Lecture 1.

Analyticity, Unitarity and Crossing: An Introduction

Dispersion Relation Theory and Phenomenology in Field Theory. Essentials and Applications.

These methods are based on general principles and are of great use even today when dealing with the strong interactions at low energies.

Standard Model of electro-weak and strong interactions.

electro-weak $\longrightarrow$ weak + electromagnetic

$SU(2) \times U(1)$ $\qquad$ $U(1)$ e.m. (Maxwell)

weak interactions - $\begin{pmatrix} e^- \\ \nu_e \end{pmatrix}_L \begin{pmatrix} \mu^- \\ \nu_\mu \end{pmatrix}_L \begin{pmatrix} \tau^- \\ \nu_\tau \end{pmatrix}_L$ $\quad e^-_R \; \mu^-_R \; \tau^-_R$

$W^\pm, Z^0$ (Glashow Weinberg Salam)

$\begin{pmatrix} u \\ d \end{pmatrix}_L \begin{pmatrix} c \\ s \end{pmatrix}_L \begin{pmatrix} t \\ b \end{pmatrix}_L$ $\quad \begin{matrix} u_R & c_R & t_R \\ d_R & s_R & b_R \end{matrix}$

Higgs $\quad h$

't Hooft & Veltman

Coulomb γ $\qquad$ electromagnetic charged leptons and quarks

Lagrangian Field Theory.
Perturbative expansion. Feynman diagram

QCD strong interactions.

$SU(3)_c$ colored quarks and gluons

Confined phase $\qquad$ asymptotically free.
(Gross, Wilczek, Politzer)

QCD spectrum: mesons $\qquad$ baryons

HADRONS $\qquad q_1\bar{q}_2 \qquad q_1 q_2 q_3$

exotics. $\qquad$ tetra-quarks $\qquad$ penta-quarks

Many of our studies in this course will be based on properties of pions — stable when weak interactions are switched off and in absence of anomalies. Can be extended to kaons.

Gell-Mann.

3 kinds of pions $\pi^{\pm}, \pi^0$
4 kinds of kaons $K^0, \bar{K}^0, K^+, K^-$
1 η

Pseudo-scalar octet
$J^P = 0^-$

Today these are considered (approximate) Goldstone bosons of spontaneously broken

NAMBU axial-vector symmetry of QCD Hamiltonian that arises when $m_u, m_d, m_s \to 0$

$m_u \simeq m_d$ $\quad \hat{m} = \frac{1}{2}(m_u + m_d)$ $\quad$ Iso-spin symmetry (Heisenberg)

$m_s \gg \hat{m}$

Weinberg, Gasser & Leutwyler systematic theory (chiral perturbation theory - Dashen R.D. Weinstein)

$m_{\pi^+} \sim 139$ MeV $\quad m_{\pi^0} \sim 135$ MeV

$m_K \sim 495$ MeV

Unitarity of the S-matrix. Implies relations between Re and Im parts of matrix elements. Provides important constraints.

Analyticity implies various physical quantities such as scattering amplitudes and decay amplitudes (form factors) which are complex functions of complex energies are boundary values for physical (real) energies on cuts in the complex plane. Analyticity is a consequence of causality and the finiteness of the speed of light. Implies a relationship between Re and Im parts analogous to Kramers-Krönig relation for the refractive index of light - dispersive and absorptive parts.

Crossing relates seemingly different and ~~un~~related processes

Iso-spin invariance very useful.

$$A + B \rightarrow C + D$$

Established in field theory Bros, Epstein, Glaser.

gets related to e.g., $A + \bar{C} \rightarrow \bar{B} + D$

$$A + \bar{D} \rightarrow C + \bar{B}$$

with corresponding swaps of c.m. energy^2 (s) and square of c.m. momentum transfer (t)

Taken together, provides powerful framework for analysis of scattering information, form factor analysis.

Dispersion relations arise when Cauchy's Theorem is applied to functions of kinematic invariant(s).

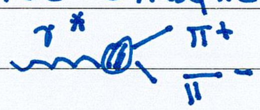
Established rigorously in field theory
[Gell-Mann, Goldberger, Thirring ; Bremermann, Oehme, Taylor]

Fixed-t dispersion relations and range of validity — Lehmann ellipse
extended by A. Martin.

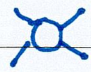
Mandelstam — double dispersion relations in s and t inspired by perturbation theory.

Frossart — 'subtractions' in dispersion relations by establishing the bound.

We will introduce electromagnetic form factor of the pion.



π-π scattering amplitudes — simplest (?)
use all the ingredients above.



Key ideas will require partial wave projections and amplitudes. (Legendre polynomials $P_\ell(\cos\theta)$)

Recall $\pi^\pm, \pi^0$ form an iso-vector $I=1$ each

π-π amplitude will be $I = 0, 1, 2$

Project on to partial waves of $\ell = 0, 1, 2, \ldots$

$I = 0, 2$ $\quad$ $\ell = 0, 2, 4, 6$
$\quad$ S D G

$I = 1$ $\quad$ $\ell = 1, 3, 5$
$\quad$ P F

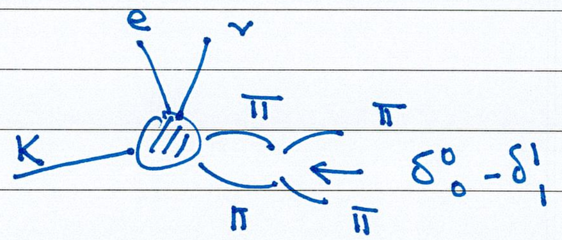
Partial waves $\quad f^I_\ell(s)$, $\quad \delta^I_\ell(s)$, $\quad \eta^I_\ell(s)$

amplitude $\quad$ phase-shift $\quad$ elasticity $= 1$ in elastic region.

