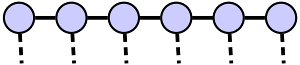
Hamiltonian approach to Parton Distribution Functions in the Schwinger model

Manuel Schneider
manuel.schneider@nycu.edu.tw



NATIONAL YANG MING CHIAO TUNG UNIVERSITY

理論與計算物理中心
Center for Theoretical and Computational Physics

[arXiv:2504.07508]
[arXiv:2409.16996]

EPS-HEP Conference 2025
Marseille
11 July 2025

Collaborators

Mari Carmen Bañuls

Krzysztof Cichy

C.-J. David Lin

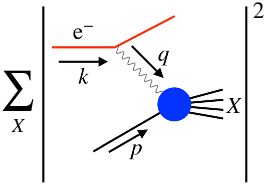
Outline

Parton Distribution Functions

- PDF: probability of constituent with momentum fraction ξ

Parton Distribution Functions

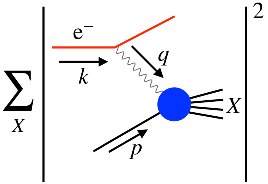
- PDF: probability of constituent with momentum fraction ξ
- test: deep inelastic scattering (DIS)
- large momentum transfer $Q^2 = -q^2$
- Bjorken parameter $\xi = \frac{Q^2}{2P \cdot q}$

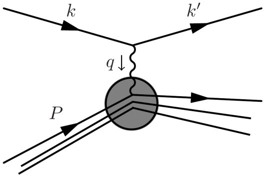


Parton Distribution Functions

- PDF: probability of constituent with momentum fraction ξ
- test: deep inelastic scattering (DIS)
- large momentum transfer $Q^2 = -q^2$
- Bjorken parameter $\xi = \frac{Q^2}{2P\cdot q}$
- factorization:

$$\underset{\text{experiment}}{\sigma(\xi, Q^2)} = \underset{\text{perturbative}}{\hat{\sigma}(\xi, Q^2)}\ \underset{\text{non-perturbative}}{f_\Psi(\xi)}$$





[Schwartz 2014]

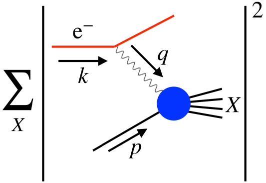
Parton Distribution Functions

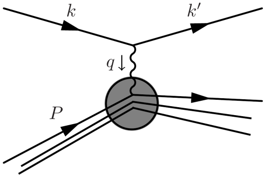
- PDF: probability of constituent with momentum fraction ξ
- test: deep inelastic scattering (DIS)
- large momentum transfer $Q^2 = -q^2$
- Bjorken parameter $\xi = \frac{Q^2}{2P\cdot q}$
- factorization:

$$\sigma(\xi, Q^2) = \hat{\sigma}(\xi, Q^2)\ f_\Psi(\xi)$$

experiment perturbative non-perturbative

$$\rightarrow \boxed{f(\xi) = \int dz^- e^{-i\xi P^+ z^-} \langle P | \bar{\psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\psi(0) | P \rangle}$$





[Schwartz 2014]

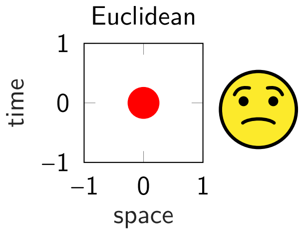
Parton Distribution Functions

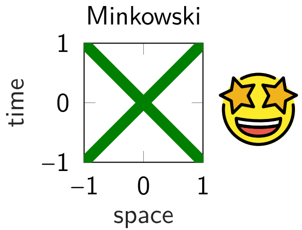
- PDF: probability of constituent with momentum fraction ξ
- test: deep inelastic scattering (DIS)
- large momentum transfer $Q^2 = -q^2$
- Bjorken parameter $\xi = \frac{Q^2}{2P\cdot q}$
- factorization:

$$\underset{\text{experiment}}{\sigma\left(\xi, Q^2\right)} = \underset{\text{perturbative}}{\hat{\sigma}\left(\xi, Q^2\right)}\ \underset{\text{non-perturbative}}{f_\Psi\left(\xi\right)}$$

$$\rightarrow \boxed{f(\xi) = \int dz^- e^{-i\xi P^+ z^-} \left\langle P \middle| \bar{\psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\psi(0) \middle| P \right\rangle}$$

- z^-: lightcone coordinate
- lattice QCD in euclidean space: lightcone $\rightarrow$ point
- Hamiltonian formalism: lightcone in Minkowski space
- Hamiltonian formalism
  $\rightarrow$ use tensor network states/quantum devices





Outline

Tensor Network States

- generic state scales exponentially

$$|\psi\rangle = \sum_{s_1, s_2, \ldots, s_N} \Psi^{s_1 s_2 \ldots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \ldots \otimes |s_N\rangle$$

Tensor Network States

- generic state scales exponentially

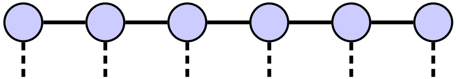
$$|\psi\rangle = \sum_{s_1, s_2, \ldots, s_N} \Psi^{s_1 s_2 \ldots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \ldots \otimes |s_N\rangle$$

- tensor network state as ansatz

$$\Psi^{s_1 s_2 \ldots s_N} = \sum_{\{i_x\}} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \ldots A^{N,s_N}_{i_{N-1}}$$

- 1d: matrix product state (MPS)

$=$



Tensor Network States

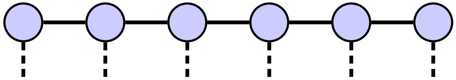
- generic state scales exponentially

$$|\psi\rangle = \sum_{s_1, s_2, \dots, s_N} \Psi^{s_1 s_2 \dots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \dots \otimes |s_N\rangle$$

