Joint Bayesian calibration and map-making for intensity mapping experiments

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What this talk is not

Not an end-to-end Bayesian model - by design

Reasons

- Component separation is not necessary/essential for gain/noise calibration
- No reliable ground-truth sky model $\rightarrow$ model dependence becomes a liability

Not: "*We model this way to minimise degeneracy*"

Instead

- Model ***degeneracies*** when they exist
- Model ***ignorance*** with general, safe representations
- Stay true to what we exactly know

What this talk is about

A Bayesian conservative's framework for calibration and map-making

Timestream of Data

System Temperature

$$\underbrace{d(t_a)}_{\text{Timestream of Data}} = \underbrace{G(t_a)}_{\text{Instrumental gain}} \underbrace{T_{\mathrm{sys}}(t_a)}_{\text{System Temperature}} \left(1 + \hat{w}\right),$$

Instrumental gain

White noise: $\hat{w} \sim \mathcal{N}(0, 1/(\mathcal{T}\Delta\nu))$

Key ingredients

- Instrumental gain modelling (smooth component & $1/f$ noise)
- Multiplicative noise treatment (no noise model conditionals)
- Joint inference of smooth gain, $1/f$ noise, and system temperature components
- Experimental setup for demonstration: MeerKAT-type observation in mind

Instrumental Gain Modelling

1/f noise
(stochastic fractional gain)

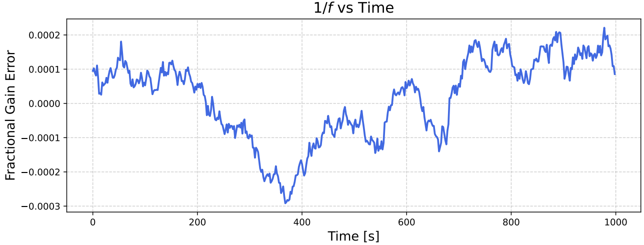
$$G(t_a) = g(t_a)\,(1 + \hat{\epsilon}\,)$$

Smooth gain

$$g(t_a) = \sum_{n=0}^{4} a_n\, P_n(x), \qquad x = \frac{2(t_a - t_{\min})}{t_{\max} - t_{\min}} - 1.$$

Flicker Noise

$$P_{\hat{\epsilon}}(f) = \begin{cases} 0, & |f| < f_c, \\ \left(f_0/|f|\right)^{\alpha}, & |f| \geq f_c. \end{cases}$$



Stochastic Gain Model

$1/f$ noise
(stochastic fractional gain)

$$G(t_a) = g(t_a)\,(1 + \hat{\epsilon})$$

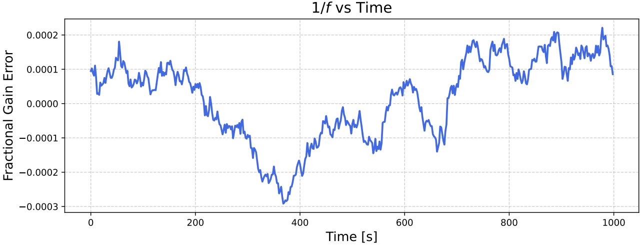
Smooth gain

$$g(t_a) = \sum_{n=0}^{4} a_n\, P_n(x), \qquad x = \frac{2(t_a - t_{\min})}{t_{\max} - t_{\min}} - 1.$$

Flicker Noise **(Complete Model)**

Gaussianity and stationarity.

$$P_{\hat{\epsilon}}(f) = \begin{cases} 0, & |f| < f_c, \\ \left(f_0/|f|\right)^{\alpha}, & |f| \geq f_c. \end{cases}$$

