^1(2.4226) \\ &< f_0^0(0.2937) - f_0^0(2.4226) - f_0(0.2937) + f_0(2.4226). \end{aligned} \tag{II.45}$$

$$\begin{aligned} &0.6146 f_1^1(0.2937) + 2.510 f_1^1(2.4226) \\ &> f_0(2.4226) - f_0^0(0.2937) + \frac{2}{3} f_0(0.2937). \end{aligned} \tag{II.46}$$

II.3 Constraints on integrals of partial wave amplitudes

Balachandran and Nuyts (B2) obtained necessary and sufficient conditions for crossing symmetry in the form a denumerable set of equality constraints involving integrals

We proceed to prove (II.24) and (II.25). Due to Bose-symmetry,

$$F_0(s) = \frac{2}{4-s} \int_0^{(4-s)/2} dt\, F(s,t,4-s-t). \tag{II.31}$$

Interchanging s and t in (II.22) we deduce

$$F(s,t) < F(s,0), \quad 0 \leq s < 4, \quad 0 \leq t \leq \frac{4-s}{2}. \tag{II.32}$$

Hence,

$$f_0(s) < F(s,0), \quad 0 \leq s < 4 \tag{II.33}$$

Combining this with (II.23) we have (II.24). Starting from

$$f_0(s) = 2 \int_0^{1/2} dx\, F(s, x(4-s)), \tag{II.34}$$

we have,

$$\frac{df_0(s)}{ds} = 2 \int_0^{1/2} dx \left[\left(\frac{\partial F(s,x(4-s))}{\partial s} \right)_t - x \left(\frac{\partial F(s,x(4-s))}{\partial t} \right)_s \right] \tag{II.35}$$

Further,

$$\left(\frac{\partial F}{\partial s} \right)_t > 0 \text{ for } 0 < t < 4 \text{ and } 4 > s > 2 > 2 - \frac{t}{2},$$
$$\left(\frac{\partial F}{\partial t} \right)_s < 0 \text{ for } 0 < s < 4 \text{ and } 0 < t < \frac{4-s}{2}.$$

Hence (II.25) follows. For the remaining results we start from the fixed-t dispersion relation (II.4) and project out the S-wave,

$$f_0(t) = c(t) + \frac{1}{\pi} \int_4^{\infty} \frac{ds'}{s'^2} \left\{ (t-4-2s') + \frac{2s'^2}{(t-4)} \ell n \left(\frac{s'+t-4}{s'} \right) \right\} A(s',t), \quad |t| < 4. \tag{II.36}$$

Thus the subtraction constant $c(t)$ in the fixed-t relation can be eliminated in favour of shape of the S-wave below threshold, and then indicate the methods of proof, the details of which are to be found in the original papers (M8, A3, A4, B12, C1, C3, G1, G2, J1, P1).

Jin and Martin (J1) obtained the results

$$f_0(s) < f_0(4), \quad 0 < s < 4 \tag{II.24}$$

and

$$\frac{df_0(s)}{ds} > 0, \quad 2 < s < 4. \tag{II.25}$$

Martin (M8) improved (II.25) to obtain,

$$\frac{df_0(s)}{ds} > 0, \quad 1.7 \leq s < 4 \tag{II.26}$$

Auberson (A3) obtained

$$\frac{df_0(s)}{ds} < 0, \quad 0 < s < 1.127 \tag{II.27}$$

Grassberger (G1) improved this result to obtain,

$$\frac{df_0(s)}{ds} < 0, \quad 0 < s < 1.217 \tag{II.28}$$

Common (C1) has derived the important result

$$\frac{d^2 f_0(s)}{ds^2} > 0, \quad 0 < s < 1.7 \tag{II.29}$$

From (II.24) to (II.29) it follows that $f_0(s)$ has a unique minimum in the range $0 < s < 4$, located somewhere between $s = 1.217$ and $s = 1.7$. The shape of the S-wave thus suggested is pictured in Fig. 1.

Martin (M8) has derived a class of inequalities of the form $f_0(s_1) < f_0(s_2)$ where $0 \leq s_{1,2} \leq 4$. For example, we quote,

$$f_0(0) > f_0(3.189), \quad f_0(3.205) > f_0(0.2134) > f_0(2.9863). \tag{II.30}$$

These inequalities have been improved by Brander (B12) and by Grassberger (G1), and generalized to iso-spin combinations other than the $\pi^0\pi^0 \to \pi^0\pi^0$ by Auberson et al (A4).

