Alien operators for PDF evolution

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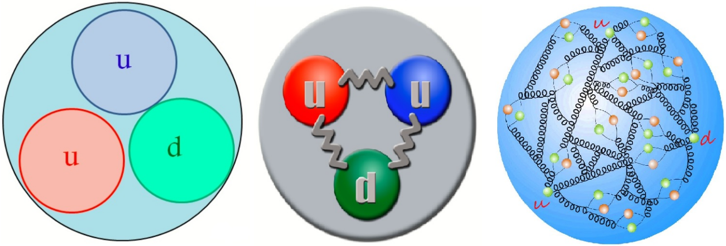
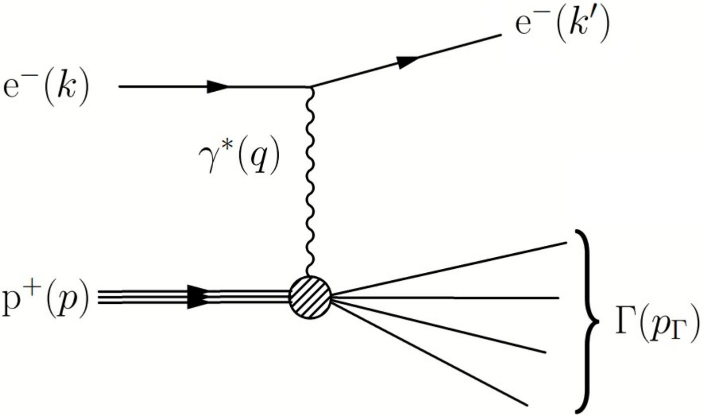
How to gain insight into the structure of hadrons

- Hadrons such as the proton are a mess of many interacting quarks/gluons!



- Nevertheless, they have well-defined physical properties such as mass, spin etc.
  $\Rightarrow$ How can we explain these in terms of the properties of the constituent partons?
- Experimentally: Perform high-energy **scattering experiments** that can resolve the inner hadron structure (e.g. scatter electrons off a proton)

Scattering experiments: Deeply-inelastic scattering



Assumptions:

- Photon highly virtual, $Q^2 \equiv -q^2 \gg p^2$
- $s \gg m_p^2$

Description of scattering experiments

- Hard scale $\Rightarrow$ Factorization between short-range and long-range physics

$$\hat{\sigma}(e^- p^+ \to e^- \Gamma) = \sum_a \int_0^1 \mathrm{d}x_a\, f_a(x_a, \mu_F^2) \sigma(e^- p_a \to e^- \Gamma; \mu_F^2)$$

- Short-range physics characterized by the perturbative partonic cross section σ
- Long-range physics described by non-perturbative parton distributions **PDFs**
- Through application of the **OPE**, the **moments** of the distributions are related to hadronic matrix elements of composite QCD operators

$$f_N(\mu^2) \equiv \int_0^1 \mathrm{d}x\, x^{N-1}\, f(x, \mu^2) \sim \langle P | O^{(N)} | P \rangle$$

Leading-twist operators

The OPE is dominated by leading-twist operators. Based on the representations of the QCD flavour group, we can distinguish two sets of such operators

$$\mathcal{O}^{(N)}_{q\,\mathrm{NS};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\lambda^\alpha\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$
$$\mathcal{O}^{(N)}_{g\,\mathrm{S};\mu_1\ldots\mu_N}(x) = \frac{1}{2}\mathcal{S}\left[\,F^{a_1}{}_{\mu\mu_1}\,D^{a_1a_2}_{\mu_2}\ldots D^{a_{N-2}a_{N-1}}_{\mu_{N-1}}F^{a_{N-1};\mu}{}_{\mu_N}\,\right]$$
$$\mathcal{O}^{(N)}_{q\,\mathrm{S};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc}A^b_\mu A^c_\nu$$
$$D_\mu = \partial_\mu - ig_s T^a A^a_\mu$$
$$D^{ac}_\mu = \partial_\mu\delta^{ac} + g_s f^{abc}A^b_\mu$$

f^{abc} are the standard QCD structure constants.

PDF scale dependence

Scale evolution of PDFs is set by the **DGLAP** equation [Gribov and Lipatov, 1972], [Altarelli and Parisi, 1977], [Dokshitzer, 1977]

$$\frac{\mathrm{d}f_i(x,\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y} P_{ij}(y) f_j\left(\frac{x}{y},\mu^2\right)$$

with P_{ij} the QCD splitting functions. These are perturbative quantities and can be computed as the anomalous dimensions of the leading-twist operators that define the PDFs

$$\frac{\mathrm{d}[\mathcal{O}_i]}{\mathrm{d}\ln\mu^2} = \gamma_{ij}[\mathcal{O}_j], \quad \gamma_{ij} \equiv a_s\gamma_{ij}^{(0)} + a_s^2\gamma_{ij}^{(1)} + \ldots$$

$$\gamma_{ij} = -\int_0^1 \mathrm{d}x\, x^{N-1} P_{ij}(x)$$

The anomalous dimensions are extracted from the renormalization of the operators. One way to do this is by considering off-shell operator matrix elements.

