Variational Quantum Eigensolver for (2+1)-Dimensional QED at Finite Density

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Motivation for this Theory

(2+1)D

Spatial   Temporal

Quantum Electrodynamics (QED)

Motivation for this Theory

(2+1)D Quantum Electrodynamics (QED)

Spatial Temporal

- Similarities to (3+1)D Quantum Chromodynamics
  - Shows confinement
  - Has a mass gap
- Related topological effects
  - Maxwell-Chern Simons theory (see Peng et al. (2024)
- Based on previous work (see Crippa et al. (2024)

Motivation for this Theory

(2+1)D → Spatial, Temporal

Quantum Electrodynamics (QED)

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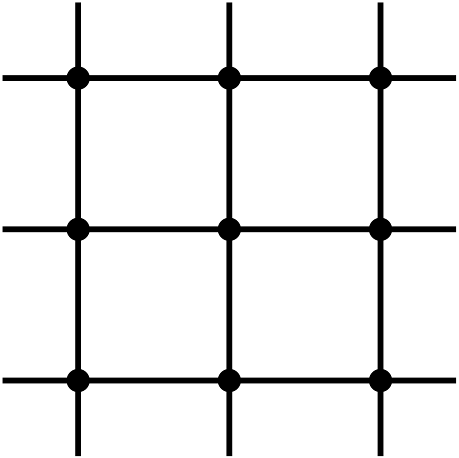
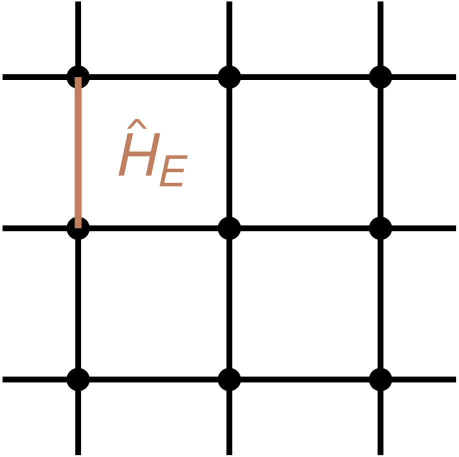
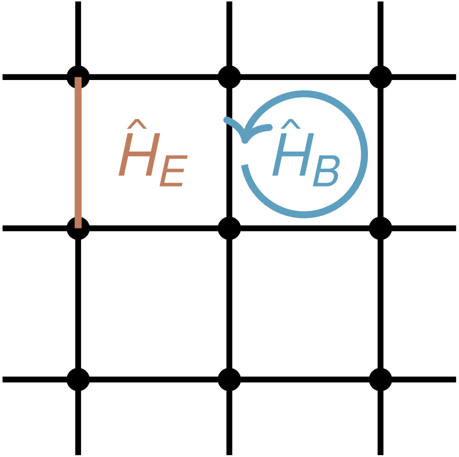
PyPI: qclatticeh

QED on the Lattice




$\hat{H}_E$

QED on the Lattice



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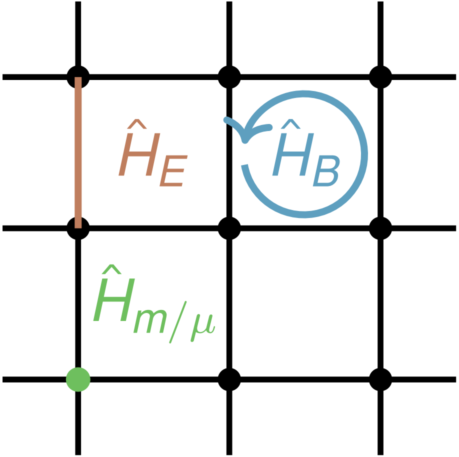

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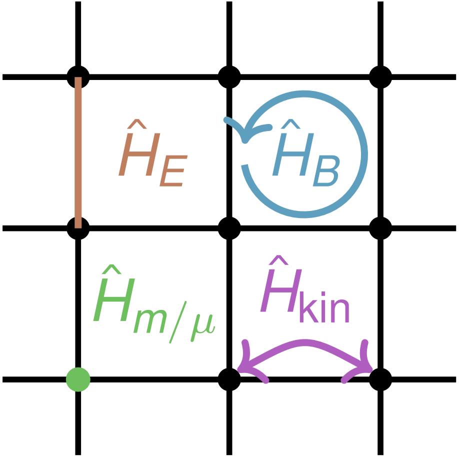

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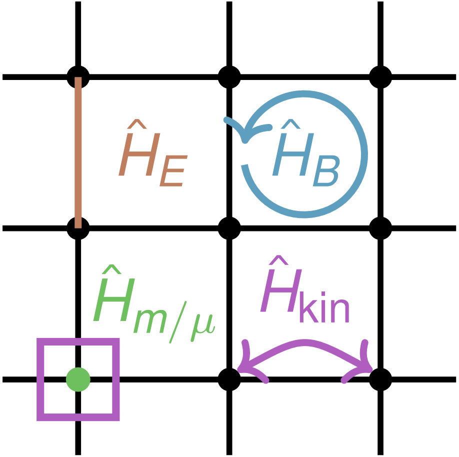
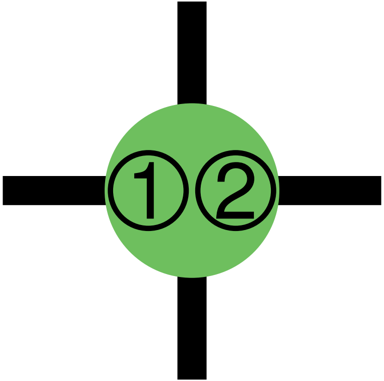
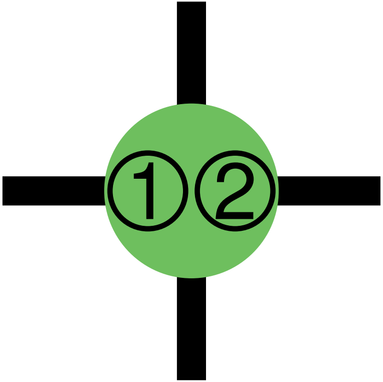