Very useful for unitarity $\quad f^I_\ell(s)$ complex for $s \gtrsim 4m^2$

as we will see.

Measured indirectly



Electromagnetic form factor

$$e^+e^- \rightarrow \gamma^* \rightarrow \pi^+\pi^-$$

$F(t)$
dominated by ρ.

$m_\rho \sim 770$ MeV
$\Gamma_\rho \sim 150$ MeV

$m_{\pi^+} = m_{\pi^-}$ CPT Theorem

But why is $m_{\pi^0} \simeq m_{\pi^+}$? Because $m_u \simeq m_d$ and $q_u \neq q_d$ (e.m. mass splitting)

Ignoring $(m_u - m_d)$ gives a lot of simplification. And in this course we will keep using.

The 3 (massless) quarks lead to $SU(3)_L \times SU(3)_R$ symmetry of QCD.

The vacuum breaks this to $SU(3)_V$ $\quad V = \frac{L+R}{2}$
via quark condensate

$SU(3)_A$ spontaneously broken

8 Goldstone bosons

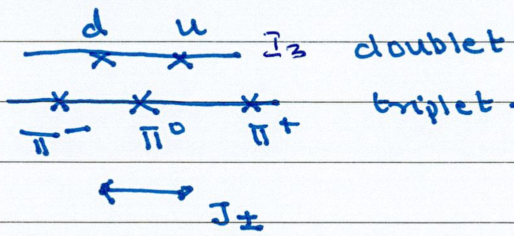
Retain the character even when $m_q \neq 0$.

Organized in the form of the $I_3 - Y$ plane

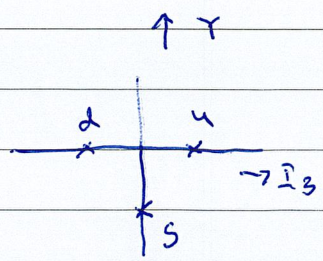
$SU(3)$: 8 generators $\quad \lambda_i$, $i = 1 \ldots 8$
Gell-Mann. $\quad$ 2 diagonal (I_3, Y)

$SU(2)$: 3 generators $\quad \sigma_i$ $\quad i = 1, 2, 3$
Pauli $\quad$ 1 diagonal

$SU(2)$
Iso-spin subgroup of $SU(3)$

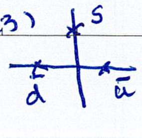


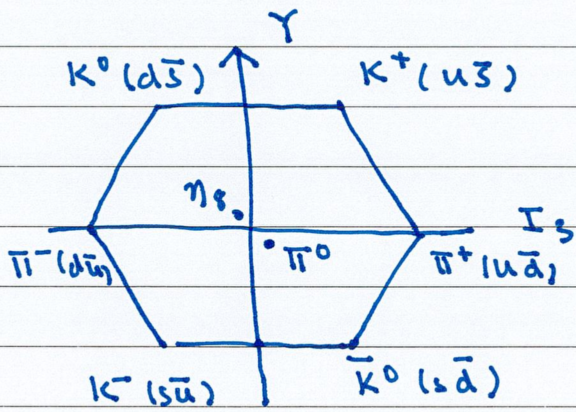
2 of $SU(2)$



3 of $SU(3)$

$\bar{3}$ of $SU(3)$





$m_{\pi^+} = 139.6\,\text{MeV}$

$\tau = 2.6033 \times 10^{-8}\,s$

$m_{\pi^0} = 135.0\,\text{MeV}$

$\tau = 8.5 \times 10^{-17}\,s$

$m_{K^-} = 493.7\,\text{MeV}$

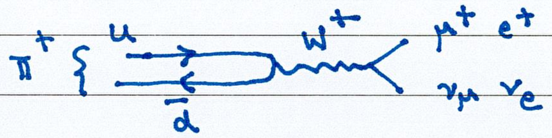
$\tau = 1.23 \times 10^{-8}\,s$

$m_{K^0} = 497.6\,\text{MeV}$

$m_\eta = 547.9\,\text{MeV}$

$m_{\eta'} = 957.8\,\text{MeV}$

Pion decay:



$$\tau = \frac{8\pi}{f_\pi^2 G^2 m_\mu^2 m_\pi} \left(\frac{m_\pi^2}{m_\pi^2 - m_\mu^2} \right)^2 \times \frac{1}{|V_{ud}|^2}$$

$V_{ud} = 0.974$

$m_\pi = 0.13957\,\text{GeV}$

$m_\mu = 0.1057\,\text{GeV}$

$G = 1.166 \times 10^{-5}\,\text{GeV}^{-2}$

$\tau = 2.6 \times 10^{-8}\,s$

$\hbar \quad 6.582 \times 10^{-25}\,\text{GeV-s}$

$f_\pi = 0.132\,\text{GeV}$

$$F_\pi = \frac{f_\pi}{\sqrt{2}} = 93.4\,\text{GeV}.$$

The fact that the pion decays suggests that it can be considered a goldstone boson.

This can be expressed as

$$\langle 0 | A_\mu^j(0) | \pi^k(p) \rangle = i F_\pi p_\mu \delta^{jk}$$

Theorem of (Callan) - Coleman - Wess - Zumino can fix the leading order effective Lagrangian

3x3 traceless Hermitian matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Concentrate on SU(2) sector

π^1, π^2, π^3 $\quad\quad \sigma_1, \sigma_2, \sigma_3$

$\pi^3 \sim \pi^0$ $\quad\quad \pi^\pm \sim (\pi^1 \pm i\pi^2)$

The principles of iso-spin invariance, spontaneously broken axial-vector symmetry fixes the lowest order $\mathcal{L}$. Can be used to compute tree-level amplitudes which we will use. We will do a lot more with this.

