Primordial black holes beyond spherical symmetry: Improved compaction function

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Motivations

What they are:

- Primordial black holes (PBH) are black hole formed in the early universe
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- Depending on their abundance, can affect the cosmological observables:
  $\rightarrow$ CMB characteristic, non-gaussianities for instance
- Source of gravitational waves: $\rightarrow$ affect the expected stochastic GW background
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- Crucial to understand condition for their formation
- Their interaction with the environment
- Their characteristic signal : GW radiation, Hawking evaporation

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Key challenges

- Most of our knowledge on BH comes from the stationary exact solutions of GR :
  $\rightarrow$ asymptotically flat, no-hair theorem $\rightarrow$ uniqueness of Schwarzschild / Kerr solutions
- PBH are by definition asymptotically FLRW (versus asymptotically flat)

PBH abundance from black hole evaporation

- Black hole are thermal objects

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- Need exact PBH solutions and study concretely the evaporation process for asymptotically FRW BH : very challenging !

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- Cosmology: power spectrum and mass distribution function (statistical approach)

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- New-solution generating method in GR to obtain exact asymptotically FRW axi-symmetric BH
- New definition of quasi-local energy allowing to provide a new criteria for PBH formation beyond spherical symmetry

A warm-up example: the Schwarzschild black hole

What is the definition of a black hole ?

- Consider the Schwarzschild black hole with the metric

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2 \qquad f(r) = 1 - \frac{2m}{r} \tag{4}$$

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- Expansions of null rays θ_ℓ and θ_n tell us how spherical light front expand/contract

$$\theta_\ell\theta_n = \frac{2}{r^2}(r-2m) \qquad \rightarrow \qquad \theta_\ell\theta_n = 0 \qquad \text{at} \qquad r = 2m \tag{6}$$

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  - $\rightarrow$ cosmological anti-trapping horizon: $\theta_\ell > 0$ and $\theta_n = 0$
  - $\rightarrow$ beyond the cosmological horizon: $\theta_\ell > 0$ and $\theta_n > 0$
- Definition purely quasi-local: a black hole is a trapped region (possibly dynamical)

What about time-dependent spherically symmetric black holes ?

Kodama miracle for time-dependent spherically symmetric geometries

- Consider a spherically symmetric spacetime

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b + R^2\mathrm{d}\Omega^2 = -f(t,r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(t,r)} + R^2(t,r)\mathrm{d}\Omega^2 \tag{7}$$

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- Notion of mass allows one to build a criteria for condition under which a (spherically symmetric) PBH will form

Criteria for PBH formation in spherical symmetry

Compaction function

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$\rightarrow$ starting point : find a generalization of the Kodama vector and Misner-Sharp mass

Criteria for PBH formation in spherical symmetry

Compaction function

- Sasaki and Shibata proposed a way to criteria for PBH formation based on the excess in the Misner-Sharp mass from an inhomogeneity
- Compaction function

$$\mathcal{C}_{\rm SS}(t,r) = \frac{2\left(\mathcal{M}_{\rm MS}(t,r) - \bar{\mathcal{M}}_{\rm MS}(t,r)\right)}{R(t,r)} \tag{11}$$

  where $\bar{\mathcal{M}}_{\rm MS}(t,r)$ is the mass of the patch of universe without inhomogeneity
- Given one cosmic history, find the maximal value of the compaction function
- Other proposals for criteria of PBH formation: finding the more robust is an open debate

Key limitations

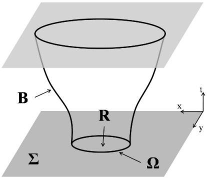
- Notion of quasi-local mass is not unique in GR: many proposals
- Restricted to spherical symmetry

→ need a generalization beyond spherical symmetry
→ starting point : find a generalization of the Kodama vector and Misner-Sharp mass
→ build a definition of the compaction function from this generalized mass notion

Improved compaction function beyond spherical symmetry

Generalization the Kodama vector beyond spherical symmetry

- Consider a region of spacetime $\mathcal{V}$ with boundary $\partial\mathcal{V} = \Sigma_i \cup \mathcal{B} \cup \Sigma_f$ and 2-sphere $\Omega = \Sigma \cup \mathcal{B}$



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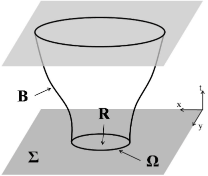

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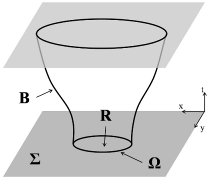

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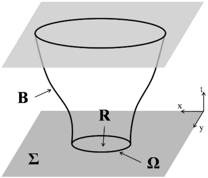

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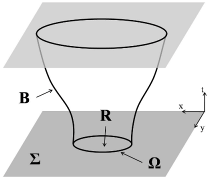

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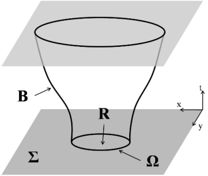

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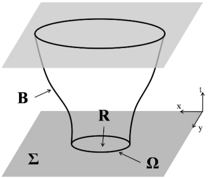

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- Key point: With this general notion of energy, we can construct a generalized compaction function beyond spherical symmetry

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How can we test this general formalism on a concrete example ?
→ find exact solution