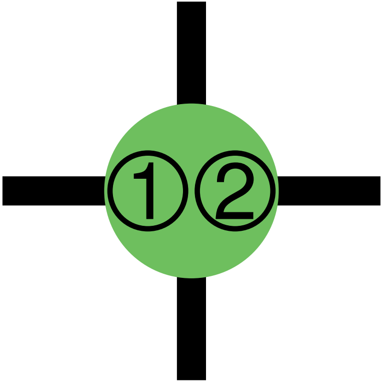
QED on the Lattice





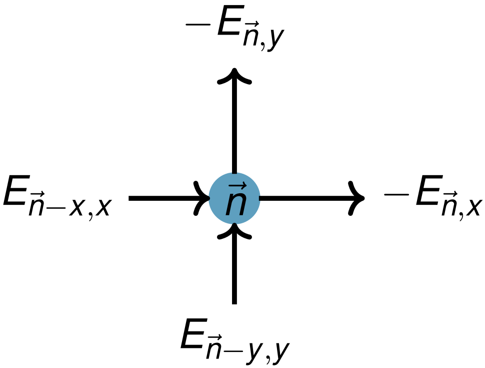
- Each flavor has mass m_i
- Each flavor has chemical potential μ_i

QED on the Lattice



- Each flavor has mass $m_i \to \bar{\psi}\psi$
- Each flavor has chemical potential $\mu_i \to \psi^\dagger\psi$

Gauss' Law on the Lattice



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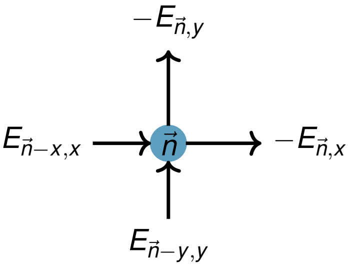

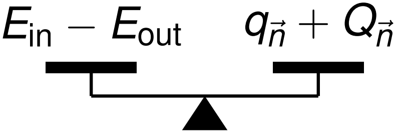

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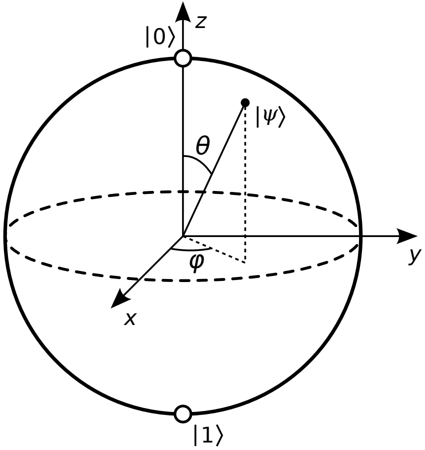
Exploit quantum-mechanical phenomena:

- Superposition
- Entanglement

Based on **qubits**:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$



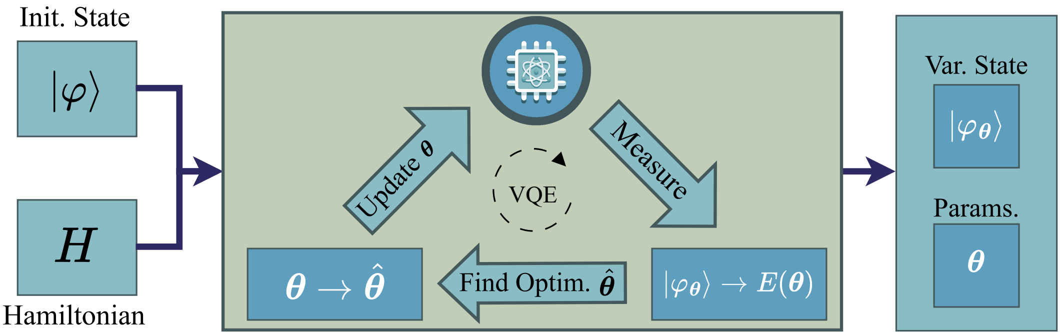
By Smite-Meister - Own work, CC BY-SA 3.0

Currently: **NISQ** = Noisy-Intermediate-Scale-Quantum

Quantum-**Classical** Hybrid Algorithm

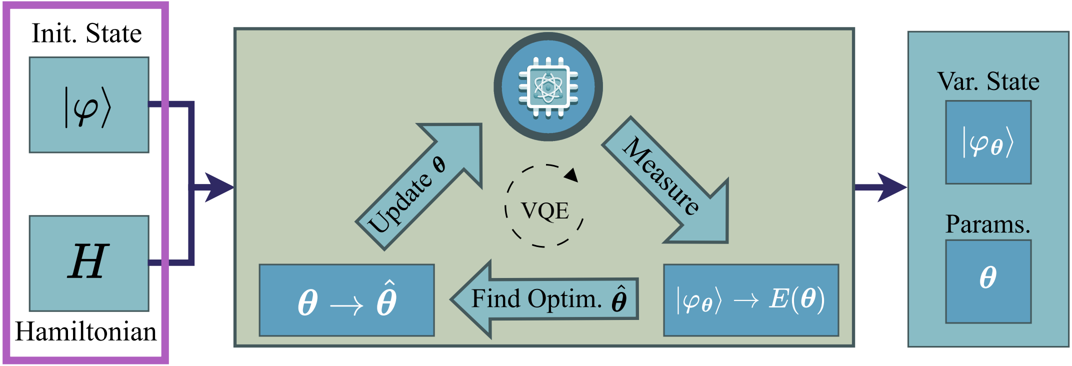
Measure the energy $\langle E \rangle$

Optimize parametrized circuit



$\Rightarrow$ Find the **ground state**!