Pion decays implies $\langle 0 | A_\mu^j(0) | \pi^k(p) \rangle = i F_\pi p_\mu \delta^{jk}$

Assemble the GB fields into an SU(3) octet

$$U = \exp\left(i \frac{\vec{\lambda}\cdot\vec{\varphi}}{F} \right)$$

$$\vec{\lambda}\cdot\vec{\varphi} = \begin{bmatrix} \frac{\pi^0}{2} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{2} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta_8. \end{bmatrix}$$

$$\mathcal{L}_2 = \frac{F_\pi^2}{4}\left(\mathrm{Tr}\, D_\mu U D^\mu U^\dagger \right) + \frac{F_\pi^2}{4} \mathrm{Tr}(\chi U^\dagger + \chi^\dagger U)$$

$$SU(2) : U = \exp(i \vec{\tau}\cdot\vec{\pi})/F_\pi)$$

$$\chi = 2B_0(s + ip)$$

$$s = \hat{m} \qquad M_\pi^2 = 2B_0 \hat{m}$$

Let us now look at general principles.

Best motivated by standard text-books

Martin & Spearman Metric $(- + + +)$

S-matrix : $S_{fi} = \langle f | S | i \rangle$

Conservation of probability - unitarity of S-matrix

$$\sum_n S^{\dagger}_{fn} S_{ni} = \delta_{fi}$$

see ch. 4, §5

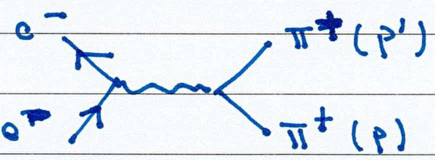
If there is no interaction $S = \mathbb{1}$. Otherwise $S = \mathbb{1} + iT$

Insert into unitarity $T - T^{\dagger} = i T^{\dagger} T$

$$\text{Im}\, T_{fi} = \frac{1}{2} \sum_n T^{\dagger}_{fn} T_{ni}$$

Let us consider the pion form factor appearing in $e^+e^- \to \pi^+\pi^-$ (production). First scattering :

Also related to $e^-\pi^+ \to e^+\pi^-$ One can be obtained from the other (scattering)



$q = p' - p$

$t = -q_\mu q^\mu = -\cancel{(p_\mu + p'_\mu)})$

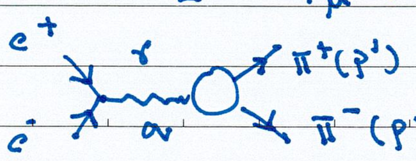
$t \leq 0$ for scattering

$$\langle p' | j_\mu | p \rangle = (p_\mu + p'_\mu)\, F(t)$$

since $(p' - p)_\mu j^\mu = 0$ (current conservation)

For production t $(p \to -p)$

$$t = -q_\mu^2 = -(p_\mu + p'_\mu)^2 \geq 4m^2$$

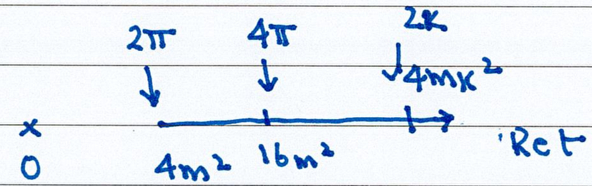


Physical regions of the 2 processes are different.

Scattering $t \leq 0$ Production $t \geq 4m^2$

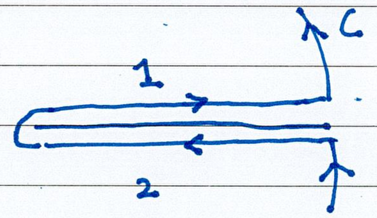
Related if it is possible to make an analytic continuation.

Electron pion Scattering



Pion pair production

can be achieved via dispersion relations. $\underline{t}$



Contributions from C drop out

On 1 $t+i\varepsilon$

on 2 $t-i\varepsilon$

Let us recall Cauchy's theorem

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0}\, dz$$

Schwarz Reflection $F(z^*) = (F(z))^*$

Let us apply it to $\dfrac{F(t') - F(0)}{(t'-0)(t'-t)}$

Real analytic function

Kanazawa-Sugawara theorem

$$F(t) = F(0) + \frac{t}{\pi} \int_{4m^2}^{\infty} \frac{\operatorname{Im} F(t')}{t'(t'-t)}\, dt'$$

Recall again $S_{fi} = \langle f|S|i\rangle$ $\quad \sum_n S^\dagger_{fn} S_{ni} = \delta_{fi}$

$S = \mathbb{1} + iT \qquad T - T^\dagger = i\,TT^\dagger$

$$\mathrm{Im}\, T_{fi} = \frac{1}{2} \sum T^\dagger_{fn} T_{ni}$$

The unitarity relation applied to $f = |\pi^+\pi^-\rangle$, $i = \gamma^*$

$$\mathrm{Im}\, F(t) = \mathrm{Im}\langle \pi^+\pi^-|T|\gamma\rangle$$
$$= \sum_n \langle \pi^-\pi^+|T^\dagger|n\rangle\langle n|T|\gamma\rangle$$

in the elastic region $4m^2 \le t \le 16\,m^2$

only cont. comes from $|n\rangle = |\pi\pi\rangle$ with $I=1,\ J=1$

$$\mathrm{Im}\, F(t) = T^*_{\pi\pi}(t)\, F(t)$$

Another way of arriving is to consider

$$F(t) = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\mathrm{Im}\, F(t')}{t'-t}\, dt'$$

Aside: Plemij formula

$$= \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{Im}\, F(t')\left[\frac{1}{t'-t} - \frac{1}{t'} + \frac{1}{t'}\right] dt'$$

Re part given in terms of Cauchy Principal value Hilbert transform.