Alien operators

When doing so, it is well-known that mixing with non-gauge-invariant (**alien**) operators needs to be taken into account [Dixon and Taylor, 1974, Kluberg-Stern and Zuber, 1975a, Kluberg-Stern and Zuber, 1975b, Joglekar and Lee, 1976, Joglekar, 1977a, Joglekar, 1977b].

These receive contributions from

- ghost fields and
- the field equations of motion (EOM)

Recently, G. Falcioni and F. Herzog derived a method to consistently construct the aliens to any loop-order [Falcioni and Herzog, 2022].

- In their approach, the aliens are derived using **generalized BRST symmetry** of the QCD Lagrangian.
- Each alien operator features a coupling constant obeying certain **constraint relations**, which were solved for fixed $N \leq 20$ in [Falcioni and Herzog, 2022, Falcioni et al., 2024a]

  $\rightarrow$ All 4-loop splitting functions now known to $N = 20$ [Falcioni et al., 2023b, Falcioni et al., 2023a, Gehrmann et al., 2024a, Falcioni et al., 2024c, Falcioni et al., 2024a, Falcioni et al., 2025]

Can we solve the relations for **arbitrary** values of N?

Alien relations

$$\mathcal{O}_{\mathrm{EOM}}^{(N),II} = g_s \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi \right) f^{abc} \sum_{i+j=N-3} \kappa_{ij} (\partial^i A^b)(\partial^j A^c),$$

$$\mathcal{O}_{c}^{(N),II} = -g_s f^{abc} \sum_{i+j=N-3} \eta_{ij} (\partial \overline{c}^a)(\partial^i A^b)(\partial^{j+1} c^c)$$

$$\kappa_{ij} + \kappa_{ji} = 0,$$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N) \binom{i+j+1}{i},$$

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = 0$$

NOTE: Bottom relation = conjugation!

Alien relations

A second application of the sum leads to

$$\sum_{t=0}^{i}(-1)^{t+j}\binom{t+j}{j}\eta_{(i-t)(j+t)} = -\sum_{t=0}^{i}(-1)^{t+j}\binom{t+j}{j}\sum_{s=0}^{i-t}(-1)^{s+j+t}\binom{s+j+t}{j+t}\eta_{(i-t-s)(j+t+s)}$$

and hence

$$\eta_{ij} = \sum_{t=0}^{i}\binom{t+j}{j}\sum_{s=0}^{i-t}(-1)^{s}\binom{s+j+t}{j+t}\eta_{(i-t-s)(j+t+s)}.$$

- Already encountered in the computation of the anomalous dimensions of leading-twist operators in non-forward kinematics, see e.g. [Moch and Van Thurenhout, 2021, Van Thurenhout, 2024]
- Great predictive power!
- Valuable information about the function space

Solving conjugation relations

- To take full advantage of conjugation relations, one needs to be able to evaluate them analytically
- Use principles of symbolic summation: Gosper's algorithm [Gosper, 1978] and (creative) telescoping [Zeilberger, 1991]
- Fully automated in `Mathematica` packages `Sigma` [Schneider, 2004, Schneider, 2007] and `EvaluateMultiSums` [Schneider, 2013, Schneider, 2014]

Alien relations

Another neat feature of the alien relations is that they show a **bootstrap**: Complicated higher-order couplings can be related to simpler lower-order ones

$$\eta^{(1)}_{ijkl} + \eta^{(1)}_{jilk} + \eta^{(1)}_{lkji} + \eta^{(1)}_{klij} = 2[\kappa^{(1)}_{ij(k+l+1)} + \kappa^{(1)}_{(k+l+1)ji}]\binom{k+l+1}{k} + \text{permutations}$$

$$\eta^{(1)}_{ijk} + \eta^{(1)}_{kij} + \eta^{(1)}_{jki} = 2\kappa_{i(j+k+1)}\binom{j+k+1}{j} + 2\kappa_{k(i+j+1)}\binom{i+j+1}{i} + 2\kappa_{j(i+k+1)}\binom{i+k+1}{k}.$$

$$\eta_{ij} + \eta_{ji} = \eta(N)\left[\binom{i+j+1}{i} + \binom{i+j+1}{j}\right]$$

The coupling $\eta(N)$ is known to $O(a_s^3)$

[Dixon and Taylor, 1974, Hamberg and van Neerven, 1992, Gehrmann et al., 2023]

Alien relations

The power of all these relations is that they allow us to write the alien couplings in terms of a **small** number of unknowns. Consider e.g. the most complicated aliens that are needed for the 4-loop renormalization of the physical operators

$$\mathcal{O}_{\mathrm{EOM}}^{(N),IV} = g_s^3 \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi\right) (f\ f\ f)^{abcde} \sum_{\substack{i+j+k+l \\ =N-5}} \kappa_{ijkl}^{(1)} (\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e),$$

$$\mathcal{O}_{c}^{(N),IV} = -g_s^3 \, (f\ f\ f)^{abcde} \sum_{\substack{i+j+k+l \\ =N-5}} \eta_{ijkl}^{(1)} (\partial \overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^{l+1} c^e)$$

A priori $>$ 500 parameters to fix!