Variational Quantum Eigensolver (VQE)



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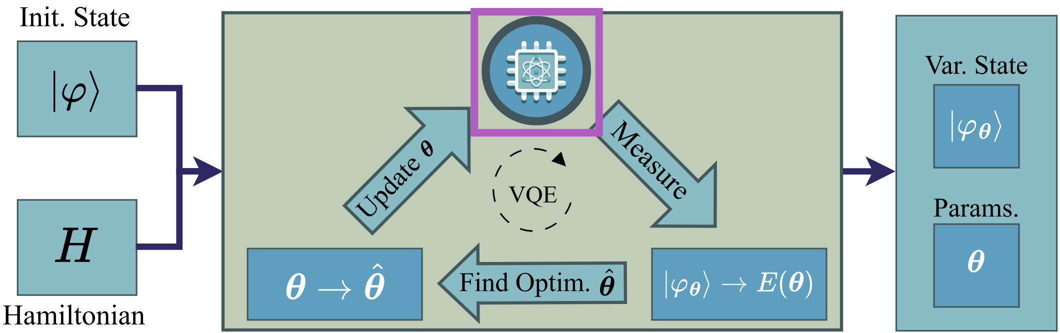

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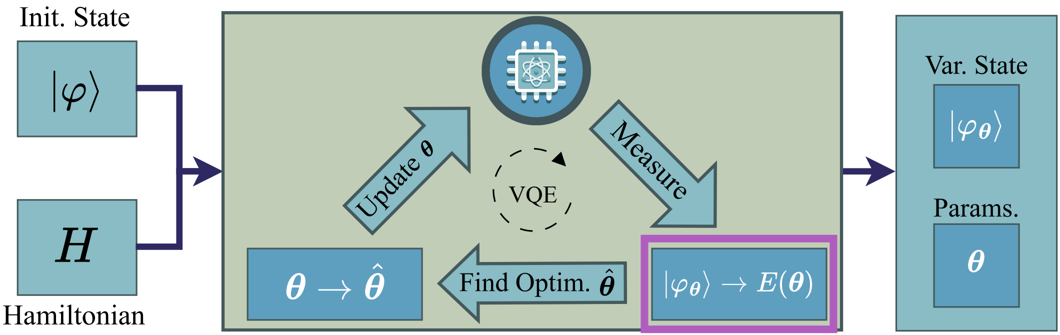
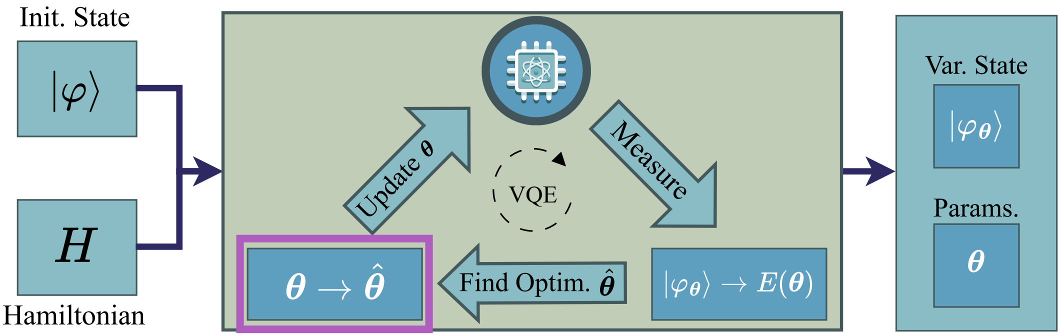
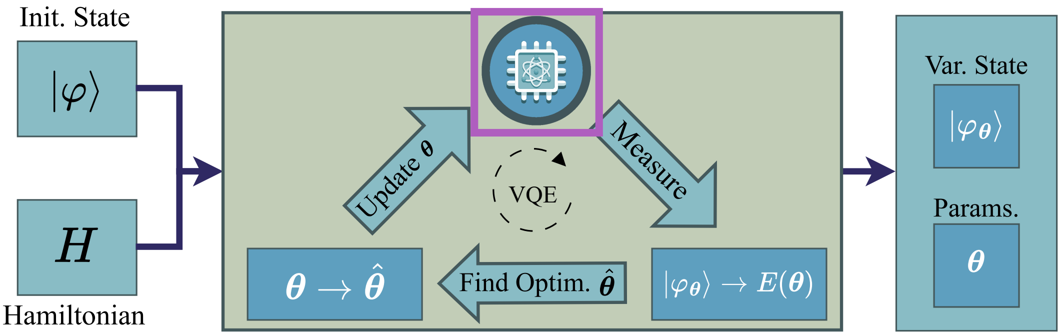


$\Rightarrow$ Find the **ground state**!



$\Rightarrow$ Find the **ground state**!

