Three-loop jet function for boosted top quarks

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In collaboration with: A.M. Clavero, R. Brüser and M. Stahlhofen, based on JHEP 04 (2025) 040

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Outline

- Motivation
- Jet function computation
- Main results
- Summary

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The top quark is the heaviest particle found so far

Only quark capable of escaping infrared slavery

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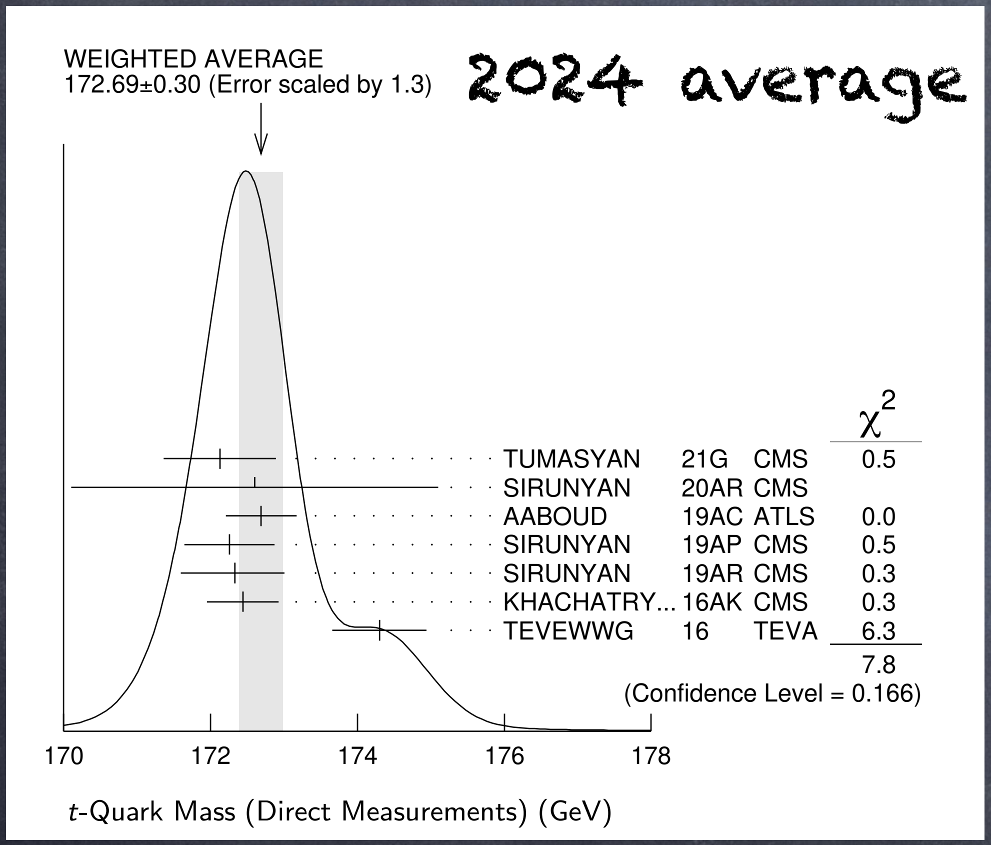

not the pole mass

related to a short-distance mass?

Current world average

$$m_t^{\mathrm{MC}} = (172.56 \pm 0.31)\,\mathrm{GeV}$$

[PRL 117 (2016) 23, 232001, JHEP 12 (2023) 065]

Calibration: make relation quantitative

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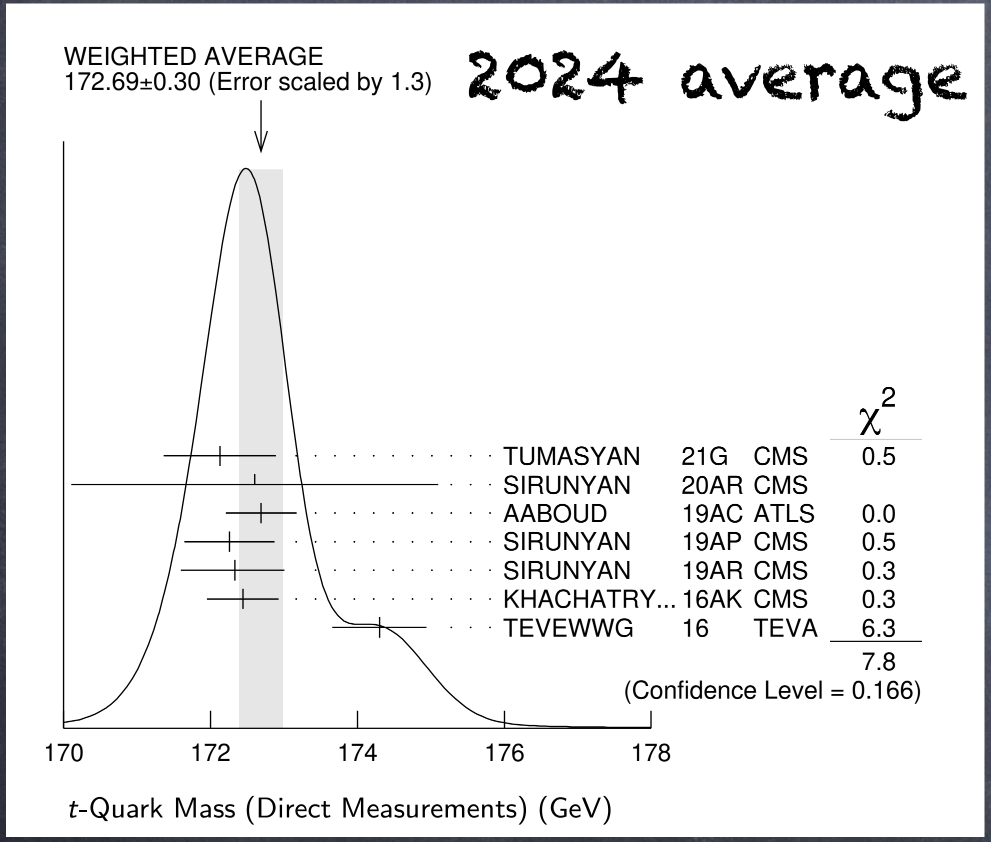

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Calibration: make relation quantitative

This talk: compute piece necessary for N³LL′ calibration

or for determination at a future e^+e^- collider

Measuring/calibrating the top mass

Tops decay very fast and, if boosted, form jets

At e^+e^- collider m_t measured with precision better than $\Lambda_{\rm QCD}$ using event shapes

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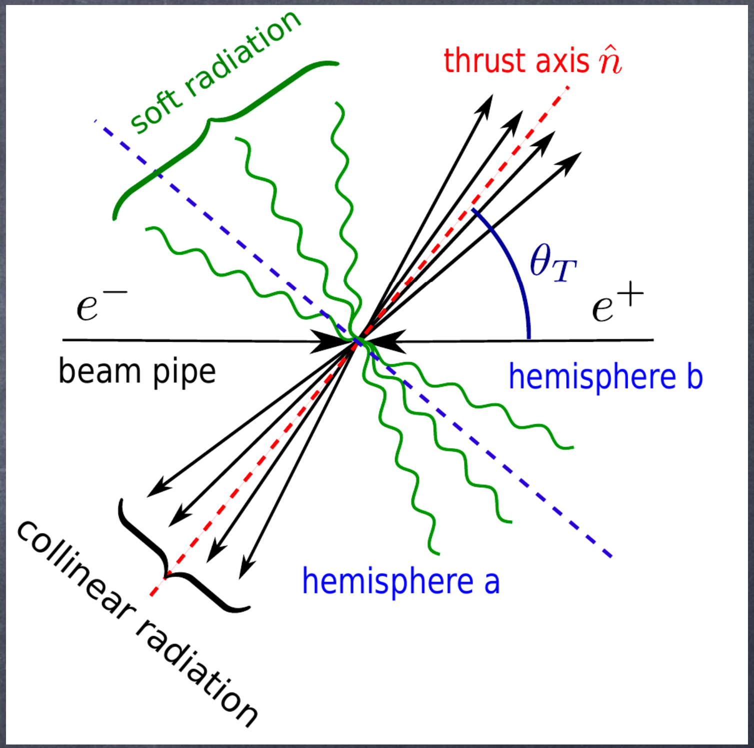
Several relevant energy scales in the process