Using all relations: Only 8 remain in the end!

→ Can be fixed by a few fixed-N OME computations
→ Explicit expressions in [Falcioni et al., 2024b]

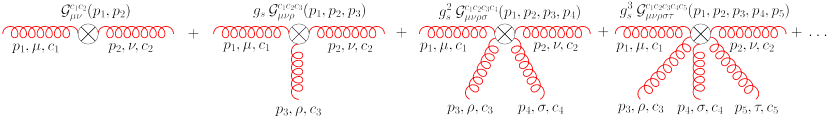
Application: Alien Feynman rules

With the couplings known, one can derive the Feynman rules of the alien operators from **first principles**

- The Feynman rules for the gauge-invariant quark and gluon operators, with up to 6 external legs, can be found e.g. in [Floratos et al., 1977, Floratos et al., 1979, Mertig and van Neerven, 1996, Kumano and Miyama, 1997, Hayashigaki et al., 1997, Bierenbaum et al., 2009, Klein, 2009, Blümlein, 2001, Moch et al., 2017, Gehrmann et al., 2023] and references therein. The generalization to an arbitrary number of legs can be found in [Somogyi and Van Thurenhout, 2024] [1]
- The alien rules were computed up to 3 external legs in [Hamberg and van Neerven, 1992],[Matiounine et al., 1998],[Blümlein et al., 2022], and extensions to the 4- and 5-point vertices were recently presented in [Gehrmann et al., 2023] and [Gehrmann et al., 2024b]

[1] Note that the latter also presents the corresponding rules for the operators with total derivatives, relevant for non-zero momentum flow through the operator vertex.

Application: Alien Feynman rules



$$
\begin{aligned}
\mathcal{G}^{c_1c_2c_3c_4c_5}_{\mu\nu\rho\sigma\tau}(p_1,p_2,p_3,p_4,p_5) = {} & \frac{1+(-1)^N}{2}\, i^{N-1} f^{c_1c_2x} f^{xc_3y} f^{yc_4c_5} \Bigg\{ \\
& - g_{\mu\rho}\Delta_\nu\Delta_\sigma\Delta_\tau \sum_{i+j=N-3} \kappa_{ij}(\Delta\cdot p_4)^i(\Delta\cdot p_5)^j + \Delta_\rho\Delta_\sigma\Delta_\tau[(p_1+2p_2)_\mu\Delta_\nu \\
& - (\Delta\cdot p_2)g_{\mu\nu}] \sum_{i+j+k=N-4} \kappa^{(1)}_{ijk}(\Delta\cdot p_3)^i(\Delta\cdot p_4)^j(\Delta\cdot p_5)^k + [p_1^2\Delta_\mu \\
& - p_{1\mu}(\Delta\cdot p_1)]\Delta_\nu\Delta_\rho\Delta_\sigma\Delta_\tau \sum_{i+j+k+l=N-5} \kappa^{(1)}_{ijkl}(\Delta\cdot p_2)^i(\Delta\cdot p_3)^j(\Delta\cdot p_4)^k(\Delta\cdot p_5)^l \Bigg\} \\
& + \frac{1+(-1)^N}{2}\, i^{N-1} d_{4f}^{c_1c_2c_3c_4c_5} \Bigg\{ \Delta_\mu\Delta_\nu\Delta_\rho[(p_4+2p_5)_\sigma\Delta_\tau \\
& - (\Delta\cdot p_5)g_{\sigma\tau}] \sum_{i+j+k=N-4} \kappa^{(2)}_{ijk}(\Delta\cdot p_1)^i(\Delta\cdot p_2)^j(\Delta\cdot p_3)^k + [p_1^2\Delta_\mu \\
& - p_{1\mu}(\Delta\cdot p_1)]\Delta_\nu\Delta_\rho\Delta_\sigma\Delta_\tau \sum_{i+j+k+l=N-5} \kappa^{(2)}_{ijkl}(\Delta\cdot p_2)^i(\Delta\cdot p_3)^j(\Delta\cdot p_4)^k(\Delta\cdot p_5)^l \Bigg\} \\
& + \text{permutations}
\end{aligned}
$$