Variational Quantum Eigensolver (VQE)



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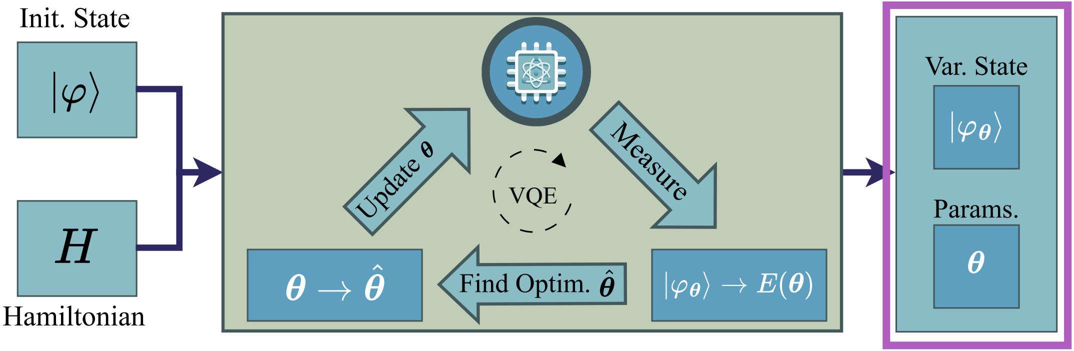
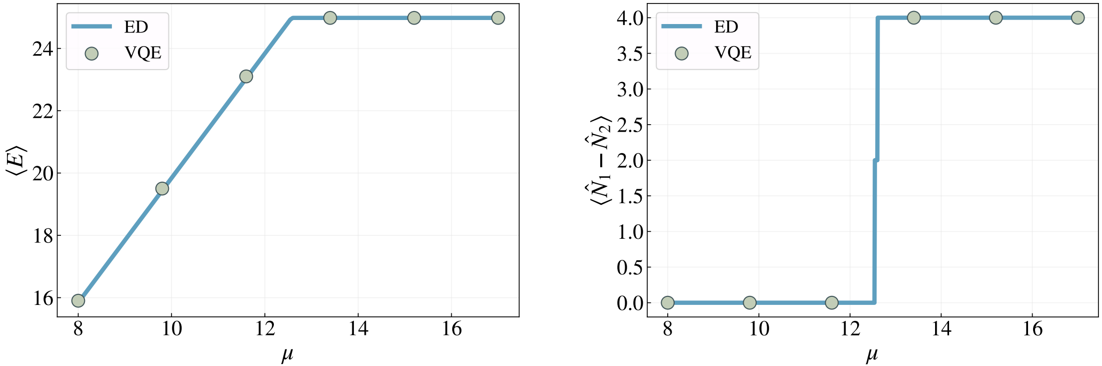


$\Rightarrow$ Find the **ground state**!

VQE Results

Lattice: 2x2 with obc, $n_{\text{flavor}} = 2$, $g = 5$, $\mu_i = [0, \mu]$



Noise-free classical simulator of quantum hardware, lowest-energy result out of 10 runs

Inference Run

VQE: classical simulation

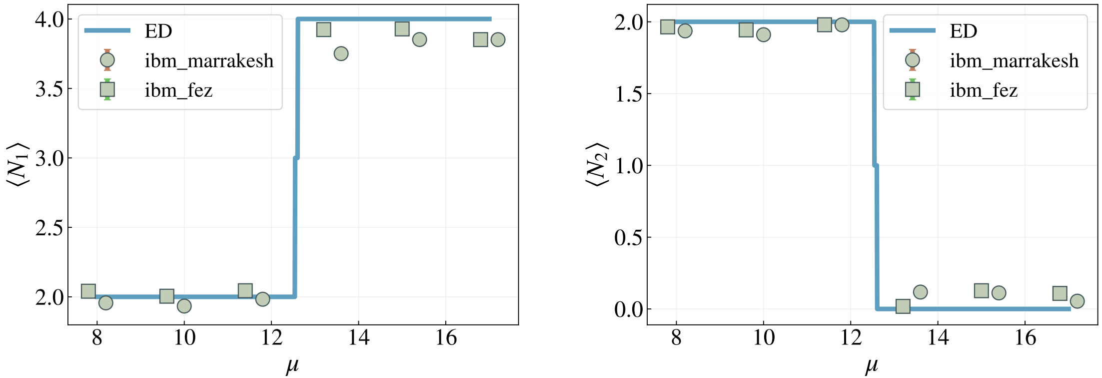
↓

Optimal set $\hat{\theta}_{\text{opt.}}$

↓

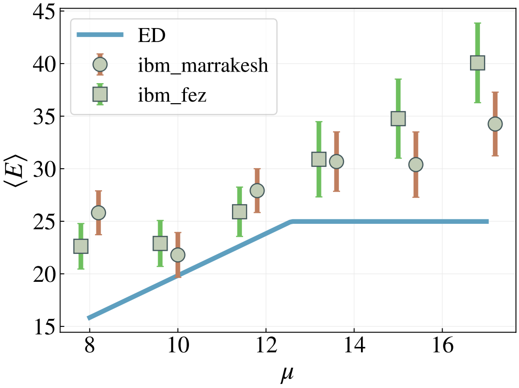
Measure with $\hat{\theta}_{\text{opt.}}$ on real device

Inference Run on Quantum Hardware



Points obtained on ibm_fez and ibm_marrakesh with ZNE and $N_{\text{shots}} = 8192$

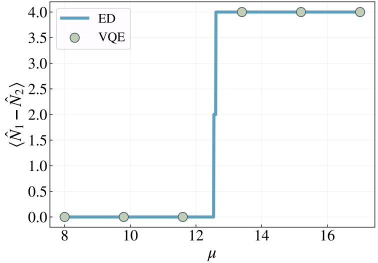
Inference Run on Quantum Hardware

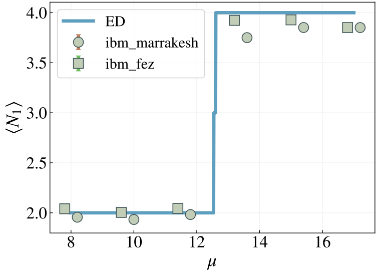


Points obtained on ibm_fez and ibm_marrakesh with ZNE and $N_{\text{shots}} = 8192$

