

Constraints on inflation from the gravitational path integral?

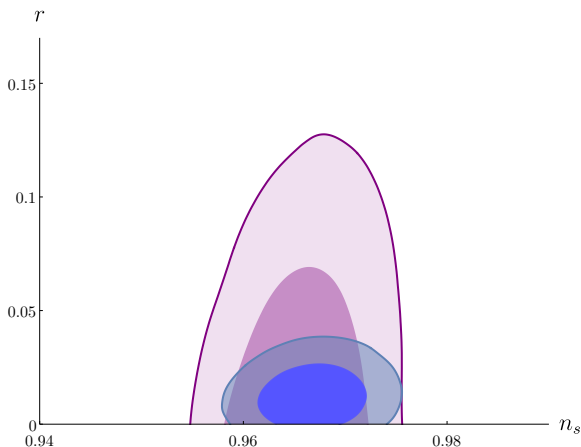
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December 11, 2025

a contribution to Strings & Cosmology Meeting

CMB constraints on precision observables

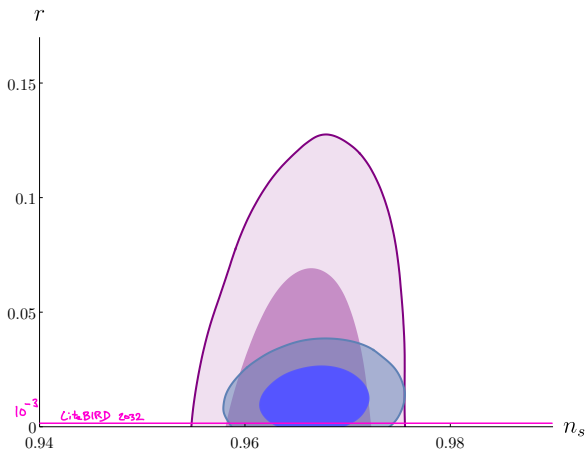
Inflation, like any other theory, requires some parameters as input that need to be measured



[Planck + BICEP/Keck + BAO, 2018]

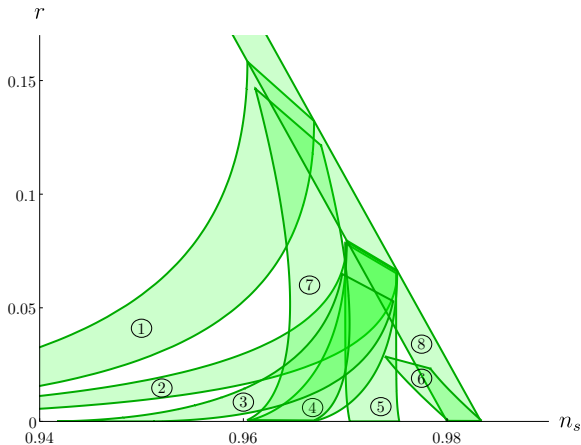
CMB constraints on precision observables

There are exciting near-future experimental prospects (hopefully)



A zoo of models

On the theory side, there are many models. Furthermore, it is far from clear that measuring (n_s, r, \dots) will tell us what mechanism drove inflation



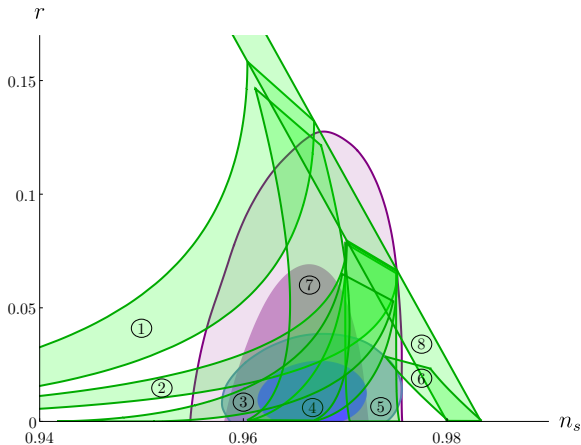
A zoo of models

$$V(\phi) = c_n \cdot \textcircled{n}$$

- ① $1 + \cos(\phi/f)$
- ② $1 - \phi^2/\mu^2$
- ③ $1 - \phi^4/\mu^4$
- ④ $1 - \exp(-q\phi)$
- ⑤ $1 - \mu^2/\phi^2$
- ⑥ $1 + \alpha \log \phi$
- ⑦ $[1 - \exp(-\sqrt{2}\phi/\sqrt{3\alpha})]^2$
- ⑧ ϕ^p

In each model the parameter is varied in some interval, e.g. $p \in [1/2, 7/2]$, and c_n is fixed so $\Delta T/T \sim 10^{-5}$

A zoo of models



We would benefit from (more) guide(s) of the theory landscape

Our motivation

Is there a *theoretical* prior on inflation models?

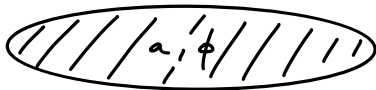
Are some initial conditions for inflation preferred, or perhaps ruled out?

Some approaches:

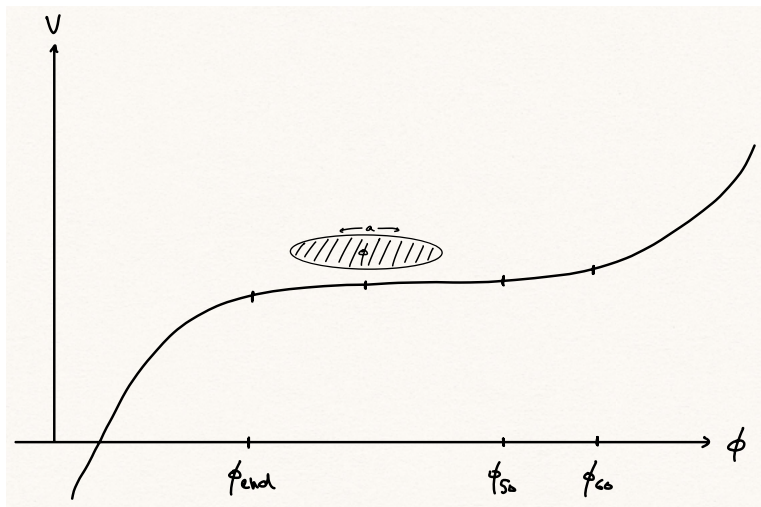
1. Top \rightarrow down: study string inflation
2. Bottom \rightarrow up: not every EFT has a good UV completion
3. **Gravitational path integral**

The GPI of inflation

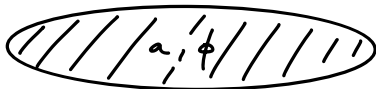
We will consider a time slice of an inflationary history for a closed universe, which has some size a and is covered by a scalar field at value ϕ



The GPI of inflation



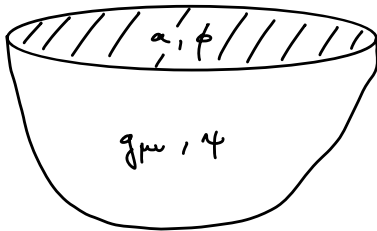
The GPI of inflation



The quantum amplitude for this slice to occur is $\Psi(a, \phi)$, with probability $|\Psi(a, \phi)|^2$. If Ψ is a semiclassical state, the momenta π_a, π_ϕ are also fixed to high accuracy, so then this can be viewed as a probability of initial (or final) conditions for classical evolution

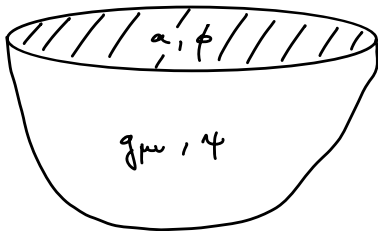
The GPI of inflation

A proposal [Hartle and Hawking '83] for $\Psi(a, \phi)$, in the limit $G_N \rightarrow 0$, is by having (a, ϕ) be the boundary values of a regular solution to the equations of motion, on a manifold with only one boundary, with amplitude set by the classical action



$$\begin{aligned}\Psi_{\text{HH}}(a, \phi) &\sim \sum_{M: \partial M = S^3} \int_{(g_{\mu\nu}, \psi)|_{\partial M = (a, \phi)}} \mathcal{D}g_{\mu\nu} \mathcal{D}\psi e^{iS[g, \psi; G_N]} \\ &\sim \sum_J P_J e^{iS[g_J, \psi_J; G_N]} [1 + \mathcal{O}(G_N)] \quad \text{as } G_N \rightarrow 0\end{aligned}$$

The GPI of inflation



This is certainly not the only possibility, but it has some generality to it, and some mathematical elegance.

