Invariants of CP violation for Majorana neutrinos
in the seesaw mechanism

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- ♦ **What can we learn from Quantum Mechanics**
- ♦ **Invariants of leptonic CP violation on seesaw**
- ♦ **Heavy flavor decays / light flavor oscillations**

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The commutator language borrowed from QM

★ 100 years ago, 3 important papers marked the birth of quantum mechanics in its matrix form:

- **W. Heisenberg**, ZPC 33 (1925) 879
- **M. Born**, **P. Jordan**, ZPC 34 (1925) 858
- **M. Born**, **W. Heisenberg**, **P. Jordan**, ZPC 35 (1926) 557

$$\boxed{\left[\hat{x}(t), \hat{p}(t)\right] = \mathrm{i}\hbar}$$

★ The two observables are represented by the Hermitian operators, and their *nonzero* commutator describes their incompatibility, i.e., they cannot be simultaneously and exactly observed.

★ The commutator of up- and down-quark mass matrices can be similarly introduced, to describe if their mass eigenstates are simultaneously identical to their flavor eigenstates (C. Jarlskog 1985).

$$C_q \equiv \mathrm{i}\left[M_\mathrm{u}M_\mathrm{u}^\dagger , M_\mathrm{d}M_\mathrm{d}^\dagger\right] \longrightarrow \det C_q = -2\mathcal{J}_q \left(m_u^2 - m_c^2\right)\left(m_c^2 - m_t^2\right)\left(m_t^2 - m_u^2\right)\left(m_d^2 - m_s^2\right)\left(m_s^2 - m_b^2\right)\left(m_b^2 - m_d^2\right)$$

Jarlskog invariant: $\mathcal{J}_q = \frac{1}{8} \sin 2\vartheta_{12} \sin 2\vartheta_{13} \sin 2\vartheta_{23} \cos \vartheta_{13} \sin \delta_q$

★ The non-commutativity of up- and down-quark mass matrices in mathematics is indeed a natural consequence of flavor mixing and CP violation in physics in the SM.

The invariant of CP violation in the quark sector

★ **In the mass basis of quarks, it’s the 3×3 CKM matrix that describes flavor mixing and CP violation in the SM. The only but powerful constraint on this matrix is unitarity:**

$$-\mathcal{L}_{\rm CC} = \frac{g}{\sqrt{2}}\,\overline{(u,\quad c,\quad t)_{\rm L}}\,\gamma^\mu \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{\rm L} W^+_\mu + {\rm h.c.}$$

N. Cabibbo (1963), M. Kobayashi, T. Maskawa (1973)

unitarity

$$\left\{ \begin{array}{l} \displaystyle\sum_i V_{\alpha i} V^*_{\beta i} = \delta_{\alpha\beta} \\ \displaystyle\sum_\alpha V_{\alpha i} V^*_{\alpha j} = \delta_{ij} \end{array} \right.$$

★ **The strength of CP violation in the SM is characterized by a universal rephasing invariant:**

$$\mathcal{J}_q \sum_\gamma \varepsilon_{\alpha\beta\gamma} \sum_k \varepsilon_{ijk} = {\rm Im}\left(\underline{V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}}\right)$$

(C. Jarlskog 1985, D.d. Wu 1986)

The commutator and the J-invariant contain the same information on CP violation (H. Fritzsch, ZZX, 1999)

★ **The CKM quartets are rephasing-invariant as they are insensitive to the phase transformations:**

$$\alpha \to e^{{\rm i}\phi_\alpha}\alpha\,, \quad \beta \to e^{{\rm i}\phi_\beta}\beta\,; \quad i \to e^{{\rm i}\phi_i} i\,, \quad j \to e^{{\rm i}\phi_j} j\,.$$

$$V_{\alpha i} \to e^{{\rm i}(\phi_i - \phi_\alpha)} V_{\alpha i}\,, \quad V_{\beta j} \to e^{{\rm i}(\phi_j - \phi_\beta)} V_{\beta j}\,, \quad V_{\alpha j} \to e^{{\rm i}(\phi_j - \phi_\alpha)} V_{\alpha j}\,, \quad V_{\beta i} \to e^{{\rm i}(\phi_i - \phi_\beta)} V_{\beta i}\,.$$

These phases cancelled out

The commutator in the lepton sector

★ Assuming the 3×3 Pontecorvo-Maki-Nakagawa-Sakata (MNS) lepton flavor mixing matrix to be unitary, one may similarly use a commutator of lepton mass matrices to describe CP violation:

Jarlskog invariant: $\mathcal{J}_\nu = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \cos\theta_{13} \sin\delta_\nu$

$$C_\nu \equiv \mathrm{i}\left[M_l M_l^\dagger , M_\nu M_\nu^\dagger\right] \longrightarrow \det C_\nu = -2\mathcal{J}_\nu \left(m_e^2 - m_\mu^2\right)\left(m_\mu^2 - m_\tau^2\right)\left(m_\tau^2 - m_e^2\right)\left(m_1^2 - m_2^2\right)\left(m_2^2 - m_3^2\right)\left(m_3^2 - m_1^2\right)$$

★ A beautiful application of this description is to establish an elegant relation between the Jarlskog invariant in vacuum and its effective counterpart in matter for neutrino oscillations.

