NNLOCAL: completely local subtraction for color-singlet production in hadron collisions

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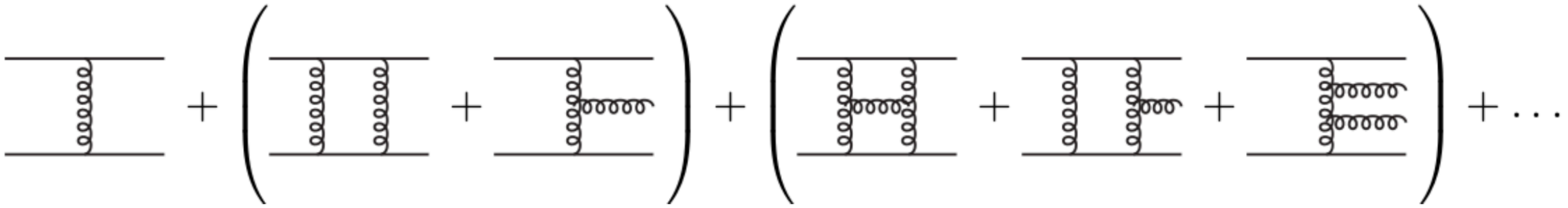
Precision physics

- **The Standard Model (SM) of particle physics** provides a **very successful description** of elementary particles and their interactions.
- Data collected at LHC give us an incredible confirmation of the SM!
- Despite its success, we know that the **SM is not the ultimate theory** of fundamental interactions.
- **High precision experiments** (like the high-luminosity phase at LHC) become crucial to test the validity of the SM.
- From a theoretical point of view, one of the more important aspects of increasing precision is **adding higher order correction** to physical observables.



Cross-section for hadron collisions

- The cross-section for a QCD process involving hadrons in the initial state can be computed using the formula

$$\sigma(p_A, p_B) = \sum_{a,b} \int_0^1 dx_a \, \boxed{f_{a/A}(x_a, \mu_F^2)} \int_0^1 dx_b \, \boxed{f_{b/B}(x_b, \mu_F^2)} \, \boxed{\hat{\sigma}_{a,b}(x_a p_A, x_b p_B, \mu_F^2)}$$

Parton distribution functions (non-perturbative)

Partonic cross-section (perturbative)

- We are interested in computing higher-order QCD corrections of the cross-section. The partonic cross-section can be expanded in as a power series in the strong coupling constant α_s.

$$\hat{\sigma}_{a,b}(p_a, p_b; \mu_F^2) = \alpha_s^\ell \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^k \hat{\sigma}_{a,b}^{\mathrm{N^kLO}}(p_a, p_b; \mu_F^2, \mu_R^2)$$

QCD at NNLO

- The computation of higher-order terms in perturbation theory involves the evaluations of Feynman diagrams with a higher number of loops and/or legs.

- In this talk I will focus on the problem of treating Infra-Red (IR) divergences that arise in diagrams that involve multiple emissions. The QCD NNLO correction for a processes with m jets at the Born level is given by

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} d\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} d\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} d\sigma^{\mathrm{VV}}_{m} J_{m}$$

- While the sum of these three contribution is finite for IR-safe observables, the three integrals are separately divergent in $d = 4$ dimensions in the limits in which one or more partons become unresolved.

- We have to regularize the integrals if we want to perform numerical computations. This problem can be solved using **local subtraction**.

Colorful subtraction formula

- Colorful subtraction[Del Duca, Somogyi, Trocsanyi]: the idea is to construct local counterterms that have the same IR behavior as the real matrix elements.

- This is done exploiting well-known limits of QCD amplitudes in which one or more partons become unresolved.

