UNIVERSITY OF LIVERPOOL

LEVERHULME TRUST

Phokhara at the frontier of NNLO

Pau Petit Rosàs

On behalf of the Phokhara development team,
based on [PPR, Torres Bobadilla (2507.XXXX)]

LIVERPOOL THEORETICAL PHYSICS

Introduction – The new $(g\text{-}2)_\mu$ landscape

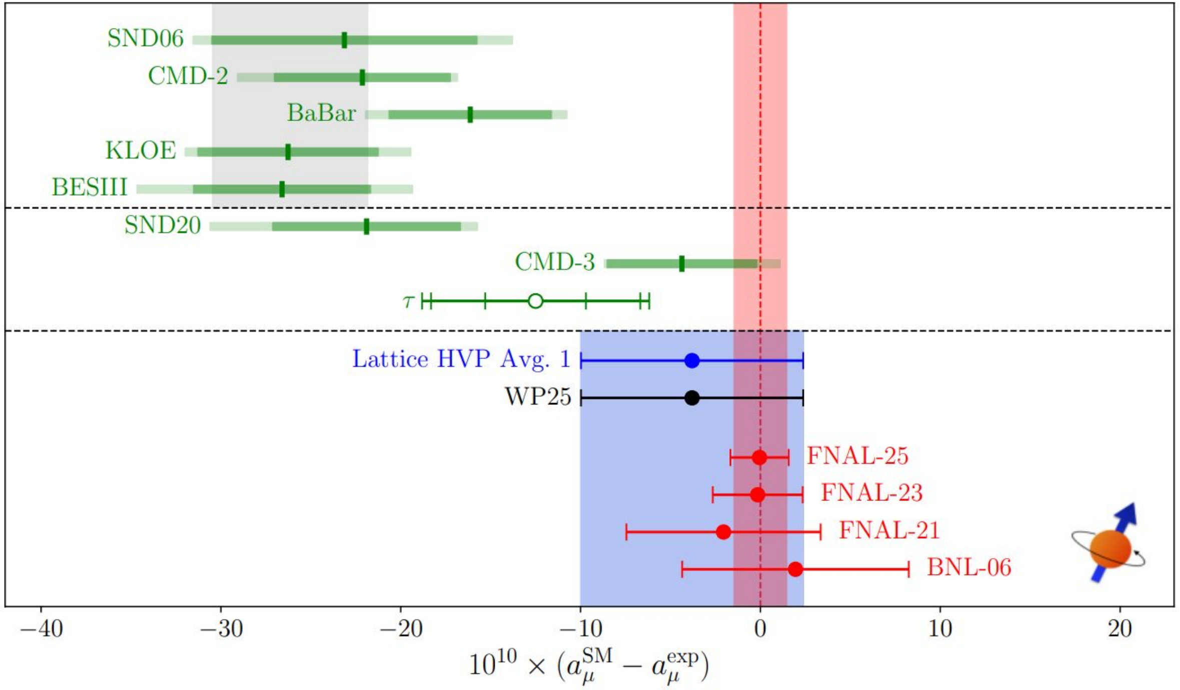

Figure from WP2025 [2505.21476]

No tension in **WP2025**, but

- ➔ **Disagreement** between data-driven and Lattice QCD.
- ➔ Friction between data of different e^+e^- experiments.
- ➔ Need for **improved MC** tools to solve conflict.
- ➔ Going **beyond NLO** in full mass $e^+e^- \to \mu^+\mu^-\gamma$ & $e^+e^- \to \pi^+\pi^-\gamma$.

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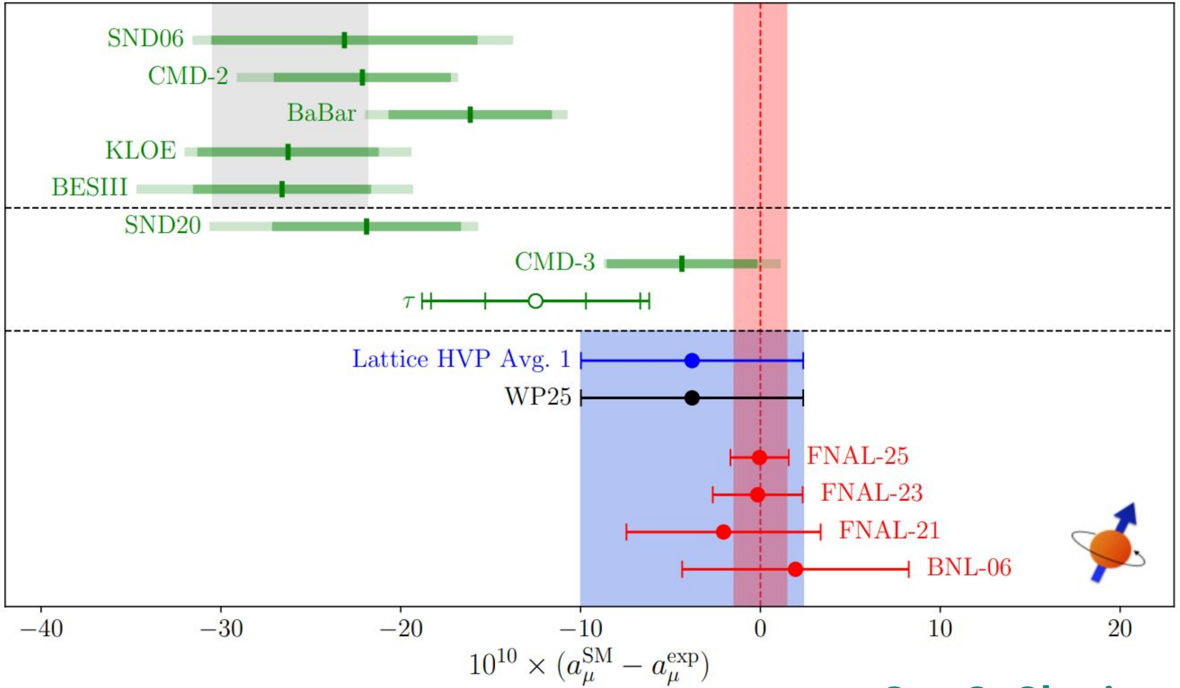