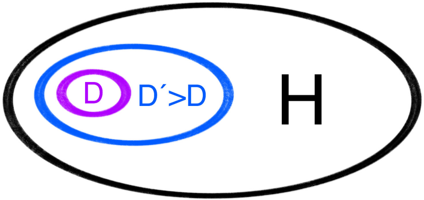
- tensor network state as ansatz

$$\Psi^{s_1 s_2 \dots s_N} \approx \sum_{\{i_x\}=1}^{D} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \dots A^{N,s_N}_{i_{N-1}}$$

- 1d: matrix product state (MPS)

$=$ 

- truncation to bond dimenion D
- polynomial resource scaling



Tensor Network States

- generic state scales exponentially

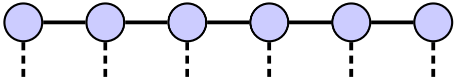
$$|\psi\rangle = \sum_{s_1, s_2, \ldots, s_N} \Psi^{s_1 s_2 \ldots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \ldots \otimes |s_N\rangle$$

- tensor network state as ansatz

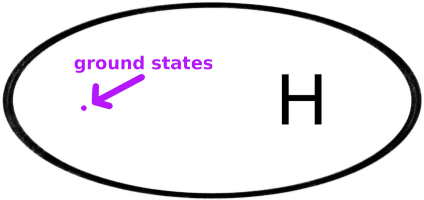
$$\Psi^{s_1 s_2 \ldots s_N} \approx \sum_{\{i_x\}=1}^{D} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \ldots A^{N,s_N}_{i_{N-1}}$$

- 1d: matrix product state (MPS)

=



- truncation to bond dimenion D
- polynomial resource scaling
- good for ground states and low excited states [Hastings 2007]



Outline

Schwinger Model [Hamer et al. 1997]

- quantum electrodynamics in 1+1 dimensions, $U(1)$ symmetry
- fermion couples to gauge boson → partons
- bound states → hadrons [Bañuls et al. 2013]
- scattering → PDF [Dai et al. 1995]
- Lagrange density:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - g\not{A} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Schwinger Model [Hamer et al. 1997]

- quantum electrodynamics in 1+1 dimensions, $U(1)$ symmetry
- fermion couples to gauge boson → partons
- bound states → hadrons [Bañuls et al. 2013]
- scattering → PDF [Dai et al. 1995]
- Lagrange density:

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - gA\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- for TN/QC: transform action into spin-model Hamiltonian:

$$\boxed{H = x\sum_{n=0}^{N-2}\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right] + \frac{\mu}{2}\sum_{n=0}^{N-1}\left[1 + (-1)^n\sigma_n^z\right] + \sum_{n=0}^{N-2}\left[\frac{1}{2}\sum_{k=0}^{n}\left((-1)^k + \sigma_k^z\right)\right]^2}$$

$$\left(x = \frac{1}{a^2g^2}, \mu = \frac{2m}{ag^2}\right)$$

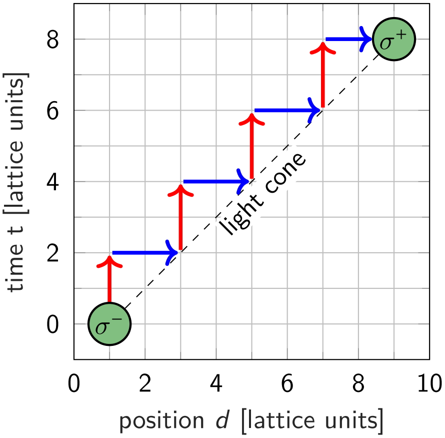
Lightfront matrix elements

$$\left\langle P \,\middle|\, \bar{\Psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\Psi(0) \,\middle|\, P \right\rangle$$
$$\rightarrow \langle P| \, \sigma^+(z^-) \, W_{z^- \leftarrow 0} \sigma^-(0) \, |P\rangle$$

Lightfront matrix elements

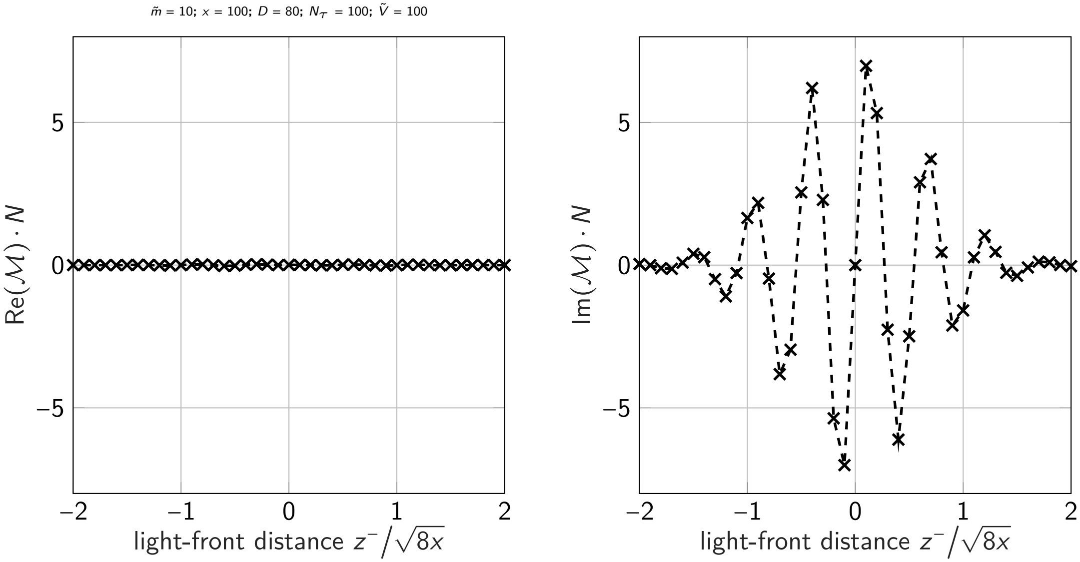
$$\left\langle P \,\middle|\, \bar{\Psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\Psi(0) \,\middle|\, P \right\rangle$$
$$\rightarrow \langle P|\,\sigma^+(z^-)\,W_{z^-\leftarrow 0}\sigma^-(0)\,|P\rangle$$