Application: Alien Feynman rules

- Ghost vertices:
  (a) Agreement with [Gehrmann et al., 2023] for 0- and 1-gluon vertices and $(f\,f)$, d_4 parts of the 2-gluon vertex
  (b) $d_{\widehat{4ff}}$ part of 2-gluon vertex new!
  (c) 3-gluon vertex new! [Recently also obtained in [Gehrmann et al., 2024b], exact agreement!]
- Alien gluon vertices:
  (a) Agreement with [Blümlein et al., 2022, Gehrmann et al., 2023] for 2- and 3-gluon vertices; agreement with [Gehrmann et al., 2023] for $(f\,f)$, d_4 parts of the 4-gluon vertex
  (b) $d_{\widehat{4ff}}$ part of 4-gluon vertex new!
  (c) 5-gluon vertex new! [Recently also obtained in [Gehrmann et al., 2024b], exact agreement!]
- Alien quark vertices:
  (a) Agreement with [Gehrmann et al., 2023] for 0-, 1- and 2-gluon vertices
  (b) 3- and 4-gluon vertices new! [3-gluon vertex recently also obtained in [Gehrmann et al., 2024b], exact agreement!]

Conclusions and outlook

- One way to reconstruct the functional form of the alien operators is based on the use of generalized BRST symmetry
- One then finds classes of EOM and ghost operators, the couplings of which obey interesting consistency relations
- Bootstrap: Complicated higher-order couplings in terms of simpler lower-order ones
- We used these relations to reconstruct the full N-dependence of the 1-loop alien couplings necessary to perform the operator renormalization to 4 loops
- This should be useful in the reconstruction of the full N-dependence of the 4-loop splitting functions!
- Generalization to higher orders in perturbation theory?

End [2]

Thank you for your attention!

[2] Part of this work has been supported by grant K143451 of the National Research, Development and Innovation Fund in Hungary.

Appendices and references

Construction of the alien operators

The complete gauge-fixed QCD action is written as

$$S = \int \mathrm{d}^D x \, \left(\mathcal{L}_0 + \mathcal{L}_{\mathrm{GF+G}}\right) .$$

Here $\mathcal{L}_0$ represents the classical part of the QCD Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{n_f} \overline{\psi}^f (i\not{D} - m_f)\psi^f ,$$

with

$$\mathcal{L}_{\mathsf{GF+G}} = -\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - \overline{c}^a \, \partial^\mu D^{ab}_\mu \, c^b$$

and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$$
$$D_\mu = \partial_\mu - i g_s T^a A^a_\mu$$
$$D^{ac}_\mu = \partial_\mu \delta^{ac} + g_s f^{abc} A^b_\mu$$

f^{abc} are the standard QCD structure constants.

Construction of the alien operators

The QCD Lagrangian can be extended to also include the leading-twist spin-N gauge-invariant operators, which we define as

$$\mathcal{O}_{\mathrm{g}}^{(N)}(x) = \frac{1}{2} F_{\nu}(x)\, D^{N-2} F^{\nu}(x)\,,$$
$$\mathcal{O}_{\mathrm{q}}^{(N)}(x) = \overline{\psi}(x) \not{\Delta}\, D^{N-1} \psi(x)\,.$$

Here Δ_{μ} is a lightlike vector and we introduced the notation

$$F^{\mu;a} = \Delta_{\nu}\, F^{\mu\nu;a}, \qquad A^{a} = \Delta_{\mu} A^{\mu;a}, \qquad D = \Delta_{\mu}\, D^{\mu}, \qquad \partial = \Delta_{\mu} \partial^{\mu}\,.$$

These physical operators now mix under renormalization with aliens, which are (a) proportional to the field EOMs and (b) contain ghosts. Schematically the complete Lagrangian is then

$$\widetilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_{\mathsf{GF+G}} + w_{\mathrm{i}}\, \mathcal{O}_{\mathrm{i}} + \mathcal{O}_{\mathsf{EOM}}^{(N)} + \mathcal{O}_{c}^{(N)}$$

Construction of the alien operators

The most general form of the EOM operator is [Falcioni and Herzog, 2022]

$$\mathcal{O}_{\mathrm{EOM}}^{(N)} = \left(D \cdot F^a + g_s \overline{\psi} T^a \not{\Delta} \psi\right) \mathcal{G}^a(A^a, \partial A^a, \partial^2 A^a, ...)$$

with $\mathcal{G}^a$ a generic local function of the gauge field and its derivatives. Expanding $\mathcal{G}^a$ in a series of contributions with an increasing number of gauge fields then leads to

$$\mathcal{O}_{\mathrm{EOM}}^{(N)} = \mathcal{O}_{\mathrm{EOM}}^{(N),I} + \mathcal{O}_{\mathrm{EOM}}^{(N),II} + \mathcal{O}_{\mathrm{EOM}}^{(N),III} + \mathcal{O}_{\mathrm{EOM}}^{(N),IV} + \ \ldots$$