Effective Hamiltonian: $$\mathcal{H}_\mathrm{m} \equiv \underbrace{\frac{1}{2E}\widetilde{U}\begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2 \end{pmatrix}\widetilde{U}^\dagger}_{\text{the matter term}} = \underbrace{\frac{1}{2E}U\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}U^\dagger}_{\text{the vacuum term}} + \underbrace{\begin{pmatrix} V_\mathrm{cc} + V_\mathrm{nc} & 0 & 0 \\ 0 & V_\mathrm{nc} & 0 \\ 0 & 0 & V_\mathrm{nc} \end{pmatrix}}_{\text{the matter potential}}$$

Diagonal matter potential yields: $$\left[D_l^2 , \mathcal{H}_\mathrm{m}\right] = \frac{1}{2E}\underline{\left[D_l^2 , \widetilde{M}_\nu \widetilde{M}_\nu^\dagger\right]} = \frac{1}{2E}\underline{\left[D_l^2 , M_\nu M_\nu^\dagger\right]}$$

The Naumov relation: $$\boxed{\widetilde{\mathcal{J}}_\nu \Delta\widetilde{m}_{21}^2 \Delta\widetilde{m}_{31}^2 \Delta\widetilde{m}_{32}^2 = \mathcal{J}_\nu \Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}$$

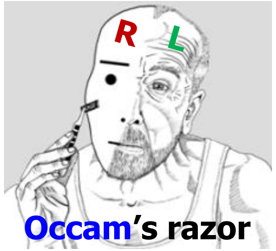
V.A. Naumov 1992
P.F. Harrison, W.G. Scott 2000; ZZX 2000

Seesaw: a most natural extension of the SM

- Neutrinos surely have the *right* to be *right* (-handed) to keep a similar kind of left-right symmetry as charged leptons and quarks —— small animals' fair play?

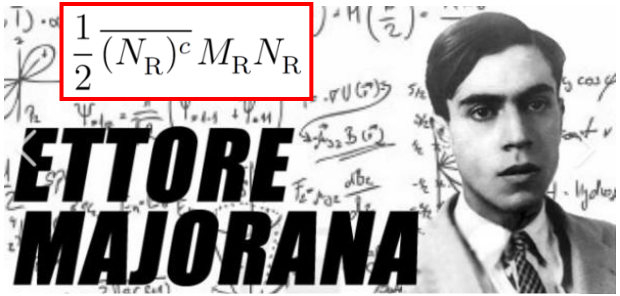
$$\begin{pmatrix} u_{\rm L} \\ d_{\rm L} \end{pmatrix} \begin{array}{c} \longleftrightarrow \\ \longleftrightarrow \end{array} \begin{array}{c} u_{\rm R} \\ d_{\rm R} \end{array}$$

$$\begin{pmatrix} \nu_{e\rm L} \\ e_{\rm L} \end{pmatrix} \begin{array}{c} \longleftrightarrow \\ \longleftrightarrow \end{array} \begin{array}{c} ? \\ e_{\rm R} \end{array}$$



Occam's razor

- Then neutrinos are allowed to couple to the SM Higgs doublet —— the Yukawa interactions. Why not?
- But the gender of neutrinos (neutral) makes it very fair to add a Majorana mass term with *N* and *N*^{c}, which is fully *harmless* to all the fundamental symmetries of the SM.



$$\boxed{\frac{1}{2}\overline{(N_{\rm R})^c} M_{\rm R} N_{\rm R}}$$

- Then we are led to seesaw (**P. Minkowski** 1977), a mechanism consistent with **Weinberg**'s SMEFT (1979).

$$-\mathcal{L}_{\rm lepton} = \overline{\ell_{\rm L}}\, Y_l H\, l_{\rm R} + \overline{\ell_{\rm L}}\, Y_\nu \widetilde{H} N_{\rm R} + \frac{1}{2}\overline{(N_{\rm R})^c} M_{\rm R} N_{\rm R} + {\rm h.c.}$$

$$= \overline{l_{\rm L}}\, Y_l l_{\rm R} \underset{\uparrow}{\phi^0} + \frac{1}{2}\overline{\left[\nu_{\rm L} \quad (N_{\rm R})^c\right]} \underline{\begin{pmatrix} 0 & Y_\nu \phi^{0*} \\ Y_\nu^T \phi^{0*} & M_{\rm R} \end{pmatrix}} \begin{bmatrix} (\nu_{\rm L})^c \\ N_{\rm R} \end{bmatrix} + \overline{\nu_{\rm L}}\, Y_l l_{\rm R} \underset{\uparrow}{\phi^+} - \overline{l_{\rm L}}\, Y_\nu N_{\rm R} \underset{\uparrow}{\phi^-} + {\rm h.c.}$$

CP violation in heavy neutrino decays is crucial for leptogenesis (**M. Fukugita**, **T. Yanagiada** 1986).