- lightcone
  $\rightarrow$ small time- and space-like steps
- time evolution:
  $e^{-i\tau H} \approx \left(e^{-i\delta\tau H_{eo}}\, e^{-i\delta\tau H_{oe}}\, e^{-i\delta\tau H_L}\right)^{N_\tau}$
- spatial evolution:
  change electric field along the path

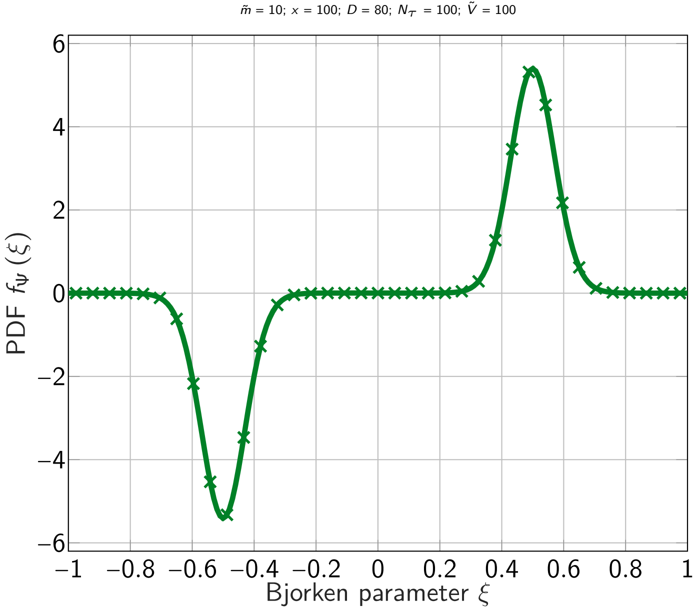


Outline

Results: matrix elements



Results: PDF

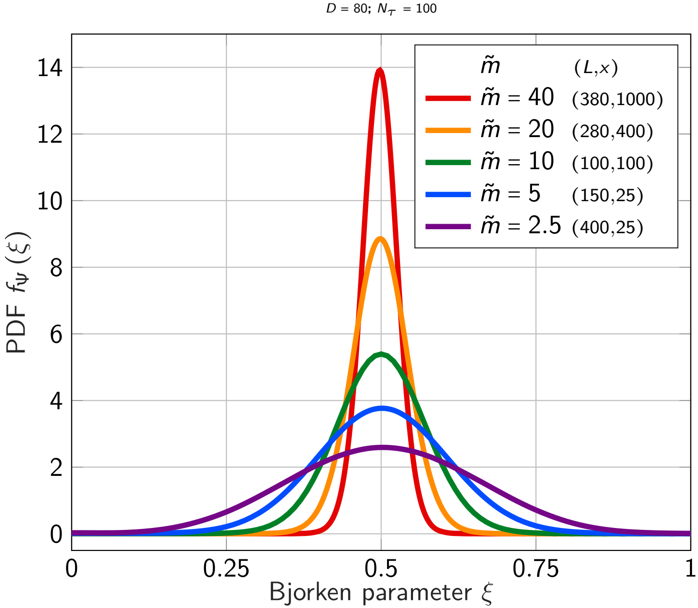


- $\xi > 0$: $f_\psi \approx$ symmetric around $\xi = 0.5$
- antifermion PDF:

$$f_{\overline{\psi}}(\xi) = -f_\psi(-\xi)$$

- $f_{\overline{\psi}}(\xi) = f_\psi(\xi)$
  $\Rightarrow$ meson ✓

Results: PDF

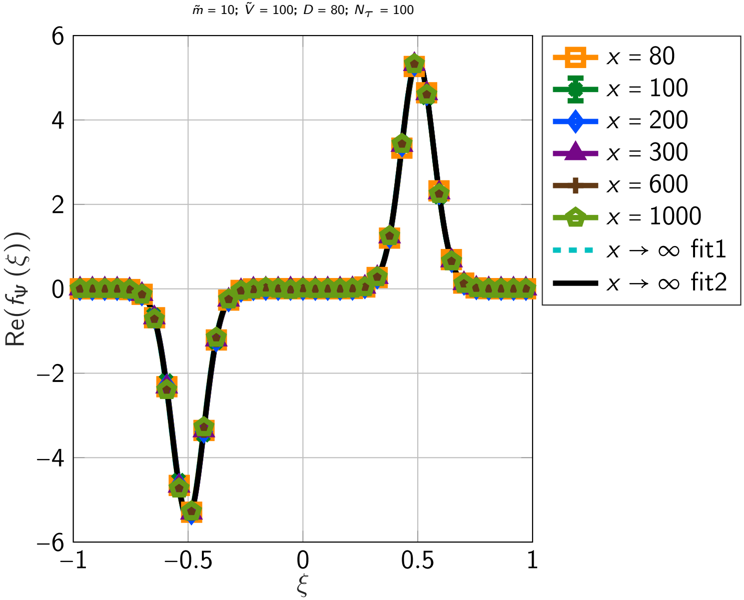


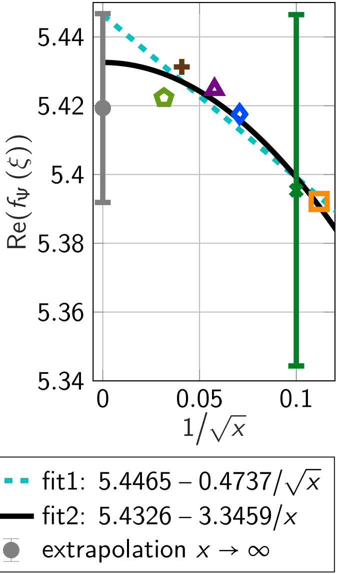
- $\xi > 0$: $f_\psi \approx$ symmetric around $\xi = 0.5$
- antifermion PDF:

$$f_{\overline{\psi}}(\xi) = -f_\psi(-\xi)$$

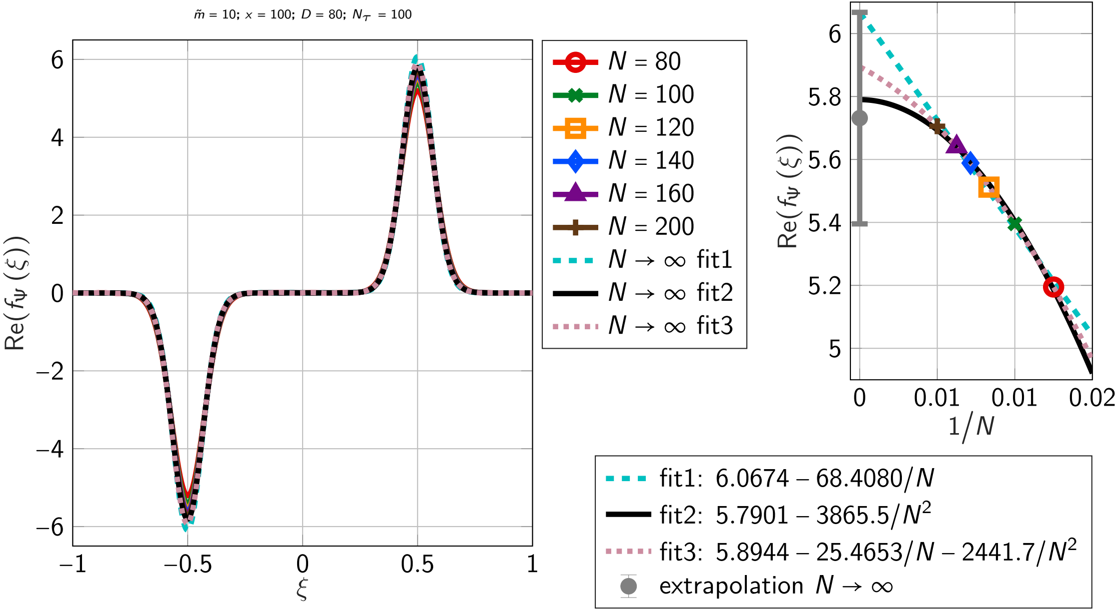
- $f_{\overline{\psi}}(\xi) = f_\psi(\xi)$
  $\Rightarrow$ meson ✓
- peak broadens with decreasing fermion mass ✓

Results: x-dependence

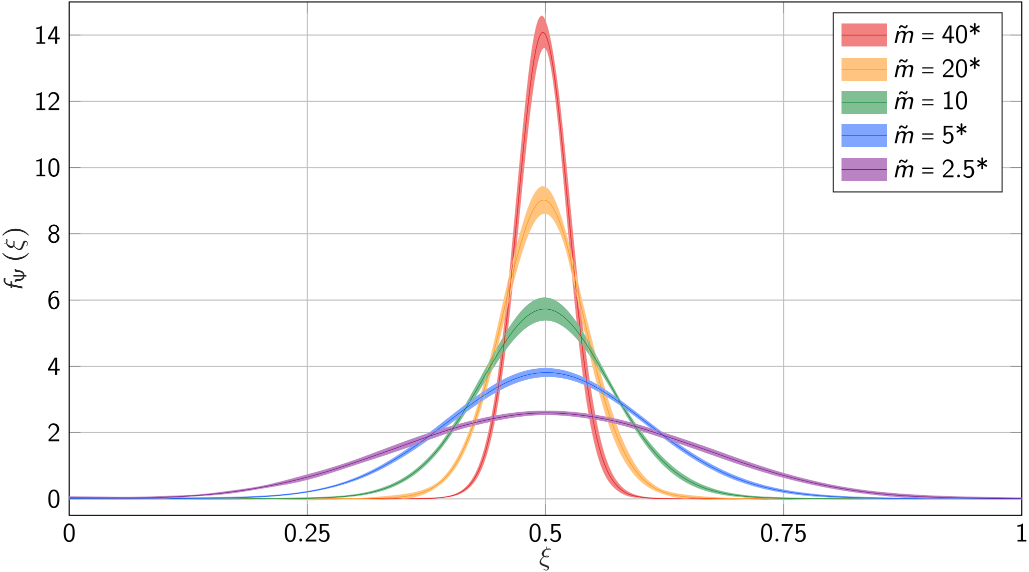




Results: N-dependence



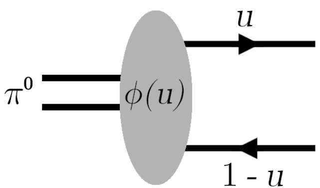
Results: PDF [*preliminary]



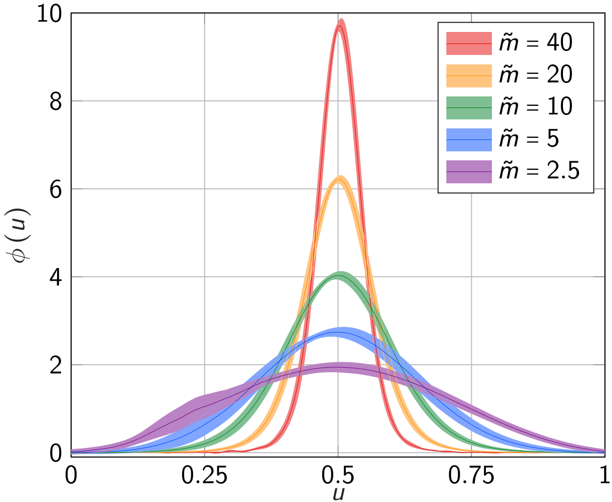
Outline

Outlook: lightcone distribution amplitude (LCDA) [preliminary]

