The Three-loop hadronic vacuum polarization in chiral perturbation theory

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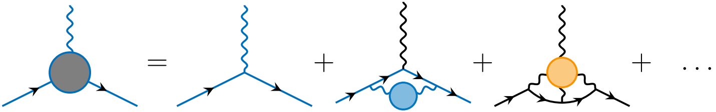
(Budapest-Marseille-Wuppertal lattice QCD collaboration)

$$\vec{\mu}_\mu = \pm g_\mu \frac{e}{2m_\mu}\vec{S}; \qquad a_\mu = \frac{(g-2)_\mu}{2}$$

Anomalous magnetic moment of the electron is one of the crowning achievements of Quantum Field Theory



Experimental and theoretical calculations agree to about one part in a trillion.

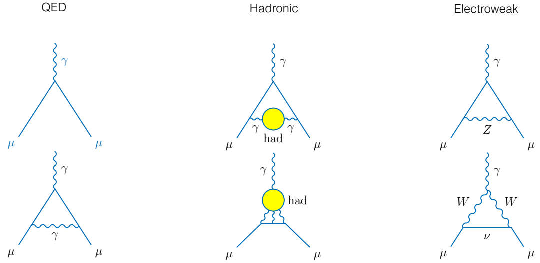
The anomalous magnetic moment of the muon is sensitive to possible new states beyond the Standard Model, then provides an interesting way to probe the new physics.

To compare theory and experiment, highly technical calculations are involved. QED calculations have to be performed at high order, and contributions from hadronic physics becomes important at such high precision.

The standard model calculation

At the needed precision all the three interactions and all the standard model particles contribute the the a_μ

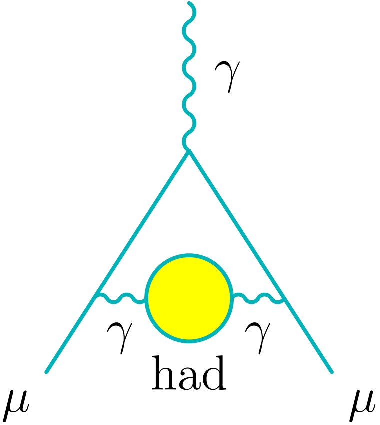
$$a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm hadronic} + a_\mu^{\rm EW}$$
$$= O\left(\frac{\alpha}{2\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\frac{\alpha}{16\pi\sin^2(\theta_W)}\left(\frac{m_\mu}{M_W}\right)^2\right)$$
$$= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right)$$



The hadronic contribution

The uncertainty of the SM prediction is dominated by the hadronic contributions

$$a_\mu^{\rm exp} - a_\mu^{\rm QED} - a_\mu^{\rm EW} = 718.6(2.2) \times 10^{-10}$$

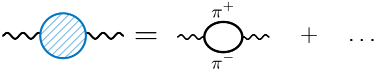


The hadronic contributions can be approached by

1. Lattice QCD (see the talks by [Lellouch] and [Lupo] at this meeting)
2. Effective QFT with hadrons such as chiral perturbation theory [this talk]
3. Dispersion relations and experimental data

The chiral perturbation theory

From now we will focus on the hadronic contributions at low-energy. This is subleading but this is where the most uncertainly arise.



We work with chiral perturbation theory which is a low-energy effective field theory for QCD

$$\mathcal{L}^{\rm QCD}(q, \bar{q}, A) \to \mathcal{L}^{\chi PT}(U, \partial_\mu U, \cdots) = \frac{F_0^2}{4} \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \text{higher-order}$$