Two types of lepton flavor mixing

◆ **A basis transformation to obtain Majorana neutrino masses and flavor mixing before or after SSB.**

$$\mathbb{U}^\dagger \begin{pmatrix} 0 & Y_\nu \phi^{0*} \\ Y_\nu^T \phi^{0*} & M_{\rm R} \end{pmatrix} \mathbb{U}^* = \begin{pmatrix} D_\nu & 0 \\ 0 & D_N \end{pmatrix}$$

working masses:

$$\begin{cases} D_\nu \equiv {\rm Diag}\{m_1, m_2, m_3\} \\ D_N \equiv {\rm Diag}\{M_4, M_5, M_6\} \end{cases}$$

Decomposition: $$\mathbb{U} = \underbrace{\begin{pmatrix} I & 0 \\ 0 & U_0' \end{pmatrix}}_{\text{sterile (unitary)}} \underbrace{\begin{pmatrix} A & R \\ S & B \end{pmatrix}}_{\text{Yukawa (interplay)}} \underbrace{\begin{pmatrix} U_0 & 0 \\ 0 & I \end{pmatrix}}_{\text{active (unitary)}}$$

A block parameterization (ZZX, 1110.0083)

◆ **Weak charged-current interactions of leptons in the seesaw mechanism:**

***U* = *AU*$_0$: the PMNS matrix**

***R* : an analogue for heavy**

$$U D_\nu U^T = ({\rm i}R)\, D_N\, ({\rm i}R)^T$$

$$-\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \overline{(e \quad \mu \quad \tau)_{\rm L}} \gamma^\mu \left[U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} + R \begin{pmatrix} N_4 \\ N_5 \\ N_6 \end{pmatrix}_{\rm L} \right] W_\mu^- + {\rm h.c.}$$

oscillations, LNV ← light **heavy → collider, LNV, LFV**

$UU^\dagger + RR^\dagger = I$ **(unitarity relation)**

- **The PMNS matrix *U* is not exactly unitary in the seesaw scenario**
- **But non-unitarity of *U* is constrained to be very small**

The full Euler-like parametrization

◆ The **1st** full **Euler-like** parametrization of $U = AU_0$ and R is useful for calculating flavor structures.

$$U_0 = \begin{pmatrix} c_{12}c_{13} & \hat{s}^*_{12}c_{13} & \hat{s}^*_{13} \\ -\hat{s}_{12}c_{23} - c_{12}\hat{s}_{13}\hat{s}^*_{23} & c_{12}c_{23} - \hat{s}^*_{12}\hat{s}_{13}\hat{s}^*_{23} & c_{13}\hat{s}^*_{23} \\ \hat{s}_{12}\hat{s}_{23} - c_{12}\hat{s}_{13}c_{23} & -c_{12}\hat{s}_{23} - \hat{s}^*_{12}\hat{s}_{13}c_{23} & c_{13}c_{23} \end{pmatrix}$$

derivable from the parameters of A and R

$$A = \begin{pmatrix} c_{14}c_{15}c_{16} & 0 & 0 \\ \begin{array}{l} -c_{14}c_{15}\hat{s}_{16}\hat{s}^*_{26} - c_{14}\hat{s}_{15}\hat{s}^*_{25}c_{26} \\ -\hat{s}_{14}\hat{s}^*_{24}c_{25}c_{26} \end{array} & c_{24}c_{25}c_{26} & 0 \\ \begin{array}{l} -c_{14}c_{15}\hat{s}_{16}c_{26}\hat{s}^*_{36} + c_{14}\hat{s}_{15}\hat{s}^*_{25}\hat{s}_{26}\hat{s}^*_{36} \\ -c_{14}\hat{s}_{15}c_{25}\hat{s}^*_{35}c_{36} + \hat{s}_{14}\hat{s}^*_{24}c_{25}\hat{s}_{26}\hat{s}^*_{36} \\ +\hat{s}_{14}\hat{s}^*_{24}\hat{s}_{25}\hat{s}^*_{35}c_{36} - \hat{s}_{14}c_{24}\hat{s}^*_{34}c_{35}c_{36} \end{array} & \begin{array}{l} -c_{24}c_{25}\hat{s}_{26}\hat{s}^*_{36} - c_{24}\hat{s}_{25}\hat{s}^*_{35}c_{36} \\ -\hat{s}_{24}\hat{s}^*_{34}c_{35}c_{36} \end{array} & c_{34}c_{35}c_{36} \end{pmatrix}$$