Construction of the alien operators

$$\mathcal{O}^{(N),I}_{\text{EOM}} = \eta(N)\ \left(D\cdot F^a + g_s\overline{\psi}\,\not{\Delta}\,T^a\psi\right)\ \left(\partial^{N-2}A^a\right),$$

$$\mathcal{O}^{(N),II}_{\text{EOM}} = g_s\left(D\cdot F^a + g_s\overline{\psi}\,\not{\Delta}\,T^a\psi\right)\sum_{\substack{i+j\\=N-3}} C^{abc}_{ij}(\partial^i A^b)(\partial^j A^c),$$

$$\mathcal{O}^{(N),III}_{\text{EOM}} = g_s^2\ \left(D\cdot F^a + g_s\overline{\psi}\not{\Delta}T^a\psi\right)\sum_{\substack{i+j+k\\=N-4}} C^{abcd}_{ijk}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d),$$

$$\mathcal{O}^{(N),IV}_{\text{EOM}} = g_s^3\ \left(D\cdot F^a + g_s\overline{\psi}\not{\Delta}T^a\psi\right)\sum_{\substack{i+j+k+l\\=N-5}} C^{abcde}_{ijkl}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e).$$

Construction of the alien operators

The coefficients $C^{a_1 \ldots a_n}_{i_1 \ldots i_{n-1}}$ can be written in terms of a set of independent colour tensors, each of them multiplying an associated coupling constant, as follows

$$C^{abc}_{ij} = f^{abc} \kappa_{ij},$$
$$C^{abcd}_{ijk} = (f\, f)^{abcd} \kappa^{(1)}_{ijk} + d^{abcd}_{4} \kappa^{(2)}_{ijk} + d^{abcd}_{\widehat{4ff}} \kappa^{(3)}_{ijk},$$
$$C^{abcde}_{ijkl} = (f\, f\, f)^{abcde} \kappa^{(1)}_{ijkl} + d^{abcde}_{4f} \kappa^{(2)}_{ijkl}$$

To avoid overcounting: κ-couplings inherit properties of the colour structures they multiply, e.g. $\kappa_{ij} = -\kappa_{ji}$

The standard gauge transformations leave $\mathcal{L}_0$ and $\mathcal{O}_i$ invariant, but not $\mathcal{O}^{(N)}_{\text{EOM}}$

$\Rightarrow$ generalized gauge transformation

$$A^a_\mu \to A^a_\mu + \delta_\omega A^a_\mu + \delta^\Delta_\omega A^a_\mu$$

Construction of the alien operators

$$A_\mu^a \to A_\mu^a + \delta_\omega A_\mu^a + \delta_\omega^\Delta A_\mu^a$$

$$\delta_\omega A_\mu^a = D_\mu^{ab}\omega^b(x),$$

$$\delta_\omega^\Delta A_\mu^a = -\Delta_\mu \Bigg[\eta(N)\,\partial^{N-1}\omega^a + g_s \sum_{\substack{i+j\\=N-3}} \widetilde{C}_{ij}^{aa_1a_2}\,\left(\partial^i A^{a_1}\right)\,\left(\partial^{j+1}\omega^{a_2}\right)$$

$$+ g_s^2 \sum_{\substack{i+j+k\\=N-4}} \widetilde{C}_{ijk}^{aa_1a_2a_3}\,\left(\partial^i A^{a_1}\right)\,\left(\partial^j A^{a_2}\right)\,\left(\partial^{k+1}\omega^{a_3}\right)$$

$$+ g_s^3 \sum_{\substack{i+j+k+l\\=N-5}} \widetilde{C}_{ijkl}^{aa_1a_2a_3a_4}\,\left(\partial^i A^{a_1}\right)\,\left(\partial^j A^{a_2}\right)\,\left(\partial^k A^{a_3}\right)\,\left(\partial^{l+1}\omega^{a_4}\right) + \mathcal{O}(g_s^4)\Bigg]$$

Construction of the alien operators

$$
\begin{aligned}
\widetilde{C}_{ij}^{abc} &= f^{abc}\eta_{ij}, \\
\widetilde{C}_{ijk}^{abcd} &= (f\,f)^{abcd}\eta_{ijk}^{(1)} + d_4^{abcd}\eta_{ijk}^{(2)} + d_{\widehat{4ff}}^{abcd}\eta_{ijk}^{(3)}, \\
\widetilde{C}_{ijkl}^{abcde} &= (f\,f\,f)^{abcde}\eta_{ijkl}^{(1)} + d_{4f}^{abcde}\eta_{ijkl}^{(2a)} + d_{4f}^{aebcd}\eta_{ijkl}^{(2b)}.
\end{aligned}
$$