$$= \underbrace{\frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\mathrm{Im}\, F(t')}{t'}}_{} + \frac{t}{\pi}\int_{4m^2}^{\infty} \frac{\mathrm{Im}\, F(t')}{t'(t'-t)}\, dt'$$

$$= F(0)\ (=1)$$

Schwarz Reflection principle: $\mathrm{Im}\, F(t)$

$$= \frac{1}{2i}\left[F(t+i\epsilon) - F(t-i\epsilon)\right]$$

Schwarz Reflection Principle 1

Then†

$$\langle \theta\phi; \gamma | T_{\mathscr{P}}^{\dagger} | 00; \gamma \rangle = \langle 00; \gamma | T_{\mathscr{P}} | \theta\phi; \gamma \rangle^*$$

$$= \langle -\theta\phi; \gamma | T_{\mathscr{P}} | 00; \gamma \rangle^* \tag{6.5}$$

$$= \frac{q}{4W} T^*(s, t, u) .$$

The second step here, where $-\theta$ is introduced, is explained in the footnote to eq. (4.138). Eq. (6.2) may now be written as

$$\operatorname{Im} T(s, t, u) = \frac{2W}{q} \sum_{\lambda} \int dQ \, \langle \theta\phi; \gamma | T_{\mathscr{P}}^{\dagger} | \alpha \rangle \langle \alpha | T_{\mathscr{P}} | 00; \gamma \rangle . \tag{6.6}$$

The spectrum of physical intermediate states will have real, positive values of s. We assume that the particles involved have non-zero mass and suppose, for example, that there is one single-particle state of mass M with the right quantum numbers, that the lowest energy two-particle state consists of two of the original (mass m) particles, then that states of three of these mass m particles can occur with the right quantum numbers and so on. For such an energy spectrum of intermediate states we can say that the imaginary part of the amplitude $T(s, t, u)$ will be non-zero along the range of the real s-axis for which the intermediate states occur and further that as far as the unitarity relation *for this particular channel* is concerned Im $T(s, t, u)$ can be zero elsewhere on the real s-axis. We cannot argue that Im $T(s, t, u)$ definitely will be zero for values of s for which there are no intermediate states since unitarity is only demanded as a physical law in the actual physical region for the process, which is for s greater than $4m^2$. Also, we shall see that the unitarity relation for processes in the u-channel (i.e. processes for which u is the square of the total c.m. energy) implies that Im $T(s, t, u)$ will be non-zero along parts of the negative s-axis.

In order to proceed we will make the simplest possible assumption, that Im $T(s, t, u)$ will be zero along the real s-axis except where unitarity in one of the three channels says that it is non-zero. Also the validity of eq. (6.6) will be accepted even for an intermediate state $|\alpha\rangle$ whose mass is less than $2m$, which is the physical threshold for the two-particle scattering process. Similarly we will accept that the right-hand side of eq. (6.6) may be non-

† The notation used here and elsewhere in this book is that $T^*(s, t, u)$ means $[T(s, t, u)]^*$.

Schwarz Reflection
Principle 3

provided s and s^* both lie inside the domain of analyticity in s. We have omitted the variable u just for convenience: in all cases it is given from s and t by eq. (6.1). Remember that t, for the moment, is assumed to be real.

An immediate consequence of eq. (6.8) is that the domain of analyticity of $T(s, t)$ cannot extend over the entire s-plane. For suppose it did, then eq. (6.8) would tell us that Im $T(s, t)$ must be zero for all real s. From unitarity we know that this is not so, that Im $T(s, t)$ is non-zero on all except a finite segment of the real axis. So $T(s, t)$ must possess some singularities in the s-plane and the simplest possibility is to suppose that the s-plane is cut along the real axis from $4m^2$ to $+\infty$ and from $-\infty$ to $-t$. $T(s, t)$ will also be singular at the points $s = M^2$ and $s = 4m^2 - M^2 - t$; we shall see later that these singularities arising from single-particle intermediate states are simple poles. Unitarity does not demand that $T(s, t)$ for fixed real t, need have any further singularities in s. So we will suppose that for a fixed real value of t, $T(s, t)$, is analytic in the entire s-plane except for cuts along the real axis from $4m^2$ to $+\infty$ and from $-\infty$ to $-t$, and for poles at M^2 and $4m^2 - M^2 - t$ corresponding to a single-particle intermediate state of mass M.

Since $T(s, t)$ tends to different values on the real axis in the region of the cut depending on whether we approach the axis from above or below it is important to specify clearly which value of $T(s, t)$ gives the physical amplitude. For a physical process s, t and u will all have real values. The convention that is adopted for specifying the physical amplitude is to give a small positive imaginary part ε to whichever of the variables s, t or u is associated with the energy of the physical process and then let ε tend to zero. Thus if s, t and u have real values which lie in the physical region for the s-channel (see Fig. 4.5) the physical value of the amplitude is just

$$\lim_{\varepsilon \to 0+} T(s + \mathrm{i}\varepsilon, t) \, .$$

Notice that if s, t, u refer to a point in the u-channel physical region, the physical amplitude is

$$\lim_{\varepsilon \to 0+} T(s, t, u + \mathrm{i}\varepsilon) = \lim_{\varepsilon \to 0+} T(s - \mathrm{i}\varepsilon, t) \tag{6.9}$$

since replacing u by $u + \mathrm{i}\varepsilon$ corresponds to subtracting $\mathrm{i}\varepsilon$ from s by eq. (6.1). Thus the physical amplitude for an s-channel process is the value of the amplitude $T(s, t, u)$ just above the cut in the s-plane, but the physical amplitude for a u-channel process is the value of $T(s, t, u)$ just below the cut in the s-plane.