ZZX
0709.2220/1110.0083

$$R = \begin{pmatrix} \hat{s}^*_{14}c_{15}c_{16} & \hat{s}^*_{15}c_{16} & \hat{s}^*_{16} \\ \begin{array}{l} -\hat{s}^*_{14}c_{15}\hat{s}_{16}\hat{s}^*_{26} - \hat{s}^*_{14}\hat{s}_{15}\hat{s}^*_{25}c_{26} \\ +c_{14}\hat{s}^*_{24}c_{25}c_{26} \end{array} & -\hat{s}^*_{15}\hat{s}_{16}\hat{s}^*_{26} + c_{15}\hat{s}^*_{25}c_{26} & c_{16}\hat{s}^*_{26} \\ \begin{array}{l} -\hat{s}^*_{14}c_{15}\hat{s}_{16}c_{26}\hat{s}^*_{36} + \hat{s}^*_{14}\hat{s}_{15}\hat{s}^*_{25}\hat{s}_{26}\hat{s}^*_{36} \\ -\hat{s}^*_{14}\hat{s}_{15}c_{25}\hat{s}^*_{35}c_{36} - c_{14}\hat{s}^*_{24}c_{25}\hat{s}_{26}\hat{s}^*_{36} \\ -c_{14}\hat{s}^*_{24}\hat{s}_{25}\hat{s}^*_{35}c_{36} + c_{14}c_{24}\hat{s}^*_{34}c_{35}c_{36} \end{array} & \begin{array}{l} -\hat{s}^*_{15}\hat{s}_{16}c_{26}\hat{s}^*_{36} - c_{15}\hat{s}^*_{25}\hat{s}_{26}\hat{s}^*_{36} \\ +c_{15}c_{25}\hat{s}^*_{35}c_{36} \end{array} & c_{16}c_{26}\hat{s}^*_{36} \end{pmatrix}$$

The latest stringent *bounds* on possible PMNS nonunitarity.
M. Blennow et al. 2023

$$\left\{ \begin{array}{l} \theta_{1j} < 2.92^\circ \\ \theta_{2j} < 0.27^\circ \\ \theta_{3j} < 2.56^\circ \\ [j = 4, 5, 6] \end{array} \right.$$

ZZX, J. Zhu, 2412.17698

Non-unitarity of the PMNS matrix

- **The PMNS matrix $U = AU_0$ in the seesaw mechanism is *non-unitary*, but this effect is rather small.**

$$A = I - \frac{1}{2}\begin{pmatrix} s_{14}^2 + s_{15}^2 + s_{16}^2 & 0 & 0 \\ 2\hat{s}_{14}\hat{s}_{24}^* + 2\hat{s}_{15}\hat{s}_{25}^* + 2\hat{s}_{16}\hat{s}_{26}^* & s_{24}^2 + s_{25}^2 + s_{26}^2 & 0 \\ 2\hat{s}_{14}\hat{s}_{34}^* + 2\hat{s}_{15}\hat{s}_{35}^* + 2\hat{s}_{16}\hat{s}_{36}^* & 2\hat{s}_{24}\hat{s}_{34}^* + 2\hat{s}_{25}\hat{s}_{35}^* + 2\hat{s}_{26}\hat{s}_{36}^* & s_{34}^2 + s_{35}^2 + s_{36}^2 \end{pmatrix} + \mathcal{O}\left(s_{ij}^4\right)$$

So $U = AU_0 = U_0 + \text{nonunitarity corrections } (\lesssim 10^{-3})$**.**

- **The strength of Yukawa interactions is proportional to *R*. And CP violation in heavy Majorana neutrino decays and thermal leptogenesis are determined by nonzero *R*.**

$$R = \begin{pmatrix} \hat{s}_{14}^* & \hat{s}_{15}^* & \hat{s}_{16}^* \\ \hat{s}_{24}^* & \hat{s}_{25}^* & \hat{s}_{26}^* \\ \hat{s}_{34}^* & \hat{s}_{35}^* & \hat{s}_{36}^* \end{pmatrix} + \mathcal{O}\left(s_{ij}^3\right)$$

$$\varepsilon_{j\alpha} \equiv \frac{\Gamma(N_j \to \ell_\alpha + H) - \Gamma(N_j \to \overline{\ell_\alpha} + \overline{H})}{\sum_\alpha \left[\Gamma(N_j \to \ell_\alpha + H) + \Gamma(N_j \to \overline{\ell_\alpha} + \overline{H})\right]}$$

$$D_j \equiv \left|R_{ej}\right|^2 + \left|R_{\mu j}\right|^2 + \left|R_{\tau j}\right|^2 \quad (j = 4, 5, 6)$$

$$\simeq \frac{1}{8\pi\langle H\rangle^2 D_j}\sum_{j'=4}^{6}\left\{M_{j'}^2\,\mathrm{Im}\left[\underline{\left(R_{\alpha j}^* R_{\alpha j'}\right)}\sum_\beta\left[\underline{\left(R_{\beta j}^* R_{\beta j'}\right)}\,\xi(x_{j'j}) + \underline{\left(R_{\beta j} R_{\beta j'}^*\right)}\,\zeta(x_{j'j})\right]\right]\right\}$$