The generalized gauge symmetry implies that the couplings $\eta_{n_1\ldots n_j}^{(k)}$ are related to $\kappa_{n_1\ldots n_j}^{(k)}$

Construction of the alien operators

$$
\begin{aligned}
\eta_{ij} &= 2\kappa_{ij} + \eta(N)\binom{i+j+1}{i}, \\
\eta^{(1)}_{ijk} &= 2\kappa_{i(j+k+1)}\binom{j+k+1}{j} + 2[\kappa^{(1)}_{ijk} + \kappa^{(1)}_{kji}], \\
\eta^{(2)}_{ijk} &= 3\kappa^{(2)}_{ijk}, \\
\eta^{(3)}_{ijk} &= 2[\kappa^{(3)}_{ijk} - \kappa^{(3)}_{kji}], \\
\eta^{(1)}_{ijkl} &= 2[\kappa^{(1)}_{ij(l+k+1)} + \kappa^{(1)}_{(l+k+1)ji}]\binom{l+k+1}{k} + 2[\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{ilkj} + \kappa^{(1)}_{likj} + \kappa^{(1)}_{lkij}], \\
\eta^{(2a)}_{ijkl} &= 3\kappa^{(2)}_{ij(k+l+1)}\binom{k+l+1}{k} + 2\kappa^{(2)}_{ijkl}, \\
\eta^{(2b)}_{ijkl} &= 2\kappa^{(2)}_{lijk}.
\end{aligned}
$$

Construction of the alien operators

The generalized gauge transformation can now be promoted to a generalized BRST (gBRST) transformation

$$A_\mu^a \to A_\mu^a + \delta_c A_\mu^a + \delta_c^\Delta A_\mu^a$$

The ghost operator is now generated by the action of gBRST on a suitable ancestor operator [Falcioni and Herzog, 2022], giving

$$\mathcal{O}_c^{(N)} = \mathcal{O}_c^{(N),I} + \mathcal{O}_c^{(N),II} + \mathcal{O}_c^{(N),III} + \mathcal{O}_c^{(N),IV} + \ \ldots$$

Construction of the alien operators

$$\mathcal{O}_c^{(N),I} = -\eta(N)(\partial\overline{c}^a)(\partial^{N-1}c^a),$$

$$\mathcal{O}_c^{(N),II} = -g_s \sum_{\substack{i+j\\=N-3}} \widetilde{C}_{ij}^{abc}(\partial\overline{c}^a)(\partial^i A^b)(\partial^{j+1}c^c),$$

$$\mathcal{O}_c^{(N),III} = -g_s^2 \sum_{\substack{i+j+k\\=N-4}} \widetilde{C}_{ijk}^{astu}(\partial\overline{c}^a)(\partial^i A^s)(\partial^j A^t)(\partial^{k+1}c^u),$$

$$\mathcal{O}_c^{(N),IV} = -g_s^3 \sum_{\substack{i+j+k+l\\=N-5}} \widetilde{C}_{ijkl}^{abcde}(\partial\overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^{l+1}c^e).$$

Construction of the alien operators

In fact, there is another, and equivalent, approach to generate the ghost operators. Namely, we could also start from anti-gBRST, for which $\omega^a(x)$ in the generalized gauge transformation should be replaced by the anti-ghost field $\overline{c}^a(x)$

$$A^a_\mu \to A^a_\mu + \delta_{\overline{c}} A^a_\mu + \delta^{\Delta}_{\overline{c}} A^a_\mu$$

$\to$ This should lead to the same operators!
$\to$ Nevertheless, the functional form of the resulting operators is different from those derived from gBRST

$\Rightarrow$ Non-trivial identities for the η-couplings!

Construction of the alien operators

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = 0,$$

$$\eta^{(1)}_{ijk} = \sum_{m=0}^{i} \sum_{n=0}^{j} \frac{(m+n+k)!}{m!\, n!\, k!} (-1)^{m+n+k} \eta^{(1)}_{(j-n)(i-m)(k+m+n)},$$

$$\eta^{(1)}_{ijkl} = -\sum_{s_1=0}^{i} \sum_{s_2=0}^{j} \sum_{s_3=0}^{k} \frac{(s_1+s_2+s_3+l)!}{s_1!\, s_2!\, s_3!\, l!} (-1)^{s_1+s_2+s_3+l} \eta^{(1)}_{(k-s_3)(j-s_2)(i-s_1)(s_1+s_2+s_3+l)}.$$