“The boundary conditions for the universe are that there are none”

The GPI of inflation

Standard inflationary correlator calculations work in the same way [Maldacena '02, '24]. In flat slicing:

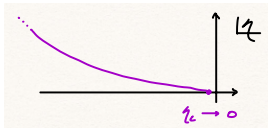
- Spatial metric $d\Sigma^2 = a^2 e^{2\zeta(\mathbf{x})} d\mathbf{x}^2$, we want $\Psi_0[\zeta(\mathbf{x})]$ for $|\zeta| \ll 1$
- Write $ds^2 = -N^2 dt^2 + a(t)^2 e^{2\tilde{\zeta}(t,\mathbf{x})} \delta_{ij} (N^i dt + dx^i) (N^j dt + dx^j)$, $\phi(t)$
Then (\dots)

$$S[\tilde{\zeta}] = \int d\eta d^3\mathbf{x} a(\eta)^2 \varepsilon(\eta) \left(\tilde{\zeta}'^2 - (\nabla \tilde{\zeta})^2 \right) + \mathcal{O}(\tilde{\zeta}^3)$$

where $\varepsilon = -\dot{H}/H^2$, $H = \dot{a}/a$. Conformal time: $dt = a d\eta$

- Tree-level approximation obtained by action of *complex* classical solution with “outgoing” boundary conditions:

$$|\Psi_0[\zeta]|^2 \sim \exp \left(- \int_{\mathbf{k}} \frac{2\varepsilon_*}{H_*^2} k^3 \zeta_{-\mathbf{k}} \zeta_{\mathbf{k}} + \dots \right), \quad k_{\text{phys}} = \frac{k}{a_*} \sim H_*$$

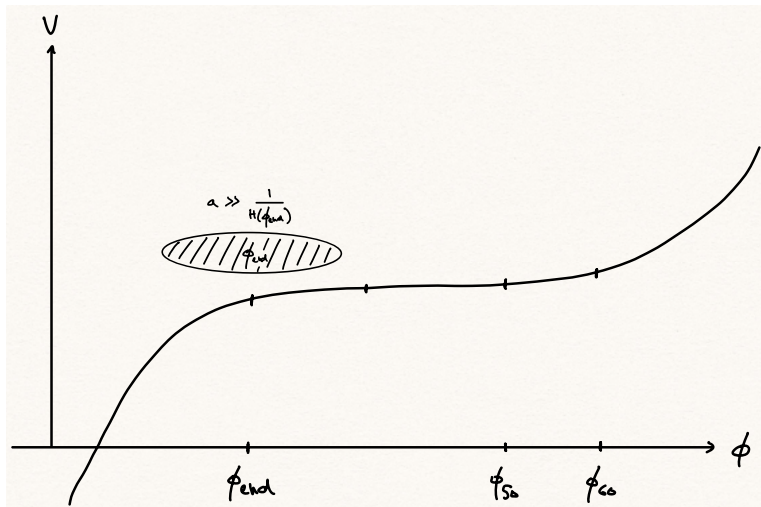


The GPI of inflation

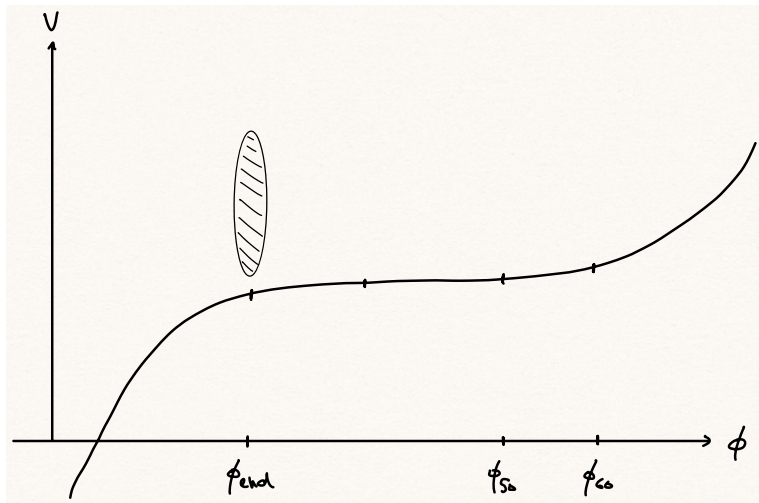
↪

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle = \frac{\delta(\mathbf{k} + \mathbf{k}')}{k^3} A_s \left(\frac{k}{k_*} \right)^{n_s - 1}, \quad n_s = 1 - 2\varepsilon_* - \left. \frac{\dot{\varepsilon}}{\varepsilon H} \right|_*$$

