Lattice calculation of the hadronic light-by-light contribution to the muon $g-2$

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EPS-HEP conference 2025 - Marseille

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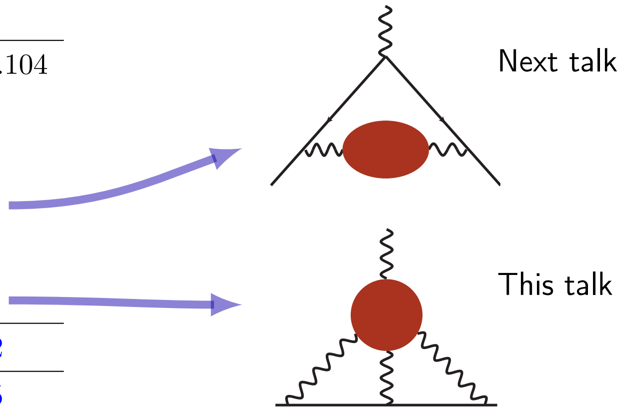
anr agence nationale de la recherche

CPT Centre de Physique Théorique

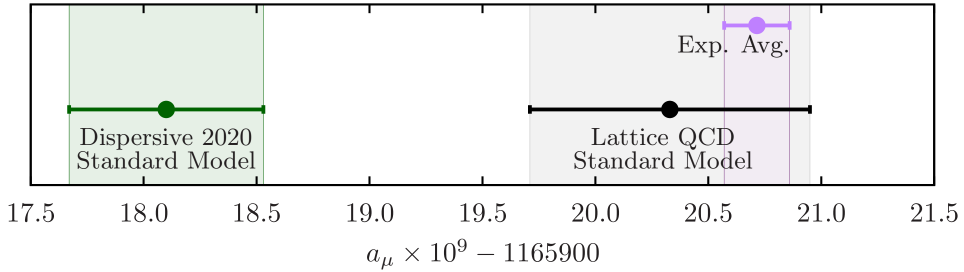
- Magnetic moment of charged leptons :

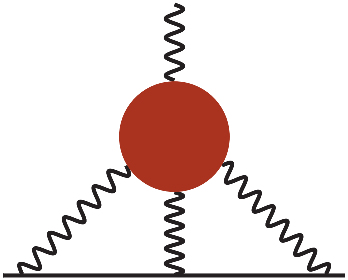
$$\vec{\mu} = g_\ell \left(\frac{Qe}{2m_\ell} \right) \vec{S} \qquad \Rightarrow \qquad a_\ell = \frac{g_\ell - 2}{2} = \frac{\alpha}{2\pi} + O(\alpha^2)$$

Contribution	$a_\mu \times 10^{11}$
- **QED** (10^{th} order)	$116\ 584\ 718.931 \pm 0.104$
- **Electroweak**	153.6 ± 1.0
- **Strong interaction**	
HVP (LO)	$7\ 041 \pm 61$
HVP (NLO + NNLO)	-85.9 ± 0.7
HLbL	112.6 ± 9.6
Standard Model	$116\ 592\ 033 \pm 62$
Experiment	$116\ 592\ 072 \pm 15$



- Status in 2025 (Lattice QCD average from [2505.21476 [hep-ph]])





- Challenging

  $\rightarrow$ hadronic light-by-light tensor $\Pi_{\mu\nu\lambda\sigma}(p_1, p_2, p_3) = \int_{x,y,z} \Pi_{\mu\nu\lambda\sigma}(x,y,z) e^{-i(q_1 x + q_2 y + q_3 z)}$

  $\rightarrow$ depends on four momenta, multi-scale system

- Until 2016 : mostly based on model estimates ($\sim 30 - 40\%$ errors)

  $a_\mu^{\mathrm{HLbL}} = 105(26) \times 10^{-11}$ [Prades, de Rafael, Vainshtein '09]

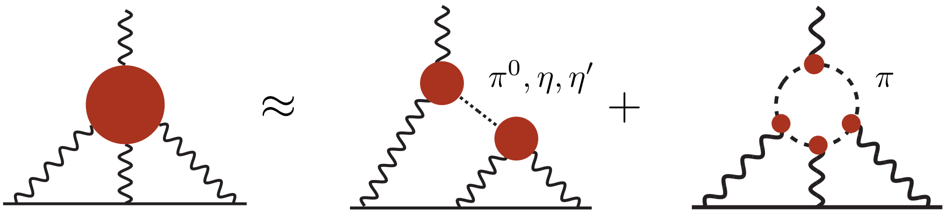
  $a_\mu^{\mathrm{HLbL}} = 116(39) \times 10^{-11}$ [Jegerlehner, Nyffeler '09]