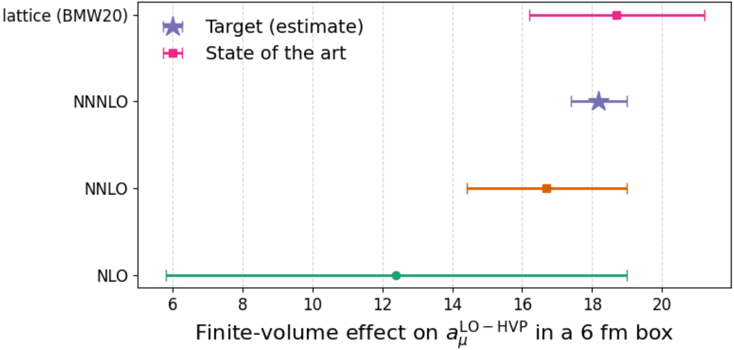
The pions enter in the parametrisation of U

$$\mathcal{L}^{\chi PT} = \mathcal{L}^{\chi PT}_{O(p^2)} + \mathcal{L}^{\chi PT}_{O(p^4)} + \mathcal{L}^{\chi PT}_{O(p^6)} + \mathcal{L}^{\chi PT}_{O(p^8)} + \cdots$$

- The EFT lagrangian is ordered in powers of the momentum
- At each order arise new low-energy constants (from UV counter-terms) that are determined by matching physical quantities [Gasser, Leutwyler], [Bijnens, Colangelo, Ecker], [Bijnens, Hermansson-Truedsson, Wang]

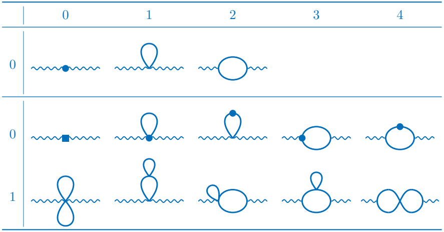
Why do this ?

An quick estimation of the contributions to the hadronic vacuum polarisation in perturbation shows that the (N^3LO) three-loop contribution will bring the theoretical uncertainly from finite volume effects to match the experimental uncertainty



(NLO) One- and (N^{2}LO) two-loop order

We work in the two-flavor chiral perturbation theory in the isospin limit, describing an $\mathrm{SU}(2)$ triplet of pion fields of mass M coupled to an external, non-dynamic photon field A_μ.



There is no (LO) tree-level contributions because we only have the pions. At (NLO) 1-loop and (N^{2}LO) 2-loop we only have contributions from two pions production $\mu^+\mu^- \to 2\pi + \gamma$

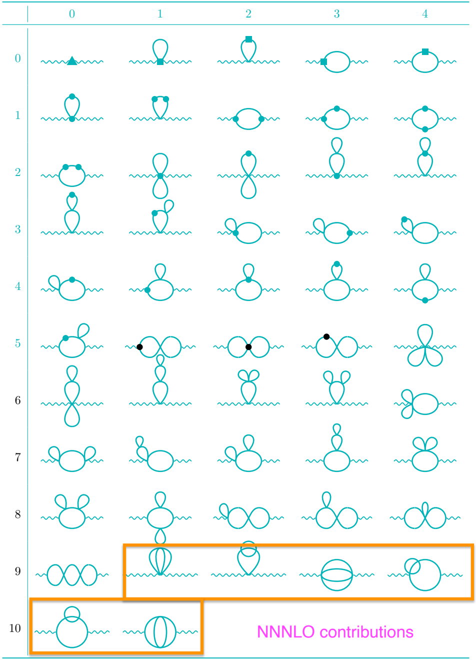
This is known from the works of [Gasser, Leutwyler], [Bijnens]

The new contribution at (N^3LO) three loops and the main topic of this talk and work to appear with members of the BMW collaboration

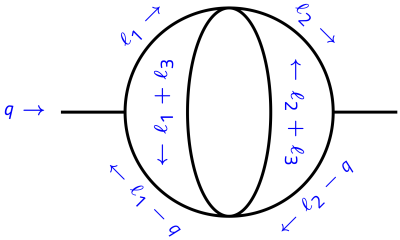
[Lellouch, Lupo, Portelli, Sjö, Szabo, Vanhove]