LCDA: decay or hadronization in inclusive processes



$$\pi^0 \to q(u)\bar{q}(1-u) \to \gamma\gamma*$$

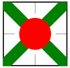


$$\boxed{if\phi(u) = \int dz^- e^{iuP^+z^-} \langle 0 | \bar{\psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\psi(0) | P \rangle}$$

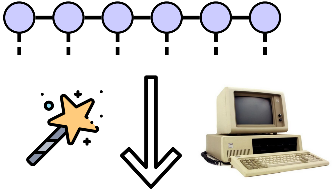
Summary

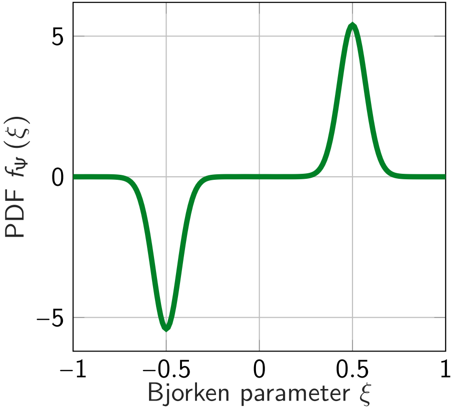
Summary:

- PDF → universal structure of hadrons
- Euclidean space: lightcone → point



- ⇒ use tensor network states / quantum devices
- Schwinger model:
  fermion- and anti-fermion-PDF for the vector meson





Outlook:

- further lightcone observables
- same analysis for QCD 🙂

[arXiv:2504.07508]
[arXiv:2409.16996]

[1] M. C. Bañuls, K. Cichy, C. J. D. Lin, and M. Schneider, "Parton distribution functions in the schwinger model from tensor network states," arXiv e-prints, arXiv:2504.07508 (2025) doi:10.48550/arXiv.2504.07508.

[2] M. Schneider, M. C. Bañuls, K. Cichy, and C.-J. D. Lin, "Parton Distribution Functions in the Schwinger Model with Tensor Networks," in Proceedings of the 41st international symposium on lattice field theory — pos(lattice2024), Vol. 466 (2025), p. 024, doi:10.22323/1.466.0024.

[3] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, (Cambridge University Press, Mar. 2014), ISBN: 978-1-107-03473-0, 978-1-107-03473-0, doi:10.1017/9781139540940.

[4] M. B. Hastings, "An area law for one-dimensional quantum systems," Journal of Statistical Mechanics: Theory & Exp. **2007**, 08024 (2007) doi:10.1088/1742-5468/2007/08/P08024.

[5] M. C. Bañuls, K. Cichy, J. I. Cirac, and K. Jansen, "The mass spectrum of the schwinger model with matrix product states," JHEP **2013**, 158 (2013) doi:10.1007/JHEP11(2013)158.

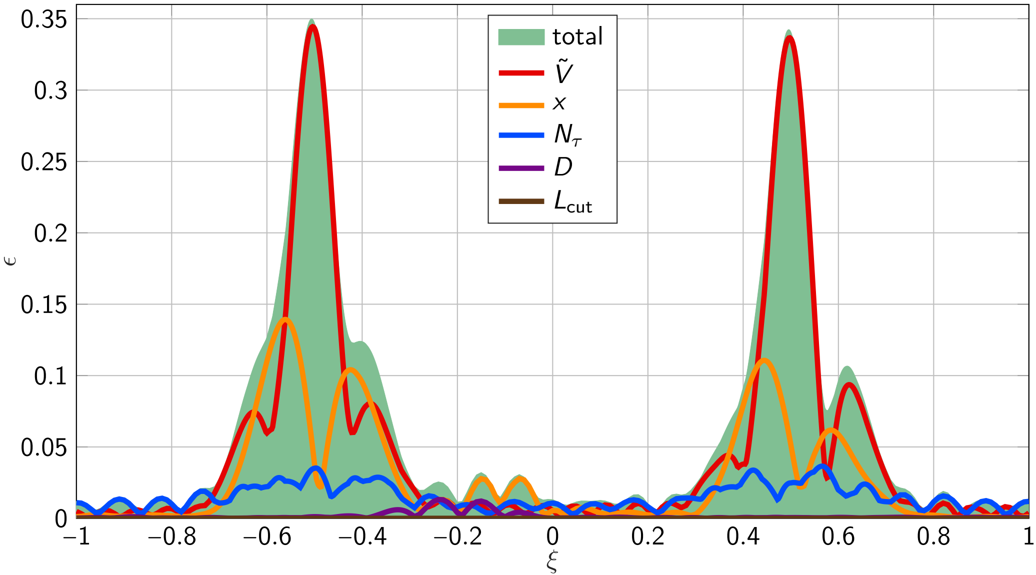
[6] J. Dai, J. Hughes, and J. Liu, "Perturbative analysis of the massless schwinger model," Phys. Rev. D **51**, 5209–5215 (1995) doi:10.1103/PhysRevD.51.5209.

[7] C. J. Hamer, Z. Weihong, and J. Oitmaa, "Series expansions for the massive schwinger model in hamiltonian lattice theory," Phys. Rev. D **56**, 55–67 (1997) doi:10.1103/PhysRevD.56.55.