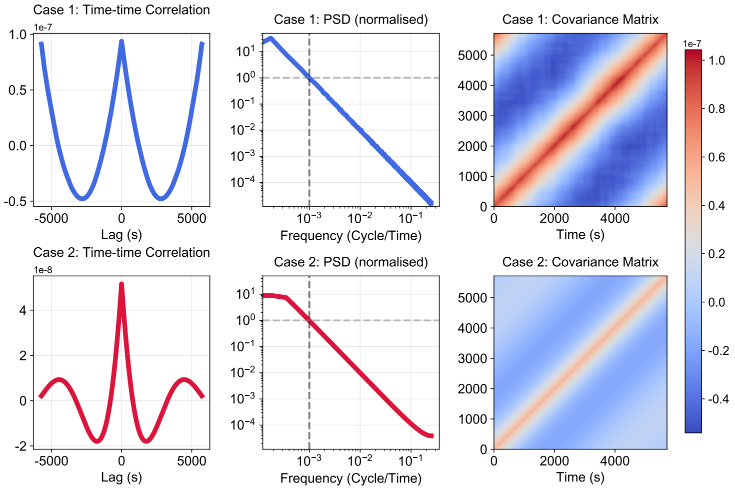


Stochastic Gain Model

But how to implement the $1/f$ noise covariance?

- **Case 1:**
  Diagonal in DFT space

- **Case 2:**
  Diagonal in CFT space



Case 1: Diagonal in FFT space; Case 2: Diagonal in CFT space

Stochastic Gain Model

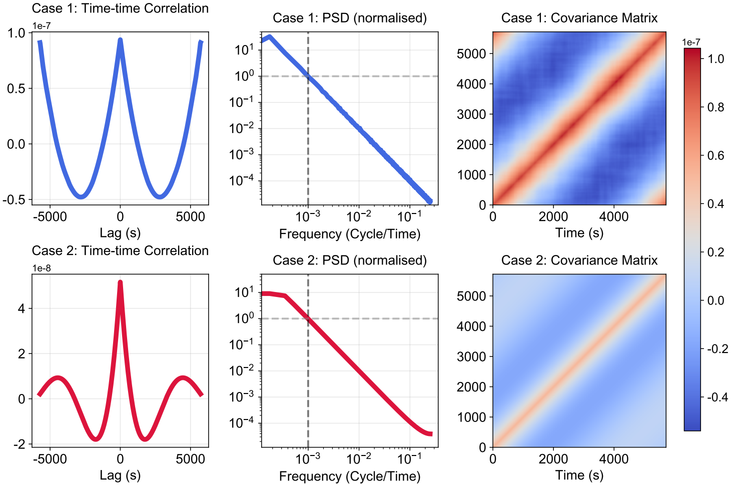
But how to implement the $1/f$ noise modelling?

- **Case 2:**
  Diagonal in CFT space

$$(\mathbf{N}_{\mathrm{corr}})_{aa'} = \xi(t_{a'} - t_a)$$

$$\xi(\tau) = \frac{\Theta_0^\alpha}{\pi\tau}\mathrm{Re}\left[\Gamma(\mu, i\Theta_c)\, e^{-i\frac{\pi}{2}\mu}\right]$$

$$\mu = 1-\alpha, \Theta_0 = \tau f_0, \Theta_c = \tau f_c$$

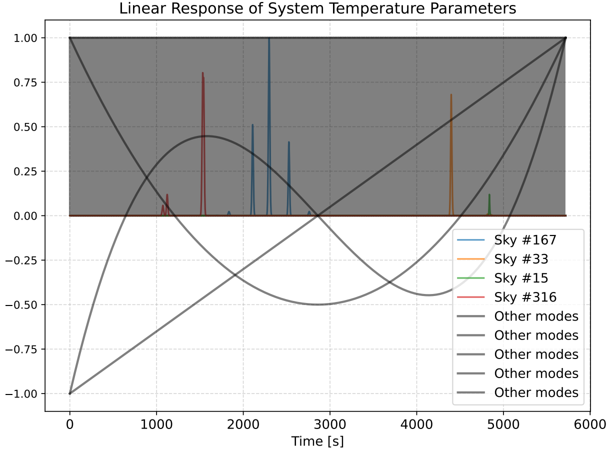


Case 1: Diagonal in FFT space; Case 2: Diagonal in CFT space

System Temperature Model

Sky + Receiver + Ground + Atmospheric + ...

$$T^{\text{sys}}_{j,p}(t_a) = \sum_X B^X_p(t_a) * T^{\text{sky},X} + T^{\text{el}}_p(t_a) + T^{\text{nd}}_{j,p}(t_a) + T^{\text{rec}}_{j,p}$$