Schwarz Reflection Principle

Γ finite segment of $\mathbb{R}$ $\qquad$ $\overline{\quad\Gamma\quad}$

$D \in \mathbb{C}$ $\qquad D \cap \mathbb{R} = \Gamma$

$f(z)$ is analytic in D

$\mathrm{Im}\, f(z) = 0$ on Γ

satisfies $\quad f(z^*) = \cancel{f^*}(f(z))^* \qquad z, z^* \in D$

$f(z)$ - real analytic in D.

$T(s,t)$ analytic in cut plane $\quad 4m^2$ to ∞

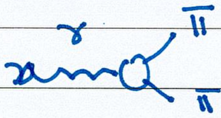
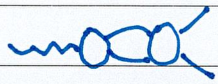
$-\infty$ to $-t$

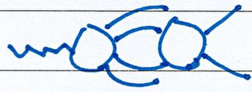
(poles at M^2, $4m^2 - M^2 - t$)

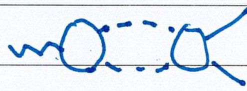
Physical value : $\quad \lim\limits_{\varepsilon \to 0+} T(s+i\varepsilon, 0)$
in s-channel

u-channel $\quad \lim\limits_{\epsilon \to 0+} T(s,t,u+i\epsilon) = \lim\limits_{\epsilon \to 0+} T(s-i\epsilon, t)$

More meaningful when we take into account unitarity.

Im [] =  2π

\+  4π

\+  $K\bar{K}$

Unitarity shows $\text{Im}\,F(t) \neq 0$ only above thresholds

$4m^2 \lesssim t \lesssim 16m^2$ only $\pi\pi$ intermediate state

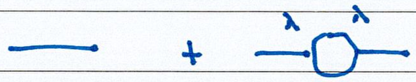
$$\text{Im}\,F(t) \sim T^*_{\pi\pi}(t)\,F(t)$$

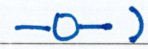
$\uparrow$

$\pi\pi$ scattering amplitude in $I=1$, $J=1$ state

Let us study some examples in perturbation theory.

From R. Zwicky Intro to Dispersion Relations.

2-point function in $\lambda\varphi^3$ theory  $+ O(\lambda^4)$

$S = 1 + iT$: $2\,\mathrm{Im}\,($  $) = \sum$ 

$$\Gamma(p^2) = \frac{1}{m^2 - p^2 - i0+} - \lambda^2 |A| \left(\frac{1}{\epsilon} + 2 - \beta \cancel{\tan} \ln\left(\frac{\beta+1}{\beta-1}\right)\right) + O(\lambda^4)$$

A - subtraction constant

$$\beta = \sqrt{1 - \frac{(4m^2 - i0)}{p^2}}$$

Gives the 'spectral function' from the Im part

$$\rho(p^2) = Z(\lambda)\,\delta(p^2 - m^2) + \lambda^2 |A|\, \beta\, \theta(p^2 - 4m^2)$$

one-loop is a universal object. Good to evaluate

$$(\mu^2)^{2-\omega} \frac{\lambda^2}{32\pi^2}\left[\frac{1}{2-\omega} + \psi(1) + 2 + \ln\frac{4\pi\mu^2}{m^2} - \sqrt{1+\frac{4m^2}{p^2}}\,\ln\left\{\frac{\sqrt{1+\frac{4m^2}{p^2}}+1}{\sqrt{1+\frac{4m^2}{p^2}}-1}\right\}\right] + O(2\omega)$$

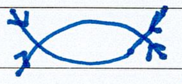
Should be rotated to Minkowski

From Ramond.

Important to do the one-loop integral.
Let us see the explicit example in the book of Ramond (done in Euclidean)

Dimensional regularization.

Ramond fish pp. 120 $\lambda\varphi^4$ theory.



P_3 P_4

$$I = \frac{1}{2}(-\lambda)^2 (\mu^2)^{4-2\omega} \times$$

$$\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{1}{(l^2+m^2)} \frac{1}{((l-p)^2+m^2)}$$

Use Feynman parametrization

$$\frac{1}{(l^2+m^2)((l-p)^2+m^2)} = \int_0^1 \frac{dx}{[l^2+m^2-2l\cdot p(1-x)+p^2(1-x)]^2}$$

Denominator $\quad l'^2+m^2+p^2x(1-x) \qquad l' = l-p(1-x)$

$$I = \frac{\lambda^2}{2}(\mu^2)^{4-2\omega}\int_0^1 dx \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{1}{[l^2+m^2+p^2x(1-x)]^2}$$

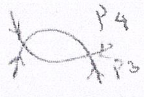
Expanding to $O(2-\omega)$ keeping $(\mu^2)^{4-2\omega}$

$$(\mu^2)^{2-\omega}\frac{\lambda^2}{32\pi^2}\int_0^1 dx\left\{\frac{1}{2-\omega}+\psi(1) - \ln\left(\frac{m^2+p^2x(1-x)}{4\pi\mu^2}\right)\right\}$$

$$\int_0^1 dx \ln\left[1+\frac{4}{a}x(1-x)\right] = -2+\sqrt{1+a}\times \ln\left[\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right]$$

To get final answer. $\qquad$ MS and $\overline{MS}$ schemes.
Rotate to Minkowski.

From Ramond (fish) pp 120.