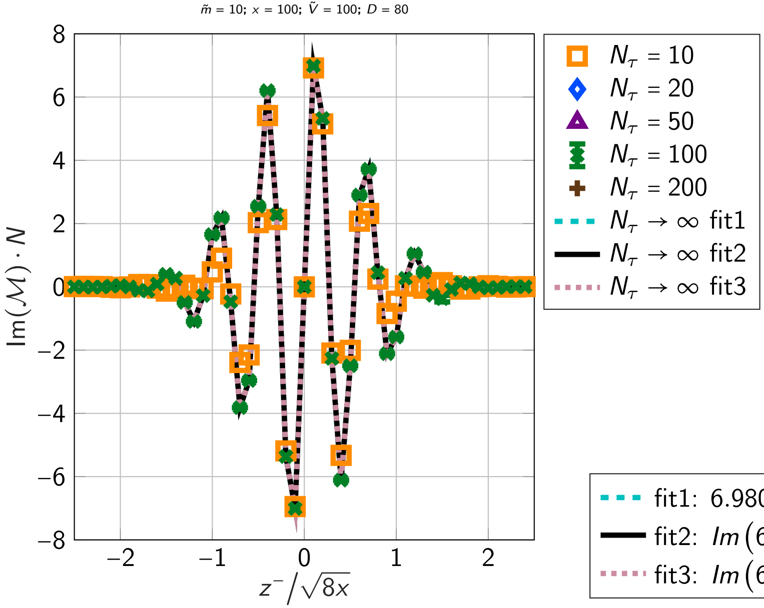
[8] Further image sources, EIC, www.computerhistory.org/timeline/1981, https://openmoji.org, www.flaticon.com/free-icons/search.

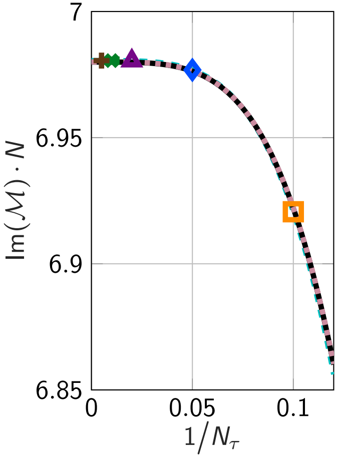
Outline

Contributions to error for $\tilde{m} = 10$



Results: N_τ-dependence



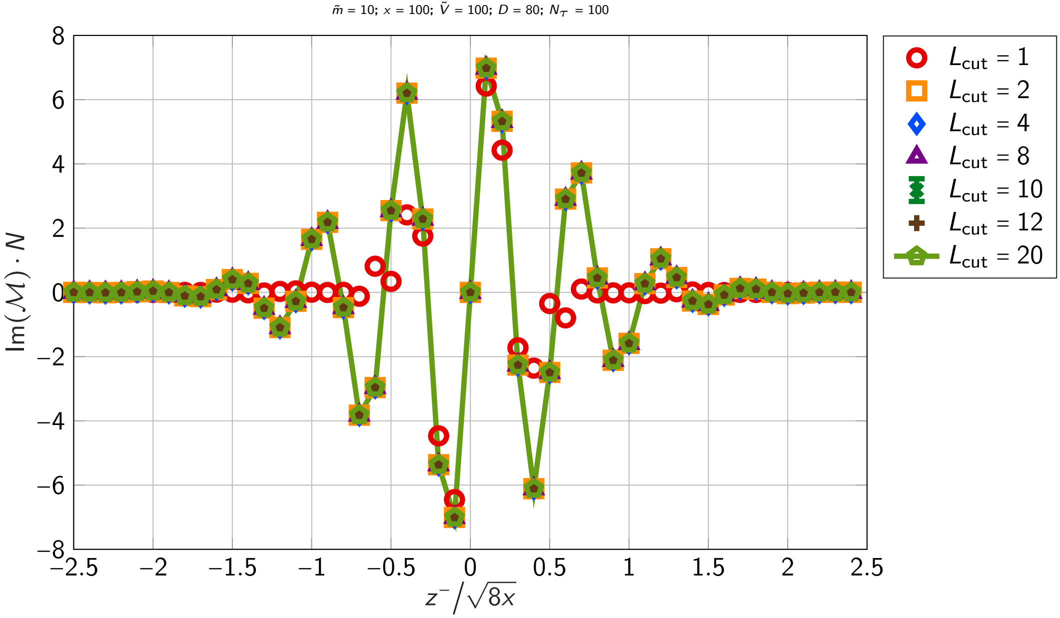


fit1: $6.9805 + 0.0179 N_\tau^{-2} - 600.9051 N_\tau^{-4}$

fit2: $Im\left(6.9800 exp\left(-6.4460 i N_\tau^{-2} d\right)\right)$

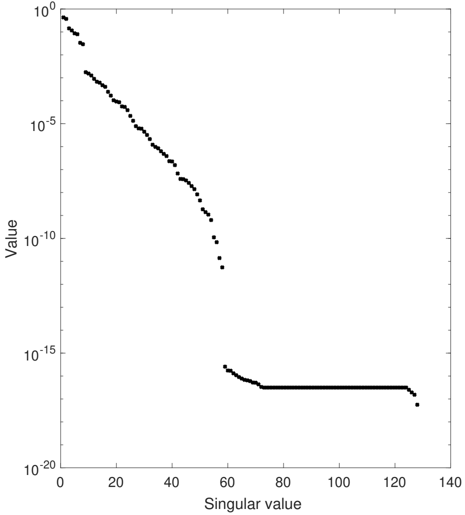
fit3: $Im\left(6.9800 exp\left(-6.4411 i N_\tau^{-2} d\right)\right)$