- Precision goal : $< 10\%$ (with controlled uncertainties)

  $\rightarrow$ requires first principle approach : data-driven dispersive framework / lattice QCD

- Hadronic contribution : low-energy physics $\rightarrow$ non-perturbative methods

$a_\mu^{\mathrm{HLbL}} =$



compute the full contribution using Lattice QCD

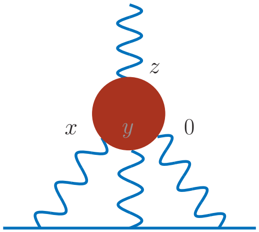
(focus of this talk)

Lattice QCD can provide inputs to the dispersive framework : π^0, η, η' transition form factors.

Dispersive framework ('21)	$a_\mu \times 10^{11}$
π^0, η, η'	93.8 ± 4
pion/kaon loops	-16.4 ± 0.2
S-wave $\pi\pi$	-8 ± 1
axial vector	6 ± 6
scalar + tensor	-1 ± 3
q-loops / short. dist. cstr	15 ± 10
charm + heavy q	3 ± 1
sum	92 ± 19

Master formula in position space :

$$a_\mu^{\mathrm{HLbL}} = \frac{m_\mu e^6}{3} \int \mathrm{d}^4 y \int \mathrm{d}^4 x \, \mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y) \, i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$$



- QED part of the diagram (muon, photons) : $\mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y)$

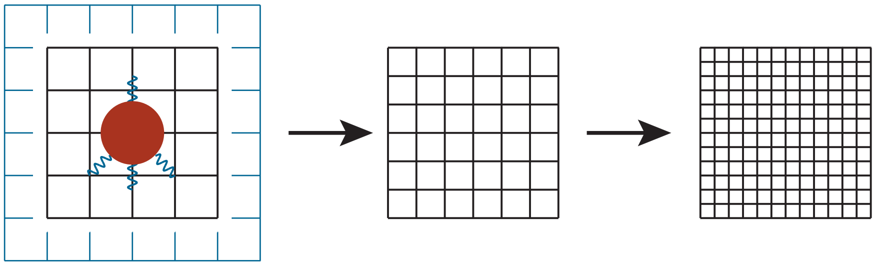
→ computed in the continuum and infinite volume

→ $\mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y)$ computed semi-analytical : Mainz group [JHEP 04 (2023) 040]

- Hadronic correlation function : $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4 z \, z_\rho \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$

⟶ Lattice QCD : numerical evaluation of the path integral in Euclidean space-time

⟶ based on Monte Carlo methods → statistical uncertainties



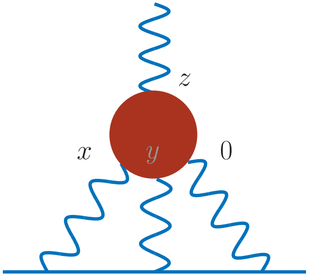
→ continuum limit extrap : $\mathcal{O}(a^2)$

→ ∞ volume extrap : $\mathcal{O}(e^{-m_\pi L})$

$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \int \mathrm{d}^4 y \int \mathrm{d}^4 x \, \mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y) \, i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$$