Four types of rephasing invariants in seesaw

★ **Given the seesaw mechanism and its weak charged-current interactions, one may define *4* types of rephasing invariants of CP violation for light and heavy Majorana neutrinos:**

$$-\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}}\, \overline{\left(e \quad \mu \quad \tau\right)_{\rm L}}\, \gamma^{\mu} \left[U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} + R \begin{pmatrix} N_4 \\ N_5 \\ N_6 \end{pmatrix}_{\rm L} \right] W^{-}_{\mu} + {\rm h.c.}$$

oscillations, LNV ← light **heavy → collider, LNV, LFV**

active and light neutrinos

- **Jarlskog-like invariants for CPV in LFV + LNV cases:** $\mathcal{J}^{ii'}_{\alpha\beta} \equiv {\rm Im}\left(U_{\alpha i} U_{\beta i'} U^*_{\alpha i'} U^*_{\beta i}\right)$ √
- **Jarlskog-like invariants for CPV in the LNV processes:** $\mathcal{V}^{ii'}_{\alpha\beta} \equiv {\rm Im}\left(U_{\alpha i} U_{\beta i} U^*_{\alpha i'} U^*_{\beta i'}\right)$?

sterile and heavy neutrinos

- **Jarlskog-like invariants for CPV in LFV + LNV cases:** $\mathcal{X}^{jj'}_{\alpha\beta} \equiv {\rm Im}\left(R_{\alpha j} R_{\beta j'} R^*_{\alpha j'} R^*_{\beta j}\right)$ √
- **Jarlskog-like invariants for CPV in the LNV processes:** $\mathcal{Z}^{jj'}_{\alpha\beta} \equiv {\rm Im}\left(R_{\alpha j} R_{\beta j} R^*_{\alpha j'} R^*_{\beta j'}\right)$ √

Analytical results in the Euler-like parametrization

★ **The invariants of CP violation for heavy neutrinos associated with the LFV and LNV processes:**

$\mathcal{X}^{45}_{e\mu} = s_{14}s_{15}s_{24}s_{25}\sin\left(\alpha_2 - \alpha_1\right)$	$\mathcal{Z}^{45}_{e\mu} = -s_{14}s_{15}s_{24}s_{25}\sin\left(\alpha_1 + \alpha_2\right)$	$\mathcal{Z}^{45}_{ee} = -s^2_{14}s^2_{15}\sin 2\alpha_1$
$\mathcal{X}^{45}_{e\tau} = s_{14}s_{15}s_{34}s_{35}\sin\left(\alpha_3 - \alpha_1\right)$	$\mathcal{Z}^{45}_{e\tau} = -s_{14}s_{15}s_{34}s_{35}\sin\left(\alpha_1 + \alpha_3\right)$	$\mathcal{Z}^{45}_{\mu\mu} = -s^2_{24}s^2_{25}\sin 2\alpha_2$
$\mathcal{X}^{45}_{\mu\tau} = s_{24}s_{25}s_{34}s_{35}\sin\left(\alpha_3 - \alpha_2\right)$	$\mathcal{Z}^{45}_{\mu\tau} = -s_{24}s_{25}s_{34}s_{35}\sin\left(\alpha_2 + \alpha_3\right)$	$\mathcal{Z}^{45}_{\tau\tau} = -s^2_{34}s^2_{35}\sin 2\alpha_3$
$\mathcal{X}^{46}_{e\mu} = s_{14}s_{16}s_{24}s_{26}\sin\left(\gamma_1 - \gamma_2\right)$	$\mathcal{Z}^{46}_{e\mu} = -s_{14}s_{16}s_{24}s_{26}\sin\left(\gamma_1 + \gamma_2\right)$	$\mathcal{Z}^{46}_{ee} = +s^2_{14}s^2_{16}\sin 2\gamma_1$
$\mathcal{X}^{46}_{e\tau} = s_{14}s_{16}s_{34}s_{36}\sin\left(\gamma_1 - \gamma_3\right)$	$\mathcal{Z}^{46}_{e\tau} = -s_{14}s_{16}s_{34}s_{36}\sin\left(\gamma_1 + \gamma_3\right)$	$\mathcal{Z}^{46}_{\mu\mu} = +s^2_{24}s^2_{26}\sin 2\gamma_2$
$\mathcal{X}^{46}_{\mu\tau} = s_{24}s_{26}s_{34}s_{36}\sin\left(\gamma_2 - \gamma_3\right)$	$\mathcal{Z}^{46}_{\mu\tau} = -s_{24}s_{26}s_{34}s_{36}\sin\left(\gamma_2 + \gamma_3\right)$	$\mathcal{Z}^{46}_{\tau\tau} = +s^2_{34}s^2_{36}\sin 2\gamma_3$
$\mathcal{X}^{56}_{e\mu} = s_{15}s_{16}s_{25}s_{26}\sin\left(\beta_2 - \beta_1\right)$	$\mathcal{Z}^{56}_{e\mu} = -s_{15}s_{16}s_{25}s_{26}\sin\left(\beta_1 + \beta_2\right)$	$\mathcal{Z}^{56}_{ee} = -s^2_{15}s^2_{16}\sin 2\beta_1$
$\mathcal{X}^{56}_{e\tau} = s_{15}s_{16}s_{35}s_{36}\sin\left(\beta_3 - \beta_1\right)$	$\mathcal{Z}^{56}_{e\tau} = -s_{15}s_{16}s_{35}s_{36}\sin\left(\beta_1 + \beta_3\right)$	$\mathcal{Z}^{56}_{\mu\mu} = -s^2_{25}s^2_{26}\sin 2\beta_2$
$\mathcal{X}^{56}_{\mu\tau} = s_{25}s_{26}s_{35}s_{36}\sin\left(\beta_3 - \beta_2\right)$	$\mathcal{Z}^{56}_{\mu\tau} = -s_{25}s_{26}s_{35}s_{36}\sin\left(\beta_2 + \beta_3\right)$	$\mathcal{Z}^{56}_{\tau\tau} = -s^2_{35}s^2_{36}\sin 2\beta_3$