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→ **See S. Charity plenary talk.**

Phokhara – Past, Present & Future

- **Phokhara**, MC generator for low energy e^+e^- colliders, with +20 years of development. **[PhysRevD.100.076004]**
- Mostly used in **2 to 3** processes with massive leptons and hadrons, widely used by the community.
- Recently updated with quality of life improvements, and **compared** against other MC in **[SciPost 10.21468].**

$e^+e^- \to$	Order	VP	VFF	Extras
$\mu^+\mu^-$	LO	alphaQED, from [320, 321] or NSK	-	Narrow resonances of J/ψ and $\psi(2S)$
$\mu^+\mu^-\gamma$	NLO with full mass dependence			
$\pi^+\pi^-$	LO	alphaQED, from [320, 321] or NSK	F×sQED choice of 3 VFF	Narrow resonances of J/ψ and $\psi(2S)$ Radiative ϕ decays
$\pi^+\pi^-\gamma$	NLO with full mass dependence			
X	$X \in 2\pi^0\pi^+\pi^-$, $2\pi^+2\pi^-$, $p\bar{p}$, $n\bar{n}$, K^+K^-, $K^0\bar{K}^0$, $\pi^+\pi^-\pi^0$, $\Lambda(\to \pi^-p)\bar{\Lambda}(\to \pi^+\bar{p})$, $\eta\pi^+\pi^-$, $\pi^0\gamma$, $\eta\gamma$, $\eta'\gamma$, $\chi_{c1} \to J/\psi(\to \mu^+\mu^-)\gamma$, $\chi_{c2} \to J/\psi(\to \mu^+\mu^-)\gamma$			

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→ **See J. Paltrinieri talk.**

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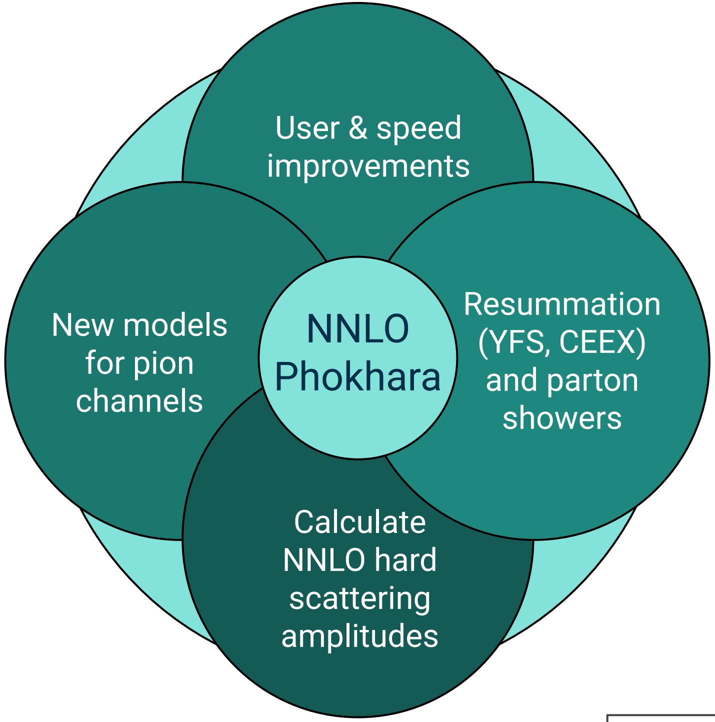
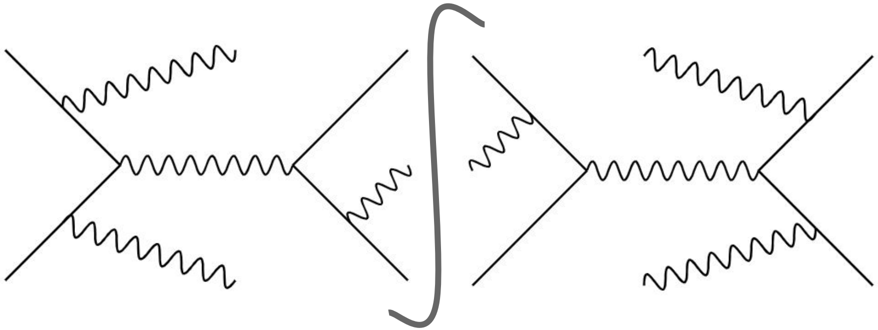