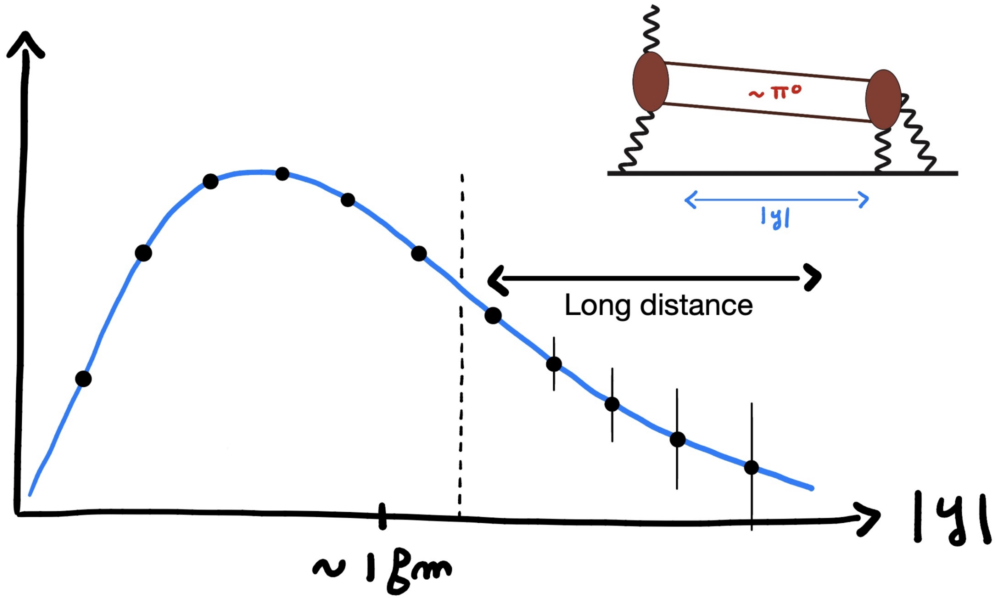
with

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4 z \, z_\rho \, \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$$



- sums over x and z are done explicitly over the lattice
- use rotational symmetry to write $\mathrm{d}^4 y = 4\pi^2 |y|^3 \mathrm{d}|y|$
- sample the remaining 1D integral over $|y|$ with $O(10)$ points

$$a_\mu(|y|) = \int_0^{|y|} \mathcal{I}(|y'|) \, \mathrm{d}|y'|$$



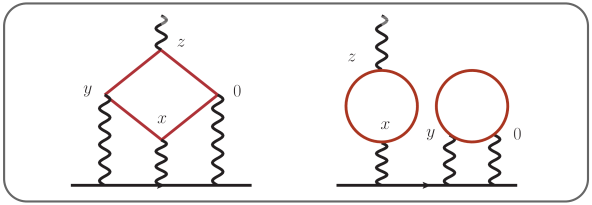
- Long distances $(|y| > 1 \text{ fm})$:
  → statistical noise increases exponentially with $|y|$ (situation even worse at $m_\pi = m_\pi^{\text{phys}}$)
  → finite-volume effects
  → dominated by the pion-exchange contribution

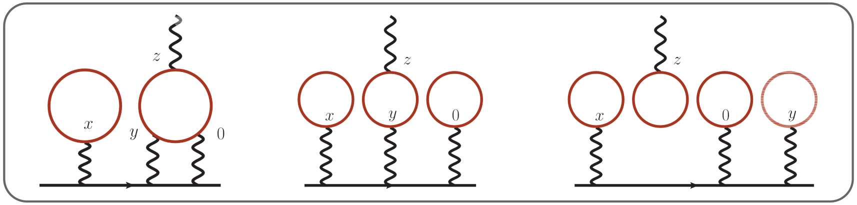
- Replace data by pion-exchange contribution, evaluated using position-space approach
  → include lattice data as much as possible to avoid systematic bias

Large cancellation between connected and disconnected contributions

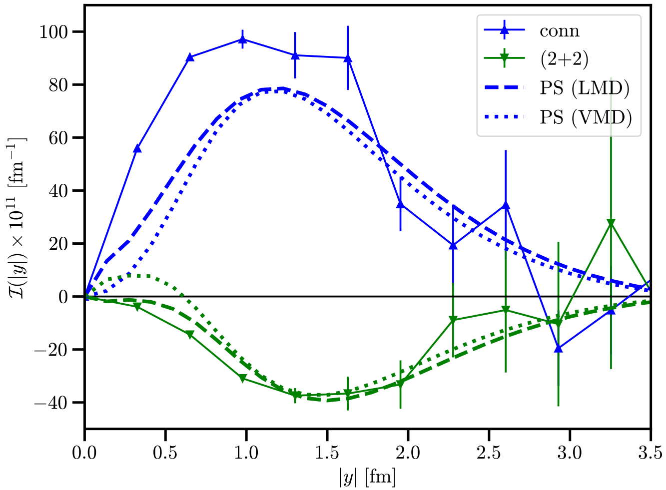
- Wick contractions : $$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4 z\, z_\rho\, \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$$