Phases: $\boxed{\alpha_i \equiv \delta_{i4} - \delta_{i5}\ , \quad \beta_i \equiv \delta_{i5} - \delta_{i6}\ , \quad \gamma_i \equiv \delta_{i6} - \delta_{i4}}$

Totally six independent CP-violating phases.

with $\alpha_i + \beta_i + \gamma_i = 0$ for $i = 1, 2, 3.$

CP asymmetries of heavy neutrino decays

★ **Flavor-dependent CPV:**

$$\varepsilon_{j\alpha} \simeq \frac{-1}{8\pi\langle H\rangle^2 D_j}\sum_{j'=4}^{6}\left[M_{j'}^2 \sum_{\beta}\left[\mathcal{Z}_{\alpha\beta}^{jj'}\xi(x_{j'j}) + \mathcal{X}_{\alpha\beta}^{jj'}\zeta(x_{j'j})\right]\right]$$

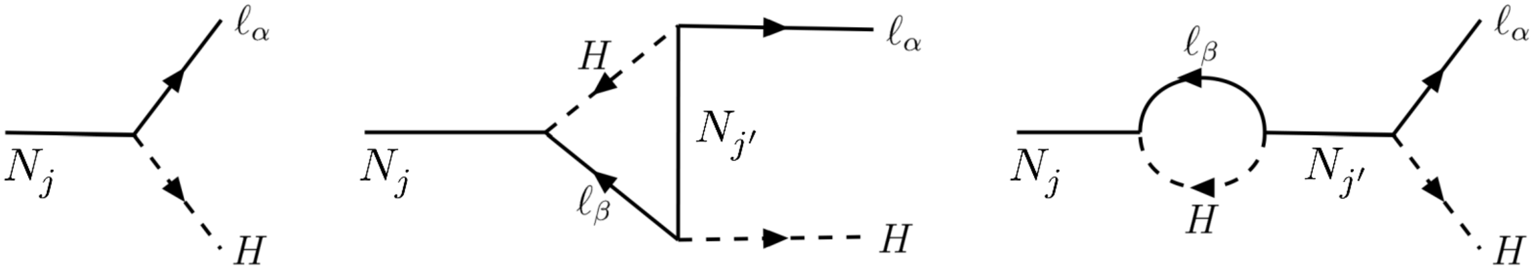
$$\underline{\varepsilon_j \equiv \varepsilon_{je} + \varepsilon_{j\mu} + \varepsilon_{j\tau}}$$

★ **Flavor-independent CPV:**

$$\varepsilon_{j} \simeq \frac{-1}{8\pi\langle H\rangle^2 D_j}\sum_{j'=4}^{6}\left[M_{j'}^2 \left(\sum_{\alpha}\mathcal{Z}_{\alpha\alpha}^{jj'} + 2\sum_{\alpha<\beta}\mathcal{Z}_{\alpha\beta}^{jj'}\right)\xi(x_{j'j})\right]$$

The loop functions:

$$\xi(x_{j'j}) = \sqrt{x_{j'j}}\left\{1 + 1/\left(1 - x_{j'j}\right) + \left(1 + x_{j'j}\right)\ln\left[x_{j'j}/\left(1 + x_{j'j}\right)\right]\right\} \text{ and } \zeta(x_{j'j}) = 1/\left(1 - x_{j'j}\right)$$



$$x_{j'j} \equiv \frac{M_{j'}^2}{M_j^2}$$

Leptogenesis

Flavor oscillations of active neutrinos

★ **Flavor oscillations of active neutrinos in the seesaw mechanism with slight PMNS non-unitarity:**