Pieces for 2→3 at NNLO



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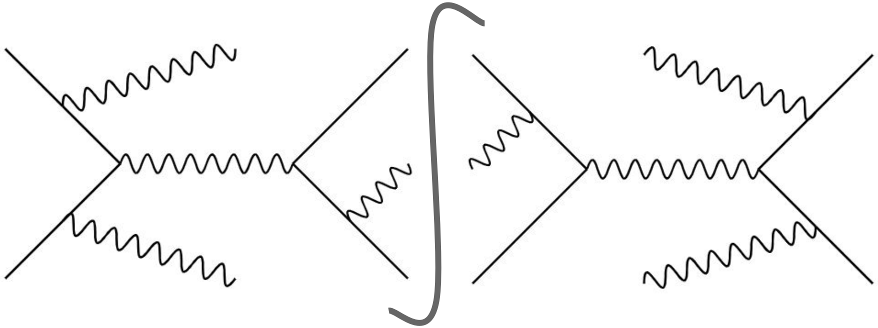

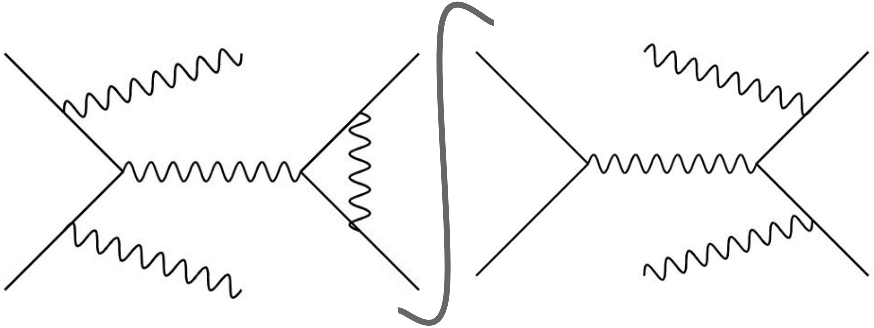

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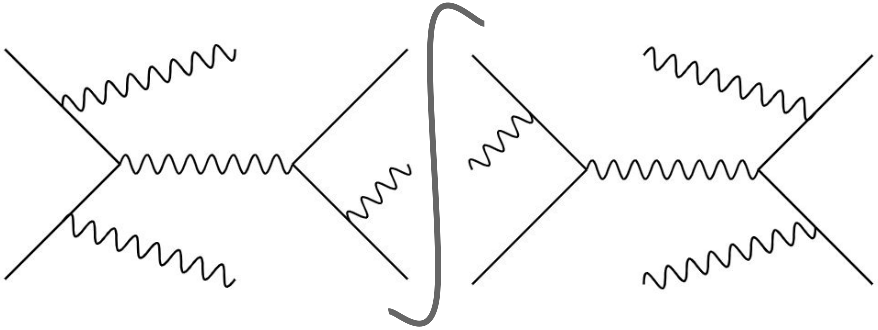