Froissart bound.

Now we turn to the Froissart bound and number of subtractions in d.r.

Use of unitarity – amplitude in terms of partial wave expansion – unitarity gives simple bound on each term – obtain a limit on number of terms that contribute effectively (as a function of s) – related to finite effective range of force – comes from analytic properties of amplitude in t-plane at fixed-s

Assume $T(s,t,u)$ is bounded for large s by some polynomial in s. [tempered distributions.]

Recall: spin-0 equal mass $T(s,t) = \sum_{l=0}^{\infty} (2l+1) f_l(s) P_l(\cos\theta)$

$\equiv x$

Martin & Spearman.

$$f_l(s) = (\eta_l e^{2i\delta_l} - 1)/2ip \qquad p = \frac{1}{2}\pi \frac{a}{\sqrt{s}}$$

δ_l - phase shift

η_l - elasticity $\quad 0 \leq \eta_l \leq 1 \qquad p\,|f_l(s)|^2 \leq \operatorname{Im} f_l(s) \leq p^{-1}$

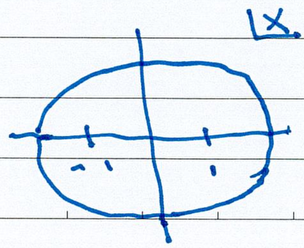
Domain of convergence of Legendre polynomial expansion

Lehmann ellipse

largest ellipse in x-plane foci ± 1 which doesnot enclose any singularity

For real s singularity will lie on real x-axis nearest singularity will correspond to fixed t_0 from lowest t-channel threshold

Titchmarsh?



$$\underset{(=x)}{\cos\theta_s} = 1 + \frac{t_0}{2q^2} \quad (\equiv x_0).$$

Optical theorem

Cross-section related to Im (amp)

$\pi\pi$ case semi-major axis $\cos\theta_0 = \left[1 + \frac{64 \cdot 4}{s(s-4)}\right]^{1/2}$

Wanders Springer Notes

imaginary part Im T semi-major axis $(2\cos^2\theta_0 - 1)$

Axiomatic. Martin showed it converges in $-28 < t < 4$ using theory of several complex variables

Compare with Mandelstam $-32 < t < 4$

M.R. Froissart bound ?

Assume for $t < t_0$ amplitude is polynomially bounded

Amplitude → boundedness

finite N $T(s,t) < s^N$ ← N is independent of s (for $s > s_0$) [N - finite]

For Froissart bound consider Im(elastic scattering amplitude) — Optical thm — cross-section

t_1: $0 < t_1 < t_0$ $\sum_l (2l+1)\, \mathrm{Im}\, f_l(s)\, P_l\left(1 + \frac{t_1}{2q^2}\right) < s^N$,

↑ Lehmann

→ asymptotic bound above s_0. for $s > s_0$

Each term is +ve. Hence

term by term → bounded

$$(2l+1)\, \mathrm{Im}\, f_l(s)\, P_l\left(1 + \frac{t_1}{2q^2}\right) < s^N \text{ for } s > s_0$$

Large l and $x > 1$ $\}$ $P_l(x) > \frac{c}{(2l+1)^{1/2}} \left\{1 + (2x-2)^{1/2}\right\}^l$

[Collins, p. 425 Intro to Regge theory]

Asymptotic behaviour.

Properties - Erdelyi, Burkhardt

DLMF (NIST, USA) [Tutorial]

Together it implies $\mathrm{Im}\, t_\ell(s) < \frac{C'}{(2\ell+1)^{1/2}} \left\{1 + \left(\frac{t_1}{q^2}\right)^{1/2}\right\}^{-\ell} s^N$

Simple algebra.

For large ℓ using binomial thm. and exponentiating

Note t_1 drops out.

$$\left\{1 + \frac{t_1}{2q^2}\right\}^{-\ell} \sim \exp\left\{-\ell\left(\frac{4t_1}{s}\right)\right\}$$

$$\mathrm{Im}\, t_\ell(s) < \exp\left\{C_1 + N \log s - C_2 \ell s^{-1/2}\right\}$$

Note t_1 drops out because $0 < t_1 < t_0$ absorbed into constant

Thus for $\boxed{\ell > N s^{1/2} \log s / C_2}$

exponential suppression.