- In the Colorful method, the NNLO cross-section for an observable J is regularized as

$$\sigma^{NNLO}[J] = \int_{m+2} \left\{ \left[d\sigma^{RR}_{m+2} J_{m+2} - d\sigma^{RR,A_2}_{m+2} J_m \right] - \left[d\sigma^{RR,A_1}_{m+2} J_{m+1} - d\sigma^{RR,A_{12}}_{m+2} J_m \right] \right\}$$

$$+ \int_{m+1} \left\{ \left[d\sigma^{RV}_{m+1} + d\sigma^{C_1}_{m+1} + \int_1 d\sigma^{RR,A_1}_{m+2} \right] J_{m+1} - \left[d\sigma^{RV,A_1}_{m+1} + d\sigma^{C_1,A_1}_{m+1} + \left(\int_1 d\sigma^{RR,A_1}_{m+2} \right)^{A_1} \right] J_m \right\}$$

$$+ \int_m \left\{ d\sigma^{VV}_m + d\sigma^{C_2}_m + \int_2 \left[d\sigma^{RR,A_2}_{m+2} - d\sigma^{RR,A_{12}}_{m+2} \right] + \int_1 \left[d\sigma^{RV,A_1}_{m+1} + d\sigma^{C_1,A_1}_{m+1} + \left(\int_1 d\sigma^{RR,A_1}_{m+2} \right)^{A_1} \right] \right\} J_m$$

Role of the counterterms

- Each counterterm is defined to regularize specific limits

$$
\left\{
\begin{array}{c}
d\sigma^{RR,A_2} \\[2em]
d\sigma^{RR,A_1} \\[2em]
d\sigma^{RR,A_{12}} \\[2em]
d\sigma^{RV,A_1} \\[2em]
d\sigma^{C_1,A_1} \\[2em]
\left(\int_1 d\sigma^{RR,A_1}\right)^{A_1}
\end{array}
\right\}
\text{ capture singular behavior of }
\left\{
\begin{array}{c}
d\sigma^{RR} \\[2em]
d\sigma^{RR} \\[2em]
d\sigma^{RR,A_2},\ d\sigma^{RR,A_1} \\[2em]
d\sigma^{RV} \\[2em]
d\sigma^{C_1} \\[2em]
\int_1 d\sigma^{RR,A_1}
\end{array}
\right\}
\text{ as }
\left\{
\begin{array}{c}
2 \\[2em]
1 \\[2em]
1,\ 2 \\[2em]
1 \\[2em]
1 \\[2em]
1
\end{array}
\right\}
\text{ partons become unresolved}
$$

[From F. Tramontano slides of HP2 2024]

Construction of the counterterms

- The counterterms are constructed following three basic principles:

1. They coincide with the QCD matrix elements in the IR limits.

2. They rely on momentum mappings such that the phase space factorizes.

3. The overlaps of singly and doubly unresolved limits is treated in a process-independent way.

- The IR limits of the QCD matrix elements cannot be directly used as counterterms, since they are well-defined only in the strict IR regions. Their definition has to be carefully extended over the whole phase space.

Features of the subtraction

- Once the counterterms are subtracted from the cross-section, each line of the subtraction formula is finite in $d = 4$ dimensions and therefore can be integrated numerically in a computer code.

- To preserve the correct result, the counterterms have to be added back integrated over the radiation phase space.

- The integration is performed analytically up to the required order in ϵ. This gives to our method a strong numerical stability.

- The integration of the counterterms A_2 and A_{12} is one of the most challenging tasks for a practical implementation of the subtraction method.

- Given the universal nature of the counterterms, their integration can be performed once and for all, and the results can be applied to different processes.

A_2 counterterm

- The counterterm A_2 regularizes the doubly-unresolved limits of matrix elements with two extra emissions.
- Its definition relies on the double and triple collinear splitting kernels, and double soft eikonal factors.