“connected” “$(2+2)$” sub-leading diagrams





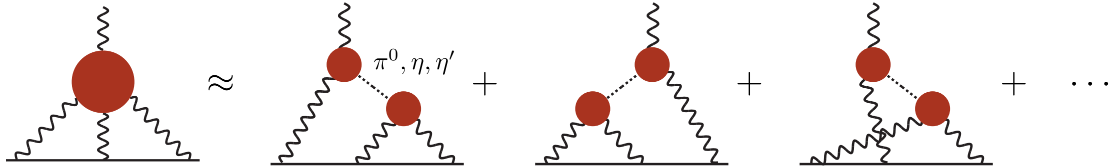
$\approx +220 \times 10^{-11}$ $\approx -100 \times 10^{-11}$ $\mathcal{O}(1 \times 10^{-11})$ (computed in this work)



- results for one lattice spacing
- pion-exchange = main contribution

$$a_\mu^{\mathrm{hlbl,conn}} \longleftarrow +\frac{34}{9} \times a_\mu^{\pi^0-\mathrm{pole}}$$
$$a_\mu^{\mathrm{hlbl,2+2}} \longleftarrow -\frac{25}{9} \times a_\mu^{\pi^0-\mathrm{pole}}$$

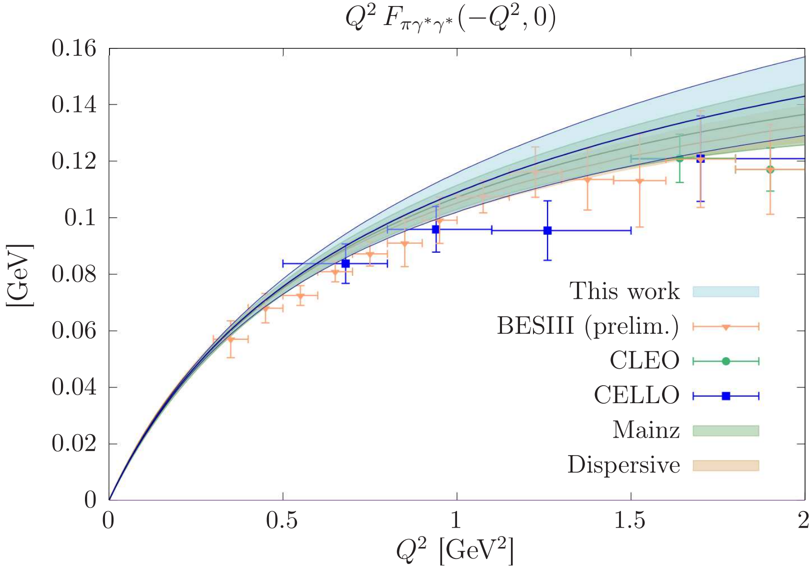
- large cancellation expected [J. Bijnens et. al ’16]



[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{P\gamma^*\gamma^*}(-Q_2^2, 0) + w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{P\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