Define $\boxed{L \equiv L(s) = C\sqrt{s}\log s,\ C > \frac{N}{C_2}}$

$$\sum_{\ell=L}^{\infty} (2\ell+1)\, \mathrm{Im}\, t_\ell(s)\, P_\ell(s) < s^{-M} \quad \text{as } s \to \infty$$

Choose C: M can be made arbitrarily large

$\boxed{\text{unitarity} \Rightarrow \text{also true when } \mathrm{Im} \to \mathrm{Re}}$

includes Re & Im.

Thus $T(s,t) \approx \sum_{\ell=0}^{L} (2\ell+1)\, t_\ell(s)\, P_\ell(x)$ as $s \to \infty$

For forward scattering using unitarity again $\mathrm{Im}\, T(s, t=0) \leq \sum_{\ell=0}^{L} \frac{(2\ell+1)}{p} \sim L^2 / p$

Needed for # of subtractions. (= 2)

$\Rightarrow \exists\ \gamma :\ \mathrm{Im}\, T(s, t=0) < \gamma s (\log s)^2$

Optical theorem $\sigma_{tot}(s) < c (\log s)^2$

Also holds for $|T(s, t=0)|$ since $|t_\ell(s)| \leq p^{-1}$

Hence $|T(s, t=0)| < \gamma s (\log s)^2$

since $|P_\ell(x)| \leq 1$ for $-1 \leq x \leq 1$

$$|\text{Im}\, T(s,t)| \leq |\text{Im}\, T(s,0)| \text{ for } t: 4m^2 - s \leq t \leq 0$$

This is in physical region of s-channel.

→ needed to prove # of subtractions needed = 2

Thus we claim that

$$T(s,t) = a(t) + b(t)s + \frac{s^2}{\pi}\int_{4m^2}^{\infty} \frac{\text{Im}T(s',t)}{s'^2(s'-s)}\,ds'$$

Physical region

$$+ \frac{s^2}{\pi}\int_{-\infty}^{-t} \frac{\text{Im}T(s',t)}{s'^2(s'-s)}\,ds'$$

Jin & Martin

is a valid dispersion relation

To show left-hand cut integral converges one needs to use crossing

$\text{Im}\, T(s',t)$ has same asymptotic behaviour for $s' \to -\infty$ as for $s' \to \infty$.

D.R. Also valid for $t < 4m^2$ $\quad T(s,t) < s^{2-\varepsilon}$
$\varepsilon > 0$

For $4m^2 - s < t < 0$ one gets a stronger inequality

$$|T(s,t)| < C\, s^{3/4} (\log s)^{3/2}$$

In order to set the stage for the Roy representation consider the $\pi^0 - \pi^0$ amplitude (Bonnier & Vinh Mau)

D.R. with 2 subtractions

$$T^{00}(s,t,u) = t^{00}(t) + \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'^2}\left[\frac{s^2}{s'-s} + \frac{u^2}{s'-u}\right] \times \operatorname{Im} T^{00}(s',t,4-s'-t)$$

Exercise.

$$T^{00}(s,t,u) = [T^{00}(s,t) - T^{00}(0,t)] + T^{00}(t,0)$$
$$= [T^{00}(s,t) - T^{00}(0,t)] + [T^{00}(t,0) - T^{00}(4,0)] + a_0^{00}$$

$$= a_0^{00} + \frac{t(t-4)}{\pi}\int_4^\infty ds' \frac{(2s'-4)\operatorname{Im} T^{00}(s',0)}{s'(s'-t)(s'+t-4)(s'-4)}$$

$$- \frac{su}{\pi}\int_4^\infty ds' \frac{(2s'+t-4)\operatorname{Im} T^{00}(s',t)}{s'(s'-s)(s'-u)(s'+t-4)}$$

Compare with (8.2.7) of Martin, Morgan & Shaw.

This is an excellent and simple illustration of how the Roy representation is obtained, the latter for the full set of iso-spin amplitudes.

Exercise for tutorials.

Very simple and elegant. For Roy representation use the same idea. Only complication is presence of iso-spin index, and crossing matrices.