Renormalization

The complete Lagrangian is now

$$\widetilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_{\mathsf{GF+G}} + w_{\mathrm{i}}\,\mathcal{O}_{\mathrm{i}} + \mathcal{O}^{(N)}_{\mathsf{EOM}} + \mathcal{O}^{(N)}_{c}$$
$$= \mathcal{L}_0(A^a_\mu, g_s) + \mathcal{L}_{\mathsf{GF+G}}(A^a_\mu, c^a, \bar{c}^a, g_s, \xi) + \sum_{\mathrm{k}} \mathcal{C}_{\mathrm{k}}\,\mathcal{O}_{\mathrm{k}},$$

where $\mathcal{C}_{\mathrm{k}}$ labels all the distinct couplings of the operators, $\mathcal{C}_{\mathrm{k}} = \{w_{\mathrm{i}}, \eta(N), \kappa^{(i)}_{n_1\ldots n_j}, \eta^{(k)}_{n_1\ldots n_j}\}$. The UV singularities associated with the QCD Lagrangian are absorbed by introducing the bare fields/parameters

$$A^{a;\mathsf{bare}}_\mu(x) = \sqrt{Z_3}A^a_\mu(x)$$
$$c^{a;\mathsf{bare}}(x) = \sqrt{Z_c}c^a(x)$$
$$\bar{c}^{a;\mathsf{bare}}(x) = \sqrt{Z_c}\bar{c}^a(x)$$
$$g_s^{\mathsf{bare}} = \mu^\epsilon Z_g g_s$$
$$\xi^{\mathsf{bare}} = \sqrt{Z_3}\xi$$

Renormalization

This is not enough to make the OMEs finite. Instead they need an additional renormalization

$$\mathcal{O}_{\mathrm{i}}^{\mathsf{ren}}(x) = Z_{\mathrm{ij}}\, \mathcal{O}_{\mathrm{j}}^{\mathsf{bare}}(x),$$

The renormalized Lagrangian becomes

$$\begin{aligned}\widetilde{\mathcal{L}} &= \mathcal{L}_0(A_\mu^{a;\mathsf{bare}}, g_s^{\mathsf{bare}}) + \mathcal{L}_{\mathsf{GF+G}}(A_\mu^{a;\mathsf{bare}}, c^{a;\mathsf{bare}}, \bar{c}^{a;\mathsf{bare}}, g_s^{\mathsf{bare}}, \xi^{\mathsf{bare}}) \\ &\quad + \sum_{\mathrm{k}} \mathcal{C}_{\mathrm{k}}^{\mathsf{bare}}\, \mathcal{O}_{\mathrm{k}}^{\mathsf{bare}}, \\ \mathcal{C}_{\mathrm{i}}^{\mathsf{bare}} &= \sum_{\mathrm{k}} \mathcal{C}_{\mathrm{k}}\, Z_{\mathrm{k\,i}},\end{aligned}$$

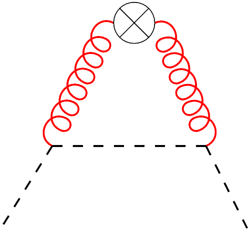
where $\mathcal{C}_{\mathrm{k}}$ is the (finite) renormalized coupling of the operator $\mathcal{O}_{\mathrm{k}}$. The UV-finite OMEs featuring a single insertion of $\mathcal{O}_{\mathrm{g/q}}^{\mathsf{ren}}$ are computed by setting the renormalized couplings $\mathcal{C}_{\mathrm{i}} = \delta_{\mathrm{i\,g/q}}$, which gives

$$\mathcal{C}_{\mathrm{i}}^{\mathsf{bare}} = Z_{\mathrm{g/q\,i}}.$$

Renormalization

$\Rightarrow$ The couplings of the bare operators $\eta^{\text{bare}}(N), \ldots$ are interpreted as the renormalization constants that mix the physical operators into the aliens

$\rightarrow$ Extracted from the direct calculation of the singularities of the OMEs, e.g.



$$\eta^{\text{bare}}(N) = Z_{\mathrm{g}\, c} = -\frac{a_s}{\epsilon}\frac{C_A}{N(N-1)} + O(a_s^2)$$

We note that this quantity is known to $O(a_s^3)$

[Dixon and Taylor, 1974, Hamberg and van Neerven, 1992, Gehrmann et al., 2023]

Colour structures

f^{abc} are the QCD structure constants. The other colour structures are in turn defined as

$$
\begin{aligned}
(f\,f)^{abcd} &= f^{abe} f^{cde}, \\
(f\,f\,f)^{abcde} &= f^{abm} f^{mcn} f^{nde}, \\
d_4^{abcd} &= \frac{1}{4!}[\mathrm{Tr}(T_A^a T_A^b T_A^c T_A^d) + \text{symmetric permutations}], \\
d_{4ff}^{abcd} &= d_4^{abmn} f^{mce} f^{edn}, \\
d_{\widehat{4ff}}^{abcd} &= d_{4ff}^{abcd} - \frac{1}{3} C_A d_4^{abcd}, \\
d_{4f}^{abcde} &= d_4^{abcm} f^{mde}.
\end{aligned}
$$

Solving conjugation relations

- To take full advantage of the anti-gBRST conjugation relations, one needs to be able to evaluate them analytically
- Use principles of symbolic summation!
- Creative telescoping [Zeilberger, 1991]: evaluate the sum of interest by rewriting it as a recursion relation using Gosper's algorithm [Gosper, 1978]
- The closed-form expression of the sum then corresponds to the linear combination of the solutions of the recursion that has the same initial values as the sum.