Exact solutions

A concrete example

- Consider a self-interacting scalar coupled to matter for simplicity

$$\mathcal{S} = \int d^4x \sqrt{|g|} \left\{ \frac{\mathcal{R}}{2\ell_P^2} - \frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu - V(\phi) \right\} \qquad \text{with} \qquad V(\phi) = V_\circ e^{k\ell_P \phi} \tag{23}$$

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- One concrete example of an exact time-dependent black hole solution

$$ds^2 = (Ct+B)^{\xi_2}\left[-\left(1-\frac{2m}{r}\right)^{\beta}dt^2 + \frac{dr^2}{(1-\frac{2m}{r})^{\beta}} + \left(1-\frac{2m}{r}\right)^{1-\beta}d\Omega^2\right], \tag{26a}$$

$$\phi = \xi_0 \ln\left(1-\frac{2m}{r}\right) + \frac{\xi_1\xi_2}{\kappa}\ln(Ct+B) \tag{26b}$$

with

$$\xi_1 = -\xi_3 = \frac{\beta}{\xi_0} \qquad (\beta^2 - 2\xi_0^2\kappa)\xi_2 = 2\xi_0^2\kappa \tag{27}$$

$$(2\xi_0^2\kappa - \beta^2)^2 V_0 = \pm\, 2\xi_0^2 C^2(\beta^2 - 6\xi_0^2\kappa), \tag{28}$$

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$$ds^2 = (Ct+B)^{\xi_2}\left[-\left(1-\frac{2m}{r}\right)^\beta dt^2 + \frac{dr^2}{(1-\frac{2m}{r})^\beta} + \left(1-\frac{2m}{r}\right)^{1-\beta} d\Omega^2\right], \tag{26a}$$

$$\phi = \xi_0 \ln\left(1-\frac{2m}{r}\right) + \frac{\xi_1\xi_2}{\kappa}\ln(Ct+B) \tag{26b}$$

with

$$\xi_1 = -\xi_3 = \frac{\beta}{\xi_0} \qquad (\beta^2 - 2\xi_0^2\kappa)\xi_2 = 2\xi_0^2\kappa \tag{27}$$

$$(2\xi_0^2\kappa - \beta^2)^2 V_0 = \pm\, 2\xi_0^2 C^2(\beta^2 - 6\xi_0^2\kappa), \tag{28}$$

- β is the scalar charge
- Can check that for $\beta = 0$, one recovers the static Schwarzschild black hole solution
- At large r, we recover the flat FLRW cosmological geometry with $a(t) = (Ct+B)^{\xi_2}$
- Can be derive from a powerful solution generating method

A concrete example

- Consider a self-interacting scalar coupled to matter for simplicity

$$\mathcal{S} = \int d^4x\sqrt{|g|}\left\{\frac{\mathcal{R}}{2\ell_P^2} - \frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu - V(\phi)\right\} \qquad \text{with} \qquad V(\phi) = V_\circ e^{k\ell_P\phi} \tag{23}$$

such that $[V_0] = L^{-4}$and $[k] = 1$.

- The field equations are given by

$$G_{\mu\nu} = \ell_P^2 T_{\mu\nu} = \ell_P^2\left[\phi_\mu\phi_\nu - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\phi_\alpha\phi_\beta + V(\phi)\right)\right] \tag{24}$$

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Dynamical horizons

- Apply the previous method: compute the mean curvature vector $H^\mu \partial_\mu$, compute its norm

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- New phenomenology from exact solutions: horizons are created and annihilated by pairs

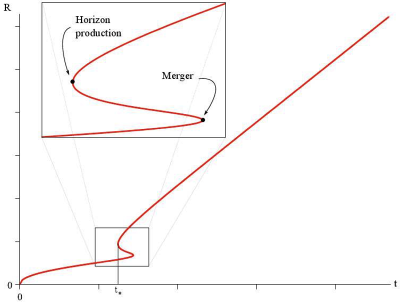

Figure: Dynamics of the cosmological and black hole horzons

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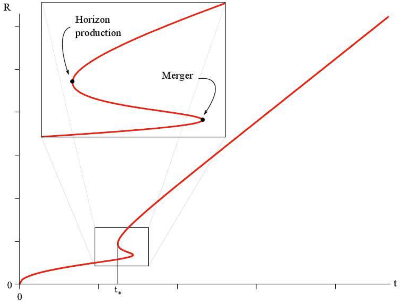

Figure: Dynamics of the cosmological and black hole horzons

- Single out the portion of the solution allowing to describe a PBH

Conclusion

Take away messages

- PBH shall play a crucial role in early and late cosmology
- Constraining their abundance, evaporation time, and their interaction with the environments require to have exact solutions
- Understanding their formation in a more realistic way requires having criteria of formation valid beyond spherical symmetry

Main results

- New proposal for the definition of the compaction function beyond spherical symmetry
- New solution-generating technique to derive exact analytical solutions of GR + matter describing PBH
- Reveal new phenomenology for the dynamics of horizons

Main goals

- Use the new definition of compaction function in numerical simulations
- Explore more realistic solutions: include rotation and spin effects
- Explore the definition of temperature for time-dependent horizons
- Merge cosmological and black hole perturbations theory to capture the emission of gravitational waves from PBH

Conclusion

Thank you