Procedure is analogous.

Roy rep for iso-spin amplitudes.

Many ways of writing them.

Roy procedure is analogous for iso-spin amplitudes.

sum over I'

$$T^{I}(s,t) = g_1^{II'} \frac{a_0^{I'}}{4m^2} + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{dx}{x^2}\left[g_2^{II'}(s,t,x)\, \mathrm{Im}\,T^{I'}(x,0) + g_3^{II'}(s,t,x)\, \mathrm{Im}\,T^{I'}(x,t)\right]$$

$$g_1(s,t,u) = s\,\mathbb{1} + t\,C_{st} + u\,C_{su}$$

$$g_2(s,t,x) = \frac{-t}{\pi x (x-4m_\pi^2)} \left(u C_{st} + s C_{st} C_{su}\right)\left(\frac{\mathbb{1}}{x-t} + \frac{C_{su}}{x-u_0}\right)$$

$[u_0 = 4m_\pi^2 - t]$

$$g_3(s,t,x) = \frac{-su}{\pi x (x-u_0)}\left(\frac{\mathbb{1}}{x-s} + \frac{C_{su}}{x-u}\right)$$

Partial wave projection 1) $\frac{1}{2}\int_{-1}^{1} dz$ $\quad z \equiv \cos\theta_s$

Note. $\longrightarrow$ or 2) $\int_0^1 dz$

dt integral 1) $-28 < 4-s \Rightarrow s < 32$
2) $-28 < \frac{4-s}{2} \Rightarrow s < 60$ $\Big\}$ upper

Axiomatic

1) $4-s < 4 \Rightarrow 0 < s$

2) $\frac{4-s}{2} < 4 \Rightarrow -4 < s$

Mandelstam upper bound 1) $-32 < 4-s \Rightarrow s < 36$
2) $-32 < \frac{4-s}{2} \Rightarrow s < 68$ $\Big\}$ upper

Crossing condition

There is a crossing condition on the absorptive parts (suppressing iso-spin indices)

$$\int_4^\infty dx \left[\{ g_2(s,t,x) - C_{tu}\, g_2(s,t,x) \} A(x,0) + [\{ g_3(s,t,x)\, A(x,t) - C_{tu}\, g_3(s,u,x)\, A(x,u) \} \right] = 0$$

Does not constrain S- and P-wave absorptive parts. Generates the Lovelace amplitude. Manifestly crossing ~~matrices~~ amplitude.

$$\frac{1}{4}(s + t\, C_{st} + u\, C_{su})\, a_0$$

$$+ \frac{1}{\pi} \int_4^\infty \frac{dx}{x(x-4)} \left\{ \left[\frac{s(s-4)}{x-s} + \frac{t(t-4)}{x-t} C_{st} + \frac{u(u-4)}{x-u} C_{su} \right] \times \operatorname{Im} f_0 \right.$$

$$\left. + 3 \left[\frac{s(t-u)}{x-s} + \frac{t(s-u)}{x-t} C_{st} + \frac{u(t-u)}{x-u} C_{su} \right] \operatorname{Im} f_1 \right\}$$

Very useful representation.

Roy equations.

Roy equations are obtained by projecting LHS on to partial waves and introducing partial-wave decomposition for absorptive parts and collecting. [Can be automated]

$$t^I_\ell(s) = k^I_\ell(s) + \sum_{I'=0}^{2} \sum_{\ell'=0}^{\infty} \int_0^\infty dx \, K^{I I'}_{\ell \ell'}(x, s) \, \mathrm{Im}\, t^{I'}_{\ell'}(x)$$

$$K^{00}_{00}(x,s) = \frac{1}{\pi(x-s)} + \frac{2}{3\pi(x-4m_\pi^2)} \ln\left(\frac{x+s-4m_\pi^2}{x}\right) - \frac{2s+5x-16m_\pi^2}{3\pi x(x-4m_\pi^2)}$$

[from ACGL]

MMS page 230.

$$\boxed{k^I_\ell(s) = a^I_0 \delta^0_\ell + \frac{s-4m_\pi^2}{4m_\pi^2}\left(\frac{1}{3}\delta^I_0\delta^0_\ell + \frac{1}{18}\delta^1_1\delta^1_\ell - \frac{1}{6}\delta^I_2\delta^0_\ell\right)}$$