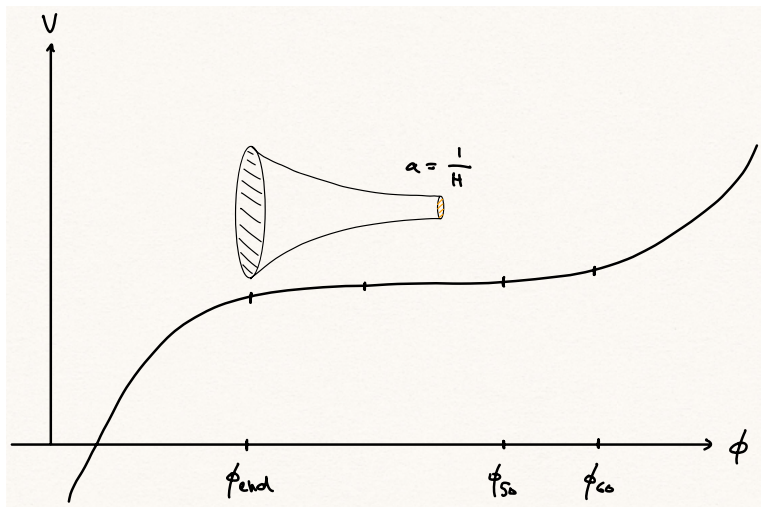
Semiclassical limit: saddle points



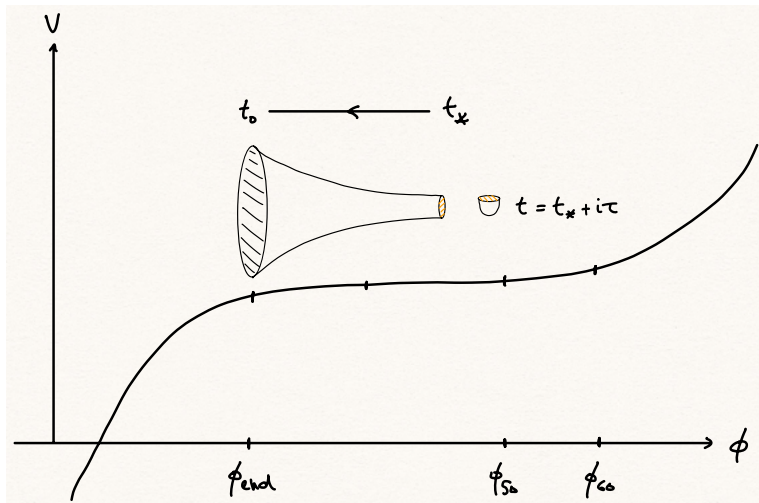
Semiclassical limit: saddle points



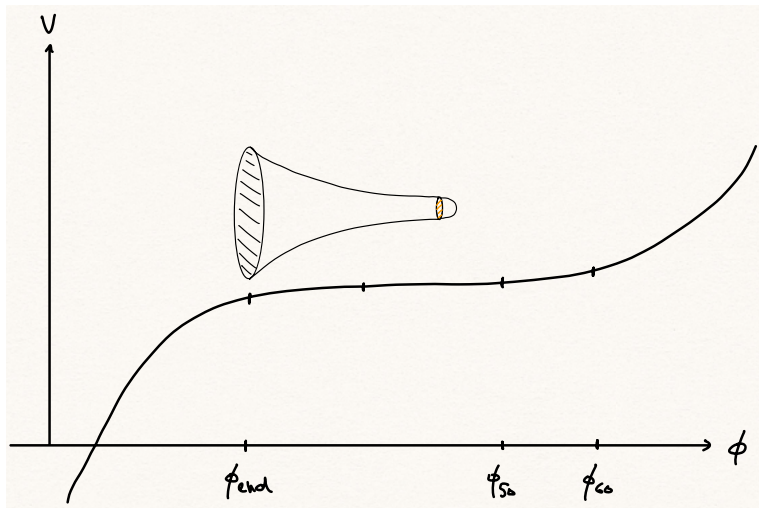
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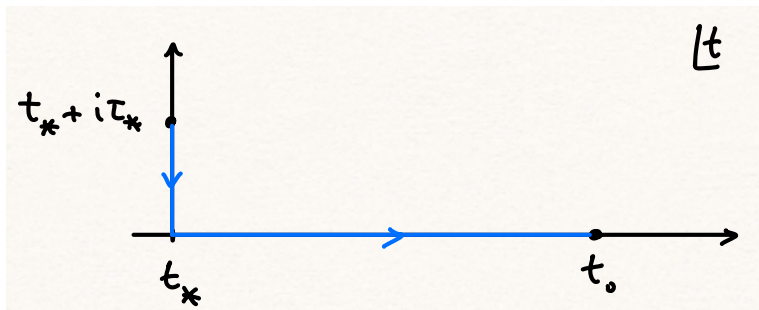
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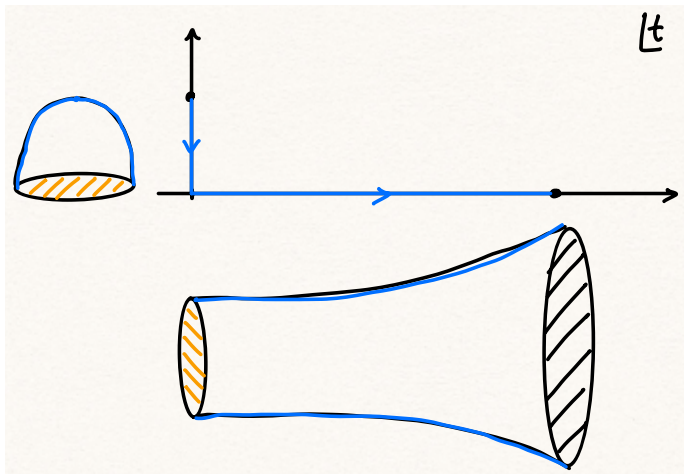


Semiclassical limit: saddle points



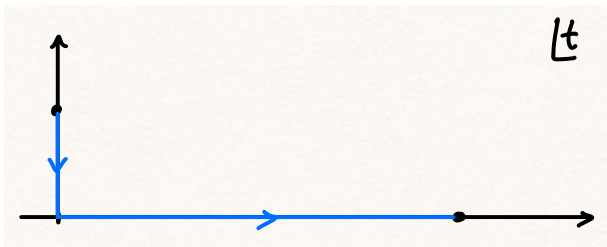
$$ds^2 = -dt^2 + a(t)^2 d\Omega_3^2, \quad \phi(t)$$

Semiclassical limit: saddle points



$$a(t) \approx \frac{1}{H_*} \cosh(H_* t), \quad \phi(t) \approx \phi_*$$

The saddles are complex



$$a(t) = \frac{1}{H_*} \cosh(H_* t) [1 + \varepsilon_* \gamma(t) + \mathcal{O}(\varepsilon^2)] , \quad \phi(t) = \phi_* + \sqrt{2\varepsilon_*} \varphi(t) + \dots$$

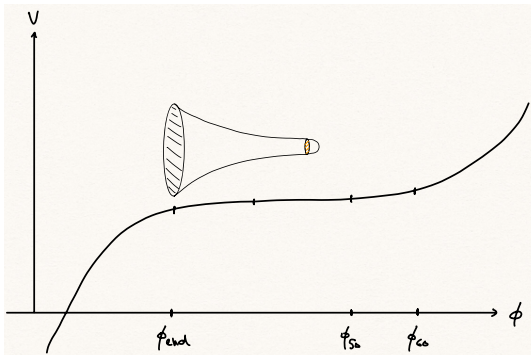
$$\begin{aligned} \varphi(t) &= \frac{1 + i \sinh t}{\cosh^2 t} - \log(1 - i \sinh t) - \frac{i\pi}{2} \\ &= -H_* t + \log 2 + 3e^{-2H_* t} - \frac{16i}{3} e^{-3H_* t} + \dots \quad \text{as } t \rightarrow \infty \end{aligned}$$

[Maldacena '24] [OJ '20, '24]

A puzzle

$$\begin{aligned} |\Psi_{\text{HH}}(a, \phi)|^2 &\sim \exp(-2 \text{Im} S) \\ &= \exp\left(\frac{\pi}{G_N H_*^2}\right) \end{aligned}$$

here H_* is the Hubble scale *at the beginning of inflation* ($aH = 1$)



A puzzle

$$\begin{aligned} |\Psi_{\text{HH}}(a, \phi)|^2 &\sim \exp(-2 \text{Im} S) \\ &= \exp\left(\frac{\pi}{G_N H_*^2}\right) \end{aligned}$$

This formula favors a small amount of inflation, with a huge pressure:

$$\begin{aligned} \frac{|\Psi_{\text{HH}}(a, \phi)|^2}{|\Psi_{\text{HH}}(a \times e, \phi)|^2} &= \exp\left[\frac{\pi}{G_N} \left(\frac{1}{H_a^2} - \frac{1}{H_{ea}^2}\right)\right] \\ &\approx \exp\left(\frac{2\pi}{G_N} \frac{\epsilon}{H_a^2}\right) \\ &\sim \exp\left(\frac{c}{A_s}\right) \sim e^{10^9} \end{aligned}$$

A puzzle

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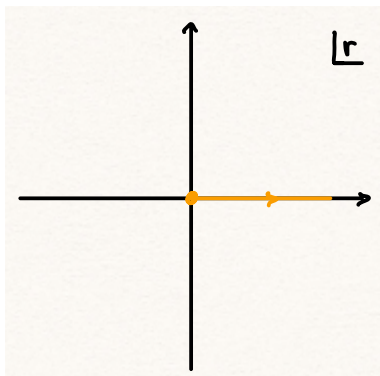
This is a *puzzle* because the Hartle-Hawking proposal gives correct predictions for small inhomogeneous fluctuations (e.g. $\zeta_{\mathbf{k}}$) around inflationary background. But it gives the wrong answer for the zero mode $\mathbf{k} = \mathbf{0}$! [Maldacena '24]

We will not solve this puzzle. Instead we will assume lots of inflation happened (say, 60 e-folds). The question we're interested in is: which inflation?