$$
\begin{aligned}
\hat{P}^{\mu\nu}_{g_1 g_2 g_3} &= C_A^2 \left\{ \frac{(1-\epsilon)}{4s_{12}^2} \left[-g^{\mu\nu} t_{12,3}^2 + 16 s_{123} \frac{z_1^2 z_2^2}{z_3(1-z_3)} \left(\frac{\widetilde{k}_2}{z_2} - \frac{\widetilde{k}_1}{z_1} \right)^\mu \left(\frac{\widetilde{k}_2}{z_2} - \frac{\widetilde{k}_1}{z_1} \right)^\nu \right] \right. \\
&- \frac{3}{4}(1-\epsilon) g^{\mu\nu} + \frac{s_{123}}{s_{12}} g^{\mu\nu} \frac{1}{z_3} \left[\frac{2(1-z_3)+4z_3^2}{1-z_3} - \frac{1-2z_3(1-z_3)}{z_1(1-z_1)} \right] \\
&+ \frac{s_{123}(1-\epsilon)}{s_{12}s_{13}} \left[2z_1 \left(\widetilde{k}_2^\mu \widetilde{k}_2^\nu \frac{1-2z_3}{z_3(1-z_3)} + \widetilde{k}_3^\mu \widetilde{k}_3^\nu \frac{1-2z_2}{z_2(1-z_2)} \right) \right. \\
&+ \frac{s_{123}}{2(1-\epsilon)} g^{\mu\nu} \left(\frac{4z_2z_3 + 2z_1(1-z_1) - 1}{(1-z_2)(1-z_3)} - \frac{1-2z_1(1-z_1)}{z_2 z_3} \right) \\
&+ \left. \left. \left(\widetilde{k}_2^\mu \widetilde{k}_3^\nu + \widetilde{k}_3^\mu \widetilde{k}_2^\nu \right) \left(\frac{2z_2(1-z_2)}{z_3(1-z_3)} - 3 \right) \right] \right\} + (5 \text{ permutations}) \; .
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_{ij}(q_1,q_2) &= \frac{(1-\epsilon)}{(q_1\cdot q_2)^2} \frac{p_i\cdot q_1\, p_j\cdot q_2 + p_i\cdot q_2\, p_j\cdot q_1}{p_i\cdot(q_1+q_2)\, p_j\cdot(q_1+q_2)} \\
&- \frac{(p_i\cdot p_j)^2}{2p_i\cdot q_1\, p_j\cdot q_2\, p_i\cdot q_2\, p_j\cdot q_1} \left[2 - \frac{p_i\cdot q_1\, p_j\cdot q_2 + p_i\cdot q_2\, p_j\cdot q_1}{p_i\cdot(q_1+q_2)\, p_j\cdot(q_1+q_2)} \right] \\
&+ \frac{p_i\cdot p_j}{2q_1\cdot q_2} \left[\frac{2}{p_i\cdot q_1\, p_j\cdot q_2} + \frac{2}{p_j\cdot q_1\, p_i\cdot q_2} \right. \\
&- \left. \frac{1}{p_i\cdot(q_1+q_2)\, p_j\cdot(q_1+q_2)} \left(4 + \frac{(p_i\cdot q_1\, p_j\cdot q_2 + p_i\cdot q_2\, p_j\cdot q_1)^2}{p_i\cdot q_1\, p_j\cdot q_2\, p_i\cdot q_2\, p_j\cdot q_1} \right) \right]
\end{aligned}
$$

A_2 counterterm integration recipe

For each counterterm, sort all the denominators into integral topologies. This requires heavy use of partial fractioning.

↓

For each topology, perform IBP reduction to master integrals (Reverse Unitarity[Anastasiou, Melnikov]).

↓

Set up differential equations for master integrals.

↓

Solve DEQs in canonical form. Boundary conditions are computed with direct integration.

Explicit example - triple collinear counterterm

- The explicit expression of the triple collinear counterterm is

$$\mathcal{C}^{IFF(0)}_{ars}(\{p\}_{m+X+2}, p_a, p_b) = (8\pi\alpha_S\mu^{2\epsilon})^2 \frac{1}{x_{a,rs}} \frac{1}{s^2_{ars}}$$
$$\langle \mathcal{M}^{(0)}_{m,(ars)b}(\{\tilde{p}\}_{m+X}, \tilde{p}_a, \tilde{p}_b) | \hat{P}^{(0)}_{f_{ars} f_r f_s}(\{x_{j,kl}, s_{jk}, k_{\perp j,kl}\}, \epsilon) | \mathcal{M}^{(0)}_{m,(ars)b}(\{\tilde{p}\}_{m+X}, \tilde{p}_a, \tilde{p}_b) \rangle$$

- This regulates the IR limit in which emitted partons with momenta p_r and p_s become collinear to initial-state parton with momentum p_a.