Simpler case of $\pi^0\pi^0$ $(m_\pi = 1)$

Kernel for S-wave scattering

$$G^{00}_0(s,x) = \frac{1}{\pi}\left[\frac{s-4}{(x-4)(x-s)} - \frac{2}{x} + \frac{2}{s-4}\ln\left(\frac{s-4+x}{x}\right)\right]$$

Roy representation for $\pi^0 - \pi^0$

$T^{00}(s,t)$

(8.2.13) See ACGL

$K'(s',s,t)$ — (8.2.16) in Martin, Morgan & Shaw

crossing condition.

Notebooks are available

cated. In that chapter we saw how to determine these kinematic singularities and how to take account of them by defining amplitudes which are free of kinematic singularities. Our concern here is not with these kinematically based effects but with the 'dynamical' singularities which appear in $f^J_{cd;ab}(s)$ as a result of the Mandelstam singularities in $T_{cd;ab}(s, t', u')$ arising directly from the s-, t- and u-channel processes. The singularities in the partial wave helicity amplitudes arising from these are determined in precisely the same way as for the spin zero case and in fact the calculation is easily seen to lead to the same results for the location of the 'dynamical' singularities in $f^J_{cd;ab}(s)$. This is clear for the effect of the s-channel singularities, which come directly from $T_{cd;ab}(s, t', u')$ in eq. (8.34). For the t'- and u'-channel singularities one can either write down a representation for $f^J_{cd;ab}(s)$ analogous to eq. (8.9) using a generalized version of Neumann's formula (eq. (9.48)) or alternatively one can use the argument that singularities occur whenever the end-points of the path of integration, $x' = \pm 1$, in eq. (8.34) correspond to values of t' or u' for which $T_{cd;ab}(s, t', u')$ has a 'dynamical' singularity. This leads immediately to equation (8.10) or (8.22).

The threshold behaviour of the amplitudes $f^J_{cd;ab}(s)$ is somewhat more complicated for particles with spin than in the spin zero case. In general the helicity amplitudes $f^J_{cd;ab}(s)$, of definite J, correspond to several different values of orbital angular momentum l. However the basic formula is still essentially eq. (8.18′)

$$f^J_{cd;ab} \sim (pq)^{l_{\min}}$$

where, when due account has been taken of parity, $l_{\min}$ is the lowest allowed value of l (see JACKSON and HITE [1968]).

§ 2. Asymptotic Bounds on Scattering Amplitudes

Using unitarity together with analyticity properties of the scattering amplitude it has been found possible to obtain asymptotic bounds on scattering amplitudes and cross-sections. The first such bound was obtained by FROISSART [1961] who deduced, using the Mandelstam representation, that

$$\sigma_{\text{tot}}(s) < c(\log s)^2 \quad \text{as } s \to \infty . \tag{8.35}$$

This result is generally known as the Froissart bound. Since Froissart's first derivation of this result a great deal of further work has been put into this problem, notably by Martin. (See, for example, MARTIN [1963, 1965], with ± 1 as foci such that it does not enclose any singular points of the amplitude. This is a well known result in complex variable theory (see, for example, TITCHMARSH [1939]). The ellipse in the x-plane is often referred to as the Lehmann ellipse. For real values of s the singularities of $T(s, t, u)$ lie on the real x-axis and the nearest singularity (which will come from either the t-channel or the u-channel so to be specific let us say t) will correspond to a *fixed* value t_0 of t coming from the lowest t-channel threshold. This gives as the nearest singularity in the x-plane

$$x_0 = 1 + \frac{t_0}{2q^2}. \tag{8.38}$$

x_0 will then be the semi-major axis of the largest ellipse with ± 1 as foci for which the expansion (8.1) is valid†.