C-metrics are weird

Consider the flat metric on \mathbb{R}^{d+1} in spherical coordinates

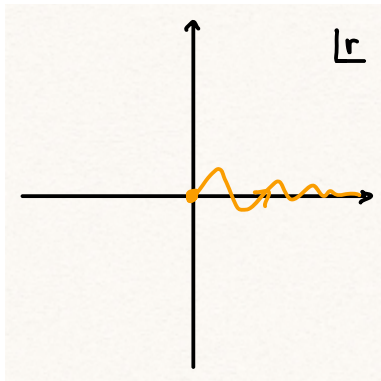
$$ds^2 = dr^2 + r^2 d\Omega_d^2$$



\mathbb{C} -metrics are weird

We can deform this into the complex r -plane by letting $r = \gamma(\ell)$, so

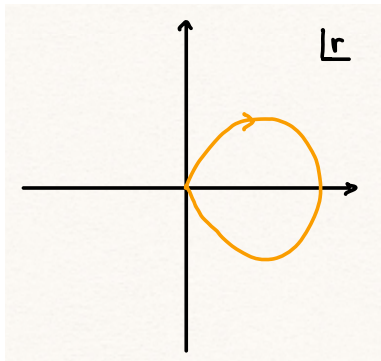
$$ds^2 = \gamma'(\ell)^2 d\ell^2 + r(\gamma(\ell))^2 d\Omega_d^2$$



This is still topologically \mathbb{R}^{d+1}

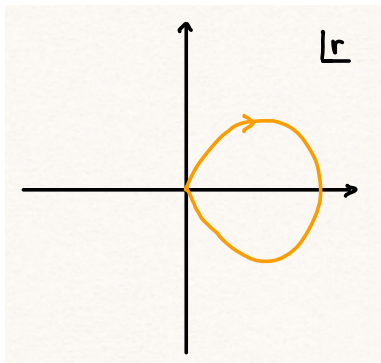
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$$ds^2 = \gamma'(\ell)^2 d\ell^2 + r(\gamma(\ell))^2 d\Omega_d^2$$



Now this is $\cong S^{d+1}$, but it's complex

\mathbb{C} -metrics are weird



This is a flat metric ($R_{\mu\nu\rho\sigma} = 0$) on the sphere. It does not satisfy Gauss-Bonnet for $d = 1$, $\int R = 0 \neq 8\pi$

But some are probably OK

Let's consider a general complex geometry (g, M) and put a massive probe scalar field on it. Let's assume we can compute its correlation functions by performing a path integral over *real* field configurations

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n) \rangle = \int_{C_{\mathbb{R}}(M)} \mathcal{D}\phi \phi(x_1)\cdots\phi(x_n) e^{-S[\phi;g]}$$

where

$$S = \frac{1}{2} \int_M d^D x \sqrt{\det g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2)$$

This manifestly converges if, for all $\phi \in C_{\mathbb{R}}(M)$ and $x \in M$,

$$\text{Re} \left[\sqrt{\det g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2) \right] > 0$$

But some are probably OK

We can also consider more general probe, free p -form matter A_p with field strength $F = dA_p$ and ask

$$\operatorname{Re}[F \wedge \star F] > 0 \quad \text{on } M, \text{ for all } A_p, p \in \{-1, 0, \dots, D\}$$

[Kontsevich & Segal '21] [Witten '21]
[Louko & Sorkin '97] [Halliwell & Hartle '90]

The idea is that QFT is well-defined on such backgrounds

KSW for a practical person

Consider a metric g on M , with components $g_{\mu\nu}(x) \in \mathbb{C}$ in your favorite real coordinate basis. KSW is equivalent to passing the following algorithm
[Benetti Genolini, OJ, Murthy; to appear]

KSW for a practical person

Consider a metric g on M , with components $g_{\mu\nu}(x) \in \mathbb{C}$ in your favorite real coordinate basis. KSW is equivalent to passing the following algorithm [Benetti Genolini, OJ, Murthy; to appear]

1. $\det g$ is nowhere negative real on M . Define $\operatorname{Re} \sqrt{\det g(x)} > 0$
2. $A \equiv \operatorname{Re} (\sqrt{\det g} g^{-1})$ is positive definite everywhere on M
- 3.

$$\sum_{i=1}^D |\arg [\lambda_i(g(x)A(x))]| < \pi$$

for all $x \in M$

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This can be used to (numerically) rule out \mathbb{C} -metrics

Recap of KSW idea

We are trying to evaluate a GPI in the semiclassical limit $G_N \rightarrow 0$. We assume that however it is defined, as $G_N \rightarrow 0$ its main contributions will come from the vicinity of solutions to the equations of motion

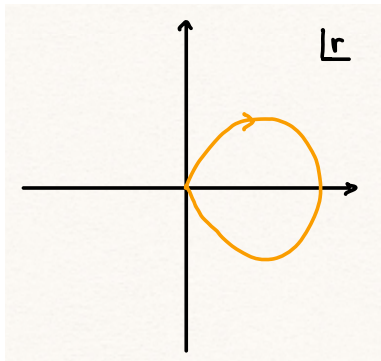
Which saddles are relevant, though, depends on what the theory is (far) away from the saddles. For quantum gravity in our universe we don't know this theory

The idea though is that we may already be able to tell which saddles *are not* relevant, by trying to integrate over other fields while fixing the gravitational part to its saddle point configuration

This may become moot when we understand quantum gravity better

Example 1: flat metric on the sphere

$$ds^2 = \gamma'(\ell)^2 d\ell^2 + r(\gamma(\ell))^2 d\Omega_d^2$$



Now this is $\cong S^{d+1}$, but it's complex. It is not KSW-allowable

Example 2: supersymmetric index

AdS/CFT is a good testing ground for KSW because we can compute the GPI by a partition function of a CFT: we know which saddles are relevant

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In supersymmetric field theories, an interesting quantity is the *index*. It is a grand canonical partition function that takes on a special form (Witten index) due to supersymmetry

$$\begin{aligned} I(\beta, \Omega_i) &= \text{Tr} e^{-\beta(H - \Omega_i J_i)} \\ &= Z_{\text{grav}}(\beta, \Omega_i) \\ &= \int_{\beta, \Omega_i} \mathcal{D}g_{\mu\nu} \mathcal{D}A_\mu^i e^{-S_{\text{grav}}[g, A]} \end{aligned}$$

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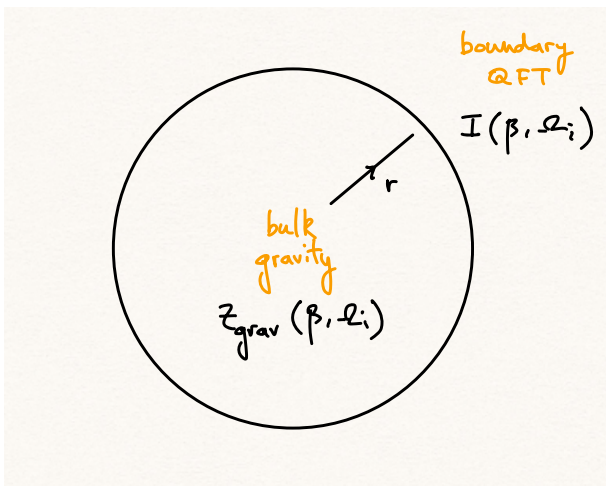
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The idea is to compare, in the semiclassical regime $G_N \rightarrow 0$ keeping $L_{\text{AdS}} = 1$ fixed, the region $R_{\text{QFT}}(\beta, \Omega_i)$ where the boundary trace converges to the region $R_{\text{grav}}(\beta, \Omega_i)$ where the corresponding bulk gravitational saddles are KSW-allowable

Example 2: supersymmetric index



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One can consider [Benetti Genolini, Murthy '25] an $\mathcal{N} = 1$ SCFT on $S^1(R = \beta) \times S^3(R = 1)$ with maximal torus subgroup $U(1) \times U(1) \subset SO(4)$ generated by J_1, J_2 , and a $U(1)_R$ symmetry generated by R