$$P(\nu_\alpha \to \nu_\beta) = \frac{1}{D_{\alpha\beta}}\left[\sum_{i=1}^{3}|U_{\alpha i}|^2|U_{\beta i}|^2 + 2\sum_{i<i'}\mathrm{Re}\left(U_{\alpha i}U_{\beta i'}U^*_{\alpha i'}U^*_{\beta i}\right)\cos\Delta_{i'i} + 2\sum_{i<i'}\mathcal{J}^{ii'}_{\alpha\beta}\sin\Delta_{i'i}\right]$$

$$P(\overline{\nu}_\alpha \to \overline{\nu}_\beta) = \frac{1}{D_{\alpha\beta}}\left[\sum_{i=1}^{3}|U_{\alpha i}|^2|U_{\beta i}|^2 + 2\sum_{i<i'}\mathrm{Re}\left(U_{\alpha i}U_{\beta i'}U^*_{\alpha i'}U^*_{\beta i}\right)\cos\Delta_{i'i} - 2\sum_{i<i'}\mathcal{J}^{ii'}_{\alpha\beta}\sin\Delta_{i'i}\right]$$

where $D_{\alpha\beta} \equiv \left(UU^\dagger\right)_{\alpha\alpha}\left(UU^\dagger\right)_{\beta\beta} = \left(AA^\dagger\right)_{\alpha\alpha}\left(AA^\dagger\right)_{\beta\beta}$, $\Delta_{i'i} \equiv \Delta m^2_{i'i}L/\left(2E\right)$ with $\Delta m^2_{i'i} \equiv m^2_{i'} - m^2_i$

CP violating asymmetries: $\mathcal{A}_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta) - P(\overline{\nu}_\alpha \to \overline{\nu}_\beta) = \dfrac{4}{D_{\alpha\beta}}\displaystyle\sum_{i<i'}\mathcal{J}^{ii'}_{\alpha\beta}\sin\Delta_{i'i}$

Example for a LBL case in vacuum:

$$\mathcal{A}_{\mu e} \simeq \boxed{-16\mathcal{J}_\nu \sin\frac{\Delta_{21}}{2}\sin\frac{\Delta_{31}}{2}\sin\frac{\Delta_{32}}{2}} - 4c_{13}\Big\{c_{12}s_{12}c_{23}\mathrm{Im}\left(\underline{a_{21}}e^{-\mathrm{i}\delta_{21}}\right)\sin\Delta_{21} + c^2_{12}s_{13}s_{23}\mathrm{Im}\left[\underline{a_{21}}e^{\mathrm{i}\left(\delta_{23}-\delta_{13}\right)}\right]\sin\Delta_{31} + s^2_{12}s_{13}s_{23}\mathrm{Im}\left[\underline{a_{21}}e^{\mathrm{i}\left(\delta_{23}-\delta_{13}\right)}\right]\sin\Delta_{32}\Big\}$$

Terrestrial matter effects are entangled with the PMNS non-unitarity effects (e.g., Y.F. Li, ZZX, J.Y. Zhu 2019).

$$a_{21} \equiv \hat{s}^*_{24}\hat{s}_{14} + \hat{s}^*_{25}\hat{s}_{15} + \hat{s}^*_{26}\hat{s}_{16}$$

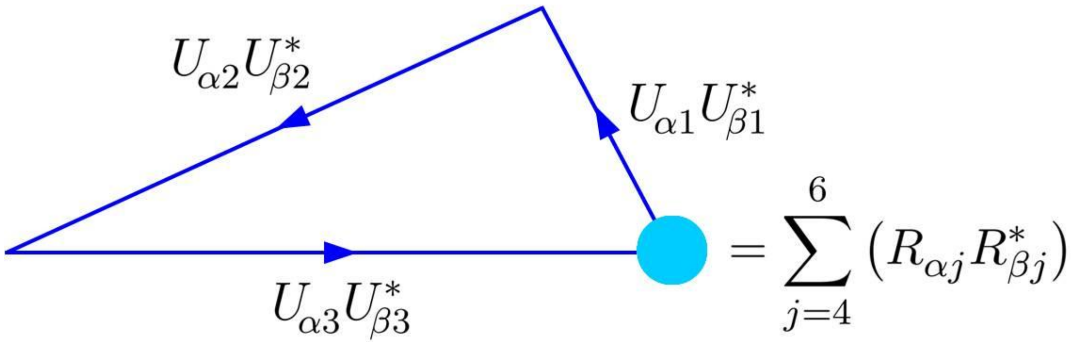
The PMNS unitarity polygons in the seesaw case

$$\mathcal{J}^{ii'}_{\alpha\beta} \equiv \mathrm{Im}\left(U_{\alpha i} U_{\beta i'} U^*_{\alpha i'} U^*_{\beta i}\right)$$

◆ **One can show that the leading terms of all these Jarlskog-like invariants are the same, coming from the unitarity limit of the PMNS matrix (ZZX, D. Zhang, 2009.09717):**

$$\boxed{\mathcal{J}^{ii'}_{\alpha\beta} = \mathcal{J}_\nu + \mathrm{corrections}}$$