Linear Model

$$\vec{T}_{\text{sys}} = \mathbf{U}_{\text{ce}}\,\vec{p}_{\text{ce}} + \mathbf{U}_{\text{res}}\,\vec{p}_{\text{res}}$$

Joint Bayesian Analysis Framework

- The data model can be rewritten as

$$d(t_a) \simeq g(t_a) T_{\rm sys}(t_a) \left[1 + \hat{w} + \hat{\epsilon}\right].$$

- the likelihood function of the data model is given by

$$L(\boldsymbol{n}|g, T_{\rm sys}, \mathbf{N}) = (2\pi)^{-\frac{N}{2}} |\mathbf{N}|^{-\frac{1}{2}} \exp\left\{\left[-\frac{1}{2}\boldsymbol{n}^T \mathbf{N}^{-1} \boldsymbol{n}\right]\right\},$$

  where $\mathbf{N} = \mathbf{N}_w + \mathbf{N}_{\rm corr}$.

- Gibbs sampling steps:

$$\begin{aligned}
g^{(i+1)} &\leftarrow \mathcal{P}_{\rm post}(g|\boldsymbol{d}, T_{\rm sys}^{(i)}, \mathbf{N}^{(i)}) \\
\mathbf{N}^{(i+1)} &\leftarrow \mathcal{P}_{\rm post}(\mathbf{N}|\boldsymbol{d}, T_{\rm sys}^{(i)}, g^{(i+1)}) \\
T_{\rm sys}^{(i+1)} &\leftarrow \mathcal{P}_{\rm post}(T_{\rm sys}|\boldsymbol{d}, g^{(i+1)}, \mathbf{N}^{(i+1)})
\end{aligned}$$

Tractable Implementation for Gain and Tsys samplers

Iterative GLS sampler

- Conditional on noise, the data model can be generally written as

$$\boldsymbol{d} = (\mathbf{U}\boldsymbol{p} + \mu) \circ (1 + \boldsymbol{n}).$$

- Rearrangement - effective additive form:

$$\boldsymbol{d}' = \mathbf{U}\boldsymbol{p} + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \Sigma),$$

where $\boldsymbol{d}' = \boldsymbol{d} - \mu$ and the noise covariance structure is given by:

$$\Sigma = \mathrm{Diag}(\mathbf{U}\boldsymbol{p}+\mu)\,\mathbf{N}\,\mathrm{Diag}(\mathbf{U}\boldsymbol{p}+\mu).$$

Estimating Σ with iterative GLS:

1. Initialize $\boldsymbol{p}^{(0)}$ using OLS:

$$\boldsymbol{p}^{(0)} = (\mathbf{U}^\top\mathbf{U})^{-1}\mathbf{U}^\top\boldsymbol{d}'.$$

2. At iteration k, compute

$$\Sigma^{(k)} = \mathrm{diag}(\mathbf{U}\boldsymbol{p}^{(k)} + \mu)\,\mathbf{N}\,\mathrm{diag}(\mathbf{U}\boldsymbol{p}^{(k)} + \mu).$$

3. Update $\boldsymbol{p}$ by solving:

$$\left[\mathbf{U}^\top\left(\Sigma^{(k)}\right)^{-1}\mathbf{U}\right]\boldsymbol{p}^{(k+1)} = \mathbf{U}^\top\left(\Sigma^{(k)}\right)^{-1}\boldsymbol{d}'.$$

4. Repeat until convergence:

$$\|\boldsymbol{p}^{(k+1)} - \boldsymbol{p}^{(k)}\| < \mathrm{tol}$$

Tractable Implementation for Gain and Tsys samplers

Gaussian Constrained Realisation (GCR) equations

Per TOD set sampling:

$$\left(\mathbf{C}^{-1} + \mathbf{U}^{\top}\Sigma^{-1}\mathbf{U}\right)\boldsymbol{p}_{\mathrm{sample}} = \mathbf{U}^{\top}\left(\Sigma^{-1}\boldsymbol{d}' + \Sigma^{-\frac{1}{2}}\omega\right) + \mathbf{C}^{-1}\bar{\boldsymbol{p}} + \mathbf{C}^{-\frac{1}{2}}\eta,$$