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2 supercharges Q, Q^\dagger are preserved by

$$ds^2 = \beta^2 dt_E^2 + d\vartheta^2 + \sin^2 \vartheta (d\phi - i\beta\Omega_1 dt_E)^2 + \cos^2 \vartheta (d\psi - i\beta\Omega_2 dt_E)^2$$

$$A_R = i\beta\Phi_R dt_E$$

$$t_E \cong t_E + 1, \quad \vartheta \in [0, \pi/2], \quad \phi \cong \phi + 2\pi, \quad \psi \cong \psi + 2\pi$$

for chemical potentials $\Omega_1, \Omega_2, \Phi_R$

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The states obtained by quantizing on S^3 are labeled by $\{H, J_1, J_2, R\}$ and we have

$$\{Q, Q^\dagger\} = H - J_1 - J_2 - \frac{3}{2}R$$

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Anti-periodicity of spinors around the S^1 is achieved by

$$\frac{\beta}{2\pi i} (1 - 2\Phi_R + \Omega_1 + \Omega_2) \in 1 + 2\mathbb{Z}$$

Example 2: supersymmetric index

With this one computes

$$\begin{aligned} I(\beta, \Omega_1, \Omega_2, \Phi_R) &= \text{Tr}_{\mathcal{H}_{S^3}} e^{-\beta(H - \Omega_1 J_1 - \Omega_2 J_2 - \Phi_R R)} \\ &= \text{Tr}_{\mathcal{H}_{S^3}} (-1)^R e^{-\beta\{Q, Q^\dagger\} + 2\pi i \sigma (J_1 + R/2) + 2\pi i \tau (J_2 + R/2)} \end{aligned}$$

where

$$\sigma = \frac{\beta}{2\pi i} (\Omega_1 - 1), \quad \tau = \frac{\beta}{2\pi i} (\Omega_2 - 1)$$

if we set

$$\frac{\beta}{2\pi i} (1 - 2\Phi_R + \Omega_1 + \Omega_2) = -1$$

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I has been computed at large N [CCMM '18, CKKN '18, Benini & Milan '20] by localization techniques, yielding

$$-\log I \sim \frac{2\pi}{27G_N} \frac{\varphi^3}{\sigma\tau}, \quad \varphi = \frac{1}{2} (1 + \sigma + \tau)$$

Example 2: supersymmetric index

On the gravity side: minimal gauged supergravity with bosonic fields g, A and action ($\Lambda = -6/L_{\text{AdS}}^2$)

$$S_{\text{grav}}[g, A] = -\frac{1}{16\pi G_N} \int \left(R + 12 - \frac{1}{3} F^2 \right) \text{vol} + \frac{8i}{27} A \wedge F \wedge F$$

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A supersymmetric solution supports a global Dirac spinor ϵ ,

$$\left[\nabla_\mu - iA_\mu - \frac{1}{2} \gamma_\mu - \frac{i}{12} (\gamma_\mu^{\nu\rho} - 4\delta_\mu^\nu \gamma^\rho) F_{\nu\rho} \right] \epsilon = 0$$

Example 2: supersymmetric index

The relevant gravitational saddles [CCLP '05] are supersymmetric, charged, rotating, asymptotically AdS₅ black holes, with coordinates $\{r, t_E, \theta, \phi, \psi\}$

$$\begin{aligned}
 ds^2 &= \frac{\Delta_\theta [(1+r^2)\beta\rho^2 dt_E + 2qiv]}{\Xi_a \Xi_b \rho^2} dt_E + \frac{2qv\omega}{\rho^2} + \frac{f}{\rho^4} \left(i \frac{\beta\Delta_\theta}{\Xi_a \Xi_b} dt_E + \omega \right)^2 \\
 &+ \frac{r^2 + a^2}{\Xi_a} \sin^2 \theta (d\phi - i\beta\Omega_1 dt_E)^2 + \frac{r^2 + b^2}{\Xi_b} \cos^2 \theta (d\psi - i\beta\Omega_2 dt_E)^2 \\
 &+ \rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) \\
 A &= -\frac{3q}{2\rho^2} \left(i \frac{\beta\Delta_\theta}{\Xi_a \Xi_b} dt_E + \omega \right) + i\beta\Phi_R dt_E
 \end{aligned}$$

Example 2: supersymmetric index

$$\begin{aligned}\nu &= b \sin^2 \theta (d\phi - i\beta\Omega_1 dt_E) + a \cos^2 \theta (d\psi - i\beta\Omega_2 dt_E) \\ \omega &= \frac{a \sin^2 \theta}{\Xi_a} (d\phi - i\beta\Omega_1 dt_E) + \frac{b \cos^2 \theta}{\Xi_b} (d\psi - i\beta\Omega_2 dt_E) \\ \Delta_r &= \frac{(r^2 + a^2)(r^2 + b^2)(1 + r^2) + q^2 + 2abq}{r^2} - 2m \\ \Delta_\theta &= 1 - a^2 \cos^2 \theta - b^2 \sin^2 \theta, \quad \rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta \\ \Xi_a &= 1 - a^2, \quad \Xi_b = 1 - b^2, \quad f = 2m\rho^2 - q^2 + 2abq\rho^2\end{aligned}$$

There are 4 arbitrary parameters (a, b, m, q) in the general solution with $(\beta, \Omega_1, \Omega_2, \Phi_R)$ functions of these. Supersymmetry requires

$$m = q(1 + a + b)$$

so that we are left with 3 parameters. We can reparametrize all of them in terms of

- a
- r_\star : horizon radius of the extremal black hole
- r_+ : horizon radius

Example 2: supersymmetric index

$$b = \frac{r_*^2 - a}{1 + a}, \quad q = -(a - ir_+)(b - ir_+)(1 - ir_+)$$

$$m = \frac{(r_+^2 + a^2)(r_+^2 + b^2)(1 + r_+^2) + q^2 + 2abq}{2r_+^2}$$

$$\frac{\beta}{2\pi} = \frac{(a + ir_+)(b + ir_+)(r_*^2 - ir_+)}{(r_+^2 - r_*^2) [2(1 + a + b)r_+ - i(r_*^2 - 3r_+^2)]}$$

$$\Omega_1 = \frac{(r_+ - i)(r_*^2 - iar_+)}{(r_*^2 - ir_+)(r_+ - ia)}, \quad \Omega_2 = \frac{(r_+ - i)(r_*^2 - ibr_+)}{(r_*^2 - ir_+)(r_+ - ib)}$$

$$\Phi_R = \frac{3(r_+^2 - ir_+)}{2(r_*^2 - ir_+)}$$

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We will restrict ourselves to the regime

$$a^2 < 1, b^2 < 1 \quad (\text{non-degenerate metric}), \quad r_+ > r_* \quad (\text{smooth})$$

(in particular all are real)

Example 2: supersymmetric index

If we change coordinates $(r, \theta) \rightarrow (z, \vartheta)$ via

$$\frac{\Xi_a \sin^2 \vartheta}{z^2} = (r^2 + a^2) \sin^2 \theta, \quad \frac{\Xi_b \cos^2 \vartheta}{z^2} = (r^2 + b^2) \cos^2 \theta$$

then we find

$$\lim_{z \rightarrow 0} ds^2 \sim \frac{dz^2}{z^2} + \frac{1}{z^2} \left[\beta^2 dt_E^2 + d\vartheta^2 + \sin^2 \vartheta (d\phi - i\beta\Omega_1 dt_E)^2 + \cos^2 \vartheta (d\psi - i\beta\Omega_2 dt_E)^2 \right]$$
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The renormalized action of these saddles matches the index computation

$$S_{\text{ren.grav}} = \frac{2\pi}{27G_N} \frac{\varphi^3}{\sigma\tau}, \quad \varphi = \frac{1}{2}(1 + \sigma + \tau)$$

Example 2: supersymmetric index

Boundary side:

$$I(\beta, \Omega_1, \Omega_2) = \text{Tr}_{\mathcal{H}_{S^3}} (-1)^R e^{-\beta\{\mathcal{Q}, \mathcal{Q}^\dagger\} + 2\pi i\sigma(J_1 + R/2) + 2\pi i\tau(J_2 + R/2)}$$

converges in

$$R_{\text{QFT}} = \{\text{Re}\beta > 0, \text{Im}\sigma > 0, \text{Im}\tau > 0\}$$

Example 2: supersymmetric index

Bulk side: for given complex metric $g(a, r_*, r_+)$ on M , we need to examine the KSW-allowability criteria at each point $p = (r, t_E, \theta, \phi, \psi) \in M$

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We find [Benetti Genolini, OJ, Murthy; to appear]

$$R_{\partial\text{grav}} = \left\{ \text{Re } \beta > 0, 0 < \text{Im } \sigma, \text{Im } \tau < \frac{\text{Re } \beta}{\pi} \right\}$$