$\sum p_i = 0$ $\quad \frac{1}{2}(-\lambda)^2 (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}\ell}{(2\pi)^{2\omega}} \frac{1}{(\ell^2+m^2)} \frac{1}{((\ell-p)^2+m^2)}$

use F.P. $\frac{1}{(\ell^2+m^2)((\ell-p)^2+m^2)} = \int_0^1 \frac{dx}{[\ell^2+m^2-2\ell\cdot p(1-x)+p^2(1-x)]^2}$

Denim. $\ell'^2+m^2+p^2x(1-x) \qquad \ell' = \ell - p(1-x)$

$$\frac{\lambda^2}{2}(\mu^2)^{4-2w}\int_0^1 dx \int \frac{d^{2w}}{(2\pi)^{2w}} \frac{1}{[\ell^2+m^2+p^2x(1-x)]^2}$$

(B-16) $$\frac{\lambda^2}{2}(\mu^2)^{4-2w}\int_0^1 dx \, \frac{\Gamma(2-w)}{(4\pi)^w} \frac{1}{[m^2+p^2x(1-x)]^{2-\omega}}$$

Expanding $O(2-\omega)$ keeping $(\mu^2)^{4-2w}$

$$(\mu^2)^{2-w} \frac{\lambda^2}{32\pi^2} \int_0^1 dx \left\{ \frac{1}{2-\omega} + \psi(1) - \ln\left(\frac{m^2+p^2x(1-x)}{4\pi\mu^2}\right) \right\}$$

$$\int_0^1 dx \ln\left[1+\frac{4}{a}x(1-x)\right] = -2 + \sqrt{1+a}\,\ln\left[\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right]$$

$a > 0$

$$(\mu^2)^{2-w}\frac{\lambda^2}{32\pi^2}\left[\frac{1}{2-w} + \psi(1) + 2 + \ln\frac{4\pi\mu^2}{m^2} - \sqrt{1+\frac{4m^2}{p^2}}\,\ln\left\{\frac{\sqrt{1+\frac{4m^2}{p^2}}+1}{\sqrt{1+\frac{4m^2}{p^2}}-1}\right\} + O(2w)\right]$$

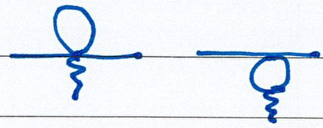
Euclidean should be rotated to Minkowski

check MS vs $\overline{\text{MS}}$ schemes.

We can then turn to 2 form factors in chiral perturbation theory.

E.M. ff of the pion. From Donoghue, Golowich, Holstein

$$H(a) \equiv - \int_0^1 \ln\left(1 - a x (1-x)\right) dx$$



$$G_\pi(q^2) = 1 + \frac{2 L_9^{(2)r}}{F_\pi^2} + \frac{1}{96\pi F_\pi^2}\left[(q^2 - 4m_\pi^2) H(q^2/m_\pi^2) - q^2 \ln \frac{m_\pi^2}{\mu^2} - \frac{q^2}{3} \right]$$

$$q^2 = t$$

The $I=1$, $\ell=1$ partial wave enters.

$$\text{chiral} \quad A(s,t,u) = \left.\frac{s - m_\pi^2}{F_\pi^2}\right| \sim 32\pi \sum f_\ell^I(s) P_\ell\left(1 + \frac{2t}{s-4m^2}\right) \qquad T^I(s,t,u)$$

$$I = 1 \quad : \quad \frac{(t-u)}{s - 4m_\pi^2} \quad \text{in} \quad s\text{-channel}$$

we get here in the t-channel the corresponding contribution.

$$96\pi \rightarrow 32\pi\,(3) \leftarrow (2\ell+1) \text{ with } \ell = 1.$$

Form factor in chiral perturbation theory

Scalar form factor of the pion

relevant had Higgs been very light

$$H \to q\bar{q} \qquad q - \text{light quark.}$$

$$\bar{J}(s) = \frac{1}{16\pi^2}\left\{\sigma(s)\ln\frac{\sigma(s)-1}{\sigma(s)+1} + 2\right\}$$

$$\sigma(s) = \left[1 - \frac{4m_\pi^2}{s}\right]^{1/2}$$

$$\operatorname{Im}\bar{J}(s) = \frac{1}{16\pi}\sigma(s)$$

Verify $\bar{J}(s) = \bar{J}(0) + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\operatorname{Im}\bar{J}(s')}{s'(s'-s)}\,ds'$

$$\Gamma(s) = 1 + \frac{s}{16\pi^2F_\pi^2}(\bar{l}_4 - 1)$$

$f_0^0 \times \Gamma_{tree.}$

$$+ \frac{2s - M_\pi^2}{2F_\pi^2}\bar{J}(s)$$

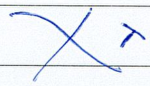
$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2} \qquad T^0(s,t,u) = 3A_s + A_t + A_u$$

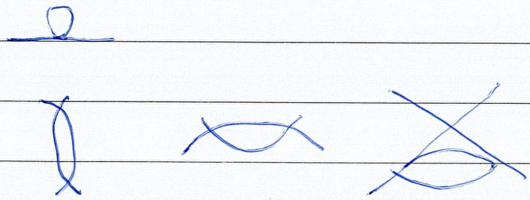
$$= \frac{3s+t+u-5M_\pi^2}{F_\pi^2}$$

$$s+t+u = 4m_\pi^2$$

$$= \frac{2s - M_\pi^2}{F_\pi^2}$$

Scattering amplitude in $\lambda\phi^4$ theory (Cheng & Li)

Chap 2. $\lambda\phi^4$ theory  scattering amp at leading order



After renormalization

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)$$

$$\tilde{\Gamma}(s) = \frac{i\lambda^2}{32\pi^2}\left\{2 + \left(\frac{4\mu^2 - s}{|s|}\right)^{1/2} \times \ln\left[\frac{\sqrt{s} - \sqrt{s-4\mu^2}}{\sqrt{s} + \sqrt{s-4\mu^2}}\right] + i\pi\right\} \qquad s > 4\mu^2$$

Note that $\Gamma_0^{(4)}$ for $s > 4\mu^2$

$\tilde{\Gamma}(t)$ and $\tilde{\Gamma}(u)$ are real.

Hence consistent with unitarity

S-wave scattering.