- We have to integrate this counterterm over the factorized radiation phase space

$$\mathrm{d}\phi_{iff} = \frac{\mathrm{d}^d p_r}{(2\pi)^{d-1}} \frac{\mathrm{d}^d p_s}{(2\pi)^{d-1}} \delta_+(c_1)\delta_+(c_2)\delta_+(c_3)\delta_+(c_4)$$

$$\begin{aligned} c_1 &= p_r^2, \quad c_2 = p_s^2\,, \\ c_3 &= (p_a + p_b - p_r - p_s)^2 - \xi_a \xi_b s_{ab}\,, \\ c_4 &= \xi_a(s_{ab} - s_{ar} - s_{as}) + \xi_b(s_{ab} - s_{br} - s_{bs})\,. \end{aligned}$$

Topologies for the triple collinear counterterm

- We find that the denominators appearing in the triple collinear counterterm can be sorted into 20 topologies

$$
\begin{aligned}
F_1(n_1,n_2,n_3) &= \int \mathrm{d}\phi_{iff} \frac{1}{s_{ar}^{n_1}(s_{ar}+s_{as})^{n_2}(s_{ab}-s_{ar}-s_{br})^{n_3}}\,,\\
F_2(n_1,n_2,n_3) &= \int \mathrm{d}\phi_{iff} \frac{1}{s_{ar}^{n_1}(s_{ar}+s_{as})^{n_2}(s_{ar}+s_{br})^{n_3}}\,,\\
F_3(n_1,n_2,n_3) &= \int \mathrm{d}\phi_{iff} \frac{1}{s_{ar}^{n_1}(s_{ab}-s_{ar}-s_{br})^{n_2}(s_{ar}+s_{as}-s_{ab})^{n_3}}\,,\\
F_4(n_1,n_2,n_3) &= \int \mathrm{d}\phi_{iff} \frac{1}{s_{ar}^{n1}(s_{ab}-s_{ar}-s_{br})^{n_2}(s_{ar}+s_{as}+s_{br}+s_{bs})^{n_3}}\,,
\end{aligned}
$$

And so on…

- For each family we perform an IBP reduction to MIs. In total, for the A_2 counterterm, we have 42 MIs. The total number of integrated counterterms is 157.

Differential equations for MIs

- The master integrals are computed as solutions of differential equations [Gehrmann, Remiddi] in the external variables ξ_a and ξ_b.

- The systems of differential equations have the form ($\vec{f}$ is a column vector containing the MIs for a given topology)

$$d\vec{f}(\xi_a, \xi_b; \epsilon) = \left(\sum_{i=a,b} A_i(\xi_a, \xi_b; \epsilon) d\xi_i \right) \vec{f}(\xi_a, \xi_b; \epsilon)$$

- The system is solved in the canonical basis [Henn], i.e. in a new basis $\vec{g} = \mathrm{T}\vec{f}$ such that the ϵ dependence in the system is factorized

$$d\vec{g}(\xi_a, \xi_b; \epsilon) = \epsilon \left(\sum_{i=a,b} B_i(\xi_a, \xi_b) d\xi_i \right) \vec{g}(\xi_a, \xi_b; \epsilon)$$

- The solutions are given in terms of multiple polylogarithms, but after an heavy usage of polylogs properties the MIs can be expressed in a very simple form involving only weight 2 functions.

Integration of the A_{12} counterterm

- The integration of the A_{12} counterterm is performed using direct integration.

$$\mathcal{IC} = \int_0^1 \mathrm{d}\eta_a \int_0^1 \mathrm{d}\eta_b \underbrace{\int_0^1 \mathrm{d}\xi_a \int_0^1 \mathrm{d}\xi_b\, f(\xi_a, \xi_b, \eta_a, \eta_b; \varepsilon)}_{\mathcal{I}(\eta_a,\eta_b;\varepsilon)} |\mathcal{M}(\eta_a p_a, \eta_b p_b)|^2$$