- Center of mass energy $Q = \sqrt{(p_{e^+} + p_{e^-})^2}$
- Jet mass $M_j^2 = \left[\sum_{i\in\text{jet}} p_i^\mu\right]^2$
- Top mass m_t
- Top width Γ_t



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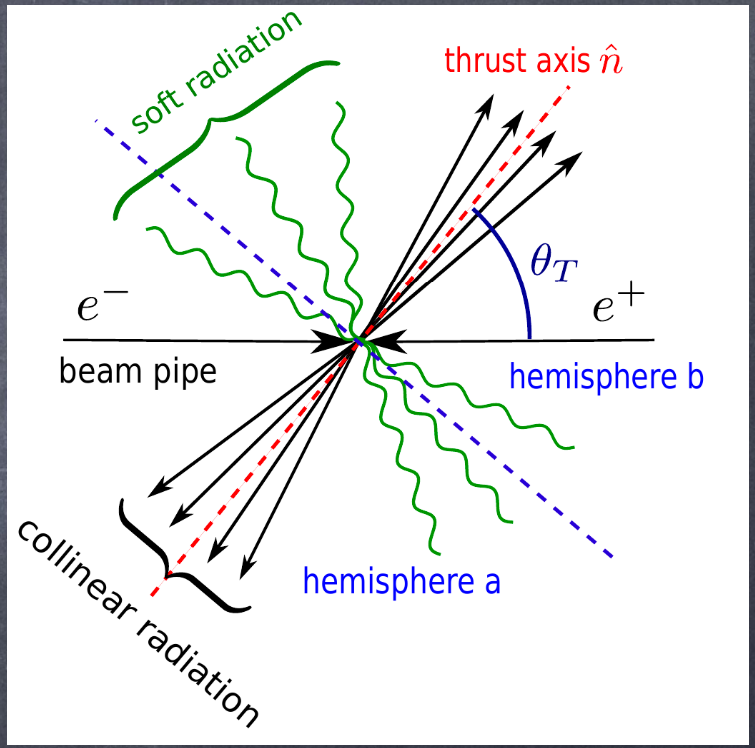
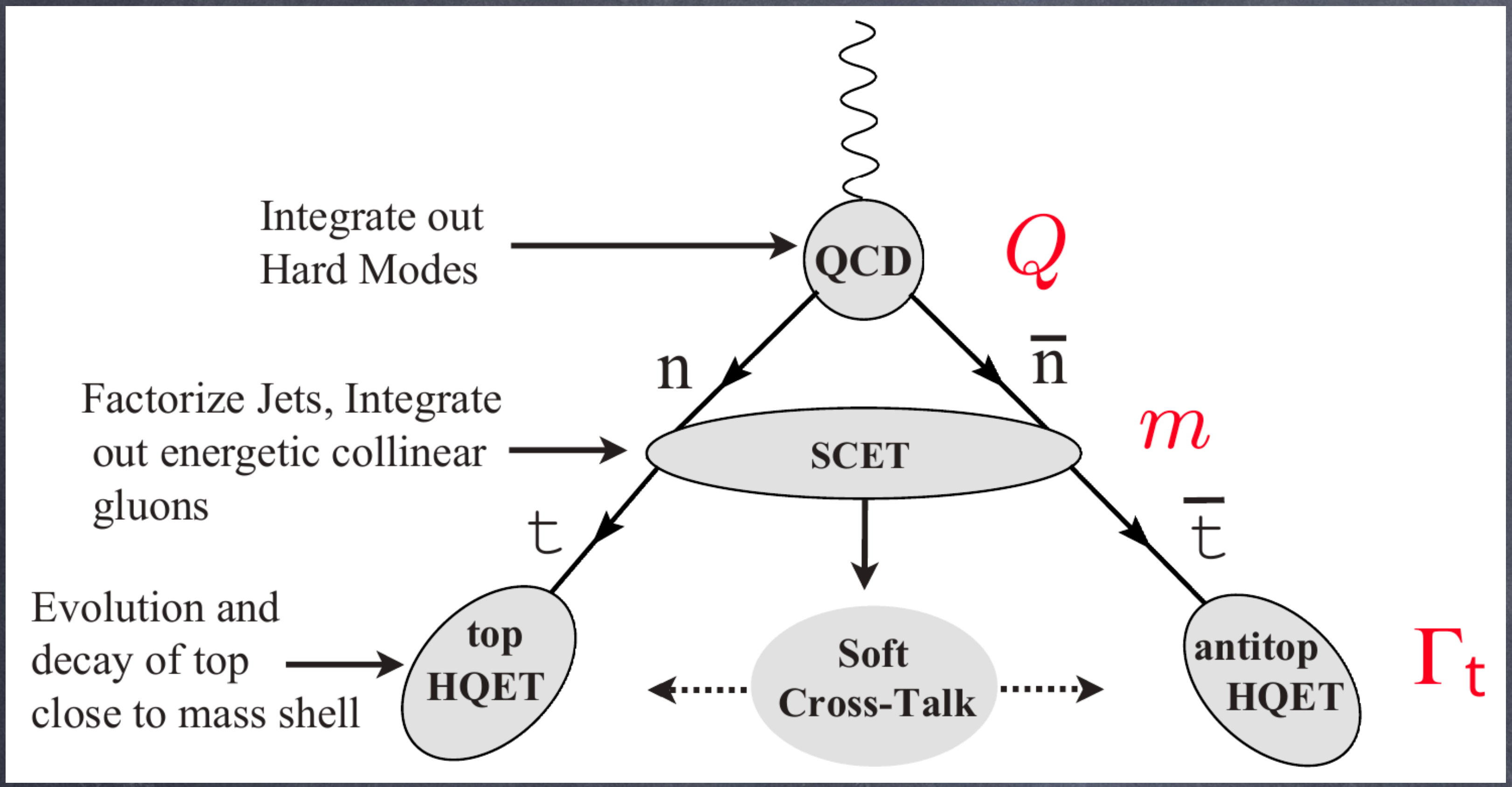


High sensitivity to top mass in peak region $\hat{s} \equiv \frac{M_j^2 - m_t^2}{m_t} \sim \Gamma_t$

Hence we have $Q \gg m_t \sim M_j \gg \Gamma_t \gg \Lambda_{\rm QCD}$ EFTs called for