Summary of Results

- Finite density analysis of (2+1)D QED
- Construction of Hamiltonian and Ansatz
- Successful VQE in limit of infinite shots
- Inference runs on real hardware





Collaborators

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DESY

Stefan Kühn

DESY

Paulo Itaborai

DESY

Action and Hamiltonian of QED

$$S_{\text{QED}} = \int d^3x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} \left(i\gamma^\mu D_\mu - m \right) \psi \right]$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $D_\mu = \partial_\mu + ieA_\mu$
ψ a spin-1/2 field with mass m
A^μ a U(1) connection

Using a Legendre-transformation:

$$H_{\text{QED}} = \int d^2x \left[-i\bar{\psi}\gamma^k \left(\partial_k + iQeA_k \right) \psi + m\bar{\psi}\psi + \frac{1}{2} \left(\vec{E}^2 + \vec{B}^2 \right) \right]$$

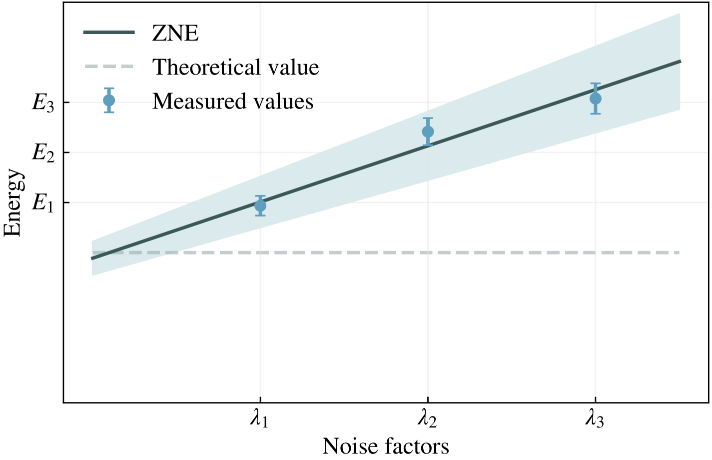
Sign Problem in QED

In lattice simulations (e.g., Monte Carlo methods):

- The fermionic degrees of freedom are integrated out, resulting in a **determinant of the fermion matrix** $\det M(\mu)$.
- At $\mu = 0$, the determinant is real and positive.
- At $\mu \neq 0$, $\det M(\mu)$ generally becomes **complex.**

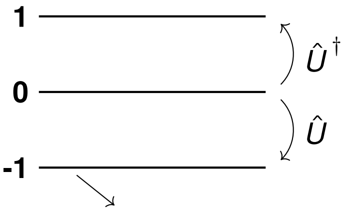
$\Rightarrow$ **Leads to sign problem!**

Zero-Noise-Extrapolation (ZNE)



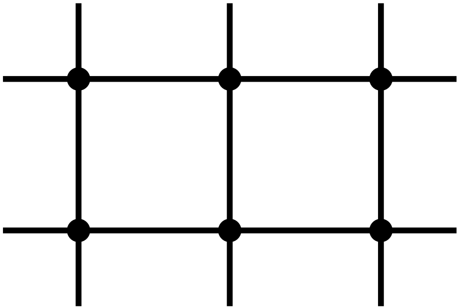
Electric field is unbounded $\Rightarrow$ $\dim(H) = \infty$

Solution: $U(1) \to \mathbb{Z}_{2l+1}$ (truncated)



$$\hat{E}\,|-1\rangle = -1 \cdot |-1\rangle$$

Lattice Quantum Electrodynamics



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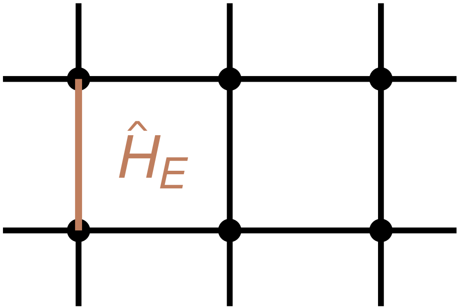

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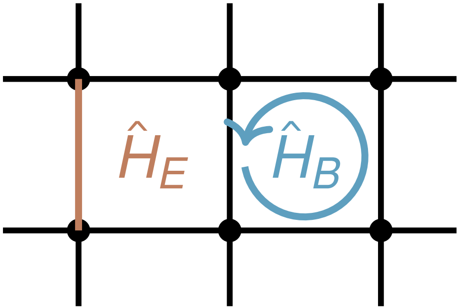
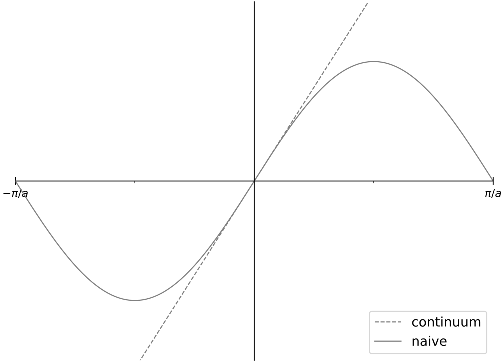


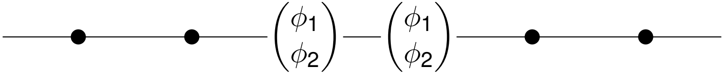
Dispersion relation of a free fermion:



By OpenScience709 - Own work, CC BY-SA 4.0

With inverse propagator: $D(p) = m + \frac{i}{a} \sum_{\mu} \gamma_{\mu} \sin(p_{\mu} a)$

Solution: Select degrees of freedom!
$\Rightarrow$ Increases lattice spacing to $2a$



Staggered Fermions (Kogut-Susskind)