L_{cut}-dependence: truncation of electric field

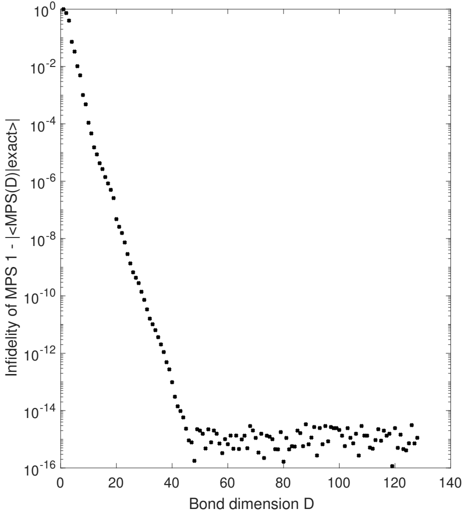


Singular values and cutoff

Schwinger model, $L = 14$, $\mu = 0.125$, $x = 10$, 2nd excitation



Singular values for cut in the middle



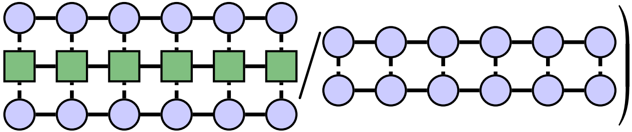
Infidelity on MPS with exact state

$1 - |\langle \Psi(D) | \Psi_{\text{exact}} \rangle|$

Efficient Tensor Network operations

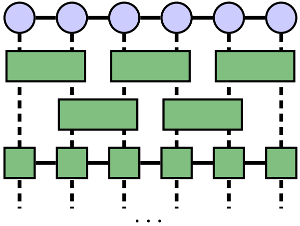
- Find groundstate and excited states

$$\min\left(E = \frac{\langle \Psi | \hat{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \right.$$

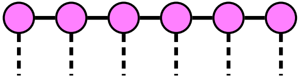


- Apply operators / time evolution

$$\hat{O}\,|\Psi\rangle =$$

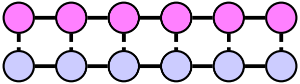


$$\longrightarrow |\Phi\rangle =$$



- Calculate overlap

$$\langle \Psi | \Phi \rangle =$$



Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - g\not{A} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_0\rho$$

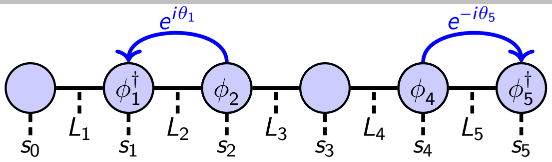
$$\mathcal{H} = -i\bar{\Psi}\gamma^1(\partial_1 - igA_1)\Psi + m\bar{\Psi}\Psi + \frac{1}{2}E^2$$

Legendre transformation
$(E = F_{01})$

Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - gA\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_0\rho$$

$$\mathcal{H} = -i\bar{\Psi}\gamma^1(\partial_1 - igA_1)\Psi + m\bar{\Psi}\Psi + \frac{1}{2}E^2$$



$$H = -\frac{i}{2a}\sum_n\left(\phi_n^\dagger e^{i\theta_n}\phi_{n+1} - \phi_{n+1}^\dagger e^{-i\theta_n}\phi_n\right) + m\sum_n(-1)^n\phi_n^\dagger\phi_n + \frac{ag^2}{2}\sum_n L_n^2$$

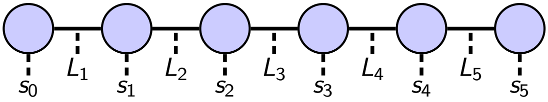
staggered fermions
$(\theta = agA_1, gL = E)$

$$\phi_n \sim \begin{cases}\Psi_{\text{upper}}(x) & \text{if } n \text{ even}\\ \Psi_{\text{lower}}(x) & \text{if } n \text{ odd,}\end{cases}$$

Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - gA\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_0\rho$$

$$\mathcal{H} = -i\bar{\Psi}\gamma^1(\partial_1 - igA_1)\Psi + m\bar{\Psi}\Psi + \frac{1}{2}E^2$$



$$H = -\frac{i}{2a}\sum_n\left(\phi_n^\dagger \cancel{e^{i\theta_n}}\phi_{n+1} - \phi_{n+1}^\dagger \cancel{e^{-i\theta_n}}\phi_n\right) + m\sum_n(-1)^n\phi_n^\dagger\phi_n + \frac{ag^2}{2}\sum_n L_n^2$$

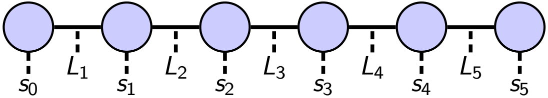
decoupling

$$\phi_n \to \prod_{k<n}\left(e^{-i\theta_k}\right)\phi_n$$

Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - gA\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_0\rho$$

$$\mathcal{H} = -i\bar{\Psi}\gamma^1(\partial_1 - igA_1)\Psi + m\bar{\Psi}\Psi + \frac{1}{2}E^2$$



$$H = -\frac{i}{2a}\sum_n \left(\phi_n^\dagger \cancel{e^{i\theta_n}} \phi_{n+1} - \phi_{n+1}^\dagger \cancel{e^{-i\theta_n}} \phi_n\right) + m\sum_n (-1)^n \phi_n^\dagger \phi_n + \frac{ag^2}{2}\sum_n L_n^2$$

$$H = \frac{1}{2a}\sum_n (\sigma_n^+\sigma_{n+1}^- + \sigma_{n+1}^-\sigma_n^+) + \frac{m}{2}\sum_n [1 + (-1)^n\sigma_n^z] + \frac{ag^2}{2}\sum_n L_n^2$$

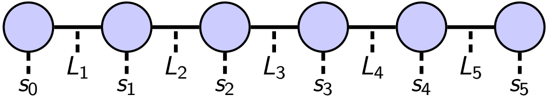
Jordan-Wigner transformation

$$\hat{\phi}_n = \prod_{k<n}(i\sigma_k^z)\,\sigma_n^-$$

Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - gA\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_0\rho$$

$$\mathcal{H} = -i\bar{\Psi}\gamma^1(\partial_1 - igA_1)\Psi + m\bar{\Psi}\Psi + \frac{1}{2}E^2$$



$$H = -\frac{i}{2a}\sum_n\left(\phi_n^\dagger \cancel{e^{i\theta_n}}\phi_{n+1} - \phi_{n+1}^\dagger \cancel{e^{-i\theta_n}}\phi_n\right) + m\sum_n(-1)^n\phi_n^\dagger\phi_n + \frac{ag^2}{2}\sum_n L_n^2$$

$$H = \frac{1}{2a}\sum_n(\sigma_n^+\sigma_{n+1}^- + \sigma_{n+1}^-\sigma_n^+) + \frac{m}{2}\sum_n[1 + (-1)^n\sigma_n^z] + \frac{ag^2}{2}\sum_n L_n^2$$