Multiple TOD sets joint fitting:

$$\left(\mathbf{C}^{-1} + \sum_{j}\mathbf{U}_j{}^{\top}\Sigma_j^{-1}\mathbf{U}_j\right)\boldsymbol{p}_{\mathrm{sample}} = \sum_{j}\mathbf{U}_j{}^{\top}\left[\Sigma_j^{-1}\boldsymbol{d}_j' + \Sigma_j^{-\frac{1}{2}}\omega_j\right] + \mathbf{C}^{-1}\bar{\boldsymbol{p}} + \mathbf{C}^{-\frac{1}{2}}\eta,$$

Tractable Implementation for Sampling $1/f$ Parameters

CoMat: An $O(n^2)$ implementation

$$\mathbf{N}_{\mathrm{corr}} = \begin{pmatrix} \xi_0 & \xi_1 & \xi_2 & \cdots & \xi_{n-1} \\ \xi_1 & \xi_0 & \xi_1 & \cdots & \xi_{n-2} \\ \xi_2 & \xi_1 & \xi_0 & \cdots & \xi_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_{n-1} & \xi_{n-2} & \xi_{n-3} & \cdots & \xi_0 \end{pmatrix}$$

- Recursive Determinant

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B).$$

- Levinson-Durbin Recursion

$$\mathbf{T}^n \begin{bmatrix} 0 \\ \vec{b}^{\,n-1} \end{bmatrix} = \begin{bmatrix} t_0 & \ldots & t_{-n+2} & t_{-n+1} \\ \vdots & & & \\ t_{n-2} & & \mathbf{T}^{n-1} & \\ t_{n-1} & & & \end{bmatrix} \begin{bmatrix} 0 \\ \\ \vec{b}^{\,n-1} \end{bmatrix} = \begin{bmatrix} \epsilon_b^n \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

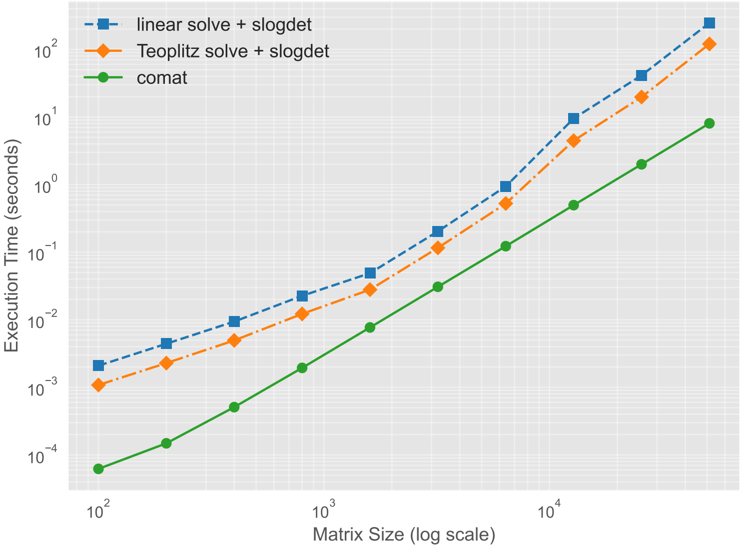
- Perturbative Scenario

$$(\Lambda + \mathbf{P})^{-1} \simeq \Lambda^{-1} - \sum_{k=0}^{m-1} \Lambda^{-1}\mathbf{P}\left[-\mathbf{P}\Lambda^{-1}\right]^k \Lambda^{-1}.$$

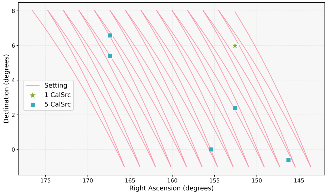
$$\ln[\det(\Lambda + \mathbf{P})] \approx \sum_{k=0}^{m-1} \frac{(-1)^k}{k+1} \mathrm{Tr}\left[\left(\mathbf{P}\Lambda^{-1}\right)^{k+1}\right] + \ln[\det(\Lambda)]$$