→ For single sums: `Sigma` [Schneider, 2004, Schneider, 2007]

→ For multiple sums: `EvaluateMultiSums` [Schneider, 2013, Schneider, 2014]

Classical telescoping and Gosper's algorithm

The telescoping algorithm is a well-known method for evaluating finite sums. Suppose we want to evaluate the following sum

$$\sum_{k=a}^{N} f(k)$$

with $a, N \in \mathbb{N}$ and $a \leq N$. Now, if we can find a function $g(N)$ such that

$$f(k) = \Delta g(k) \equiv g(k+1) - g(k)$$

then

$$\begin{aligned}\sum_{k=a}^{N} f(k) &= \sum_{k=a}^{N} g(k+1) - \sum_{k=a}^{N} g(k) \\ &= g(N+1) - g(a).\end{aligned}$$

Here, Δ represents the finite difference operator. The telescoping function $g(N)$ can be found by application of Gosper's algorithm [Gosper, 1978].

Classical telescoping and Gosper's algorithm

Suppose

$$\frac{g(N)}{g(N-1)}$$

is a rational function in N. The algorithm consists of three main steps. Assume we want to calculate the telescoping function for some sequence $\{a_N\}$

$$a_N = \Delta b(N).$$

It is assumed that $\{a_N\}$ is a hypergeometric sequence, that is

$$\frac{a_{N+1}}{a_N} = q(N)$$

with $q(N)$ a rational function of N. The steps of Gosper's algorithm can then be summarized as follows

Classical telescoping and Gosper's algorithm

1. Determine three functions $f(x), g(x)$ and $h(x)$ such that
$$q(x) = \frac{f(x+1)}{f(x)} \frac{g(x)}{h(x+1)}$$
and
$$\gcd[g(x), h(x+n)] = 1 \quad (n \in \mathbb{N}_0).$$
2. Solve the so-called Gosper equation,
$$f(x) = g(x)y(x+1) - h(x)y(x),$$
for the polynomial $y(x)$.
3. If such a polynomial solution does not exist, it means that the sum in question does not have a hypergeometric closed form. Otherwise, the telescoping function is determined by
$$t(x) = \frac{h(x)}{f(x)} y(x) \qquad \text{with } b(N) = t(N)a(N)$$

More details can e.g. be found in [Kauers and Paule, 2011]

Creative telescoping

Classical telescoping works when dealing with sequences that depend on one variable only. When we want to determine a closed form for a summation of a sequence depending on two variables, we can use the creative telescoping algorithm by Zeilberger [Zeilberger, 1991]. The idea is similar to that of classical telescoping. Suppose we want to evaluate

$$\sum_{k=a}^{b} f(N,k) \equiv S(N).$$

The way to go about this is by attempting to find d functions $c_0(N), \ldots, c_d(N)$ and a function $g(N,k)$ such that

$$g(N,k+1) - g(N,k) = c_0(N) f(N,k) + ... + c_d(N) f(N+d,k).$$

Summing both sides, and applying classical telescoping to the left-hand side then gives

$$g(N,b+1) - g(N,a) = c_0(N) \sum_{k=a}^{b} f(N,k) + ... + c_d(N) \sum_{k=a}^{b} f(N+d,k).$$

Creative telescoping

This leads to an inhomogeneous recursion relation for the original sum of the form

$$q(N) = c_0(N)S(N) + ... + c_d(N)S(N+d).$$

Typically, one starts this procedure at $d = 0$, which is equivalent to classical telescoping. The value of d is then increased stepwise until a solution is found. The creative telescoping algorithm can be applied when the sequence under consideration is holonomic. A sequence $\{a_N\}$ is said to be holonomic if there exist polynomials $p_0(x), \ldots, p_r(x)$ such that the following recursion relation is obeyed [Kauers and Paule, 2011]

$$p_0(N)a_N + p_1(N)a_{N+1} + \cdots + p_r(N)a_{N+r} = 0 \quad (N \in \mathbb{N}, p_r(N) \neq 0).$$

For example, the harmonic numbers $\{S_1(N)\}$ form a holonomic sequence as they obey

$$(N+1)S_1(N) - (2N+3)S_1(N+1) + (N+2)S_1(N+2) = 0.$$

More details on the summation algorithms reviewed here can e.g. be found in the excellent books [Graham et al., 1989, Petkovŝek et al., 1996].

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