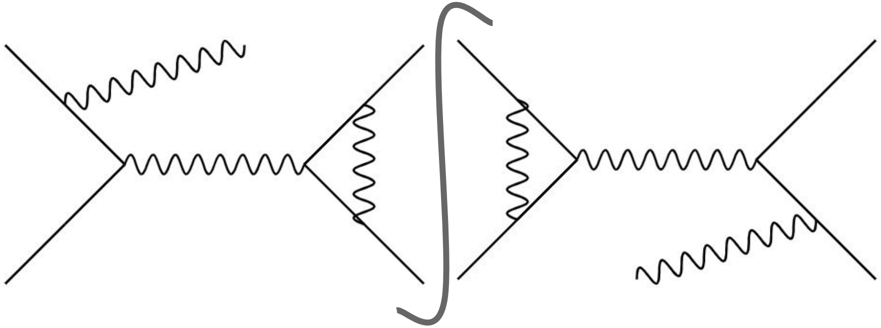

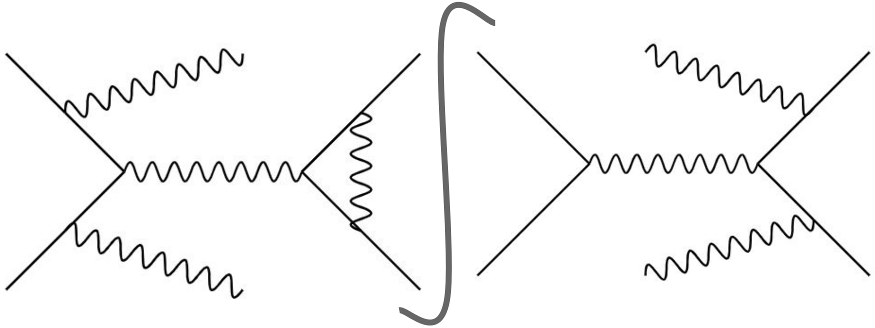

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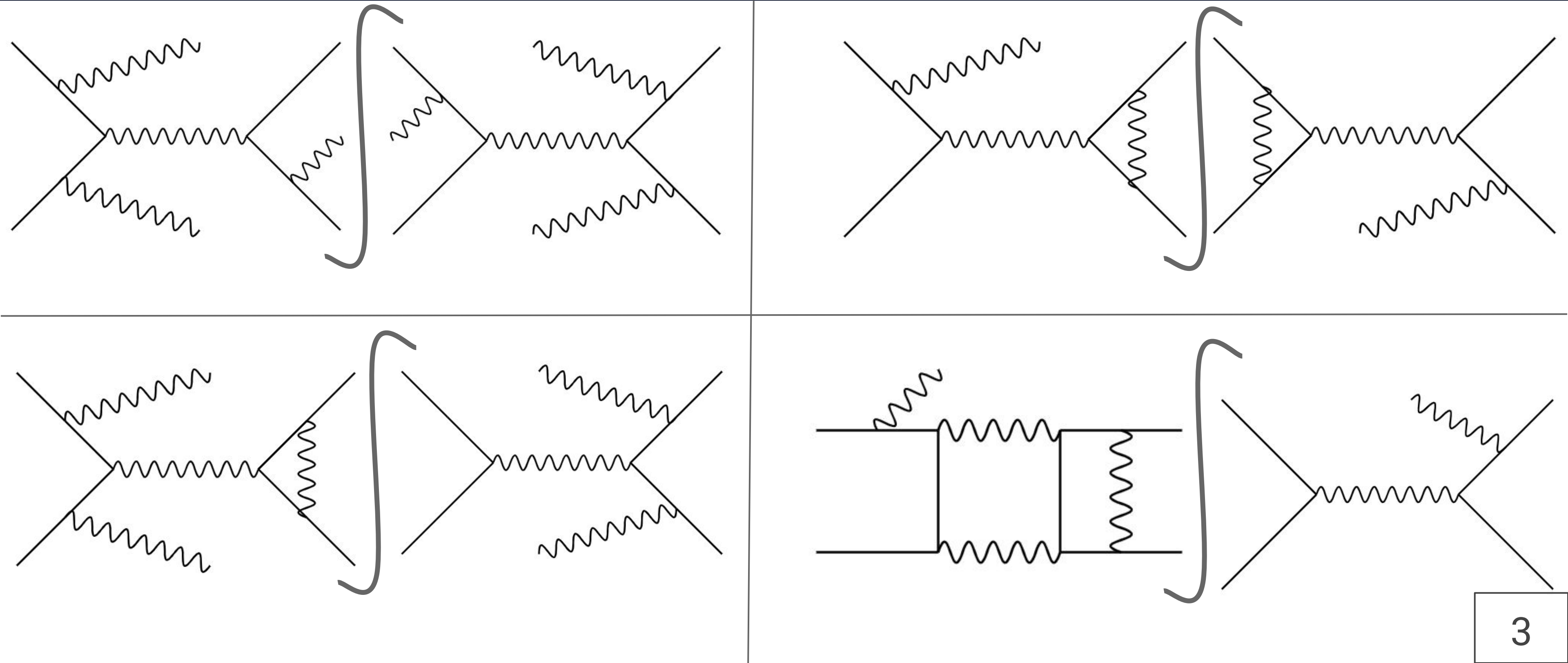