Integrand concentrated at spacelike momenta below 2 GeV

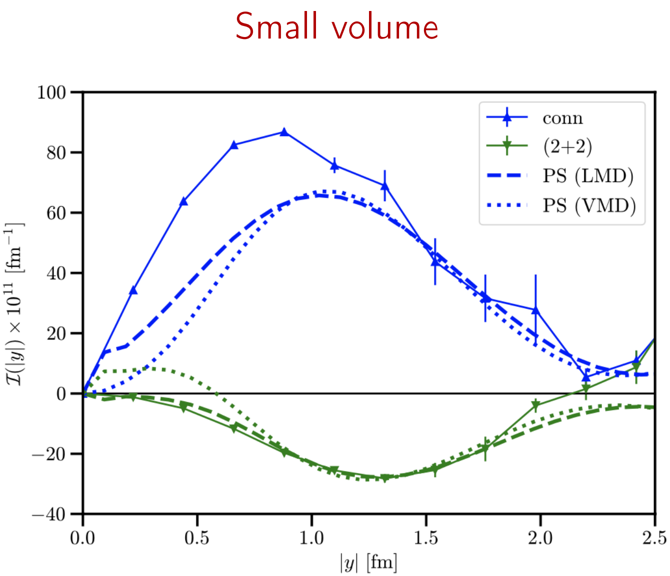


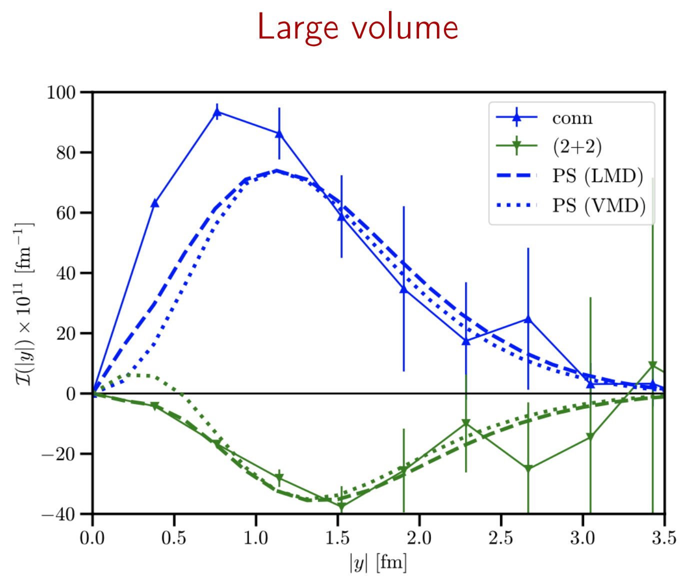
- **Transition form factors**
  [Phys.Rev.D 111 (2025) 5, 054511]

$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\; \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) = i \int \mathrm{d}^4x\, e^{iq_1 x}\, \langle 0 | T\{J_\mu(x) J_\nu(0)\} | \pi^0(\vec{p}) \rangle$$

$\rightarrow$ computed on the same set of ensembles

$\rightarrow$ all π^0, η and η' TFFs have been computed

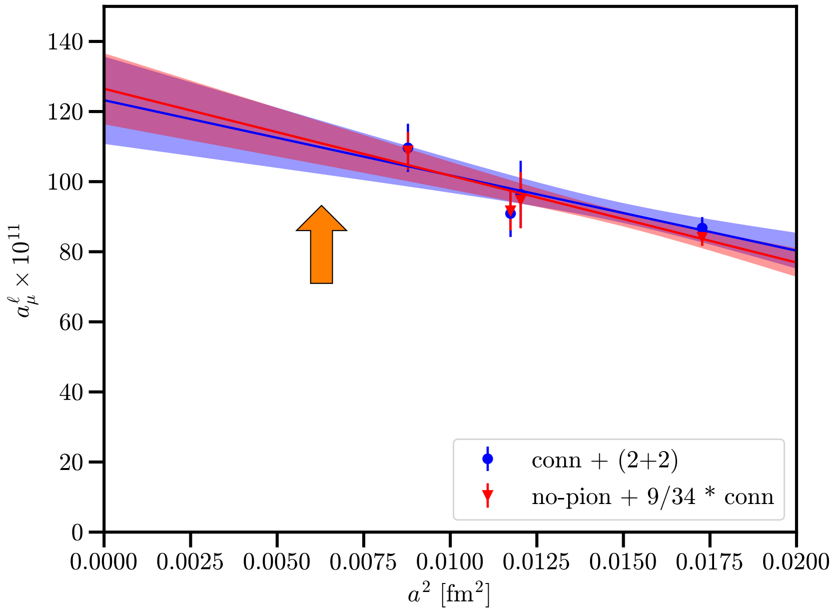




- Pion exchange computed in finite volume : good description of our data at large $|y|$
- ... both in large and small volumes

  $\rightarrow$ finite-volume effects are dominated by pion-exchange
- Strategy : replace lattice data by pion exchange for $|y| > y_{\rm cut}$

  $\rightarrow$ keep $y_{\rm cut}$ large such that this correction is small compared to statistical uncertainty.



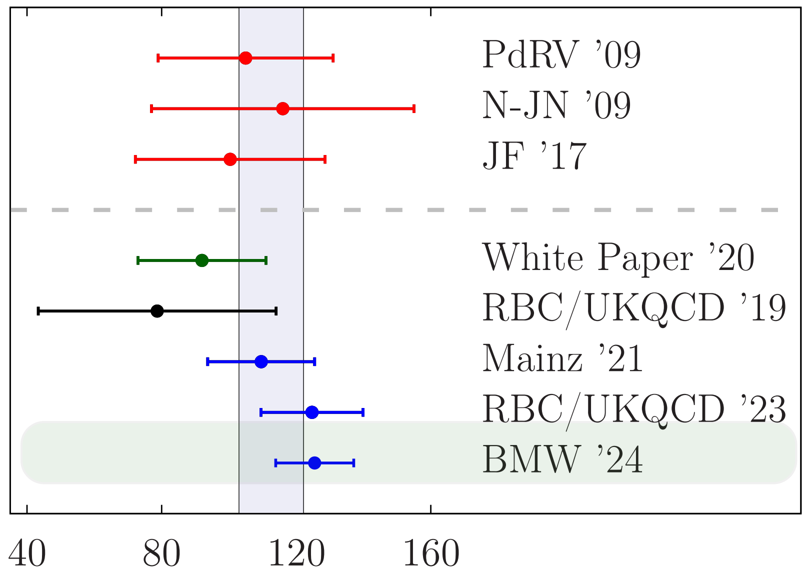
- Staggered quarks : leading discretization effects are $\mathcal{O}(a^2)$
- 3 lattice spacings - data at a fourth lattice spacing is in production
- all simulations are performed at the physical pion-mass