Tractable Implementation for Sampling $1/f$ Parameters

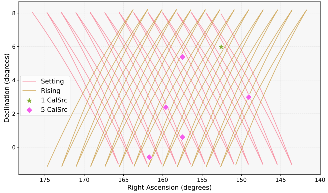
CoMat



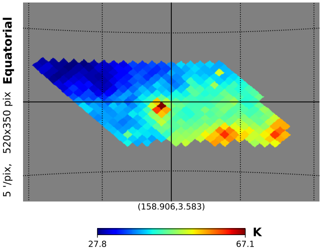
Illustrating Examples
Experimental Setup



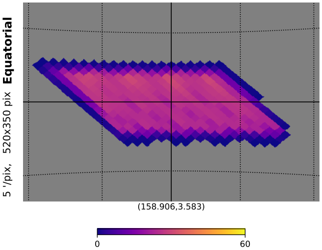
(a) 1×**TOD**: Scan



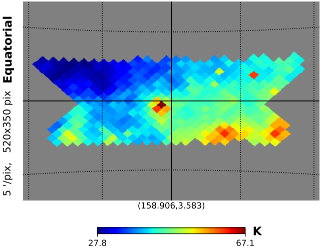
(b) 2×**TOD**: Scan



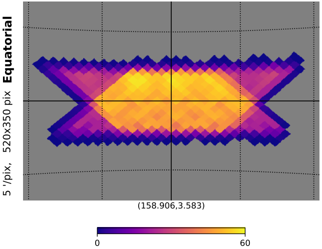
(a) 1×**TOD**: Sky



(a) 1×**TOD**: Integrated beam.



(b) 2×**TOD**: Sky



(b) 2×**TOD**: Integrated beam.

Illustrating Examples
Prior Setup

Smooth Gain Prior

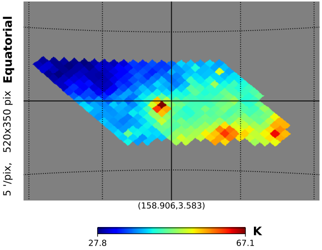
- **DC mode:**
  Prior STD $\sim 20\%$
- **Other modes:**
  Flat priors

$1/f$ Prior

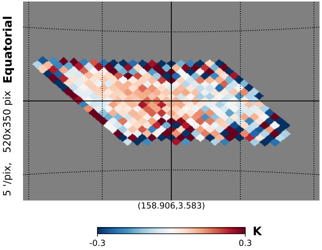
- **Power law index:**
  Flat prior
- **Reference (Knee) frequency:**
  Flat prior

System Temperature Prior (Rough Sky Knowledge + Cal. Src.)

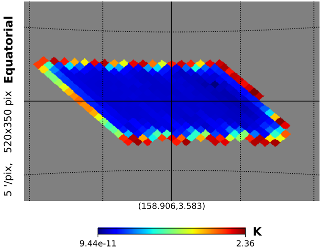
- Prior STD per pixel: 10 K
- Flux scale calibration:
  - Sharp prior on one pixel (“1 CalSrc”)
- Other components (receiver temperature, ground pickup, etc)
  - **No prior!** *If you choose to model ignorance, that’s because ignorance is your prior!*



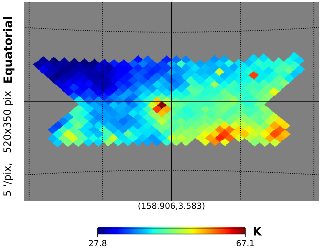
(a) Estimated (1×TOD; 1 CalSrc)



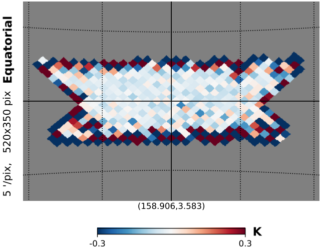
(a) Residual (1×TOD; 1 CalSrc)