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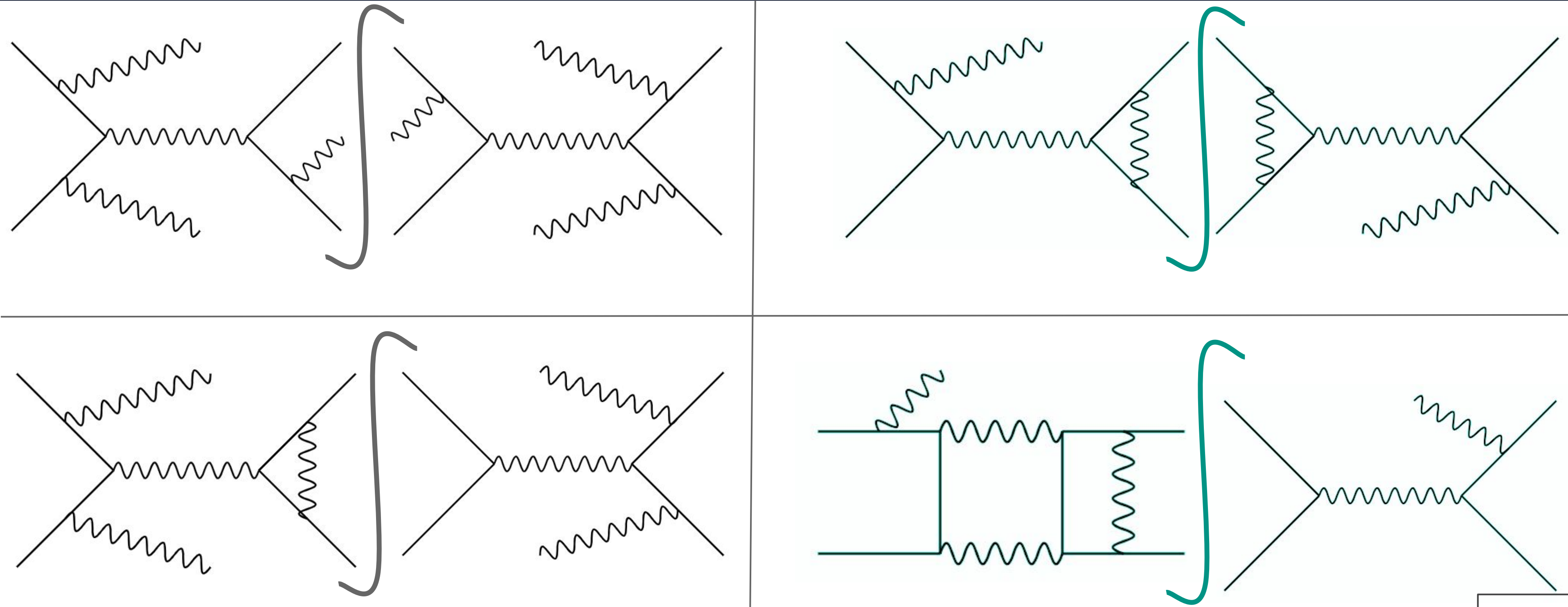
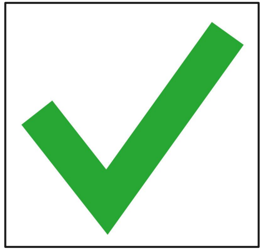


Challenges of 2→3 amplitudes

Amplitude generation

Qgraf, Recola, FeynRules, FORM, FeynCalc, Tapir,…



Challenges of $2\to3$ amplitudes

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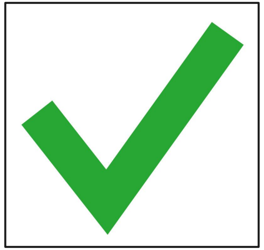
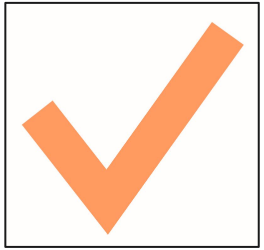
Reduction to MIs

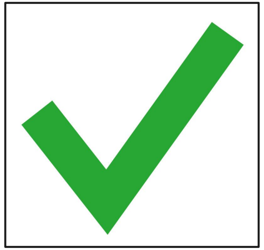
Kira, Fire, Blade, LiteRed, NeatIBP, Reduze,...

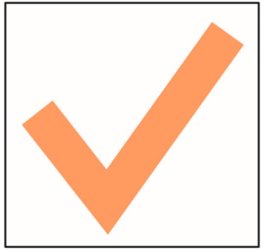


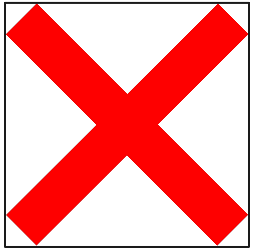


Challenges of 2→3 amplitudes

Amplitude generation	Reduction to MIs	Evaluation of MIs
Qgraf, Recola, FeynRules, FORM, FeynCalc, Tapir,...	Kira, Fire, Blade, LiteRed, NeatIBP, Reduze,...	AMFlow, pySecDec, DiffExp, Collier,SeaSyde,...





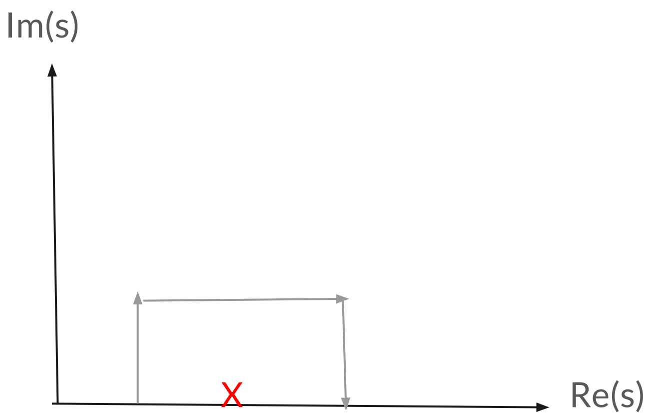


Need for an integrator or a fast generation of grids

The differential equation method

- Feynman integrals can be cast into differential form.
- Highly non trivial or impossible to obtain canonical form for some processes, but
  - once obtained, nice analytic structure.
  - Packages to solve the system of differential equations via the Frobenius method.
- However, packages are in Mathematica → slow for a MC, costly to generate grids with high dimensionality, hard to parallelize.
- What about solving the differential equations numerically?