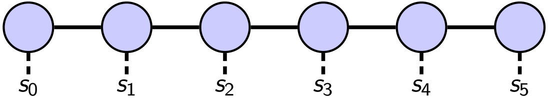
Gauss's law:
$(x = \frac{1}{a^2g^2}, \mu = \frac{2m}{ag^2})$

$$L_n - L_{n-1} = Q_n = \frac{1}{2}[(-1)^n + \sigma_n^z] + q_n$$

Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - g\not{A} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_0\rho$$

$$\mathcal{H} = -i\bar{\Psi}\gamma^1(\partial_1 - igA_1)\Psi + m\bar{\Psi}\Psi + \frac{1}{2}E^2$$



$$H = -\frac{i}{2a}\sum_n\left(\phi_n^\dagger \cancel{e^{i\theta_n}}\phi_{n+1} - \phi_{n+1}^\dagger \cancel{e^{-i\theta_n}}\phi_n\right) + m\sum_n(-1)^n\phi_n^\dagger\phi_n + \frac{ag^2}{2}\sum_n L_n^2$$

$$H = \frac{1}{2a}\sum_n(\sigma_n^+\sigma_{n+1}^- + \sigma_{n+1}^-\sigma_n^+) + \frac{m}{2}\sum_n[1 + (-1)^n\sigma_n^z] + \frac{ag^2}{2}\sum_n L_n^2$$

$$H = x\sum_{n=0}^{N-2}[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+] + \frac{\mu}{2}\sum_{n=0}^{N-1}[1 + (-1)^n\sigma_n^z] + \sum_{n=0}^{N-2}\left[\sum_{k=0}^{n} Q_k\right]^2$$

Gauss's law:
$\left(x = \frac{1}{a^2g^2}, \mu = \frac{2m}{ag^2}\right)$

$$L_n - L_{n-1} = Q_n = \frac{1}{2}[(-1)^n + \sigma_n^z] + q_n$$

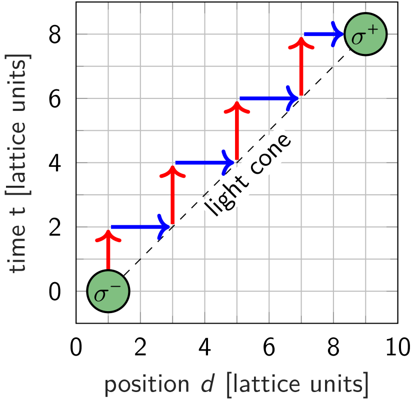
Lattice forumlation of PDF

PDF for lattice spin model:

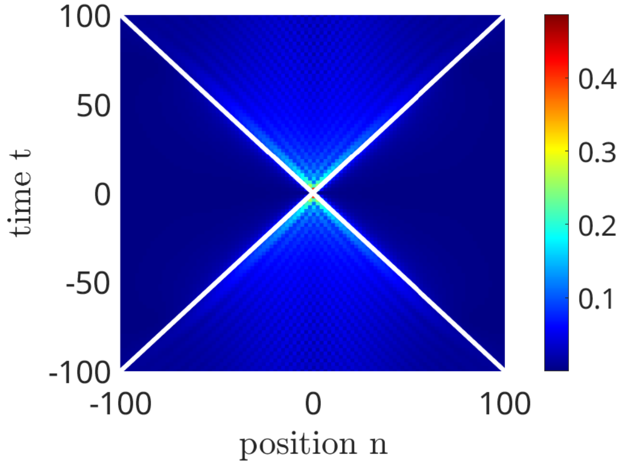
$$f_\Psi(\xi) = \frac{NM}{8\pi x} \sum_{d=0,2,4,\dots,L} e^{-i\xi \frac{Md}{2x}} \mathcal{M}(d)$$

$$\mathcal{M}(d) = \mathcal{M}_{0,0}(d) - \mathcal{M}_{0,1}(d) - \mathcal{M}_{1,0}(d) + \mathcal{M}_{1,1}(d)$$

$$\mathcal{M}_{a,b}(d) = \langle h| e^{iHt_d} \prod_{k<d+a} \left(-i\sigma_k^z\right) \sigma_{d+a}^{+} e^{-iH_{d-1}\delta t} \dots e^{-iH_3\delta t} e^{-iH_1\delta t} \prod_{k'<b} \left(i\sigma_{k'}^z\right) \sigma_b^- |h\rangle_c \,.$$



Light-cone structure



$$\langle P|\, e^{iHt} \prod_{k<n} (i\sigma_k^z)\, \sigma_n^+ e^{-iH_0 t} \prod_{k'<0} (-i\sigma_{k'}^z)\, \sigma_0^- \,|P\rangle$$

- even-to-even matrix element
- calculated to each site at each timeslice
- static charge fixed at orign

Factorization

cross section:

$$\sigma \propto L^{\mu\nu}(k,q)\, W_{\mu\nu}(q,P)$$

hadronic Tensor:

$$W_{\mu\nu}(\xi,P) = \sum_i \int_\xi^1 \frac{dz}{z} f_i(z)\, \hat{W}_{\mu\nu}\left(\frac{\xi}{z}, Q\right)$$

leading order with $\hat{W} \propto \delta\left(1-\frac{\xi}{z}\right)$:

$$W_{\mu\nu}(q,P) = 4\pi\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)F_1 + \frac{8\pi x}{Q^2}\left(P_\mu - \frac{P\cdot q}{q^2}q_\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)F_2$$

factorization (leading order):

$$F_1(\xi) = \frac{1}{2}\sum_i e_i^2 f_i(\xi)$$

$$F_2(\xi) = 2xF_1(\xi)$$