$$\begin{aligned}
&\int_0^1 \mathrm{d}\xi_a \int_0^1 \mathrm{d}\xi_b \frac{(\eta_a + \eta_b - \eta_b\xi_a + \eta_a\eta_b\xi_a - \eta_a\xi_b + \eta_a\eta_b\xi_b)^{-1+2\varepsilon}}{(-\xi_a + \eta_a\xi_a - \xi_b + \eta_b\xi_b + \xi_a\xi_b - \eta_a\xi_a\xi_b - \eta_b\xi_a\xi_b + \eta_a\eta_b\xi_a\xi_b)} \\
&\times \frac{(1+\eta_a - \xi_a + \eta_a\xi_a)^{-\varepsilon}(1-\xi_b+\eta_b\xi_b)^{2-\varepsilon}(2-\xi_b+\eta_b\xi_b)^{-1-\varepsilon}(1+\eta_b-\xi_b+\eta_b\xi_b)^{-1-\varepsilon}(2-\xi_a+\eta_a\xi_a-\xi_b+\eta_b\xi_b)^{-1+2\varepsilon}}{(\eta_a+\eta_b-\eta_a^2\eta_b-\eta_a\eta_b^2-2\eta_b\xi_a+2\eta_a\eta_b\xi_a+\eta_b\xi_a^2-2\eta_a\eta_b\xi_a^2+\eta_a^2\eta_b\xi_a^2-2\eta_a\xi_b+2\eta_a\eta_b\xi_b+\eta_a\xi_b^2-2\eta_a\eta_b\xi_b^2+\eta_a\eta_b^2\xi_b^2)} \\
&\times \Big\{[2-(1-\eta_b)\xi_b-((1-\eta_a)\xi_a(1-(1-\eta_b)\xi_b))][(1-\xi_a)(1-(1-\eta_b)\xi_b)+\eta_a(\eta_b+\xi_a-\xi_a\xi_b+\eta_b\xi_a\xi_b)] \\
&\times \big(-\eta_b^2(-1+\xi_b)(1+\xi_b)+(-1+\xi_b)(\xi_a-\xi_b)+\eta_b\xi_b(-1-\xi_a+2\xi_b)+\eta_a(\eta_b+\xi_a-\xi_a\xi_b+\eta_b\xi_a\xi_b)\big)\Big\} \\
&\times (1-\eta_a)^{1-2\varepsilon}\eta_a^{-\varepsilon}(1-\eta_b)^{-1-2\varepsilon}\eta_b^{-\varepsilon}(1-\xi_a)^{-\varepsilon}\xi_a^{-\varepsilon}(1-\xi_b)^{-1-\varepsilon}\xi_b^{-1-\varepsilon}(1-\xi_a+\eta_a\xi_a)^{-\varepsilon}(2-\xi_a+\eta_a\xi_a)^{-\varepsilon}
\end{aligned}$$

- Analytical integration seems hopeless… but actually it is not! This integral can be performed analytically after several steps of sector decomposition, partial fractioning and distributional expansions.

- A dedicated tool for partial fractioning (LinApart)[Chargeishvili, Fekésházy, Somogyi, Van Thurenhout] has been developed to speed up the computation.

NNLOCAL code

- The subtraction method has been implemented in the public code NNLOCAL.

```
************************** version  beta  ***************************
*                                                                   *
* ##    ## ##    ## ##        #######   ######     ###    ##        *
* ###   ## ###   ## ##       ##     ## ##    ##   ## ##   ##        *
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*                                                                   *
*********************************************************************
```

- As validation of our code, we applied it to the computation of NNLO cross-section for Higgs production in gluon fusion in the heavy-top limit with $n_f = 0$, i.e. in a theory without quarks.