$$d\vec{J} = \sum_{i=0}^{\infty} \epsilon^i \mathbb{A}^{(i)}(\vec{x})\vec{J}$$



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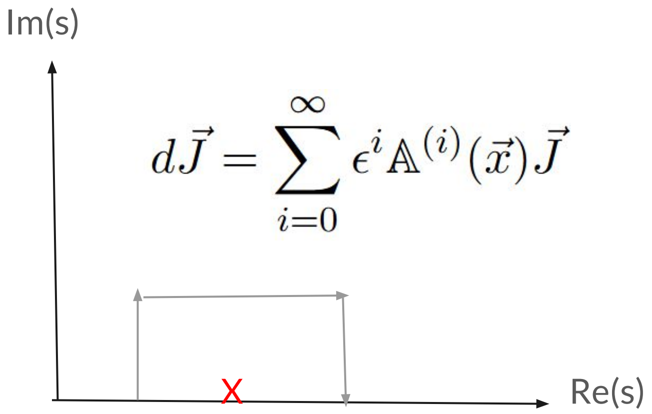

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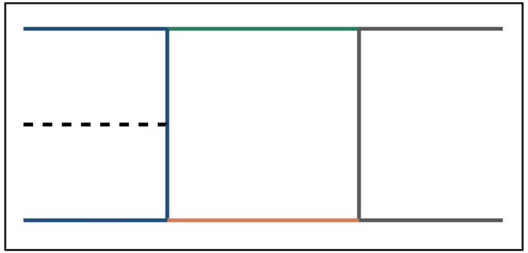
→ C++ integrator

A new integrator

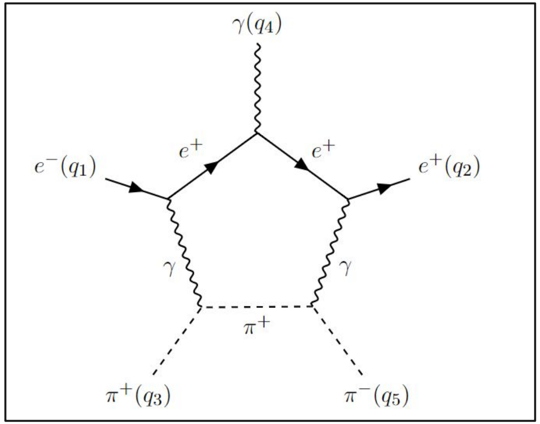
1. Input partial DE w.r.t. each kinematic variable.
2. Input boundary values for MIs at a non-singular point.
3. Input analytic expressions for singularities and branch cuts.
4. Find optimal path between origin and desired final point for each kin. var.
5. Evolve the DE variable by variable in that path:
    a. Multiply the boundary values by square roots defined in terms of the current variable.
    b. Solve the coupled partial DE with controlled stepper from Boost Odeint library.
    c. Divide out the canonical factors from the solution.
6. If desired, use the final result to go to a new final point.

A few examples

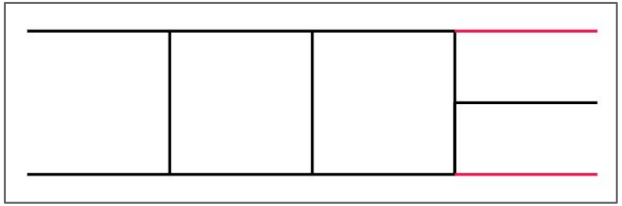
New models for pion final states



One-loop at $\Theta(\epsilon^2)$



Two-Loop five point two massive final states



[PPR, Torres Bobadilla (2507.XXXX)]

One–Loop with 9 scales

The GVMD model

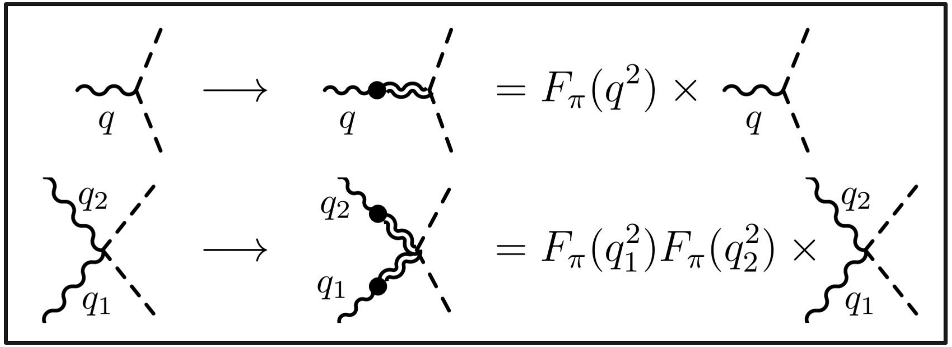
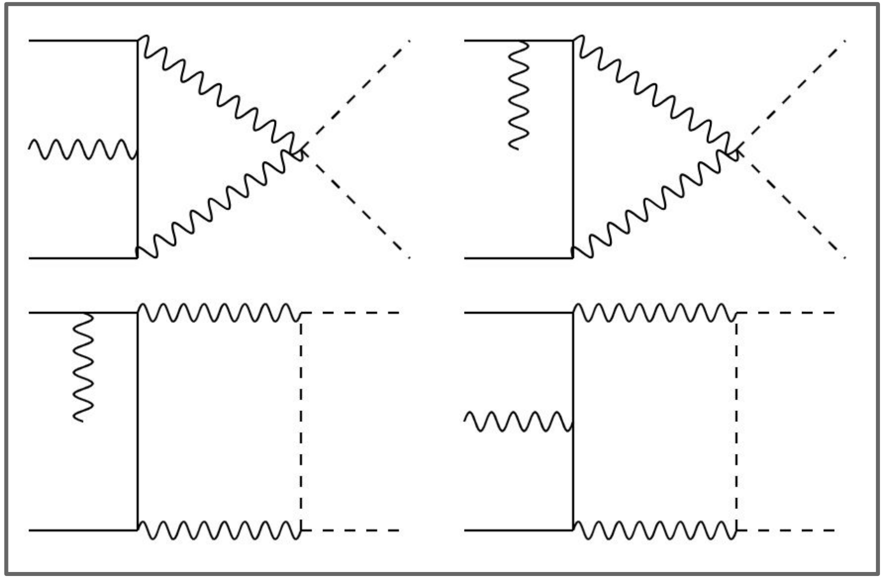