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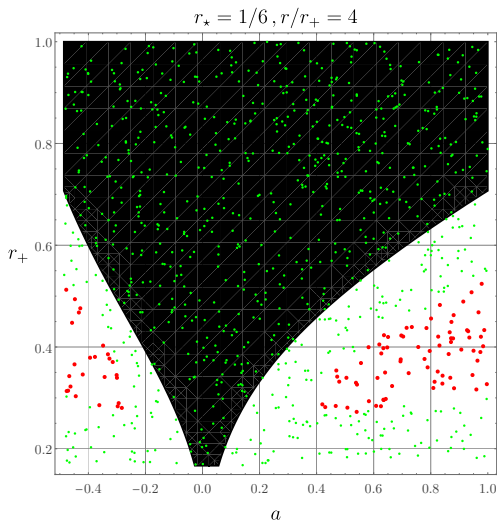
$$R_{\partial\text{grav}} = \left\{ \text{Re } \beta > 0, 0 < \text{Im } \sigma, \text{Im } \tau < \frac{\text{Re } \beta}{\pi} \right\}$$

In the regime $a^2, b^2 < 1, r_+ > r_*$, writing $\beta, \sigma, \tau(a, r_*, r_+)$, this is equivalent to

$$R_{\partial\text{grav}} = \{ \text{Re } \beta > 0, \text{Im } \sigma, \text{Im } \tau > 0 \} = R_{\text{QFT}}$$

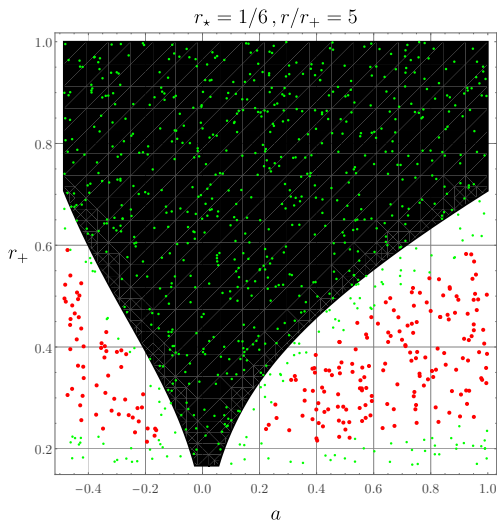
Example 2: supersymmetric index

Finally, we find *numerically* that $r < \infty$ does not introduce more constraints



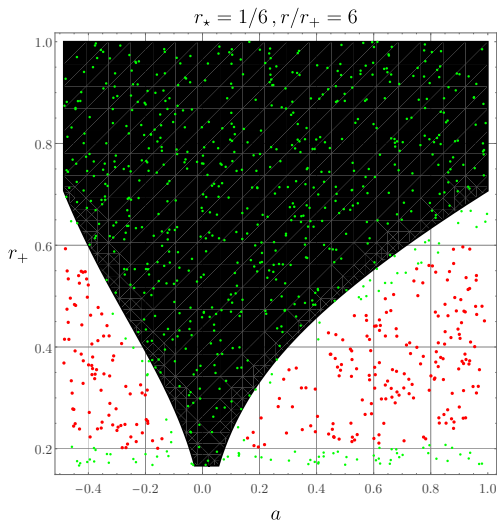
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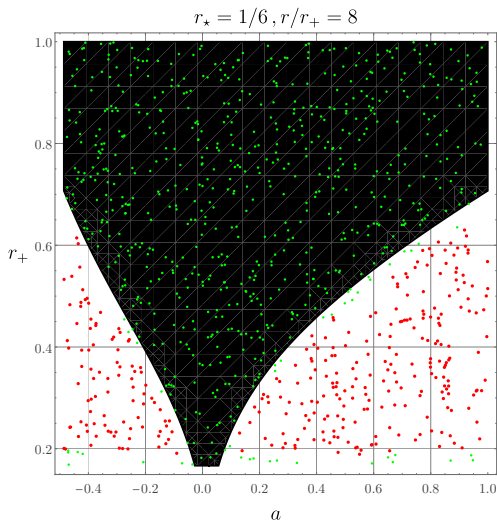
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So

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This also holds for the supersymmetric index in AF_4 , the topologically twisted index in AF_4 and the superconformal index in $AA\text{dS}_4$

Example 3: GPI of inflation

The Hartle-Hawking saddle point has an allowable representation

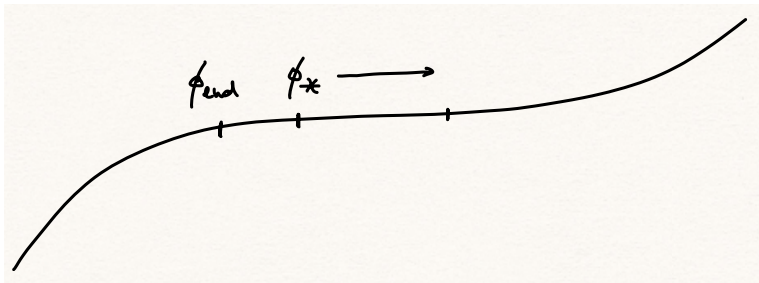


$$\frac{V'(\phi_*)}{V(\phi_*)} \int_{\phi_*}^{\phi_{\text{end}}} |d\phi| \frac{V'(\phi_*)}{V'(\phi)} < 1 + O(\varepsilon_*)$$

[OJ, '24]

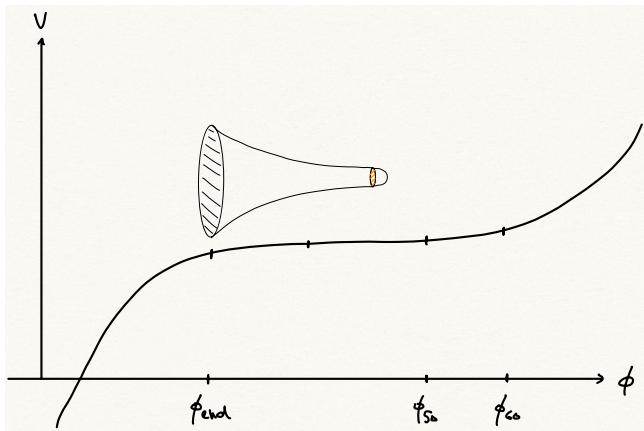
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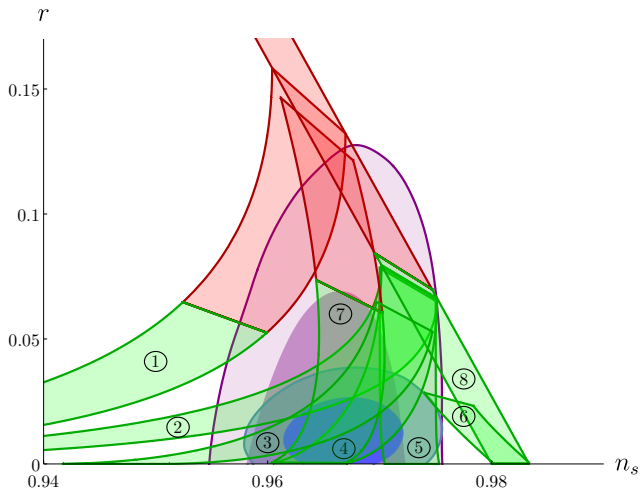


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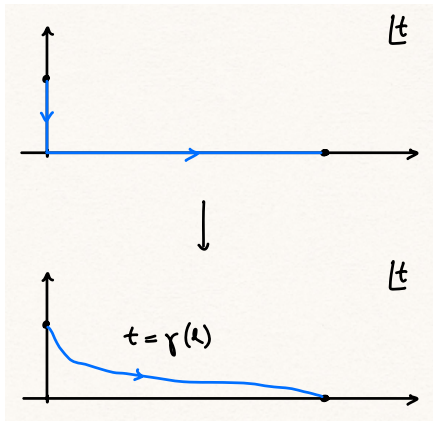


Example 3: GPI of inflation



Thank you for your attention!