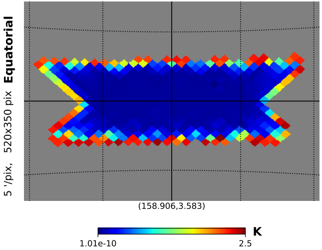
(a) Uncertainty (1×TOD; 1 CalSrc)



(b) Estimated (2×TOD; 1 CalSrc)



(b) Residual (2×TOD; 1 CalSrc)



(b) Uncertainty (2×TOD; 1 CalSrc)

An illustrating Example

Histogram of the residuals across pixels

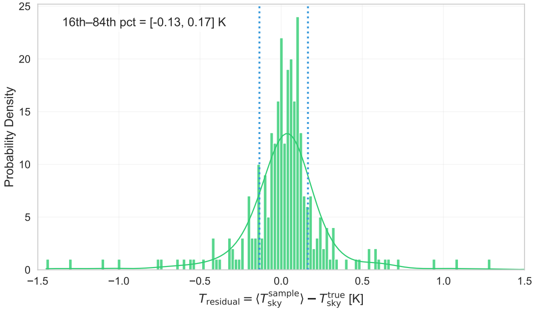


Figure: 1×TOD; 1 CalSrc

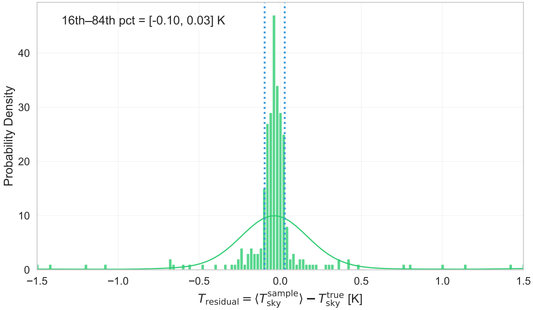


Figure: 2×TOD; 1 CalSrc

CONCLUSION & DISCUSSION

- CONSIDER MODEL IGNORANCE:

  Beyond end-to-end modelling with educated guesses, it's also valuable to explicitly model our ignorance—identifying and parameterizing what we don't know.

- LEVERAGE INTERNAL CONSTRAINTS:

  Design experiments strategically to exploit internal constraints that limit the discriminability between different components in the data model.

- THINK LIKE THE TELESCOPE:

  MeerKAT-like observations can distinguish system temperature components by their differing temporal behaviors—use this perspective to guide modelling.

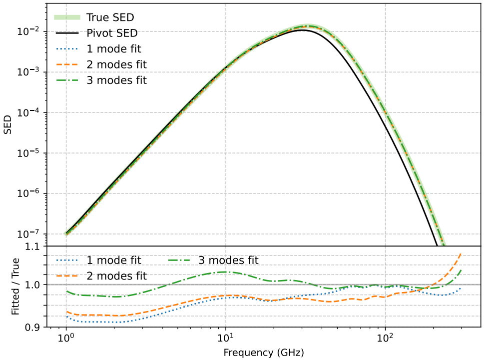
- BAYESIAN MODELLING WITH IGNORANCE SHOULD EVOLVE:

  Building a pipeline is only the beginning—learning and adapting both the prior and the model itself is essential as understanding deepens.

Echoes from Earlier Sessions I

Ad: SpyDust

- an improved and extended Python implementation for modelling spinning dust emission
  Zhang & Chluba JCAP03(2025)038
- A full Stokes implementation with reduced dimensionality (via moment expansion)
  Zhang & Chluba in prep.
- Improved Fokker-Planck treatment - better rotational statistics
  Zhang & Chluba in prep.



Echoes from Earlier Sessions II

“redshift-space distortions” (RSD)

This is CORRECT!

$$P_s(k, \mu) = (1 + \beta^2 \mu(k)^2)^2 P_r(k)$$

- Large scale, but smaller than the curvature scale: Cartesian plane wave synthesis
- $\mu(k)$: defined with the normal mode peculiar velocity
- Scalar perturbations - $\vec{u}(\vec{k}) \parallel \vec{k}$
- $\mu(k)$: NOT directly given by LOS projection of real space peculiar velocity