We now make the assumption that for $t < t_0$ the amplitude $T(s, t)$ is bounded, for large s, by a polynomial in s

$$T(s, t) < s^N, \tag{8.39}$$

for s greater than some s_0 and where N is independent of s. In deriving the Froissart bound we only need consider the imaginary part of the elastic scattering amplitude as we shall use the optical theorem to obtain the total cross-section. Taking a point t_1 such that $0 < t_1 < t_0$ we can replace $T(s, t_1)$ in (8.39) by its partial wave expansion and obtain the result

$$\sum_{l=0}^{\infty} (2l+1) \operatorname{Im} f_l(s)\, P_l\left(1 + \frac{t_1}{2q^2}\right) < s^N, \quad \text{for } s > s_0. \tag{8.40}$$

Since each term in this series is positive (eq. (8.37)) we can deduce that for $s > s_0$,

$$(2l+1) \operatorname{Im} f_l(s)\, P_l\left(1 + \frac{t_1}{2q^2}\right) < s^N. \tag{8.41}$$

Now for large l and for $x > 1$, the Legendre polynomials satisfy the inequality

$$P_l(x) > \frac{C}{(2l+1)^{\frac{1}{2}}} \{1 + (2x-2)^{\frac{1}{2}}\}^l \tag{8.42}$$

† Although in the above discussion we introduced the singularities in the x-plane as consequences of the Mandelstam singularities in $T(s, t, u)$ the assumption that $T(s, t, u)$ should be analytic in the ellipse described above is a distinctly weaker analyticity assumption than the Mandelstam representation, see for example MARTIN [1966].

Froissart's bound now follows directly from the Optical Theorem, eq. (6.49),

$$\sigma_{\text{tot}}(s) < c(\log s)^2. \tag{8.35}$$

Notice that the inequality (8.51) holds also for $|T(s, t=0)|$. This follows since, from (8.37), $|f_l(s)| \leqslant \rho^{-1}$, so that (8.50) holds with $|T(s, t=0)|$ replacing Im $T(s, t=0)$. Hence

$$|T(s, t=0)| < \gamma\, s(\log s)^2. \tag{8.52}$$

Since $|P_l(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$ we can see from (8.49) and (8.37) that

$$|\text{Im } T(s, t)| \leqslant |\text{Im } T(s, 0)| \tag{8.53}$$

for values of t in the range

$$4m^2 - s \leqslant t \leqslant 0. \tag{8.54}$$

This range is the s-channel physical region. It follows from (8.53) and (8.51) that for values of t lying in the above range not more than two subtractions are needed in the fixed t dispersion relation for $T(s, t)$. Thus

$$\begin{aligned} T(s, t) = a(t) + b(t)s + \frac{s^2}{\pi}\int_{4m^2}^{\infty} \frac{\text{Im } T(s', t)}{s'^2(s'-s)}\, ds' + \\ + \frac{s^2}{\pi}\int_{-\infty}^{-t} \frac{\text{Im } T(s', t)}{s'^2(s'-s)}\, ds' \end{aligned} \tag{8.55}$$

is a valid dispersion relation provided that t lies in the range defined by (8.54). To show that the integral over the left-hand cut converges one needs to use crossing. This shows that Im $T(s, t)$ has the same asymptotic behaviour for $s \to -\infty$ as for $s \to +\infty$. This result has been extended by JIN et al. [1964] who have shown that the twice subtracted dispersion relation is valid for positive values of t less than $4m^2$. Thus

$$|T(s, t)| < s^{2-\varepsilon} \qquad \text{for } t < 4m^2 \tag{8.56}$$

where $\varepsilon > 0$.

For t lying in the range $4m^2 - s < t < 0$, that is for $0 < \theta < \pi$, one can obtain a stronger inequality than (8.52). This comes from the property of $P_l(x)$ (see, for example, MAGNUS and OBERHETTINGER [1954], p. 71) that for $|x| < 1$,

$$|P_l(x)| < \left(\frac{2}{\pi l \sqrt{(1-x^2)}}\right)^{\frac{1}{2}}. \tag{8.57}$$

May 23, 2024

Discussion regarding the behavior of $\sum \mathrm{Re}\, f_\ell$ and $\sum \mathrm{Im}\, f_\ell$ in the Froissard bound sector.

From Martin & Spearman "... Then it is easily verified that

$$\sum_{\ell=L}^{\infty} (2\ell+1)\, \mathrm{Im}\, f_\ell(s) P_\ell(s) < s^{-M} \quad s \to \infty \qquad (8.48)$$

... Because of (8.37) it is clear that a similar result holds if we replace $\mathrm{Im}\, f_\ell(s)$ by $\mathrm{Re}\, f_\ell(s)$ in (8.48)..."