$$a_\mu^{\mathrm{conn},l} = 220.1(13.0)_{\mathrm{stat}}(3.8)_{\mathrm{syst}} \times 10^{-11}$$
$$a_\mu^{(2+2),l} = -101.1(12.4)_{\mathrm{stat}}(3.2)_{\mathrm{syst}} \times 10^{-11}$$
$$a_\mu^{l} = 122.6(11.5)_{\mathrm{stat}}(1.8)_{\mathrm{syst}} \times 10^{-11}$$

Conclusion

- Complete calculation now published [Phys.Rev.D 111 (2025) 11, 114509 - BMW collaboration]
  - → 3 lattice spacings at the physical pion mass (and two different volumes)
  - → dedicated calculation of the pion transition form factor : finite-volume effects, long-distance contribution
  - → precision <10% (target precision to match Fermilab's experimental result)
  - → outlook : add a fourth lattice spacing to improve our continuum limit extrapolation

$a_\mu^{\mathrm{HLbL}} \times 10^{11}$



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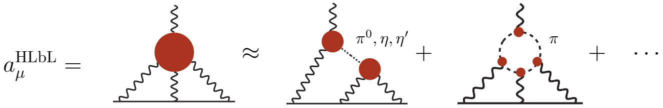
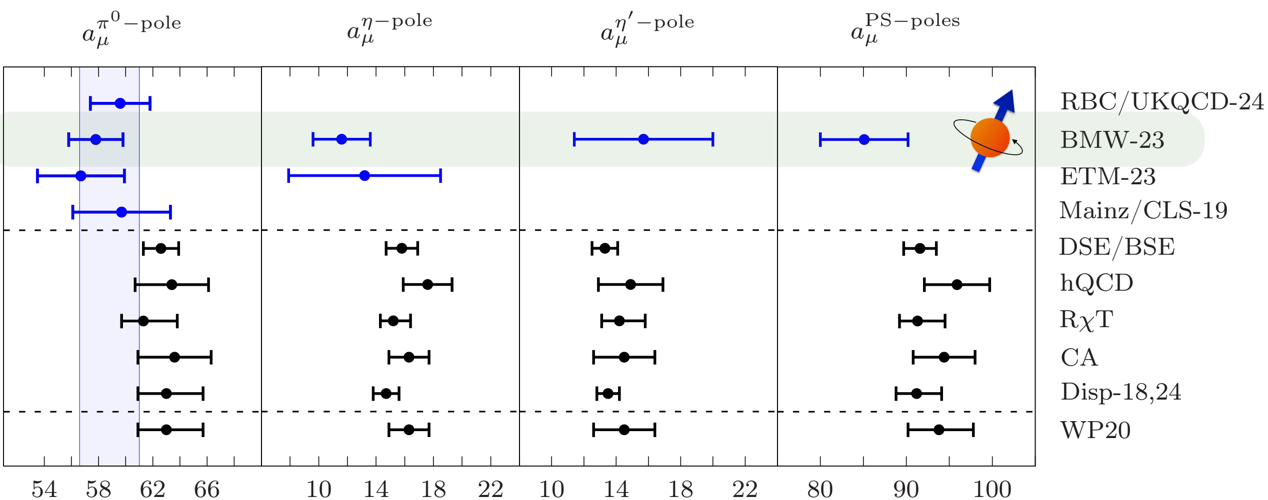


- **Light pseudoscalar exchange** : dominant contribution in dispersive framework
  - → lattice calculation of the π^0, η, η' transition form factors [Phys.Rev.D 111 (2025) 5, 054511 - BMW collaboration]



Thank you !