Sequence of EFTs in peak region

Consider the doubly differential hemisphere mass distribution



match QCD to SCET

↓ evolve

match SCET to bHQET

↓ evolve

jet functions

↓ evolve

soft function

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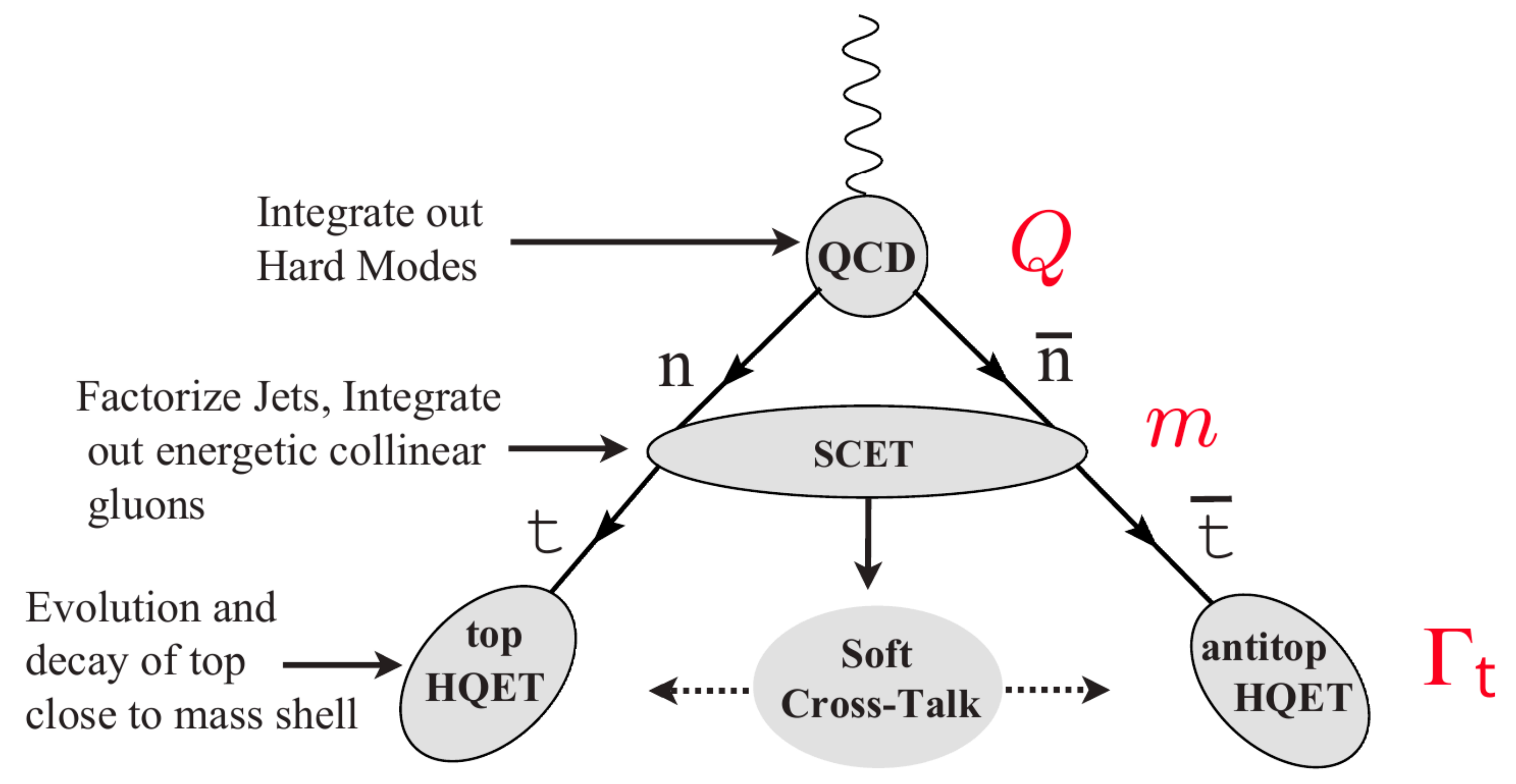

it factorizes at leading power in

$$\frac{m_t}{Q}, \frac{\Gamma_t}{m_t}, \frac{\hat{s}}{m_t}$$

$$\frac{\mathrm{d}^2\sigma^{(\mathrm{dijet})}}{\mathrm{d}M_1^2\mathrm{d}M_2^2} = \sigma_0\, H_Q^{(n_\ell+1)}(Q,\mu)\, H_m\left(m, \frac{Q}{m}, \mu\right) \int \mathrm{d}\ell^+\mathrm{d}\ell^-\, S^{(n_\ell)}(\ell^+,\ell^-,\mu)$$

$$\times\, B_n^{(n_\ell)}\left(\frac{M_1^2 - Q\ell^+}{m} - m, \Gamma_t, \mu\right) B_{\bar{n}}^{(n_\ell)}\left(\frac{M_2^2 - Q\ell^-}{m_t} - m_t, \Gamma_t, \mu\right).$$

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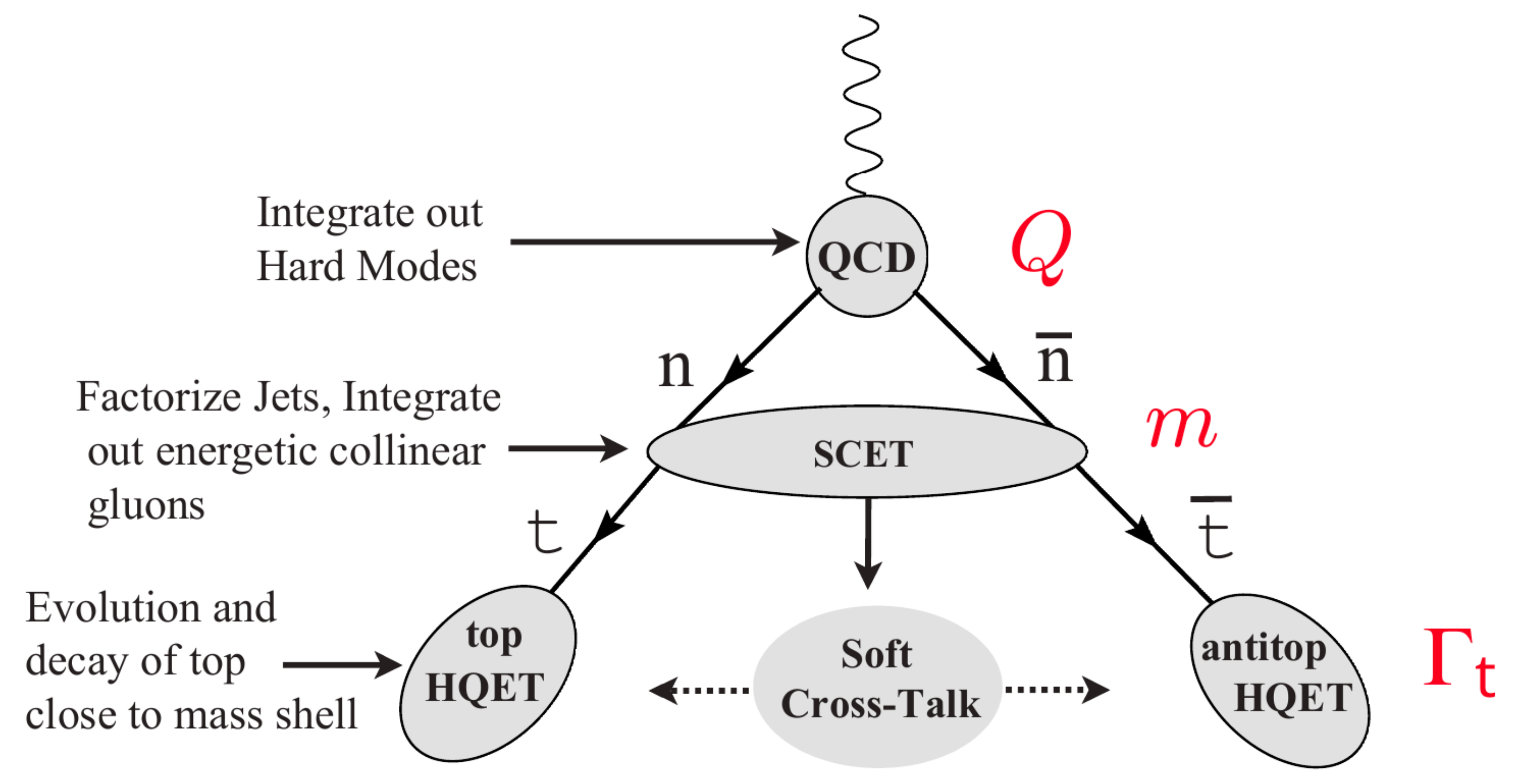

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Hard piece known at 3-loops, soft at 2/3 loops. What about jet?

Jet function for boosted tops

For $\hat{s} = 2\,v \cdot r$ and in the pole scheme, one has

$$B(\hat{s}, 0, \Gamma_t) \equiv \mathbf{Im}[\mathcal{B}(\hat{s} + i\Gamma_t)]\,, \quad \mathcal{B}(\hat{s}) \equiv \frac{-i}{4\pi N_c m_t} \int \mathrm{d}^d x\, e^{i\, r\cdot x} \langle 0 | T\{\bar{h}_v(0) W_n(0) W_n^\dagger(x) h_v(x)\} | 0 \rangle$$

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Appears in factorisation theorems for

- Doubly differential hemisphere mass
- Thrust
- C-parameter (C-jettiness)
- N-jettiness

In position (Fourier) space obeys non-abelian exponentiation

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heavy quark fields

light-like Wilson lines

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v = velocity of top quarks

n = jet direction

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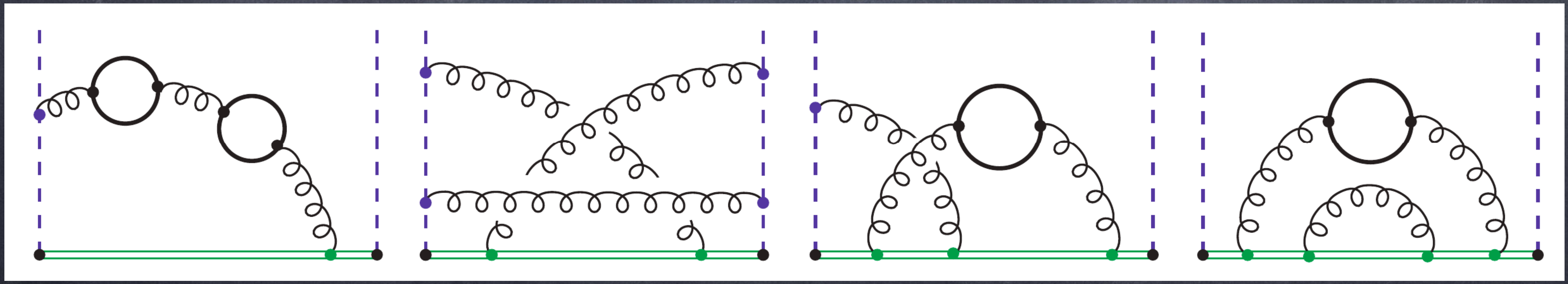
This talk: computation at 3-loops



Involves { eikonal / Heavy quark / Relativistic massless } propagators

Complexity of computation

	1-loop	2-loops	3-loops	4-loops
Feynman diagrams	4	50	1100	31000
Scalar integrals	3	70	5400	?
Master integrals	1	3	20	?

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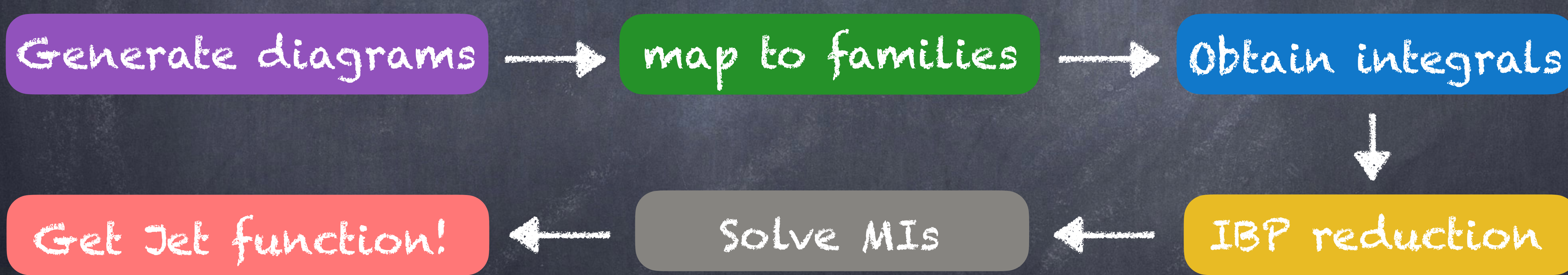

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Workflow

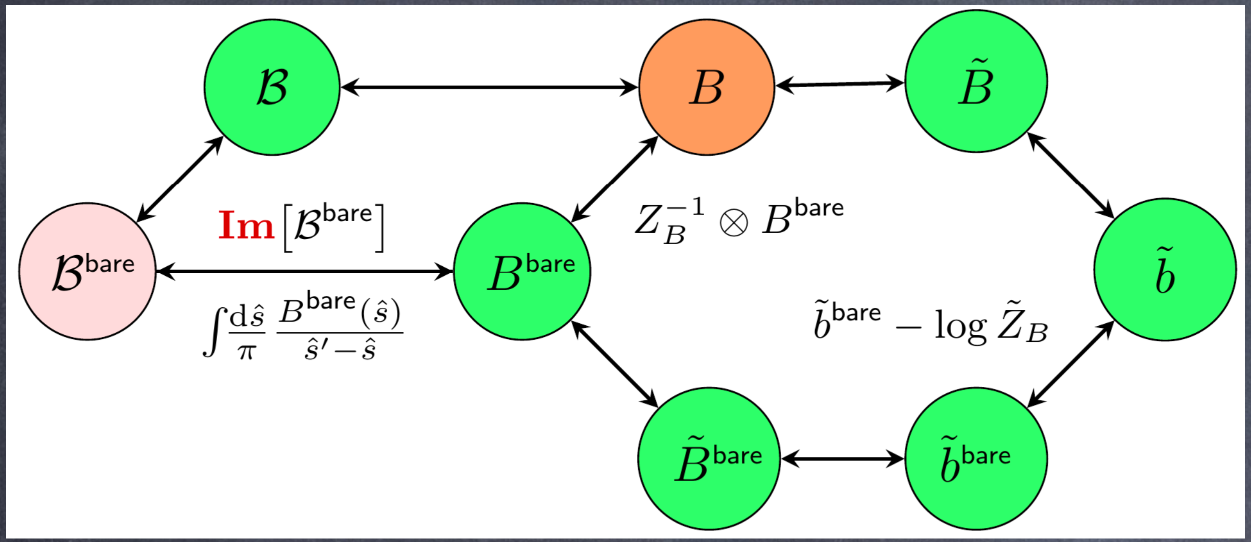
Generate diagrams → map to families → Obtain integrals
↓
IBP reduction → Solve MIs → Get Jet function!

Complexity similar to that of a soft function for heavy-to-light decays: use strategy similar to [Brüser, Liu, Stahlhofen 2020]

Different forms of jet function

Jet junction needs to be renormalised and RG-evolved

Convolutions in momentum space, products in position space



$$\tilde{B}(x) = \int d\hat{s}\, e^{-ix\hat{s}} B(\hat{s})$$

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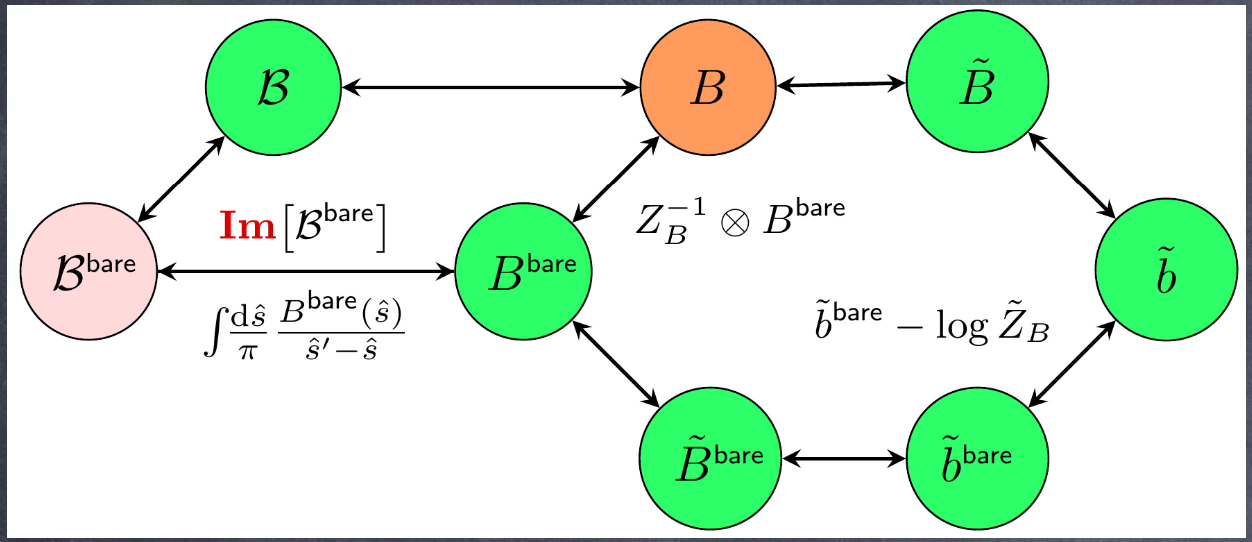

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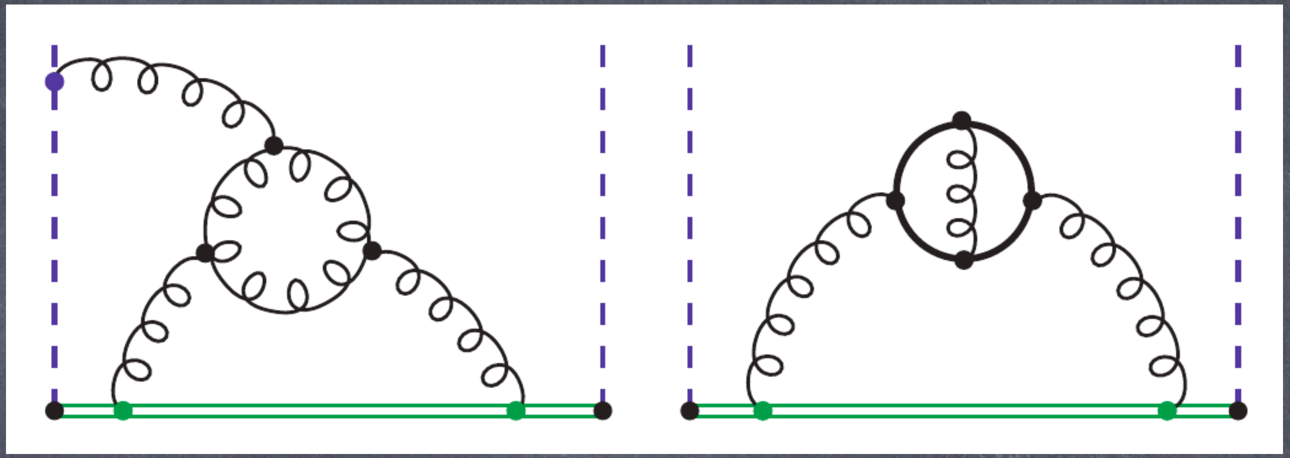
We have derived closed or recursive expressions for

- All renormalization factors
- All divergent pieces
- Relations between coefficients
- Relations to short-distance masses

$$\tilde{B}_{in} = \frac{(-1)^n}{n!} \sum_{j=k-2}^{l-1} B_{ij} |j|!\, \kappa_{j+1-n}$$

Some boring details

1100 diagrams generated with qgraf [Nogueira 1993]



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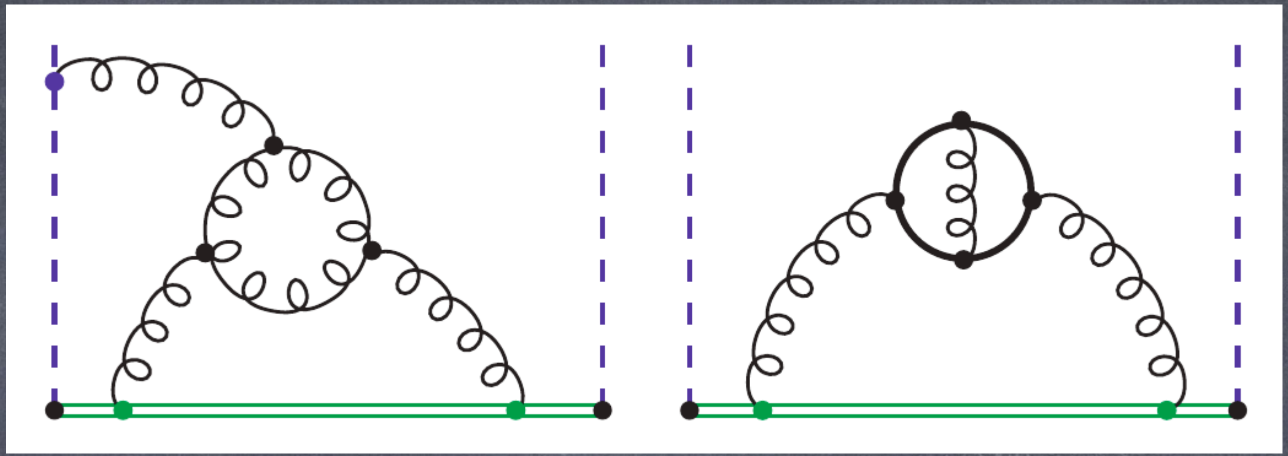

Results processed with Looping [Brüser, unpublished]

- Identify integral families [Pak 2012]
- Color and Dirac algebra with FORM and color.h
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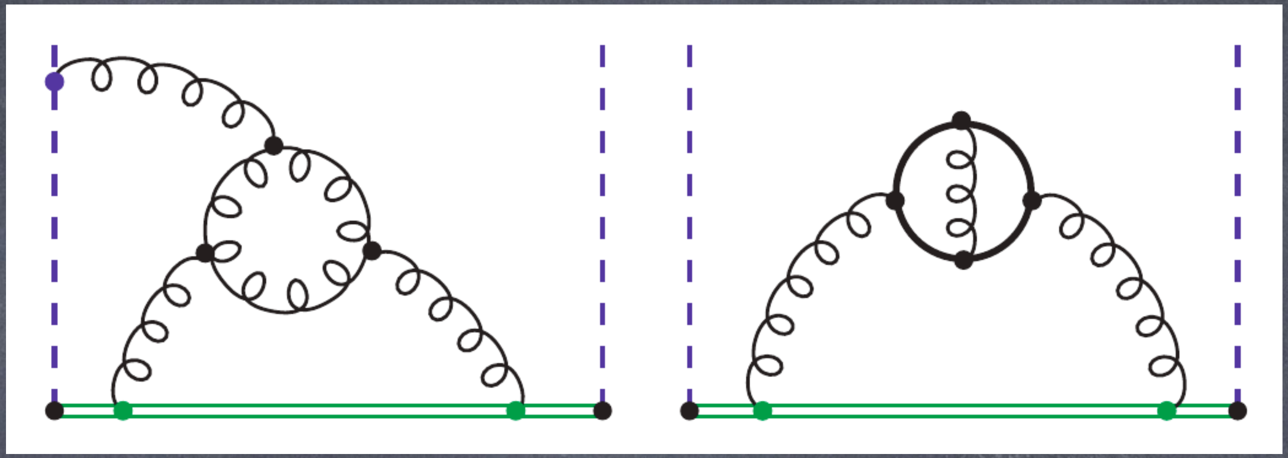

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Left with 5400 scalar integrals

Integral reduction

- IBP reduce each family with FIRE6 [Smirnov, Chuharev 2020] and LiteRed [Lee 2014]

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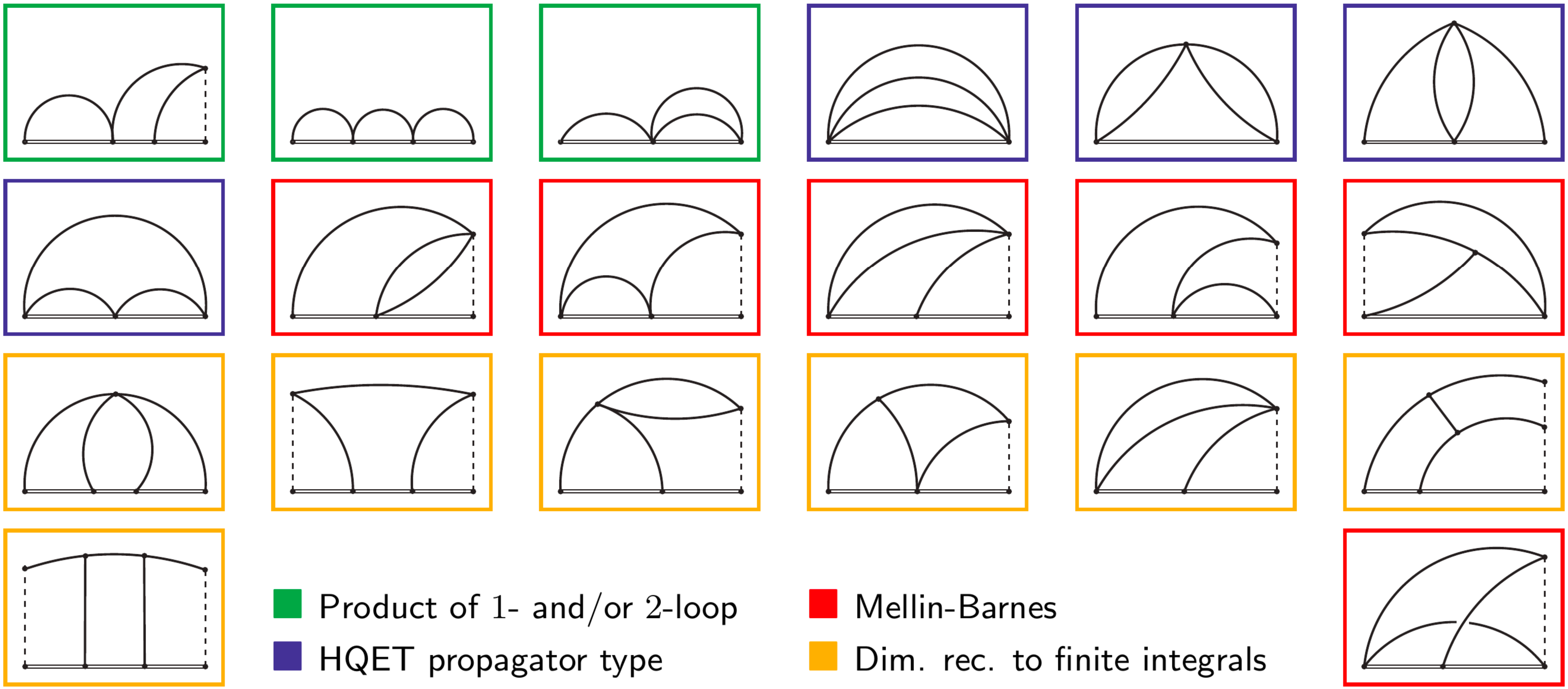
- Other packages: worse performance

Master integrals



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Mellin-Barnes

use identity

$$(A+B)^{-\lambda} = \int_{\mathcal{C}} \frac{\mathrm{d}s}{2\pi i} \frac{\Gamma(s)\Gamma(\lambda - s)}{\Gamma(\lambda)} A^{s-\lambda} B^{-s}$$

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Quasi-finite integrals

Search for finite integrals in d+2n with Reduze2 [2012]

Compute finite integrals with HyperInt [Panzer 2015]

Relate to MI with dim-shifts and IBP

expanded in ε

Result for the jet function

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new!

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Main result: non-logarithmic piece of renormalized $\tilde{b} = \log(\tilde{B})$

$$\tilde{b}_{30} = \left(\frac{203\pi^2\zeta_3}{27} - \frac{105398\zeta_3}{243} + \frac{236\zeta_3^2}{9} + \frac{902\zeta_5}{9} + \frac{31952191}{26244} + \frac{93821\pi^2}{8748} - \frac{3023\pi^4}{4860} + \frac{1031\pi^6}{10206}\right)C_F C_A^2$$
$$+ \left(\frac{3488\zeta_3}{243} + \frac{846784}{6561} - \frac{8\pi^2}{243} + \frac{52\pi^4}{1215}\right)C_F T_F^2 n_\ell^2$$
$$+ \left(\frac{10760\zeta_3}{81} + \frac{8\pi^2\zeta_3}{9} + \frac{224\zeta_5}{9} - \frac{124717}{486} - \frac{55\pi^2}{54} + \frac{148\pi^4}{405}\right)C_F^2 T_F n_\ell$$
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$C_F^3, C_F^2 C_A$ color factors missing

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logarithmic pieces are obtained from anomalous dimensions

$$\tilde{b}_{l1} = \gamma^B_{l-1} + 2\sum_{j=1}^{l-1} j\beta_{l-j-1}\,\tilde{b}_{j0} \qquad \tilde{b}_{l2} = \Gamma^c_{l-1} + \sum_{j=1}^{l-1} j\beta_{l-j-1}\,\tilde{b}_{j1} \qquad \tilde{b}_{lk} = \frac{2}{k}\sum_{j=k-2}^{l-1} j\beta_{l-j-1}\tilde{b}_{j,k-1}$$

Result in other spaces: use general relations given in paper

Z-factors

3-loop cusp and non-cusp anomalous dimensions known

[Korchemsky and Radyushkin, 1987, Moch, Vermaseren and Vogt, 2004]
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Closed form for Z-factors in terms of anomalous dimensions

$$z(\gamma, \alpha_s) \equiv \int_0^{\alpha_s} \frac{\mathrm{d}x}{x} \frac{\gamma(x)}{\varepsilon + \hat{\beta}(x)},$$

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For example

$$\alpha_s^{\mathrm{bare}} = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_\alpha[\alpha_s(\mu)]\, \alpha_s(\mu) \quad \Rightarrow \quad Z_\alpha(\alpha_s) = \exp\left[-z\left(\hat{\beta}, \alpha_s\right)\right]$$

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$$\tilde{B}^{\mathrm{bare}}(x) = \tilde{Z}_B(x,\mu)\, \tilde{B}(x,\mu) \quad \Rightarrow \quad \tilde{Z}_B(x,\mu) = C_B[\alpha_s(\mu)]\, (ix\mu e^{\gamma_E})^{z_B[\alpha_s(\mu)]}$$

$$C_B(\alpha_s) = \exp\left[\frac{1}{2} z(\gamma_B^{\mathrm{nc}} + z_B, \alpha_s)\right] \qquad z_B(\alpha_s) = z(\Gamma^c, \alpha_s)$$

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$$z(\gamma, \alpha_s) \equiv \int_0^{\alpha_s} \frac{\mathrm{d}x}{x} \frac{\gamma(x)}{\varepsilon + \hat{\beta}(x)}, \qquad \hat{\beta}(\alpha_s) = -\frac{1}{2\alpha_s} \beta_{\mathrm{QCD}}(\alpha_s).$$

For example

$$\alpha_s^{\mathrm{bare}} = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_\alpha[\alpha_s(\mu)]\, \alpha_s(\mu) \quad \Rightarrow \quad Z_\alpha(\alpha_s) = \exp\left[-z(\hat{\beta}, \alpha_s)\right]$$

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$$C_B(\alpha_s) = \exp\left[\frac{1}{2} z(\gamma_B^{\mathrm{nc}} + z_B, \alpha_s)\right]$$

$$z_B(\alpha_s) = z(\Gamma^c, \alpha_s)$$

Z-factors

3-loop cusp and non-cusp anomalous dimensions known

[Korchemsky and Radyushkin, 1987, Moch, Vermaseren and Vogt, 2004]
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Reproduce 3-loop cusp and non-cusp anomalous dimensions

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Short distance scheme

The u = 1/2 renormalon of jet function cancelled when switching to (low-scale) short-distance scheme such as MSR mass

$$B(\hat{s}, \delta m, \Gamma_t, \mu) = \mathrm{Im}\big[\mathcal{B}(\hat{s} - 2\delta m + i\Gamma_t, \mu)\big] = \exp\!\left(-2\delta m \frac{\partial}{\partial \hat{s}}\right) B(\hat{s}, 0, \Gamma_t, \mu)\,, \qquad \delta m \equiv m - m^{\mathrm{SD}}$$

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- Derivative scheme [Jain, Scimemi, Stewart 2008]

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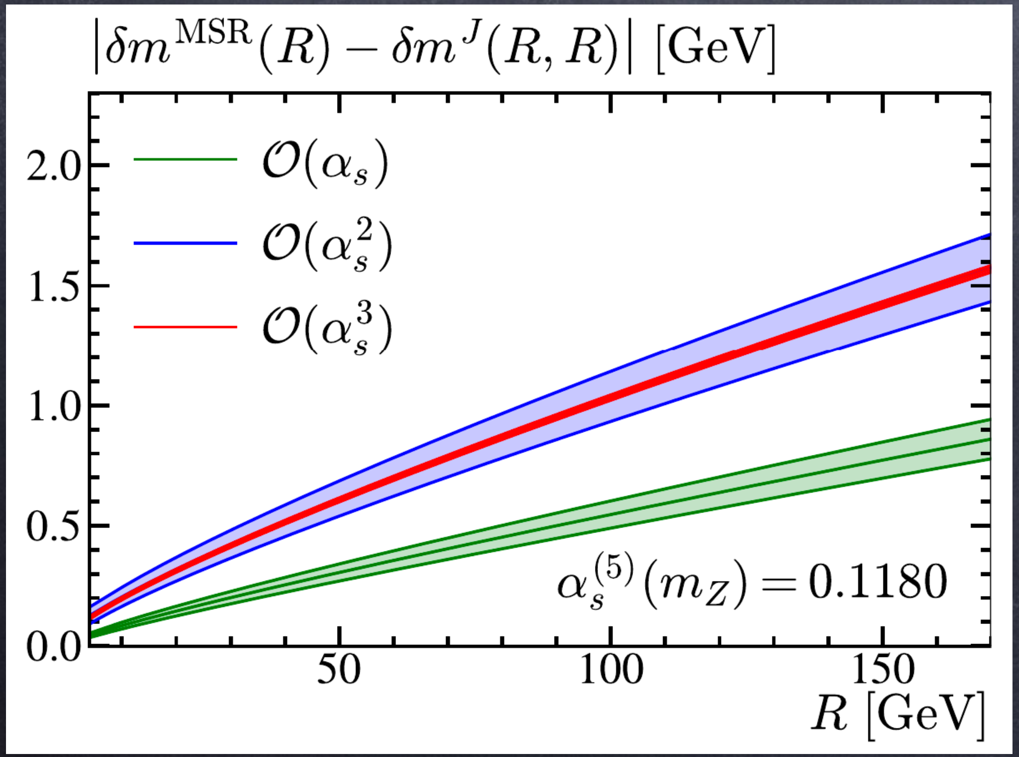

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- Non-derivative scheme [Gracia, VM 2021]

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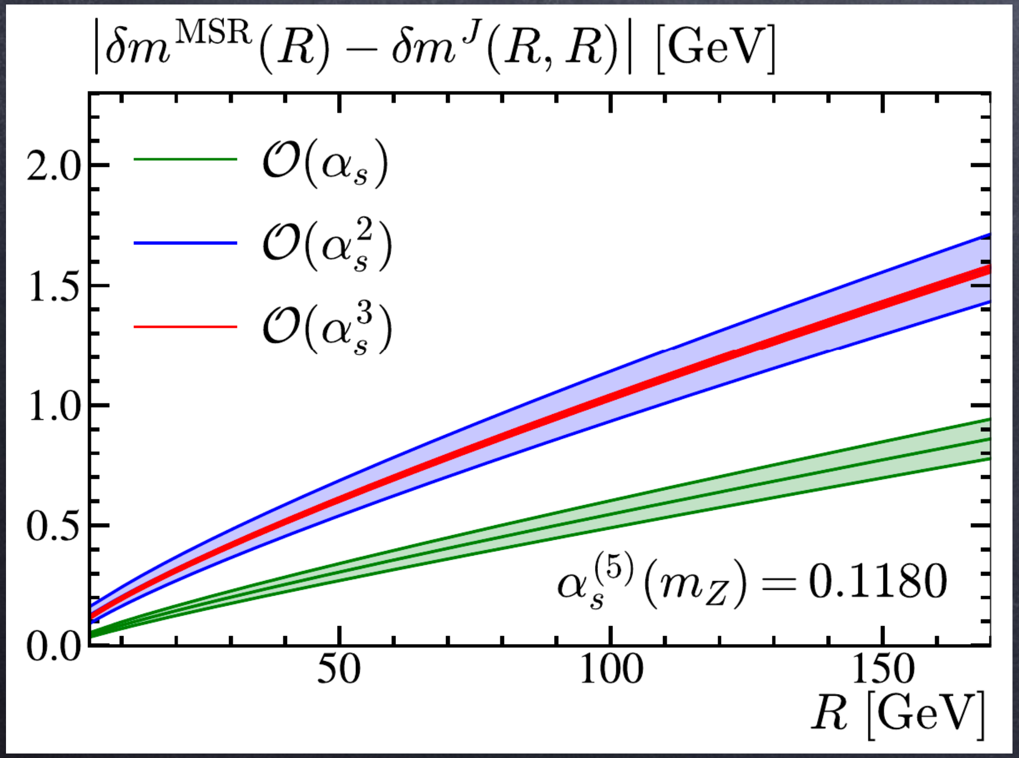
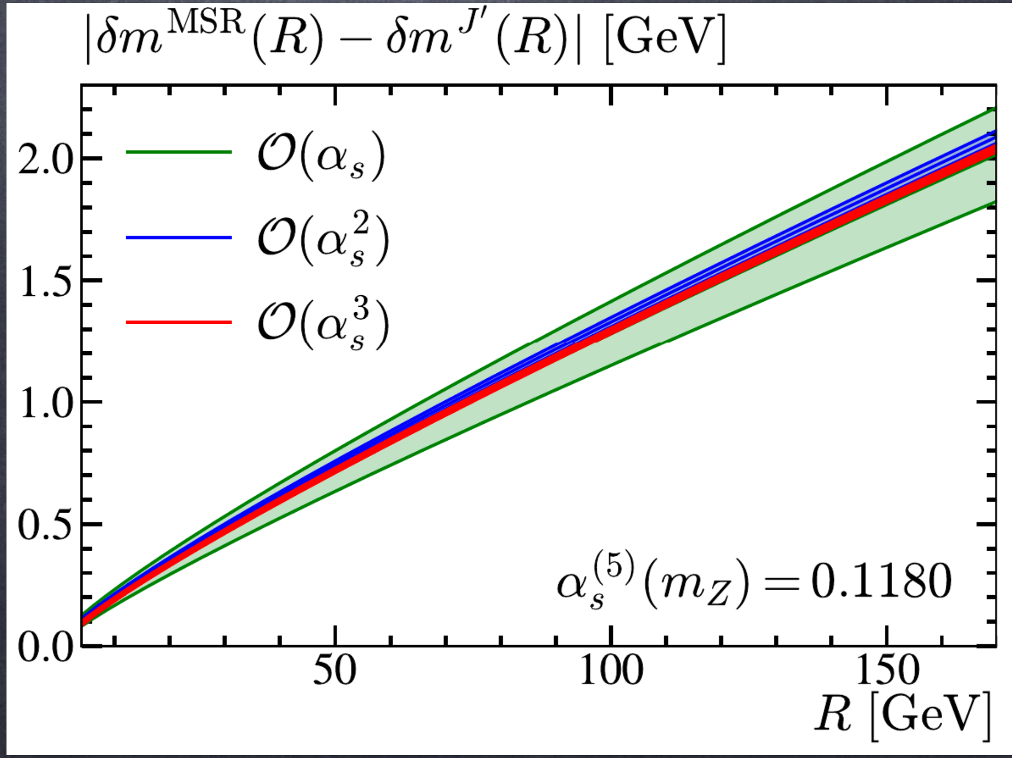
$$\delta m^{J'}(R) = \frac{R e^{\gamma_E}}{2} \log\left[m \tilde{B}\left(\frac{1}{iRe^{\gamma_E}}, R\right)\right]$$





Four loop estimate

Obtain an estimate for $\tilde{b}_{40}$ based on u = 1/2 renormalon dominance

Using R-evolution one gets the exact relation

$$\tilde{b}_{l0} = (2\beta_0)^l \sum_{k=0}^{l-1} S_k \sum_{j=0}^{l-1-k} g_j \left(1 + \frac{\beta_1}{2\beta_0^2} + l\right)_{k-j-l}$$

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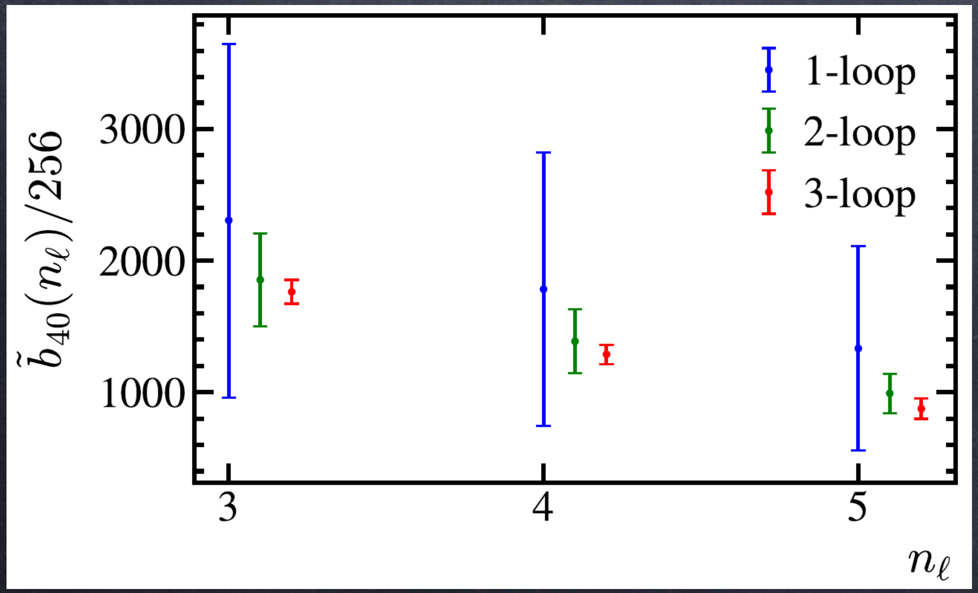
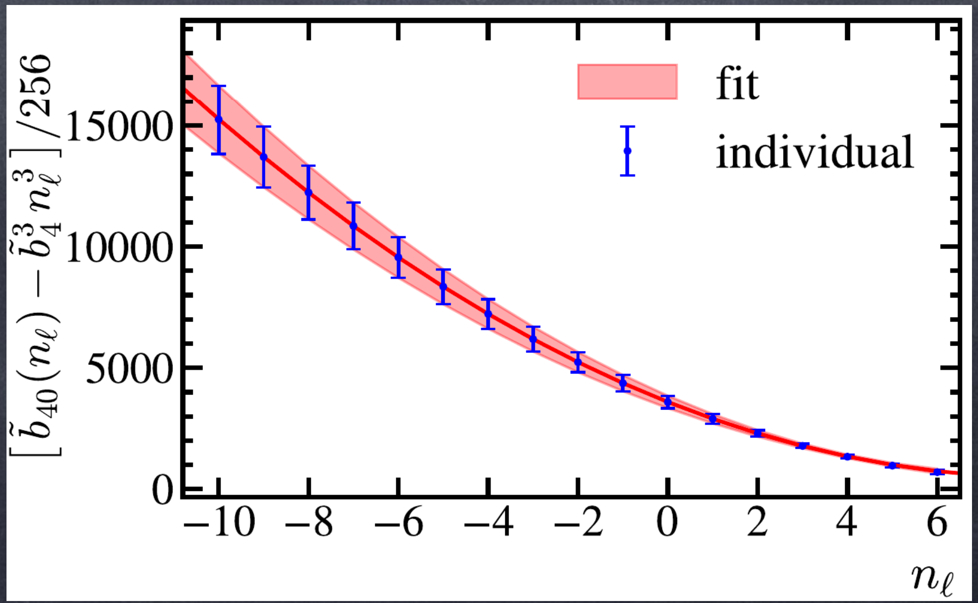
truncating red sum, estimates are obtained





Conclusions

- bHQET jet function: universal ingredient in peak-region factorization theorems for observables that probe the top mass
- Key for determining the top mass in renormalon-free scheme
- Computed to 3-loops using modern multi-loop techniques
- Paves the way for N^3LL' resummation. To be used in calibration

Outlook

- Get the last missing piece at 3 loops
- Compute 4-loop anomalous dimension