Combined use of UAC

After form factor need to extend to scattering amplitudes

1. $a + b \rightarrow c + d$

s-channel $s = (p_a + p_b)^2 > 0$; $> 4m^2$ physical region

depends on c.m. energy and scattering angle

2. $\bar{d} + b \rightarrow c + \bar{a}$ t-channel

$t > 0$ physical region $t > 4m^2$

3. $\bar{c} + b \rightarrow \bar{a} + d$ u-channel

$u > 0$ physical region $u > 4m^2$

In 1 $t < 0$: negative square of momentum transfer

Physical regions are non-overlapping

Inter-relate amplitudes (crossing)

Combining analyticity, unitarity and crossing of partial wave amplitudes $\longrightarrow$ asymptotic bounds

Forward scattering $|T(s, \cos\theta = 1)| < \text{cons}\ s(\log s)^2$
$s \to \infty$

Regge theory – half-page introduction

Profound implications

Let us consider having 2 single particle pole terms

M.R. $$T(s, \cos\theta) = \beta_1 \frac{P_{\ell_1}(\cos\theta)}{s_1 - s} + \beta_2 \frac{P_{\ell_2}(\cos\theta)}{s_2 - s} + F(s, \cos\theta)$$

F all other singularities imposed by unitarity etc.

So Froissart bound violated $P_\ell(\cos\theta) \sim t^\ell$
$\cos\theta \to \infty$

So F has to cancel.

spin $J > 1$ cannot be expressed as poles

Regge $$T(s, \cos\theta) = \beta_s \frac{P_{\alpha(s)}(\cos\theta)}{\sin \pi\alpha(s)} + B(s, \cos\theta)$$

NR $\to$ rel.

$\alpha(s)$ – trajectory complex – ℓ plane

$\ell = \alpha(s)$ $\ell \in \mathbb{Z}$ bound state poles

$B \to 0$ as $\cos\theta \to \infty$

Links many particles together. $s > 0$ passes through many poles

Schwarz Reflection Principle and Kanazawa-Sugawara theorem.

From J. A. Oller 'A brief Introduction ...'

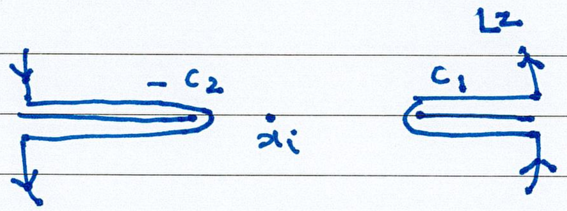
The Schwarz reflection principle states that a function $f(z)$ of a complex variable: $f(z) \in \mathbb{R}$ in a finite segment Γ of the real axis, then

$$f(z) = f(z^*)^*$$

in a domain D in complex z plane if (i) $\Gamma \subset D$ and (ii) $f(z)$ is analytic in D.

The Sugawara - Kanazawa theorem

$f(z)$ analytic everywhere in complex z plane



except for 2 cuts and poles x_i between them.

Assume: 1) $f(z)$ has finite limit $f(\infty \pm i\varepsilon)$ as $z \to \infty \pm i\varepsilon$ along the c_1 cut RHC $\varepsilon \to 0^+$

2) possible div. of $f(z)$ less strong than a finite power $N \geq 1$ of z

3) $f(z)$ has definite but (not necessarily finite) limits for $z \to -\infty \pm i\varepsilon$ along c_2 cut LHC

$$\lim_{z \to \infty} f(z) = f(\infty + i\varepsilon) \quad \operatorname{Im} z > 0$$

$$\lim_{z \to \infty} f(z) = f(\infty - i\varepsilon) \qquad \operatorname{Im} z < 0$$

and admits the dispersion relation

$$f(z) = \sum \frac{R_i}{z - x_i} + \frac{1}{\pi}\left(\left(\int_{c_1}^{\infty} + \int_{-\infty_2}^{-c_2} \right) \frac{\Delta f(x)}{x - z}\, dx \right) + \bar{f}(\infty)$$

$$\Delta f(z) = \frac{1}{2i}\left[f(x+i\varepsilon) - f(x-i\varepsilon) \right]$$

$$\bar{f}(x) = \frac{1}{2}\left[f(x+i\varepsilon) + f(x-i\varepsilon) \right]$$

contribution from infinite circle of the contour in terms of boundary values of $f(z)$ at infinity along only one of the cuts.

Why does causality imply analyticity?

LSZ formalism K_i = K.G. Operator

$$S_{fi} = \delta_{fi} + \frac{Z^{-1}}{\sqrt{2\omega_c 2\omega_a}} \frac{1}{(2\pi)^3} \int dx \int dy$$

$$\exp(i p_c y - i p_a x)$$

$$K_x K_y \langle d(p_d) | \theta(y_0 - x_0) [\varphi_a(x), \varphi_c(y)] | b(p_b) \rangle$$

$$[\varphi(x), \varphi(y)] = 0 \qquad (x-y)^2 < 0$$

$(+ - - -)$ space-like

θ fn & vanishing commutator $\Rightarrow$ analytic continuation in p_a and p_b

Consider a simple example

$$\tilde{f}(p) = \int_{-\infty}^{\infty} dx\, e^{ipx} f(x) \qquad f(x) = 0,\ x < 0$$

$$f(x) = \theta(x) f(x)$$

Then the integral converges if $p \in \mathbb{C}$ & $\mathrm{Im}(p) > 0$

$$ipx = ix \,\mathrm{Re}\, p - x \,\mathrm{Im}\, p$$

Has $-$ve real part for $x > 0$, $\mathrm{Im}\, p > 0$

$\tilde{f}(p)$ is analytic in UHP of complex p.