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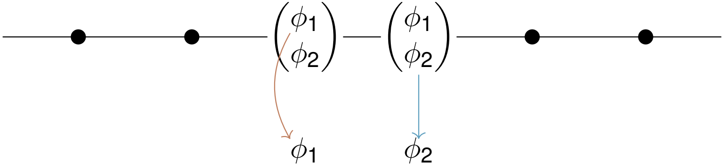

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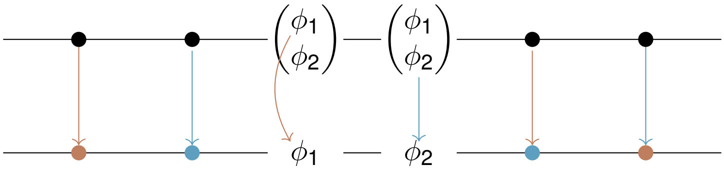

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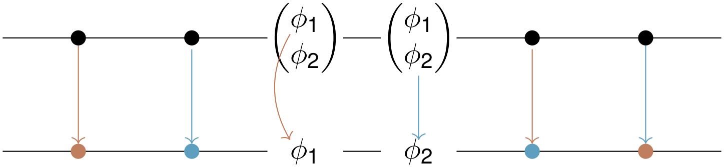
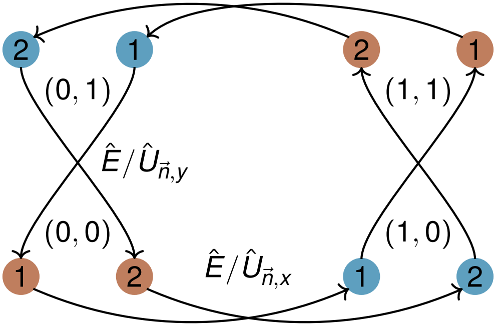


✓Removes doublers in (1+1)D
↯ Only reduces doublers in (2+1)D!

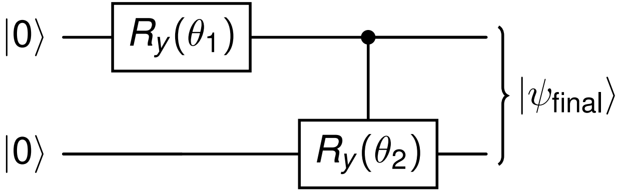
Extension to Multiple Flavors

Distribute flavors to different lattice sites
Both share one gauge link!



Quantum Circuit

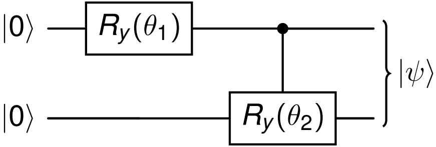
Qubits are manipulated by **gates**



Gauge Field on a Quantum Computer

Remember: $U(1) \rightarrow \mathbb{Z}_{2l+1}$

$\Rightarrow$ Uneven number of states ↯ Quantum computer has 2^n states



$$|\psi\rangle = \cos\left(\frac{\theta_1}{2}\right)|00\rangle + \sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)|10\rangle + \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)|11\rangle$$

Excludes one state!

Fermions on a Quantum Computer

Gauss' law $\Rightarrow$ Charge conservation $\Rightarrow \# |1\rangle = \# |0\rangle$

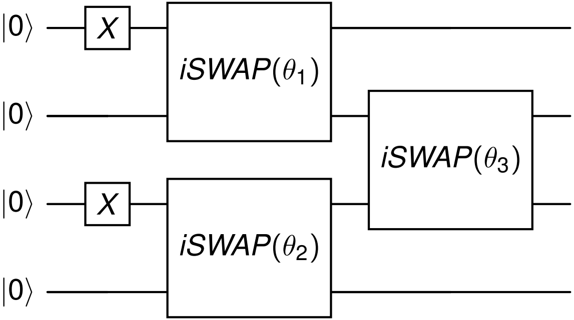
Use (parametrized) *iSWAP*-gates:

$$|11\rangle \rightarrow |11\rangle, \ |00\rangle \rightarrow |00\rangle$$

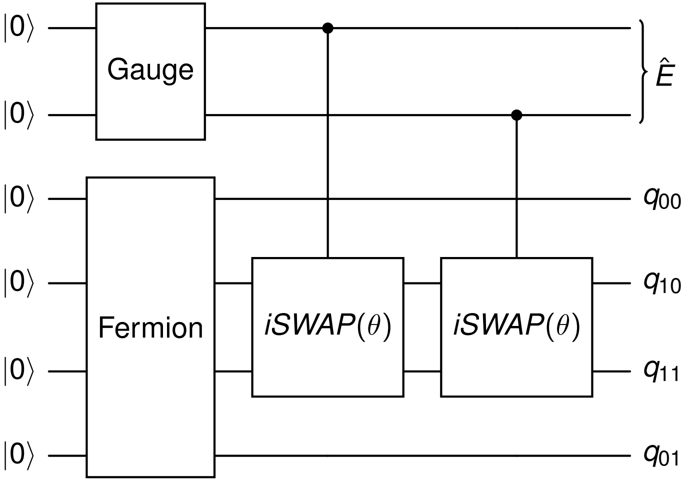
$$|01\rangle \rightarrow \frac{1}{2}\left(1 + e^{2i\theta}\right)|01\rangle + \frac{1}{2}\left(1 - e^{2i\theta}\right)|10\rangle$$

$$|10\rangle \rightarrow \frac{1}{2}\left(1 - e^{2i\theta}\right)|01\rangle + \frac{1}{2}\left(1 + e^{2i\theta}\right)|10\rangle$$