Figure from [2204.12235]

$$F_\pi(q^2) = \sum_{v=1}^{3} a_v \frac{\Lambda_v}{\Lambda_v - q^2} = \sum_{v=1}^{3} a_v \left(1 + \frac{q^2}{\Lambda_v - q^2}\right)$$



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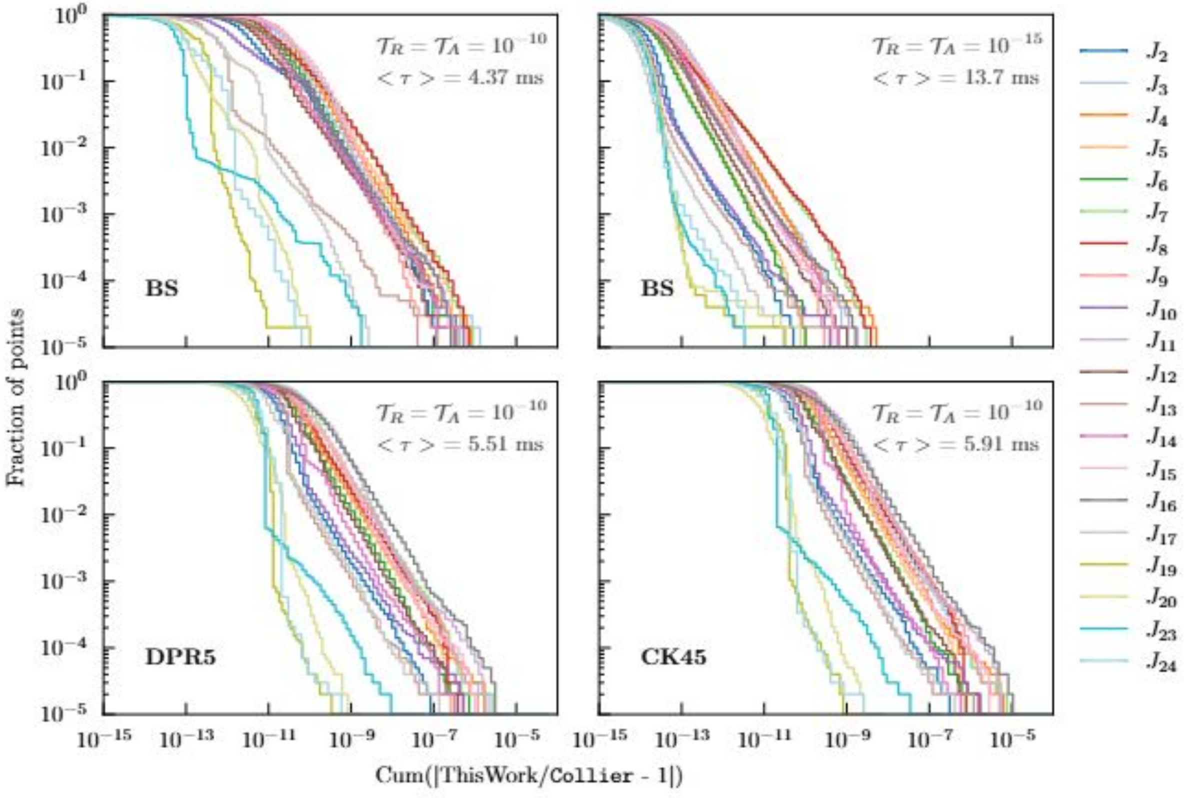
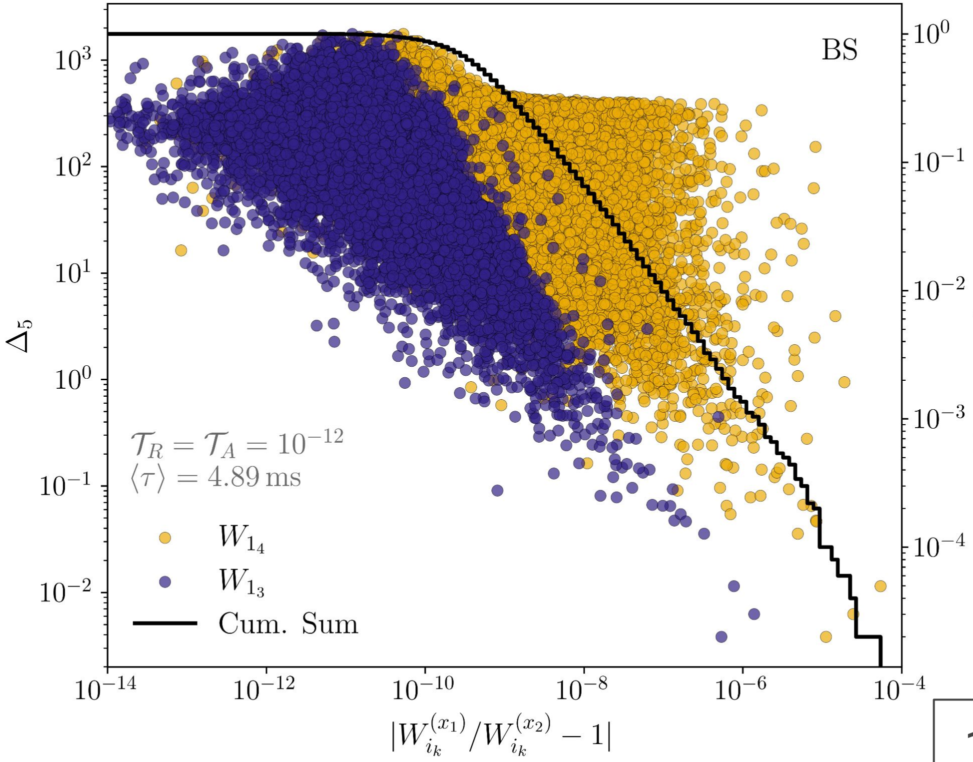