Allowable representation of Hartle-Hawking saddle



$$\begin{aligned} ds^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu \\ &= -dt^2 + a(t)^2 d\Omega_3^2 \\ &= -\gamma'(\ell)^2 d\ell^2 + a(\gamma(\ell))^2 d\Omega_3^2 \end{aligned}$$

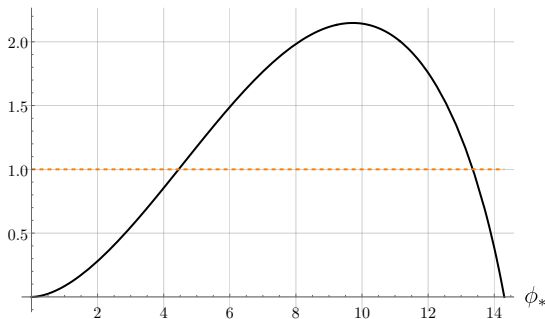
Does there exist a curve γ , connecting the south pole to the boundary, with

$$\sum_{i=1}^4 |\arg \lambda_i(\ell)| < \pi \quad \text{for all } \ell?$$

KSW & large curvature puzzle of the HH state

$$f = 5 : \quad N_e \lesssim 0.9 \text{ or } N_e \gtrsim 42$$

$$\mathcal{A}(\phi_*, \phi_{\text{end}})$$



$$V(\phi) = c_f [1 + \cos(\phi/f)]$$

$$\mathcal{A} = \frac{V'(\phi_*)}{V(\phi_*)} \int_{\phi_*}^{\phi_{\text{end}}} |d\phi| \frac{V'(\phi_*)}{V'(\phi)}$$

Example 4: Hawking-Page

Consider the partition function $Z(\beta) = \text{Tr} e^{-\beta H}$ of a CFT_d on $S^1(r = \beta) \times S^{d-1}(r = 1)$, with holographic dual

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Compute this as $g_{\text{YM}} \rightarrow 0, N \rightarrow \infty$ keeping $\lambda = g_{\text{YM}}^2 N$ fixed (bulk: $G_N \rightarrow 0$ keeping $L_{\text{AdS}} = 1$ fixed) with bulk path integral in AdS_{d+1} with boundary conditions $S^1(r = \beta) \times S_1^{d-1}(r = 1)$

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For any β we have the thermal AdS saddle

$$ds^2 = (1 + r^2)d\tau^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_{d-1}^2, \quad \tau \cong \tau + \beta, \quad r : 0 \rightarrow \infty$$

This solution “fills in” the S^{d-1} , so $M = S^1 \times D^d$

Example 4: Hawking-Page

There are also the AdS-Schwarzschild solutions

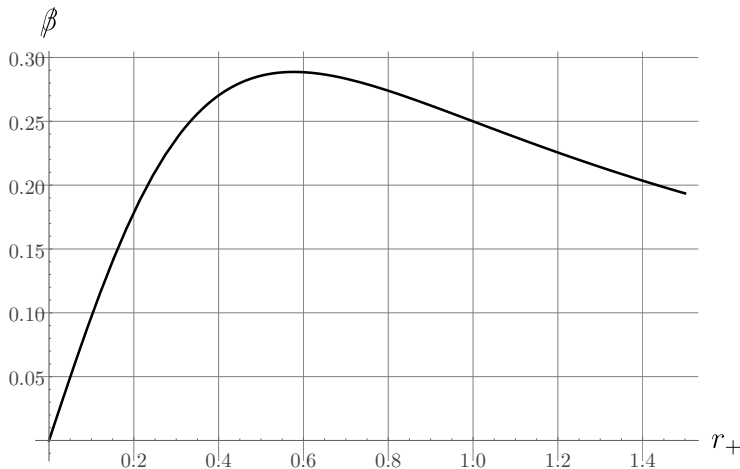
$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \quad \tau \cong \tau + \beta, \quad r : r_+ \rightarrow \infty$$

$$f(r) = 1 + r^2 - (1 + r_+^2) \left(\frac{r_+}{r} \right)^{d-2}$$

These instead fill in the S^1 , so $M = D^2 \times S^{d-1}$. To avoid a conical singularity, r_+ must be related to β via

$$\beta \equiv \frac{\beta}{4\pi} = \frac{r_+}{d-2 + dr_+^2}$$

Example 4: Hawking-Page



$$r_+^{(\pm)} = \frac{1 \pm \sqrt{1 - (\beta/\beta_C)^2}}{2d\beta}, \quad \beta_C = \frac{1}{2\sqrt{d(d-2)}}$$

Example 4: Hawking-Page

Now (naively)

$$\begin{aligned} Z(\beta) &\sim e^{-S_{\text{thermal AdS}}} + e^{-S_{\text{big BH}}} + e^{-S_{\text{small BH}}} + \dots \\ &= e^{-S_{\text{thermal AdS}}} (1 + e^{-\beta\Delta F_{\text{BBH}}} + e^{-\beta\Delta F_{\text{SBH}}} + \dots) \end{aligned}$$

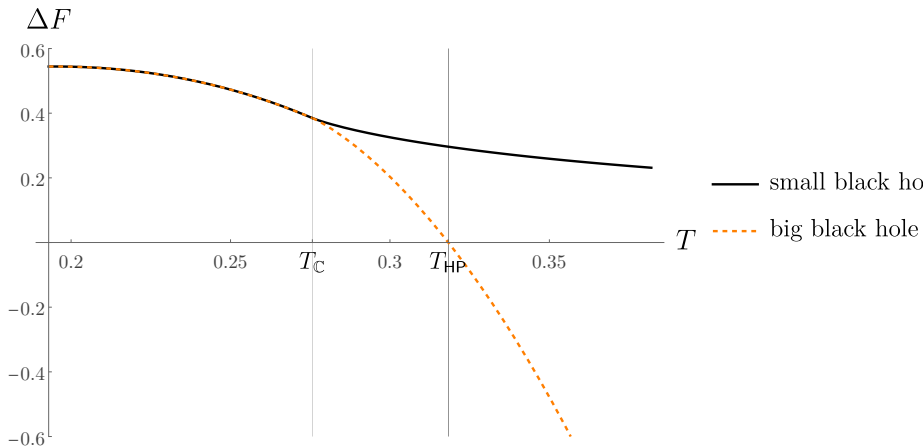
where

$$\begin{aligned} \Delta F &= F_{\text{BH}} - F_{\text{thermal AdS}} \\ &= T (S_{\text{BH}} - S_{\text{thermal AdS}}) \\ &= \frac{\text{vol}(S^{d-1})}{16\pi G_N} r_+^{d-2} (1 - r_+^2) \end{aligned}$$

When $\text{Re } \Delta F < 0$, the black holes (naively) dominate, otherwise it is the thermal AdS

Example 4: Hawking-Page

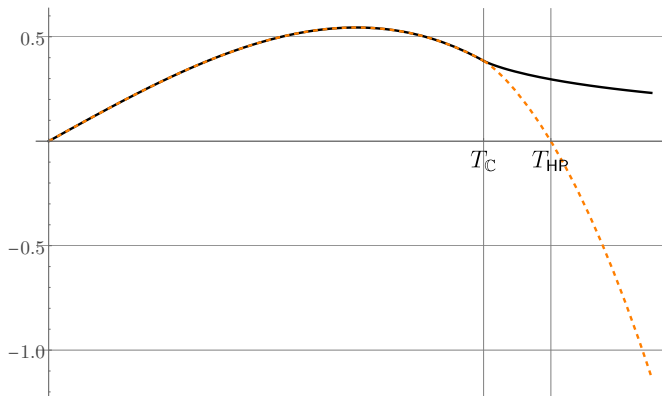
$$d = 3$$



$$\beta_{HP} = \frac{\sqrt{d(d-2)}}{d-1} \beta_C < \beta_C$$

Example 4: Hawking-Page

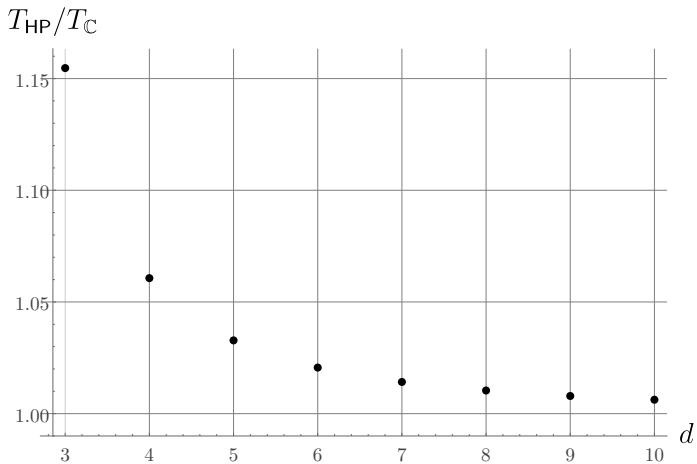
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 ΔF 