$\leq$ 3% **$<$ 0.01%**



$$\mathcal{J}^{12}_{e\mu} \simeq \mathcal{J}_\nu + c_{12} s_{12} c_{23} \mathrm{Im} a_{21}$$

$$\mathcal{J}^{12}_{\tau e} \simeq \mathcal{J}_\nu + c_{12} s_{12} s_{23} \mathrm{Im} a_{31}$$

$$\mathcal{J}^{12}_{\mu\tau} \simeq \mathcal{J}_\nu + c_{12} s_{12} c_{23} s_{23} \left(s_{23} \mathrm{Im} a_{21} + c_{23} \mathrm{Im} a_{31}\right)$$

$$\mathcal{J}^{23}_{\mu\tau} \simeq \mathcal{J}_\nu + c_{12} c_{23} s_{23} \left(s_{12} s_{23} \mathrm{Im} a_{21} + s_{12} c_{23} \mathrm{Im} a_{31} + c_{12} \mathrm{Im} a_{32}\right)$$

$$\mathcal{J}^{31}_{\mu\tau} \simeq \mathcal{J}_\nu + s_{12} c_{23} s_{23} \left(c_{12} s_{23} \mathrm{Im} a_{21} + c_{12} c_{23} \mathrm{Im} a_{31} - s_{12} \mathrm{Im} a_{32}\right)$$

$$\mathcal{J}^{23}_{e\mu} \simeq \mathcal{J}^{31}_{e\mu} \simeq \mathcal{J}^{23}_{\tau e} \simeq \mathcal{J}^{31}_{\tau e} \simeq \mathcal{J}_\nu$$

$$a_{ii'} \equiv \hat{s}^*_{i4} \hat{s}_{i'4} + \hat{s}^*_{i5} \hat{s}_{i'5} + \hat{s}^*_{i6} \hat{s}_{i'6}$$

Life is much easier in the minimal seesaw case

- ♦ **The Jarlskog invariant and the CP-violating asymmetries of heavy Majorana neutrinos depend on 3 original CP phases in a relatively simple way if the PMNS non-unitarity is neglected (ZZX 2023):**

	$\sin 2\alpha_1$	$\sin 2\alpha_2$	$\sin 2\alpha_3$	$\sin{(\alpha_1 + \alpha_2)}$	$\sin{(\alpha_1 + \alpha_3)}$	$\sin{(\alpha_2 + \alpha_3)}$	$\sin{(\alpha_1 - \alpha_2)}$	$\sin{(\alpha_2 - \alpha_3)}$	$\sin{(\alpha_3 - \alpha_1)}$
$\mathcal{J}_\nu$	√	√	√	√	√	√	√	√	√
ε_{4e}	√			√	√		√		√
$\varepsilon_{4\mu}$		√		√		√	√	√	
$\varepsilon_{4\tau}$			√		√	√		√	√
ε_{4}	√	√	√	√	√	√			
ε_{5e}	√			√	√		√		√
$\varepsilon_{5\mu}$		√		√		√	√	√	
$\varepsilon_{5\tau}$			√		√	√		√	√
ε_{5}	√	√	√	√	√	√			

- ♦ **Parameters in the minimal (3 + 2) seesaw: 7 = 2 + 3 + 2 (low) ↔ 11 = 2 + 6 + 3 (high).**

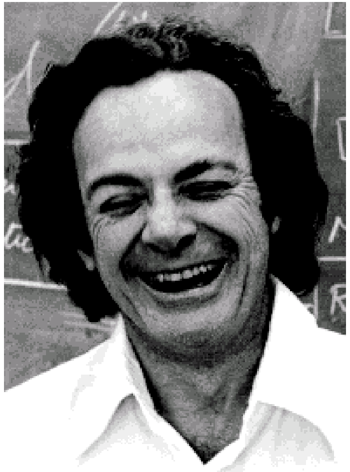
- ♦ **It is also very important to calculate the neutrino mass-squared differences, active flavor mixing angles and all LNV and LFV effects at low energies with the *original* seesaw parameters/invariants in the seesaw framework (ZZX, J.Y. Zhu, 2412.17698).**

Concluding remarks

★ **Question**: rephasing invariants (e.g., moduli of the PMNS matrix elements, the Jarlskog invariant or its analogs, angles of the unitarity triangles or polygons) and basis-dependent parametrizations, which set is more useful in the studies of neutrino physics?

★ **My personal answer**: they are two sides of the same coin, and one of them may be more useful in making the underlying physics more transparent, or making correlative relations between intrinsic model parameters and observable quantities more straightforward.

> Theories of the known, which are described by different physical ideas, may be equivalent in all their predictions and are hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the unknown. For different views suggest different kinds of modifications which might be made. I, therefore, think that a good theoretical physicist today might find it useful to have a wide range of physical viewpoints and mathematical expressions of the same theory available to him —— R.P. Feynman's Nobel Lecture.



THANK YOU FOR YOUR ATTENTION