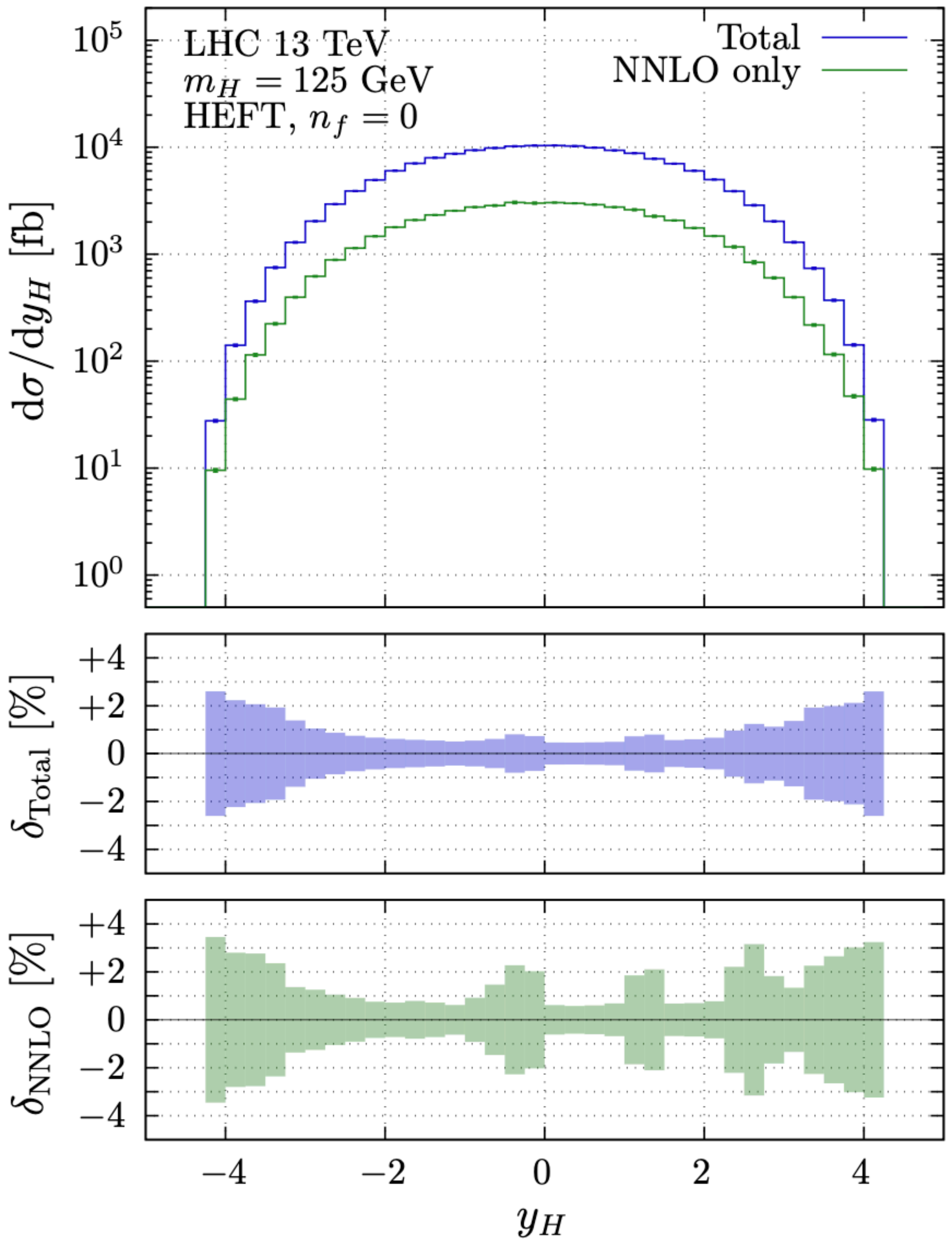
- Since the counterterms are integrated analytically, we can check IR pole cancellation explicitly.

- Our result for the total cross-section is validated against the code n3loxs[Baglio, Duhr, Mistlberger, Szafron]. We observe perfect agreement.