Let us discuss this in some detail.

(8.37) reads $\quad \rho |f_\ell(s)|^2 \leq \mathrm{Im}\, f_\ell(s) \leq \rho^{-1}$

(8.36) is also needed: $\quad f_\ell(s) = \dfrac{\eta_\ell e^{2i\delta_\ell} - 1}{2i\rho}$

$\rho = \dfrac{1}{2} \pi \dfrac{q}{\sqrt{s}}$; q^2 is eq. (4.82) which in equal mass case is $\quad q^2 = (s - 4m^2)/4$

asymptotically ρ approaches $\pi/4$ (< 1)

so $\quad |f_\ell(s)|^2 \leq \dfrac{1}{\rho} \mathrm{Im}\, \cancel{\mathrm{Im}}\, f_\ell(s) < \mathrm{Im}\, f_\ell(s)$

$\Rightarrow \quad |\mathrm{Re}\, f_\ell(s)|^2 + |\mathrm{Im}\, f_\ell(s)|^2 < \mathrm{Im}\, f_\ell(s) \qquad |f_\ell|^2 < \dfrac{1}{\rho^2} < 1$

$\Rightarrow \quad (\mathrm{Re}\, f_\ell(s))^2 < \mathrm{Im}\, f_\ell(s) \qquad \mathrm{Re}\, f_\ell < 1$

$\qquad \mathrm{Im}\, f_\ell < 1.$

and $\quad (\mathrm{Im}\, f_\ell(s))^2 < \mathrm{Im}\, f_\ell(s)$

$\Rightarrow \quad \mathrm{Im}\, f_\ell(s) < 1 \Rightarrow |\mathrm{Re}\, f_\ell(s)|^2 < 1 \qquad \mathrm{Re}\, f_\ell < 1$

$\Rightarrow \quad \mathrm{Re}\, f_\ell(s) < 1 \qquad \Rightarrow (\mathrm{Re}\, f_\ell)^2 < \mathrm{Re}\, f_\ell$

$\frac{1}{\rho} > 1$

Answers to some Questions KH 29.05.2024

* About Froissart bound

In the lecture, we saw that

$$\mathrm{Im} f_\ell(s) \lesssim \exp\left(c_1 + N \log s - c_2 \ell s^{-1/2}\right) \qquad (1)$$

This leads to $\mathrm{Im}\, T(s,0) \lesssim c\, s \log^2 s$.

We want a bound for the full amplitude, running the argument we did for $\mathrm{Re} f_\ell$ instead of $\mathrm{Im} f_\ell$ would not work as it is not sign definite.

However, from unitarity

$$|\mathrm{Re} f_\ell|^2 + \mathrm{Im} f_\ell^2 \leq \mathrm{Im} f_\ell\, \rho^{-1} \qquad \rho = \frac{\pi}{2} \frac{q}{\sqrt{s}} \xrightarrow{s \gg 1} \frac{\pi}{2}$$

$$\overset{(1)}{\rightsquigarrow} \quad \mathrm{Re} f_\ell \lesssim \exp\left(\tfrac{1}{2}\left(c_1 + N \log s - c_2 \ell s^{-1/2}\right)\right)$$

$\hookrightarrow$ for $\ell > c \sqrt{s} \log s$, f_ℓ decay exponentially

and the bound on $T(s,0)$ follow.

* Froissart to 2 subtractions DR.

Lets start by writting the DR (here scatr)

$$\frac{T(s,t)}{s^2} = \frac{1}{2\pi i} \oint_{C_s} \frac{ds'}{s'-s} \frac{T(s',t)}{s'^2}$$

$$= \frac{T(0,t)}{s^2} + \frac{\partial_s T(0,t)}{s} + \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'-s} \frac{\mathrm{Disc}_s T(s',t)}{s'^2} + \int_{-\infty}^{-4m^2-t} \frac{ds'}{s'-s} \frac{\mathrm{Disc}_s T(s',t)}{s'^2}$$

$$+ \frac{1}{2\pi i} \int_{C_\infty} \frac{ds'}{s'-s} \frac{T(s',t)}{s'^2}$$

$=0$ if $\displaystyle \lim_{|s| \to \infty} \frac{T(s,t)}{s^2} = 0$ (in all directions)