- 29 MIs and two orders in the dimensional regulator.
- 100k arbitrary points.
- Process with 9 kinematic scales, two complex masses.
- Good precision and reasonable speed, slower than Collier.
- Fast and precise enough to use for evaluating integrals at higher order in the dimensional regulator.

One-Loop at $\Theta(\epsilon^2)$

- 21 MIs with four orders in the dimensional regulator.
- 50k points generated with Phokhara for a B-factory scenario.
- When close to singular kinematic points, precision worsens.
- Might need higher precision types, extrapolation, or explore other solutions.



Two-Loop pp→ttj

- Test integrator with two-loop one mass process. 88 and 121 MIs respectively, expanded in four orders in epsilon.
- Differential equation built in **[2404.12325]**. No canonical form for PBb, but polynomial in dim. regulator.
- Fast in double precision, might not be enough.
- However, quad precision still faster than DiffExp and easy to parallelize generation of grids.

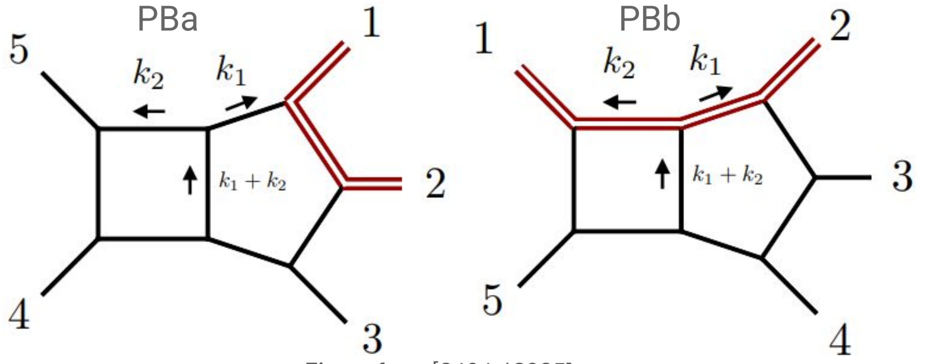


Figure from [2404.12325]

		$\mathcal{T}_A$, $\mathcal{T}_R$	$\mathcal{R}$	$\langle\tau\rangle$ [s]	DiffExp $\langle\tau\rangle$ [s]
PB_A	double	10^{-12}, 10^{-12}	10	0.0881	580.85
	quad	10^{-28}, 10^{-28}	27	51.588	795.516
PB_B	double	10^{-12}, 10^{-12}	10	0.100	555.438
	quad	10^{-28}, 10^{-28}	27	89.088	826.219

Conclusions

- **Need** for **NNLO Monte Carlos** for low energy e^+e^- colliders.
- Work ongoing for improving Phokhara at different frontiers:
  - **Resummation** of soft photon logs and QED parton showers.
  - Explore new models of the pion final states.

- Fast evaluation of Feynman integrals is needed.
- Built an **integrator** capable of evaluating such integrals, either on-the-fly or with grids.
- Several improvements possible for the integrator, targeting optimization and precision.

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