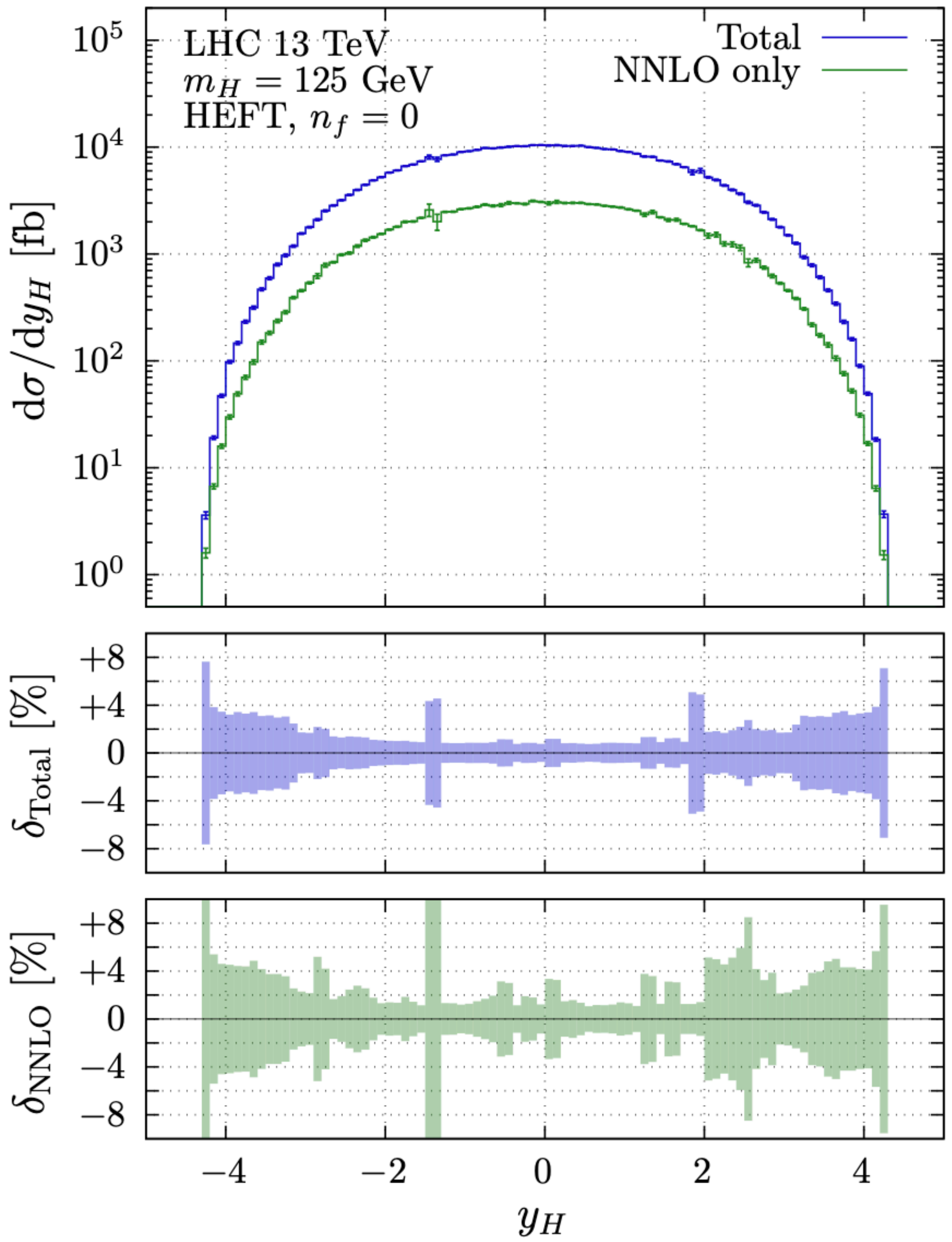
Differential results

- Our code is fully differential in all particle momenta, therefore any infrared and collinear safe quantity can be computed using NNLOCAL.

- We present results for the rapidity distribution of the Higgs boson at NNLO in the heavy-top theory with $n_f = 0$ flavors.



$\Delta y_H = 0.25$



$\Delta y_H = 0.1$

Conclusions and outlook

- One of the bottlenecks for the computation of NNLO cross-sections in proton-proton collisions is the presence of IR divergences, that need to be regularized if we want to build fully-differential tools.
- One possibility is offered by the Colorful subtraction method!
- After defining suitable counterterms, we integrate them analytically in order to have explicit pole cancellation and robust numerical control.
- We implemented our framework in the code NNLOCAL, and validated it using as benchmark the process $gg \to H$ in the heavy-top theory with $n_f = 0$ quark flavor.
- Many exciting future directions for this project: include quark channels as well as other color-singlet procesess in NNLOCAL, include jets in the final state, extensions to N3LO…