At this order opens the new four π intermediate state

$$\mu^+\mu^- \to 4\pi + \gamma$$



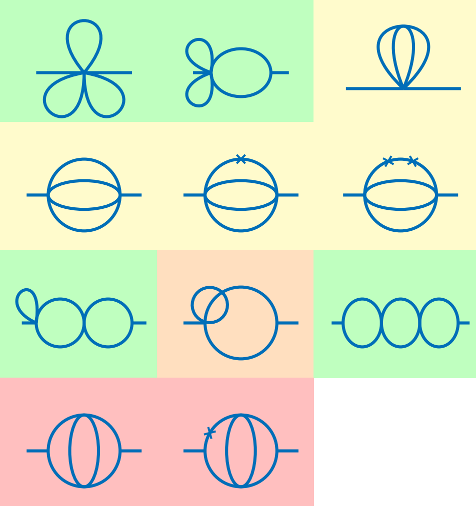
The three-loop amplitude in details



$$\mathbb{I}_{\nu_1,\cdots,\nu_9}(q^2, M^2) = \int \frac{d^D\ell_1 d^D\ell_2 d^D\ell_3}{\prod_{i=1}^9 (k_i^2 - M^2)^{\nu_i}}$$

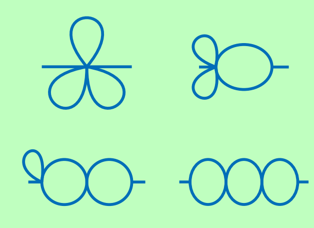
This is the top (most complicated) topology arising from the Feynman graphs that is dressed by the vertices from ChPT using a `FORM` code

The three-loop amplitude in details



The IBP integral reduction is done with `LiteRed` give 11 scalar master integrals

Masters: tadpoles and bubbles

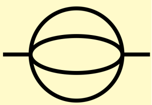


- Divergent but simple product of tadpole and bubble
- Just multiple polylogarithms function of $t = p^2/M^2$

$$I_{\ominus}(4-2\epsilon;t) = \frac{i \exp\left[\sum_{k=2}^{\infty}(-\epsilon)^k \frac{\zeta(k)}{k}\right]}{\epsilon}\left[1+\sum_{n=1}^{\infty}(-\epsilon)^n \frac{J_{\ominus}^{(n)}(t)}{n!}\right],$$

$$J_{\ominus}^{(n)}(t) = \int_0^1 (\log(1-x(1-x)t))^n dx$$

Masters: the motivic K3 period integrals



It's expansion near four dimension $D = 4 - 2\epsilon$ reads

$$E_1(4-2\epsilon; t) = \frac{2}{\epsilon^3} + \frac{23-t}{3\epsilon^2} + \frac{t^2 - 54t + 18\pi^2 + 630}{36\epsilon} + \bar{E}_1(t) + O(\epsilon)$$

The finite piece $\bar{E}_1(t)$ involves a new kind of integral obtained from to motivic relative periods of a K3 surface of Picard number 19 arising from the singularities of the 3-loop sunset integral [Bloch, Kerr, Vanhove]

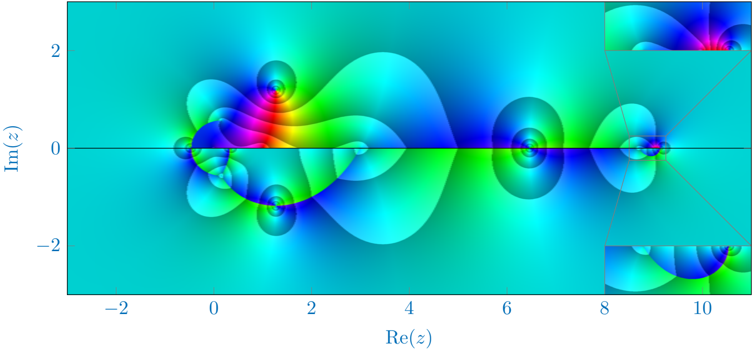
From general consideration is it expected that higher loop integrals are associated with Feynman Calabi-Yau geometry.

The finite piece in four dimensions is a combination of the masters in two dimensions

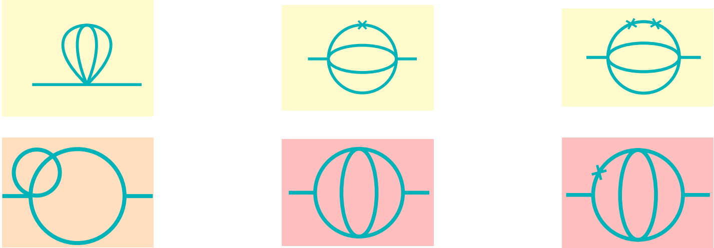
$$E_1(2;t) = \frac{\left(\eta(q^2)\eta(q^6)\right)^4}{\left(\eta(q)\eta(q^3)\right)^2}\left(\sum_{n\geqslant 1}\frac{\psi(n)}{n^3}\frac{q^n}{1-q^n} - 4(\log q)^3 + 16\zeta(3)\right)$$

The (mirror) map between $t = p^2/M^2$ and the complex structure of the K3 surface

$$t = \frac{p^2}{M^2} = -\left(\frac{\eta(q)\eta(q^3)}{\eta(q^2)\eta(q^6)}\right)^6; \qquad \eta(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$



Masters: the motivic K3 periods integrals



The other master are obtained by either differentiation or integration of E_1.
Their expansion near $D = 4$ reads

$$E_r(4-2\epsilon;t) = \frac{c_r^{(3)}(t)}{\epsilon^3} + \frac{c_r^{(2)}(t)}{\epsilon^2} + \frac{c_r^{(1)}(t)}{\epsilon} + \bar{E}_r(t) + O(\epsilon)$$

The total amplitude

The complete amplitude at 3-loop takes the form

$$\Pi^T(t) = \frac{r_2 t^2}{\epsilon^3} + \frac{C^{(2)}(t)}{\epsilon^2} + \frac{C^{(1)}(t)}{\epsilon} + \Pi^T_{\text{finite}}(t)$$

- The leading divergence $r_2 t^2$ produces a new combination of low-energy constants.
- The subleading divergences $C^{(2)}(t)$, $C^{(1)}(t)$ are cancelled by the counter-terms from the lower loop low-energy constants. These constant are known from previous work extracted from the pion radius [Bijnens et al.]
- The finite piece $\Pi^T_{\text{finite}}(t)$ is composed : polylogarithms functions and elliptic functions from the K3 motivic periods

The dispersive relation

Dispersion relation connect the imaginary part of the amplitude to $a_\mu^{\rm HPV}$

$$a_\mu^{\rm HPV} = \frac{1}{\pi}\int_{4m_{\pi^0}^2}^{\infty} \frac{dt}{t}\,\Im{\rm m}\Pi^T(t) \int_0^1 dz \frac{z^2(1-z)}{z^2+(1-z)t/m_\mu^2}$$

The imaginary part of the amplitude gives the cross-section of pions emission

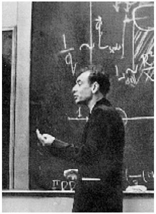
$$\Im{\rm m}\Pi^T(s) = \frac{1}{12\pi}\frac{\sigma^0(e^+e^- \to \gamma^* \to n-\pi+\gamma)}{\underbrace{\sigma^0(e^+e^- \to \mu^+\mu^-)}_{\frac{4\pi\alpha^2}{3s}}}$$

- NLO and N^{2}LO: $\mu^+\mu^- \to 2\pi+\gamma$
- N^{3}LO: a new channel opens $\mu^+\mu^- \to 4\pi+\gamma$

Conclusion and outlook

With this **complete** evaluation of the N^3LO contribution to the hadronic vacuum polarisation in chiral perturbation theory from pions allows (works to appear in [Lellouch, Lupo, Portelli, Sjö, Szabo, Vanhove])

- A fast high-precision evaluation using python based routines and `sagemath`
- An evaluation of the 4 pions cross-section production $\mu^+\mu^- \to 4\pi + \gamma$: the new low energy constants do not arise so everything needed is known
- An evaluation of the finite volume effects and reach the resquested target for uncertainty control



Метод важнее открытия, ибо правильный метод исследования приведет к новым, еще более ценным открытиям. (Л. Д. Ландау)

from http://www.famhist.ru/famhist/landau/000738b2.htm

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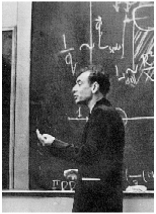

A method is better than a discovery, because a good method can lead to new results, and much more valuable discoveries. (L. D. Landau)

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