Fermion Circuit



Composed Circuit



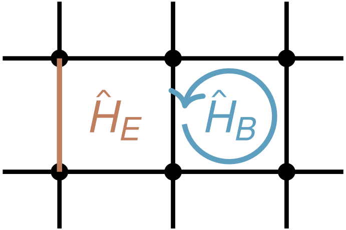
Implementation of Gauge Fields

Electric field is unbounded $\Rightarrow$ $\dim(H) = \infty$

Solution: $U(1) \rightarrow \mathbb{Z}_{2l+1}$

$$E = \begin{pmatrix} l & 0 & \dots & 0 \\ 0 & l-1 & \dots & 0 \\ 0 & \ddots & \vdots & 0 \\ 0 & \dots & 0 & -l \end{pmatrix} \qquad U = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & \dots & \dots & 0 \\ 0 & \ddots & \vdots & 0 \\ 0 & \dots & 1 & 0 \end{pmatrix}$$

Lattice Quantum Electrodynamics



$$\hat{H} = \hat{H}_E + \hat{H}_B$$

$$\hat{H}_E = \frac{g^2}{2} \sum_{\vec{r}} \left(\hat{E}^2_{\vec{r},x} + \hat{E}^2_{\vec{r},y} \right)$$

$$\hat{H}_B = -\frac{1}{2a^2 g^2} \sum_{\vec{r}} \left(\hat{P}_{\vec{r}} + \hat{P}^\dagger_{\vec{r}} \right)$$

where

$$\hat{P}_{\vec{r}} = \hat{U}_{\vec{r},x} \hat{U}_{\vec{r}+x,y} \hat{U}^\dagger_{\vec{r}+y,x} \hat{U}^\dagger_{\vec{r},y}$$

Problems with Staggered Fermions

Lagrangian:

- Wilson fermions
- Staggered fermions
  - Remove tastes (rooting)

Hamiltonian:

- Wilson fermions
  - Computationally costly
- Staggered fermions in (1+1)D
  - Remove tastes
- Staggered fermions in (2+1) and (3+1)D
  - No rooting trick
  - Cannot remove tastes

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$\Rightarrow$ Need to go to **Wilson fermions**!

Wilson Fermions

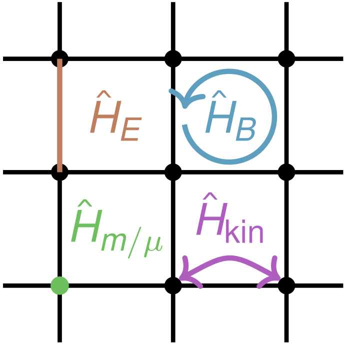
Solution to doubling problem: Add mass to the doublers!

$$\tilde{D}(p) = m + \frac{i}{a}\sum_\mu \gamma_\mu \sin(p_\mu a) + \frac{1}{a}\sum_\mu (1 - \cos(p_\mu a))$$

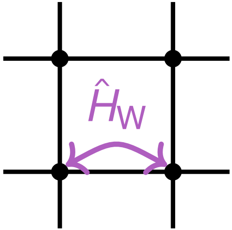
$$m_{\text{Doubler}} = m + \frac{2l}{a} \quad (l\text{: number of } p_\mu = \frac{\pi}{a})$$

$\lightning$ Violates chiral symmetry!

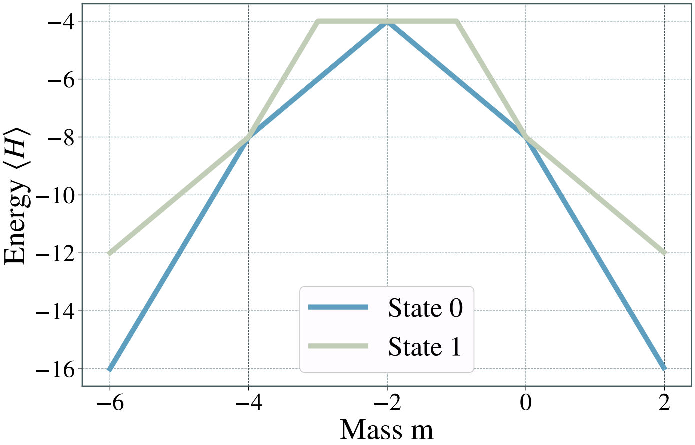
Full Setup (Wilson)



- Two-component spinors
- Added term in Hamiltonian
  - Wilson term
  - Acts as second derivative



2x2, periodic boundary condition



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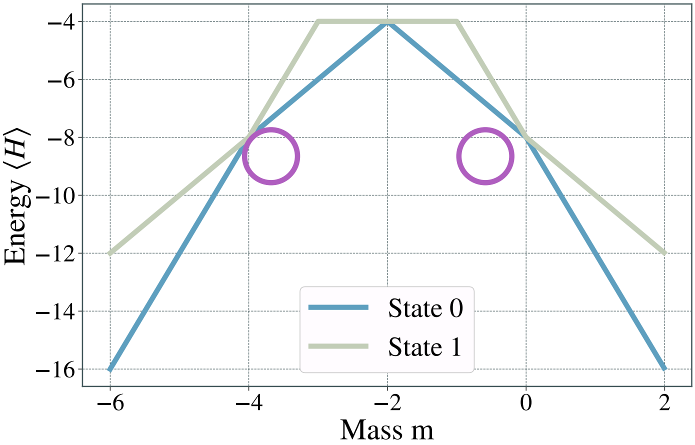


Summary and Outlook

Summary

- Implementation of (2+1)D QED
  - With Staggered fermions
  - With Wilson fermions
  - Including multiple flavors
- Inference runs on IBM-Q hardware

Outlook:

- Improving the quantum circuit
- Inference runs near phase transitions
- Connect Wilson fermions to Chern-Simons theory