T — small black hole

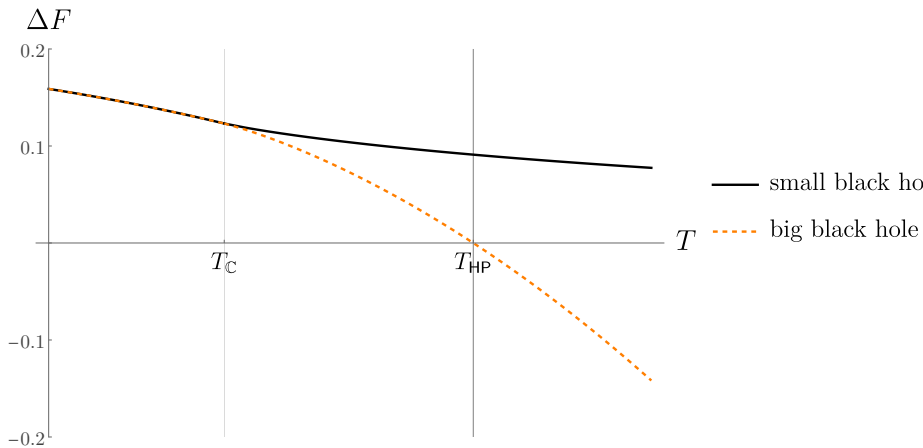
— big black hole

Example 4: Hawking-Page



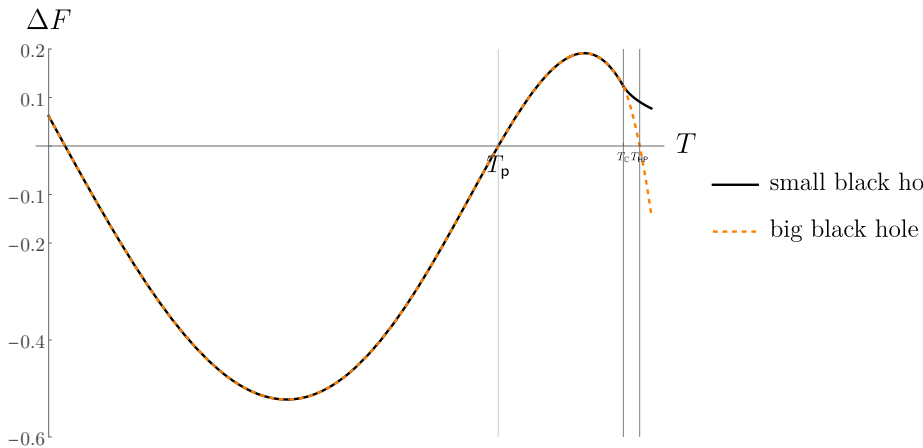
Example 4: Hawking-Page

$$d = 7$$



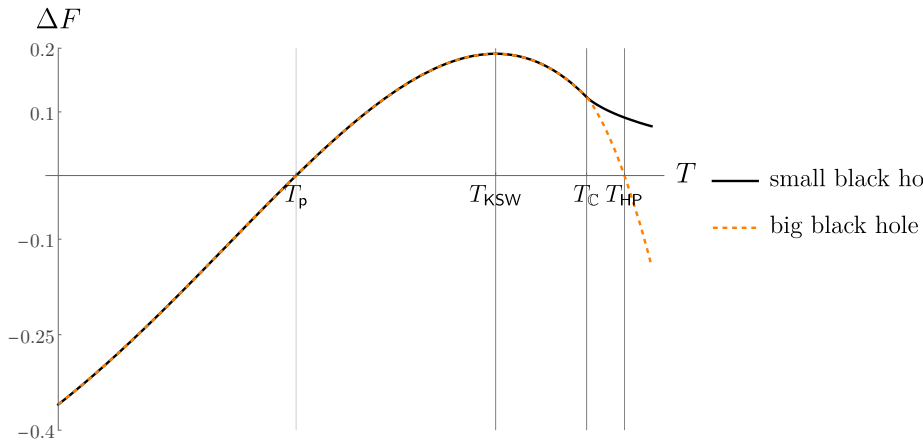
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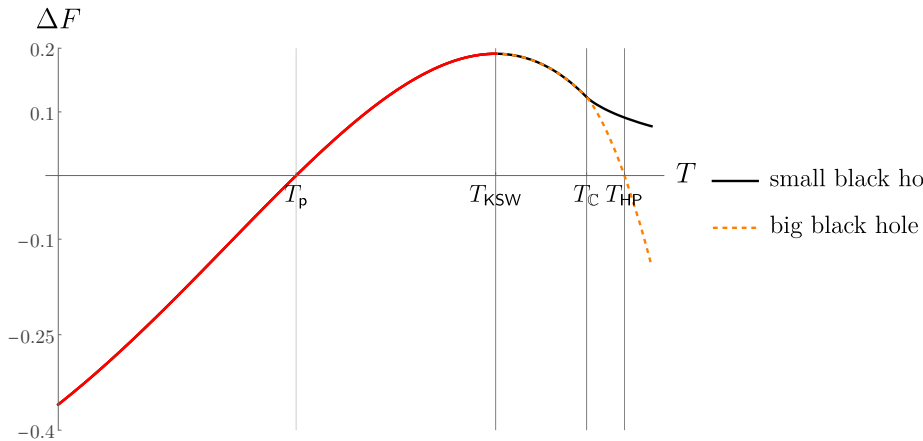
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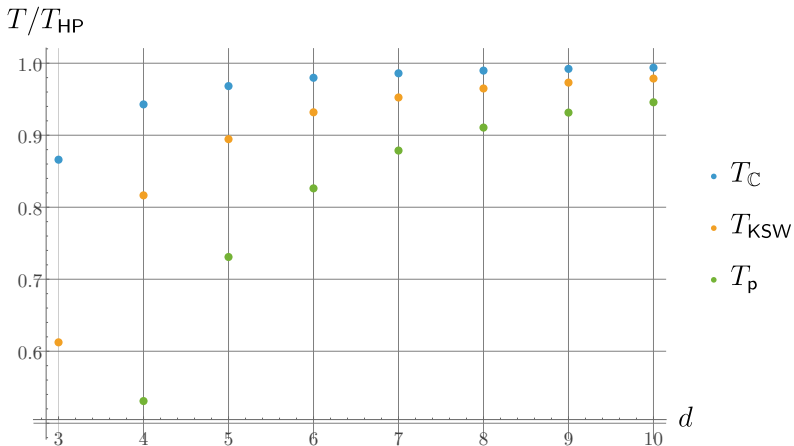


Example 4: Hawking-Page

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Example 4: Hawking-Page



Example 4: Hawking-Page

Actually we have not found T_{KSW} here, but a *lower bound*, which suffices for our purposes. You can get this simply by examining

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2$$

near $r = r_+$. An allowable metric must certainly have

$$(d-1)|\arg r_+^2| < \pi,$$

so r_+ cannot become “too complex”. We find

$$T_{\text{KSW}} > \cos\left(\frac{\pi}{2(d-1)}\right) T_{\text{C}}$$