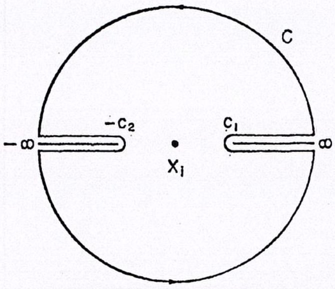


FIG. 1. Singularities of $f(z)$ in the z plane are shown. The two lines are branch cuts and the dot is a pole. The contour line C is the one to which Cauchy's theorem (4) is applied.

definitely simpler than, the one conjectured by Mandelstam.[3]

In the Appendix, we present the most technical part of our proof of the theorem. This part is, however, essential to our proof.

II. STATEMENT OF THE THEOREM

Let $f(z)$ be analytic everywhere in the complex z plane except for two cuts and poles on the real axis as shown in Fig. 1. We assume that the divergence of $f(z)$ at $|z|=\infty$ is not stronger than a large but finite power of $|z|$.[4]

If $f(z)$ has finite limits $f(\infty\pm i\epsilon)$ as $z\to\infty\pm i\epsilon$ along the c_1 cut (ϵ being a positive infinitesimal number), then the limits of $f(z)$ when z approaches infinity in any other direction are

$$\begin{aligned}\lim_{|z|\to\infty} f(z)&=f(\infty+i\epsilon),\ \text{in the upper half-plane,}\\ &=f(\infty-i\epsilon),\ \text{in the lower half-plane,}\end{aligned}\tag{1}$$

provided that $f(z)$ approaches definite (not necessarily finite) limits at $-\infty$ along the c_2 cut. The dispersion relation for $f(z)$ becomes

$$f(z)=\sum_i\frac{R_i}{z-x_i}+\frac{1}{\pi}\left(\int_{c_1}^{\infty}+\int_{-\infty}^{-c_2}\right)\frac{\Delta f(x)dx}{x-z}+\bar{f}(\infty),\tag{2}$$

where

$$\begin{aligned}\Delta f(x)&=(1/2i)[f(x+i\epsilon)-f(x-i\epsilon)],\\ \bar{f}(x)&=(1/2)[f(x+i\epsilon)+f(x-i\epsilon)],\end{aligned}\tag{3}$$

are respectively, the absorptive and dispersive parts of $f(z)$ when z approaches real x in the upper half plane and R_i is the residue at the pole at x_i. It is stressed that the behavior at the end of either cut is sufficient to yield (1) and (2). We do not have to know the limits of the $f(z)$ along both cuts simultaneously.

The proof is given in Sec. III and in the Appendix. Our proof is complete in so far as $f(z)$ satisfies the conditions implied by (9), (40), and (42) below. It is possible that the theorem is correct without these conditions. These conditions are already sufficiently weak to accommodate virtually all the cases of actual interest in physics.

III. PROOF OF THE THEOREM

The Cauchy integral theorem applied to the contour C of Fig. 1 is

$$f(z)=\sum_i\frac{R_i}{z-x_i}+\frac{1}{\pi}\int_{\text{cuts}}\frac{\Delta f(x)dx}{x-z}+\frac{1}{2\pi i}\int_{\infty}\frac{f(z')dz'}{z'-z},\tag{4}$$

where the second term is the same as the second term of (2) and the last term is the integral over the infinite circle.

Since (4) is correct for any z as long as z is inside C and, therefore, $|z'|>|z|$ in the last term of (4), we can expand $1/(z'-z)$ in a power series of z/z'. Applying the boundedness condition at $|z|=\infty$ to the integrands of the resulting series, we see that this series becomes a finite polynomial in z:

$$\frac{1}{2\pi i}\int_{\infty}\frac{f(z')}{z'-z}dz'=\sum_{n=0}^{N}\frac{z^n}{2\pi}\int_0^{2\pi}\frac{f(z')}{z'^n}d\theta=\sum_{n=0}^{N}a_nz^n,\tag{5}$$

where $z'=|z'|\exp(i\theta)$ and N is some positive integer for which $f(z)/z^N$ becomes at most finite at $|z|=\infty$.

We now see that the contribution from the infinite circle can always be eliminated by introducing $N+1$ subtractions, whereas we would need an infinite number of subtractions if the series in (5) did not terminate. Therefore, our boundedness condition is no more than is necessary in any case.[4]

We can rewrite (4) in a divergenceless form by introducing regulations,

$$\frac{1}{x-z}=\frac{z}{x(x-z)}+\frac{1}{x}=\frac{z^2}{x^2(x-z)}+\frac{z}{x^2}+\frac{1}{x}=\cdots.\tag{6}$$

If we introduce one regulation in the c_1 integral since $\Delta f(\infty)$ is finite and $N+1$ regulations in the c_2 integral and change the sign of x in all the c_2 integrals, we get

$$\begin{aligned}f(z)=\sum_i\frac{R_i}{z-x_i}&+\frac{z}{\pi}\int_{c_1}^{\infty}\frac{\Delta f(x)dx}{x(x-z)}\\ &+(-1)^N\frac{z^{N+1}}{\pi}\int_{c_2}^{\infty}\frac{\Delta f(-x)dx}{x^{N+1}(x+z)}\\ &+\left(a_0+\int_{c_1}^{\infty}\frac{\Delta f(x)dx}{\pi x}-\int_{c_2}^{\infty}\frac{\Delta f(-x)dx}{\pi x}\right)\\ &+\left(a_1+\int_{c_2}^{\infty}\frac{\Delta f(-x)dx}{\pi x^2}\right)z+\cdots\\ &+\left(a_N+(-1)^{N+1}\int_{c_2}^{\infty}\frac{\Delta f(-x)dx}{\pi x^{N+1}}\right)z^N.\end{aligned}\tag{7}$$

[3] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

[4] We do not state that $f(z)$ has no essential singularity at infinity, since the infinite point is not isolated in this case and the term, essential singularity, does not apply to such a point. The boundedness condition assumed here is equivalent to requiring that only a finite number of subtractions is necessary to obtain the exact dispersion relation, as is explained in the third paragraph of Sec. III.