Discretizing the Derivative

Naive discretization: $\partial_x \psi \Rightarrow \frac{\psi(\vec{r}+\hat{x})-\psi(\vec{r}-\hat{x})}{2a}$

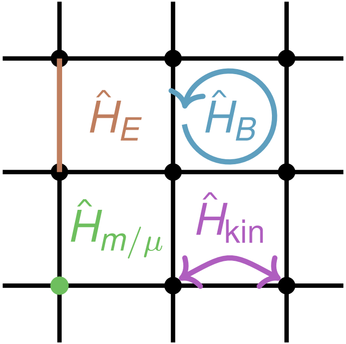
Not gauge-invariant!

$$-i\bar{\psi}\gamma^1\left(\partial_1 + iQeA_1\right)\psi$$

$$\downarrow$$

$$\frac{i}{2a}\bar{\psi}(\vec{r})\gamma^1 U_{\vec{r},1}\psi(\vec{r}+\hat{x}) + \text{h.c.}$$

Full Setup (Staggered)



$$\hat{H}_{\text{kin}} = \frac{i}{2a}\sum_{\vec{r},f}\left(\hat{\phi}^{\dagger}_{\vec{r},f}\hat{U}_{\vec{r},x}\hat{\phi}_{\vec{r}+\hat{x},f} - \text{h.c.}\right)$$

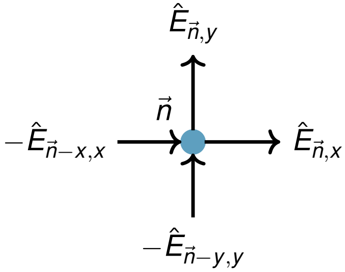
$$-\frac{(-1)^{r_x+r_y}}{2a}\sum_{\vec{r},f}\left(\hat{\phi}^{\dagger}_{\vec{r},f}\hat{U}_{\vec{r},y}\hat{\phi}_{\vec{r}+\hat{y},f} + \text{h.c.}\right)$$

$$\hat{H}_{\mu} = \sum_{f,\vec{r}}\mu_f \cdot \hat{\phi}^{\dagger}_{f,\vec{r}}\hat{\phi}_{f,\vec{r}}$$

$$\hat{H}_{\text{m}} = \sum_{f,\vec{r}} m_f \cdot (-1)^{r_x+r_y}\hat{\phi}^{\dagger}_{f,\vec{r}}\hat{\phi}_{f,\vec{r}}$$

Here: ϕ is one component!

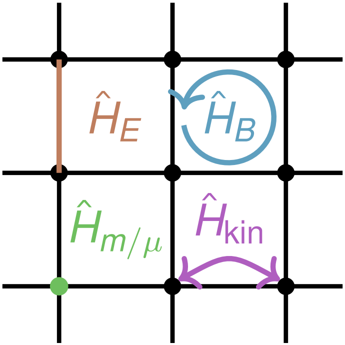
Gauss' Law on the Lattice



$$\left[\sum_{\mu=x,y}\left(\hat{E}_{\vec{r},\mu}-\hat{E}_{\vec{r}-\mu,\mu}\right)-\hat{q}_{\vec{r}}-Q_{\vec{r}}\right]|\Phi\rangle=0$$

$$\hat{q}_{\vec{r}}=\hat{\phi}^{\dagger}_{\vec{r}}\hat{\phi}_{\vec{r}}-\frac{1}{2}\left[1+(-1)^{r_x+r_y+1}\right]$$

Full Setup (Wilson)



$$\hat{H}_{\text{kin}} = \sum_{\text{sites},f,k} \frac{1}{2a}\left(i\hat{\bar{\psi}}_{x,f}\gamma^k \hat{U}_{(x,k)}\hat{\psi}_{x+\hat{k},f} + \text{h.c.}\right)$$

$$\hat{H}_{\mu} = \sum_{\text{sites},f} \mu_f \cdot \hat{\psi}^\dagger_{x,f}\hat{\psi}_{x,f}$$

$$\hat{H}_{\text{m}} = \sum_{\text{sites},f} m_f \cdot \hat{\bar{\psi}}_{x,f}\hat{\psi}_{x,f}$$

$$\hat{H}_{\text{Wil.}} = \frac{r}{2a} \sum_{\text{sites},f,k} \left(\hat{\bar{\psi}}_{x,f}\hat{U}_{(x,k)}\hat{\psi}_{x+\hat{k},f} + \text{h.c.} + 2 \cdot \hat{\bar{\psi}}_{x,f}\hat{\psi}_{x,f}\right)$$

Here: ψ is two-component!

Chemical Potential

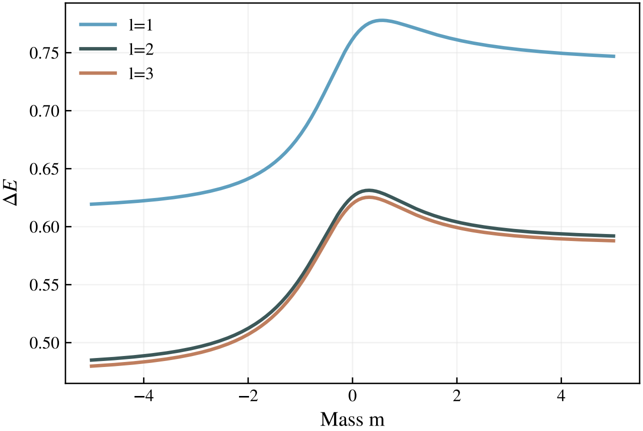
Energy for adding a fermion of one flavor

Compare:

Mass: $m \cdot \bar{\psi}\psi$

Chemical potential: $\mu \cdot \psi^{\dagger}\psi$

Convergence of Truncation



Quantum Electrodynamics (QED)

Example of a QED interaction: