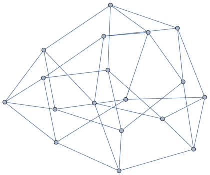
EPS HEP 2025, 8 July 2025
Palais du Pharo, Marseille, France

Statstics and asymptotics of subdivergence-free Feynman integrals in ϕ^4 theory



Paul-Hermann Balduf
University of Oxford, Mathematical Institute

Based on JHEP 11(2023), 160; JHEP 11(2024), 038 (with K. Shaban); hep-th/2412.08617 (with J. Thürigen).

Slides, related papers, data set etc. available from paulbalduf.com/research

Join Mastodon! It's like twitter, but with elephants and open source! **math**stodon.xyz/@paulbalduf

Recall: ϕ^4 theory

University of Oxford Mathematical Institute

- ϕ^4_4 theory: A single, real-valued, bosonic (i.e. commutative) field ϕ in $4-2\epsilon$ spacetime dimensions. Interacts with itself, vertex in Feynman graphs is 4-valent.
- Isn't present in the real world, but similar to Higgs field. Related to statistical physics when extrapolated to $D=3$. We only consider mass $m=0$ (=at critical point).
- ϕ^4 theory has *generic* properties of perturbative QFT, namely
  1. Is renormalizable in 4 dimensions (infinitely many divergent diagrams).
  2. Number of diagrams grows factorially with loop number L.
  3. Can use N-component field $\vec{\phi}$ with global internal $O(N)$ symmetry, large/small N limits etc.

Recall: ϕ^4 theory

UNIVERSITY OF OXFORD
Mathematical Institute

- ϕ^4_4 theory: A single, real-valued, bosonic (i.e. commutative) field ϕ in $4-2\epsilon$ spacetime dimensions. Interacts with itself, vertex in Feynman graphs is 4-valent.
- Isn't present in the real world, but similar to Higgs field. Related to statistical physics when extrapolated to $D=3$. We only consider mass $m=0$ (=at critical point).
- ϕ^4 theory has *generic* properties of perturbative QFT, namely
  1. Is renormalizable in 4 dimensions (infinitely many divergent diagrams).
  2. Number of diagrams grows factorially with loop number L.
  3. Can use N-component field $\vec{\phi}$ with global internal $O(N)$ symmetry, large/small N limits etc.
- We restrict ourselves to:
  1. Only vertex-type graphs (4 external legs).
  2. Only graphs without subdivergence (i.e. *primitive* in the renormalization Hopf algebra [Kreimer 1998]). These graphs are cyclically 6-edge connected (=do not have 4-valent or 2-valent subgraphs with loops).
  3. Consider only the scale-dependence (and not the full functional dependence on all masses and momenta). This quantity contributes to the Symanzik *beta function* (renormalization group function [Callan 1970; Symanzik 1970]).

Periods in ϕ^4 theory

UNIVERSITY OF OXFORD
Mathematical Institute

- For momentum p and reference μ, let $s := \ln \frac{p^2}{\mu^2}$ be the (energy-) scale. Feynman integral of a primitive vertex-type graph G evaluates to

$$\text{const} \cdot \mathcal{P}(G) \cdot s \quad + \quad s - \text{independent terms.}$$

Periods in ϕ^4 theory

- For momentum p and reference μ, let $s := \ln \frac{p^2}{\mu^2}$ be the (energy-) scale. Feynman integral of a primitive vertex-type graph G evaluates to

$$\text{const} \cdot \mathcal{P}(G) \cdot s \quad + \quad s - \text{independent terms.}$$

- $\mathcal{P}(G)$ is a finite number, called the *period* [Broadhurst and Kreimer 1995; Schnetz 2010]. Feynman integral in parametric form (Assign variable a_e to each edge e. $\mathcal{U}$ is the Symanzik polyomial):

$$\mathcal{P}(G) := \left(\prod_{e \in E_G} \int_0^\infty \mathrm{d}a_e \right) \delta \left(1 - \sum_{e=1}^{|E_G|} a_e \right) \frac{1}{\mathcal{U}_G^2(\{a_e\})} \in \mathbb{R}.$$

- $\mathcal{P}$ is a period in the sense of [Kontsevich and Zagier 2001]. Their number theory is interesting, but not for this talk.

Completions and decompletions

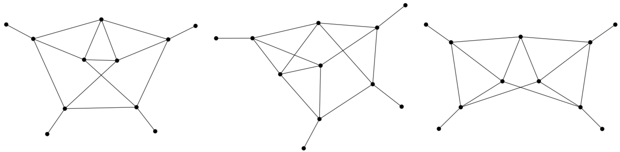
UNIVERSITY OF OXFORD
Mathematical Institute

- The period $\mathcal{P}(G)$ is defined for a vertex-type graph, i.e. G has 4 external edges. At L loops, G has $L+1$ (internal) vertices.

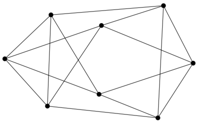
Completions and decompletions

UNIVERSITY OF OXFORD Mathematical Institute

- The period $\mathcal{P}(G)$ is defined for a vertex-type graph, i.e. G has 4 external edges. At L loops, G has $L+1$ (internal) vertices.
- Merge the 4 external edges at a new vertex. The resulting graph has $L+2$ vertices and no external edges. It is called the *completion* of G.



(3 non-isomorphic decompletions)

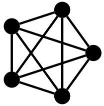


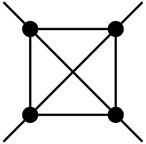
(completion)

- All decompletions G of some fixed completion have the same period $\mathcal{P}(G)$.
(this is non-trivial. The Symanzik polynomials are distinct, but the integrals evaluate to the same number)

Example: loop order $L = 3$

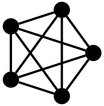
Mathematical Institute

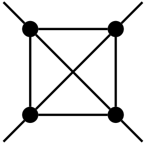
- There are no primitive graphs with $L = 2$ in ϕ^4_4-theory.
- There is exactly one primitive graph on $L + 2 = 5$ vertices, $K_5 =$ 
- All vertices in K_5 are equivalent $\Rightarrow$ removing any of them gives the same decompletion

$G =$ 

UNIVERSITY OF OXFORD
Mathematical Institute

Example: loop order $L = 3$

- There are no primitive graphs with $L = 2$ in ϕ^4_4-theory.
- There is exactly one primitive graph on $L + 2 = 5$ vertices, $K_5 =$ 
- All vertices in K_5 are equivalent $\Rightarrow$ removing any of them gives the same decompletion

$G =$ 

- Symanzik polynomial

$$\mathcal{U}_G = a_1(a_2 + a_3)(a_4 + a_5) + a_4 a_5 a_6 + a_1(a_2 + a_3 + a_4 + a_5)a_6 + a_2 a_5(a_4 + a_6) + a_3 a_4(a_5 + a_6) + a_2 a_3(a_4 + a_5 + a_6)$$

- Period integral can be solved with some effort,

$$\mathcal{P}(G) = \left(\prod_{e=1}^{6} \int_0^\infty \mathrm{d}a_e \right) \delta\left(1 - \sum_{e=1}^{6} a_e\right) \frac{1}{\mathcal{U}_G^2(\{a_e\})} = 6\zeta(3) \approx 7.212$$

How many primitive graphs are there?

UNIVERSITY OF OXFORD
Mathematical Institute

(Recall: decompletions are graphs with 4-valent vertices, cyclically 6-edge connected, with 4 external edges)

Just generate them all and count...

How many primitive graphs are there?

(Recall: decompletions are graphs with 4-valent vertices, cyclically 6-edge connected, with 4 external edges)

Just generate them all and count...

Number grows factorially.
> 1 billion at 15 loops.

L	Primitive decompletions
1	1
2	0
3	1
4	1
5	3
6	10
7	44
8	248
9	1,688
10	13,094
11	114,016
12	1,081,529
13	11,048,898
14	120,451,435
15	1,393,614,379
16	17,041,643,034

Period Symmetries

UNIVERSITY OF OXFORD
Mathematical Institute

- All decompletions of the same completion have the same period.
- There are a few other *symmetries*, where the period of non-isomorphic graphs has the same value [Schnetz 2010; Panzer 2022; Hu et al. 2022].
  - Planar dual graphs have the same period.
  - In a 3-vertex cut, the period is the product of the two sides' periods
  - In a 4-vertex cut, can take the planar dual on either side or "twist" the connection at the cut vertices.
- There are interesting combinatorial *invariants* that respect the same symmetries.

Counts of primitive graphs

UNIVERSITY OF OXFORD
Mathematical Institute

L	Vertex-type graphs "decompletions"	Vacuum graphs "completions"	independent periods
3	1	1	1
4	1	1	1
5	3	2	1
6	10	5	4
7	44	14	9
8	248	49	31
9	1,688	227	134
10	13,094	1,354	819
11	114,016	9,722	6,197
12	1,081,529	81,305	55,196
13	11,048,898	755,643	543,535
14	120,451,435	7,635,677	5,769,143
15	1,393,614,379	82,698,184	65,117,118

$\Rightarrow$ Exploiting all symmetries, still millions of independent period integrals remain.

Relation to random graphs on $n = L + 2$ vertices

UNIVERSITY OF OXFORD
Mathematical Institute

- Recall that our *primitive completions* are 4-regular and cyclically 6-edge connected graphs. They are a subset of all 4-regular simple graphs.
- $\mathcal{G}_{n,4}$ is space of random 4-regular simple (but not necessarily primitive) graphs [Bender and Canfield 1978; Bollobás 1980] on $n = L + 2$ vertices.

Relation to random graphs on $n = L + 2$ vertices

UNIVERSITY OF OXFORD
Mathematical Institute

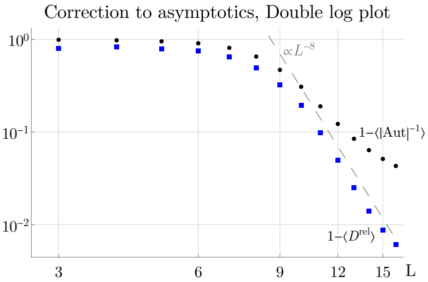
- Recall that our *primitive completions* are 4-regular and cyclically 6-edge connected graphs. They are a subset of all 4-regular simple graphs.
- $\mathcal{G}_{n,4}$ is space of random 4-regular simple (but not necessarily primitive) graphs [Bender and Canfield 1978; Bollobás 1980] on $n = L + 2$ vertices.
- Known for $G \in \mathcal{G}_{n,4}$ in the limit $n \to \infty$:
  - G is almost surely 4-edge connected [Wormald 1981],
  - asymptotic number of graphs in $\mathcal{G}_{n,4}$ coincides with asymptotic number of primitive graphs [Bender and Canfield 1978; Bollobás 1982; Borinsky 2017],
  - $\langle |\mathrm{Aut}(G)| \rangle \to 1$ [McKay and Wormald 1984],
  - Distribution of cycles of fixed length [Bollobás 1980; McKay, Wormald, and Wysocka 2004],
  - ...
- $\Rightarrow$ To leading asymptotic order $n \to \infty$, $\mathcal{G}_{n,4}$ is a good model for primitive graphs.
- We deal with graphs on $n \leq 20$ vertices. Are we in the asymptotic region?

Are we in the $n \to \infty$ asymptotic region for $\mathcal{G}_{n,4}$?

UNIVERSITY OF OXFORD
Mathematical Institute

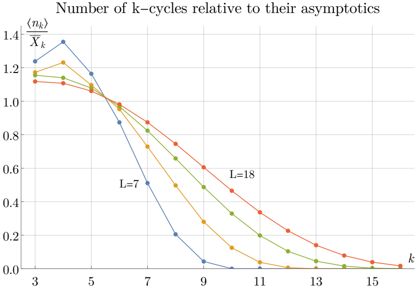
Symmetry factors $1 - \langle |\operatorname{Aut}(G)^{-1}|\rangle \to 0$?

All decompletions non-isomorphic,
$1 - \left\langle \frac{\#\text{decompletions}}{\#\text{vertices}} \right\rangle \to 0$?



$\Rightarrow$ asymptotic domain starts at $L \sim 10$.

Cycles of length k are Poisson distributed with mean $\tilde{X}_k := \frac{3^k}{2k}$?



$\Rightarrow$ only good for short cycles where $k \ll L$.

Conclusion: whether we are in the asymptotic domain depends on the quantity in question. "local" quantities are asymptotic starting from $L \approx 10$.

Computing periods numerically

University of Oxford Mathematical Institute

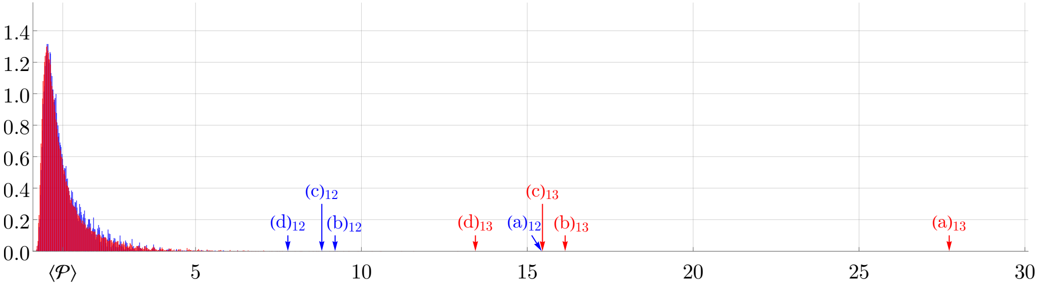
- Periods can be quickly ($\sim$ 1h/graph) computed numerically with new algorithm up to $L \approx 16$ loops [Borinsky 2023; Borinsky, Munch, and Tellander 2023], based on Hepp bound /tropicalization [Panzer 2022]. Exploit symmetries etc.
- Computed all graphs including 13 loops, incomplete samples for $L \leq 18$. Typical accuracy 4 digits ($\approx$ 100ppm). In total, $\approx 2 \cdot 10^6$ distinct completions (=vacuum graphs).

Distribution of periods

UNIVERSITY OF OXFORD Mathematical Institute

- Most periods are somewhat close to the mean $\langle \mathcal{P} \rangle$
- There are few, but extreme, outliers. Relative standard deviation $\delta(\mathcal{P}) \approx 100\%$.
- The pattern of outliers repeats at each loop order, but scaled.

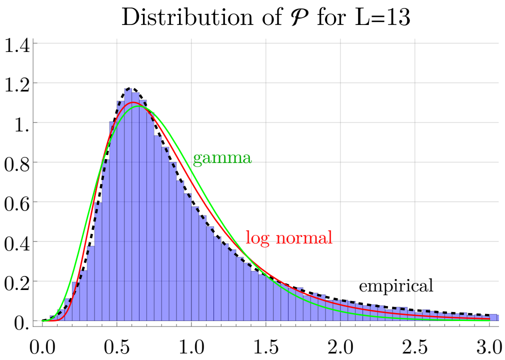
Distribution of periods, without symmetry factor, normalized to unit mean



Continuous part of the distribution

UNIVERSITY OF OXFORD
Mathematical Institute

- Shape of distribution is essentially unchanged at higher loop order, just scaled.

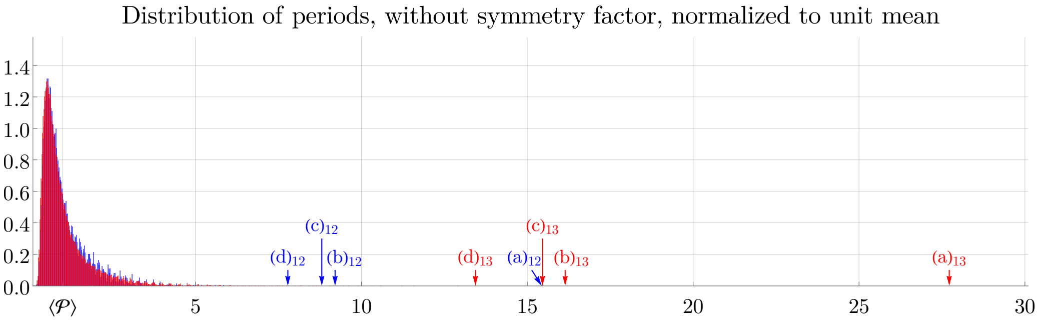


- This histogram is for *uniform* sampling of periods. If one samples non-uniformly $\propto \frac{1}{|\mathrm{Aut}|}$, histogram resembles log-gamma distribution [Borinsky and Favorito 2025].

Outliers in the distribution

UNIVERSITY OF OXFORD
Mathematical Institute

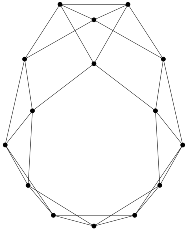
- Recall distribution: 4 distinct outliers.



Which ones are the outliers?

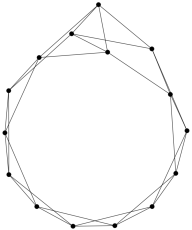
University of Oxford Mathematical Institute

- The zigzag graphs ($=(1,2)$-circulants) and their cousins.
- They look "symmetric", but that's deceptive, overall only weak correlation between $\mathcal{P}$ and symmetry factor.



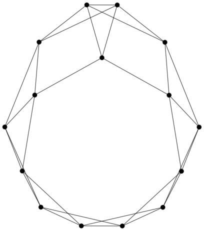
/bcdc/2467/347/578/69/8ac/9d/b/ab/dc/bc//c/c//

$\mathcal{P} \approx 403425.9$ $|\mathrm{Aut}| = 2$



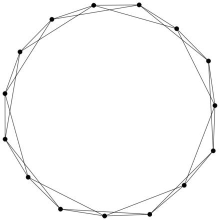
/bcdc/2359/345/47/68/9a/78d/8b/b/ac/cc/d/c/c//

$\mathcal{P} \approx 464116.5$ $|\mathrm{Aut}| = 2$



/bcdc/2478/345/459/6/9a/78d/8b/b/ac/cc/d/c/c//

$\mathcal{P} \approx 484645.5$ $|\mathrm{Aut}| = 4$



/bcdc/2346/345/67/58/8a/79/9c/ab/cc/bd/d/c/c//

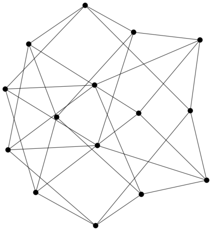
$\mathcal{P} \approx 832206.8$ $|\mathrm{Aut}| = 30$

4th largest $\longrightarrow$ largest

The smallest periods

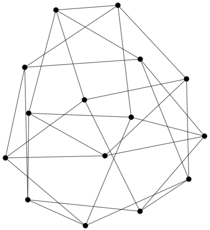
UNIVERSITY OF OXFORD
Mathematical Institute

- Smallest $\mathcal{P}$, not smallest $\frac{\mathcal{P}}{|\mathrm{Aut}|}$.
- No immediately visible structure.



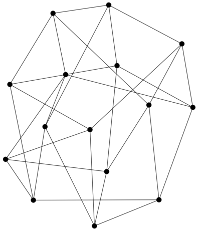
/bcdc/679a/45ac/459d/7c/6b/8c/8b/dc/cc/bd/////

$\mathcal{P} \approx 5930.41$ $|\mathrm{Aut}| = 4$



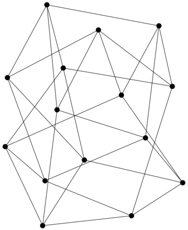
/bcdc/478c/378d/46c/5b/69d/ac/ac/9b/cc/bd/////

$\mathcal{P} \approx 6078.193$ $|\mathrm{Aut}| = 10$



/bcdc/267b/45c/456b/7c/8d/ac/9d/9ab/cc/dc/////

$\mathcal{P} \approx 6109.389$ $|\mathrm{Aut}| = 1$



/bcdc/256a/478/456b/9c/8c/7d/cc/bd/adc/bc/////

$\mathcal{P} \approx 6131.892$ $|\mathrm{Aut}| = 2$

smallest $\longleftarrow$ 4^{th} smallest

Correlations

UNIVERSITY OF OXFORD
Mathematical Institute

- We saw: The outliers are not at all random, but very special graphs.
- Can we guess the value of a period for a given graph G with some simple function $\bar{\mathcal{P}}(G)$?

Correlations

UNIVERSITY OF OXFORD Mathematical Institute

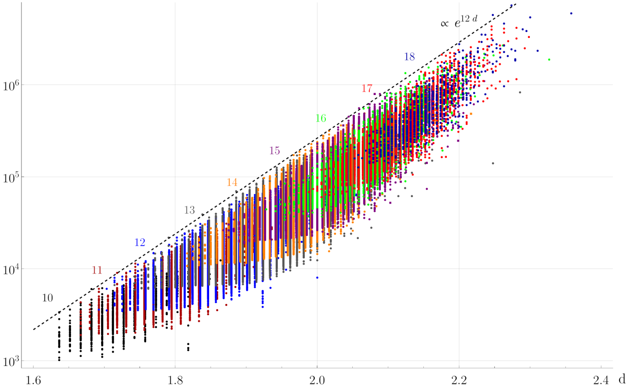
- We saw: The outliers are not at all random, but very special graphs.
- Can we guess the value of a period for a given graph G with some simple function $\bar{\mathcal{P}}(G)$?
- The period of a graph G is correlated with many properties of G, in [Balduf and Shaban 2024] we examined ≈ 150 distinct properties empirically.
- Recall: At fixed L, all 4-regular graphs have the same number of edges and vertices.

Average vertex distance

UNIVERSITY OF OXFORD
Mathematical Institute

- Average of the shortest path between all pairs of vertices, where each edge has length 1.
- Relatively fast to count, clearly correlated, but low accuracy $\delta \approx 30\%$.
- Note that the graphs of different loop order align. This correlation is universal across all loop orders.

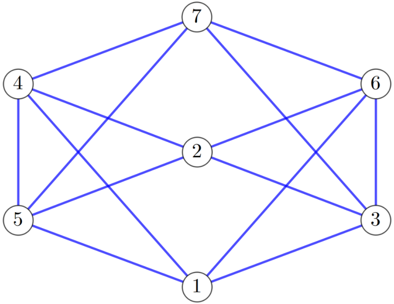
$\mathcal{P}$ as a function of the mean distance



Average resistance (Kirchhoff index)

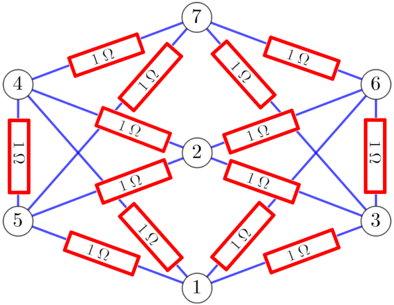
University of Oxford Mathematical Institute

- Consider a completion G.



Average resistance (Kirchhoff index)

- Consider a completion G.
- Assign unit electrical resistance to every edge.

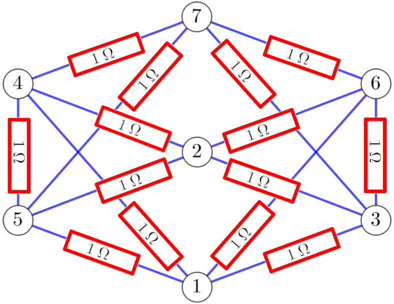


Average resistance (Kirchhoff index)

UNIVERSITY OF OXFORD
Mathematical Institute

- Consider a completion G.
- Assign unit electrical resistance to every edge.
- Resistance r_{v_i,v_j} between vertices v_i and v_j. Matrix of resistances can be computed from the pseudoinverse $\mathbb{L}^+$ of the (unlabelled) Laplacian $\mathbb{L}$,

$$r_{i,j} = \mathbb{L}^+_{i,i} + \mathbb{L}^+_{j,j} - \mathbb{L}^+_{i,j} - \mathbb{L}^+_{j,i}.$$



from vertex ...

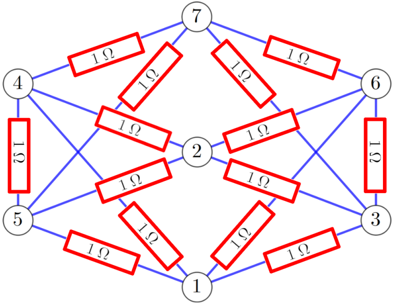
to vertex ...	1	2	3	4	5	6	7
1	0	$\frac{1}{2}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{1}{2}$
2	$\frac{1}{2}$	0	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{1}{2}$
3	$\frac{29}{30}$	$\frac{29}{30}$	0	$\frac{8}{15}$	$\frac{8}{15}$	$\frac{2}{3}$	$\frac{29}{30}$
4	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{8}{15}$	0	$\frac{2}{3}$	$\frac{8}{15}$	$\frac{29}{30}$
5	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{8}{15}$	$\frac{2}{3}$	0	$\frac{8}{15}$	$\frac{29}{30}$
6	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{2}{3}$	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{29}{30}$
7	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	0

Average resistance (Kirchhoff index)

- Consider a completion G.
- Assign unit electrical resistance to every edge.
- Resistance r_{v_i,v_j} between vertices v_i and v_j. Matrix of resistances can be computed from the pseudoinverse $\mathbb{L}^+$ of the (unlabelled) Laplacian $\mathbb{L}$,

$$r_{i,j} = \mathbb{L}^+_{i,i} + \mathbb{L}^+_{j,j} - \mathbb{L}^+_{i,j} - \mathbb{L}^+_{j,i}.$$

- *Kirchhoff index* $R(G)$ = average resistance



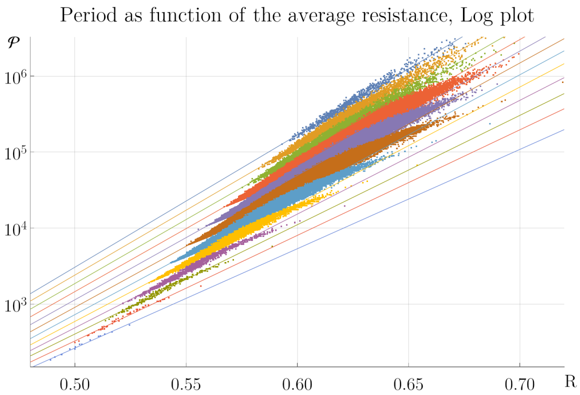
from vertex ...

to vertex ...	1	2	3	4	5	6	7
1	0	$\frac{1}{2}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{1}{2}$
2	$\frac{1}{2}$	0	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{1}{2}$
3	$\frac{29}{30}$	$\frac{29}{30}$	0	$\frac{8}{15}$	$\frac{8}{15}$	$\frac{2}{3}$	$\frac{29}{30}$
4	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{8}{15}$	0	$\frac{2}{3}$	$\frac{8}{15}$	$\frac{29}{30}$
5	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{8}{15}$	$\frac{2}{3}$	0	$\frac{8}{15}$	$\frac{29}{30}$
6	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{2}{3}$	$\frac{8}{15}$	$\frac{8}{15}$	0	$\frac{29}{30}$
7	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	$\frac{29}{30}$	0

$$\Rightarrow R(G) = \frac{71}{90}$$

University of Oxford Mathematical Institute

Average resistance



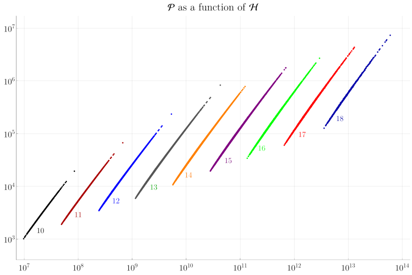
(Colors = loop orders 5 . . . 18)

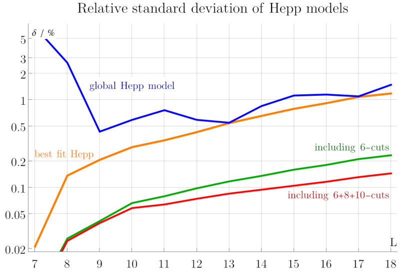
- Resistance approximation reaches $\delta \approx 5\%$ with linear fit.
- Nice interpretation: spatially "larger" (=less dense) graphs have larger periods.

Hepp bound

UNIVERSITY OF OXFORD
Mathematical Institute

- *Hepp bound* $\mathcal{H}$ [Hepp 1966; Panzer 2022] arises from "tropicalization" of period integral.
- Strongly correlated with period. Low order polynomial function, combined with edge-cuts $\ln(c_j)$, gives $\delta \approx 0.2\%$.
- Computing $\mathcal{H}$ for a graph requires iteration over *all* subgraphs (and/or caching).





Many more...

Work with **Kimia Shaban**

UNIVERSITY OF OXFORD
Mathematical Institute

- Counts of cycles and cuts of various types,
- Eigenvalues etc. of graph matrices,
- Variance, higher moments of electrical resistance etc,
- *Martin invariant* [Panzer and Yeats 2023] ($=O(N)$ symmetry factor at $N=-2$).
- Combined linear regression leads $\delta \approx 0.1\%$, quadratic even better.
- Also tried some machine learing models.

See [Balduf and Shaban 2024] for full details.

Is that good for something?

Mathematical Institute

- Tested several conjectures, and discovered new relations and conjectures that are worth investigating mathematically.

Is that good for something?

UNIVERSITY OF OXFORD
Mathematical Institute

- Tested several conjectures, and discovered new relations and conjectures that are worth investigating mathematically.
- Correlations have very concrete practical use: $\bar{\mathcal{P}}$ tells us which Feynman graphs contribute most to the sum of all periods.
- The sum of all periods is the primitive contribution to the beta function,

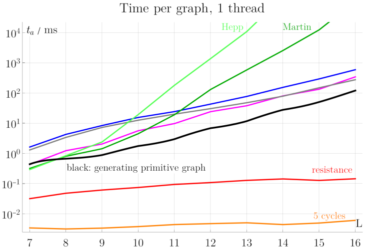
$$\beta_L^{\text{prim}} := 2 \cdot 4!(L+2) \sum_{\substack{\text{completion } G \\ L \text{ loops}}} \frac{\mathcal{P}(G)}{|\text{Aut}(G)|}.$$

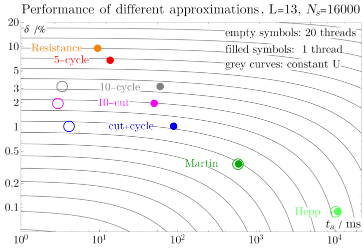
- Compute this sum with a weighted sampling algorithm [Metropolis et al. 1953; Hastings 1970].

Which of the correlations should we use?

Mathematical Institute

- Accuracy of the weighted sum is limited by speed t_a and accuracy δ of approximation function $\bar{\mathcal{P}}$.
- Implemented everything in `C++`. For all functions, t_a grows exponentially with L, but vastly different rate.
- Consider curves of equal resulting sampling accuracy. For $L \geq 13$, cut+cycle model is most suitable of all models considered.

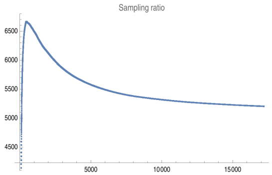


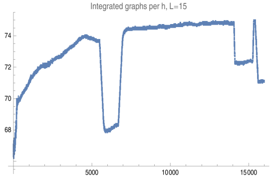


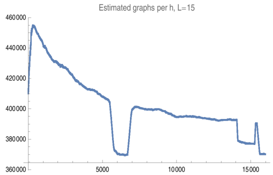
Weighted sampling implementation details

University of Oxford Mathematical Institute

- The program is largely autonomous: Allocates number of threads dynamically to various tasks, tunes parameters automatically, . . .
- Let it run on some servers in Waterloo and see:



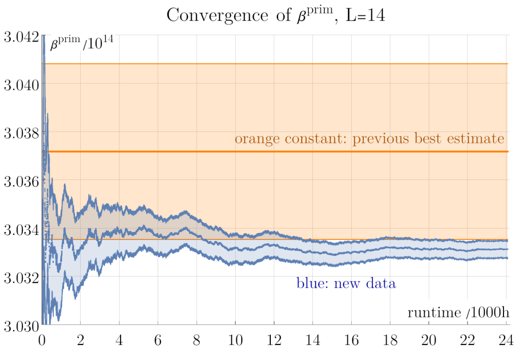




Example results: Primitive beta function for $L = 14$

UNIVERSITY OF OXFORD
Mathematical Institute

- Reached 120ppm standard deviation after 24k CPU core h (< 2 weeks walltime).
- Previous work with uniform random sampling took 400k CPU core h for 1063ppm.



Closer comparison shows: Weighted sampling is $\approx 1000\times$ faster than uniform random sampling, or reaches $\approx 32\times$ the accuracy at the same runtime.

$O(N)$ dependence and asymptotic number of graphs

UNIVERSITY OF OXFORD
Mathematical Institute

- *Circuit partition polynomial* $J(G,N)$ gives $O(N)$-symmetry factor of graphs G,

$$\beta_L^{\text{prim}}(N) := 2 \sum_{\substack{\text{completion } G \\ L \text{ loops}}} \frac{4!(L+2)\cdot J(G,N)\cdot \mathcal{P}(G)}{3^{L+2}N(N+2)|\text{Aut}(G)|}.$$

- Compute exact asymptotics of $\frac{J(G,N)}{|\text{Aut}(G)|}$ (=number of graphs weighted by symmetry factors) from 0-dimensional QFT [Balduf and Thürigen 2024].

$O(N)$ dependence and asymptotic number of graphs

University of Oxford Mathematical Institute

- *Circuit partition polynomial* $J(G,N)$ gives $O(N)$-symmetry factor of graphs G,

$$\beta_L^{\text{prim}}(N) := 2 \sum_{\substack{\text{completion } G\\ L \text{ loops}}} \frac{4!(L+2)\cdot J(G,N)\cdot \mathcal{P}(G)}{3^{L+2}N(N+2)|\text{Aut}(G)|}.$$

- Compute exact asymptotics of $\frac{J(G,N)}{|\text{Aut}(G)|}$ (=number of graphs weighted by symmetry factors) from 0-dimensional QFT [Balduf and Thürigen 2024].
- Sum of *all* vacuum graphs series $Z(\hbar,0)=\sum \hbar^n z_n$, has the asymptotics for $n\to\infty$

$$z_n \sim \frac{3^{\frac{N-1}{2}}}{\sqrt{2\pi}\Gamma(\frac{N}{2})}\left(\frac{2}{3}\right)^{n+\frac{N-1}{2}}\Gamma\left(n+\frac{N-1}{2}\right)\left(1-\frac{(N-2)(N-4)}{24(2n+N-3)}+\frac{(N-2)(N-4)(N-6)(N-8)}{1152(2n+N-3)(2n+N-5)}+\dots\right).$$

- For primitive graphs only, $\text{prim}(\hbar_{\mathcal{R}}) =: \sum p_L \hbar_{\mathcal{R}}^L$ for $L\to\infty$

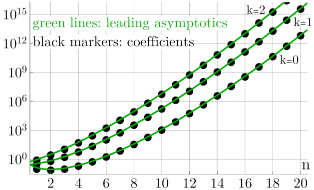
$$p_L \sim \frac{3^{\frac{N-1}{2}}e^{-\frac{12+3N}{4}}}{\frac{4}{3}\sqrt{2\pi}\Gamma(\frac{N+4}{2})}\left(\frac{2}{3}\right)^{L+\frac{N+5}{2}}\Gamma\left(L+\frac{N+5}{2}\right)\cdot\left(36-\frac{9(3N^2-4N-80)}{4(L+\frac{N+3}{2})}+\dots\right).$$

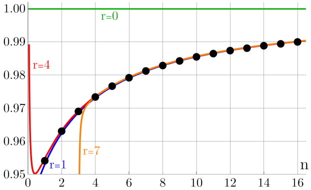
- Also known for other classes of graphs, and many more subleading terms in $\frac{1}{L}$ expansion.

0-dimensional asymptotics

UNIVERSITY OF OXFORD
Mathematical Institute

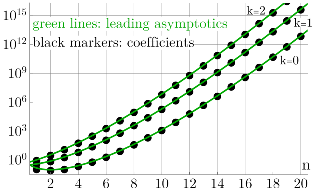
- Are these asymptotic formulas accurate at typical loop numbers? – it depends . . .
- For *all* graphs, excellent agreement even at low n (note that the asymptotic expansion itself is divergent. Including higher orders r of subleading corrections eventually makes it worse).

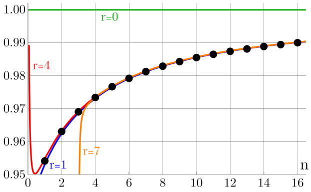


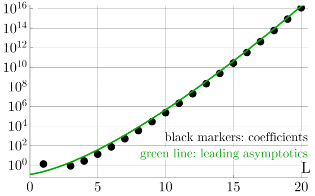


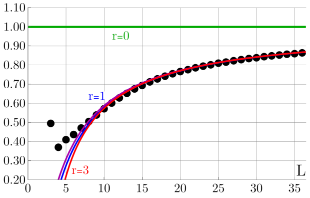
0-dimensional asymptotics

- For *all* graphs, excellent agreement even at low n.
- For primitive graphs, 20% off leading asymptotics even at 25 loops.





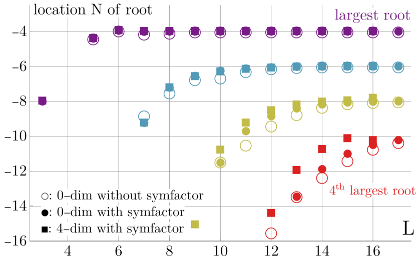




(The asymptotics is exact. This is not a problem of incorrect fit parameters etc.)

Large-loop asymptotics and N-dependence

- Use exact enumeration and asymptotics in 0-dimensional theory [Balduf and Thürigen 2024].
- E.g. asymptotics $p_L \sim \frac{1}{\Gamma\left(\frac{N+4}{2}\right)}$ implies zeros at $N \in \{-4, -6, \ldots\}$.

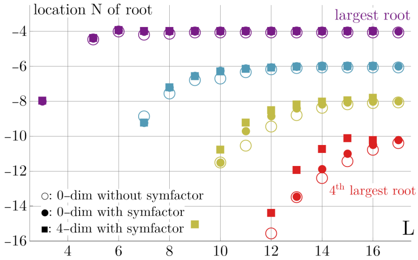


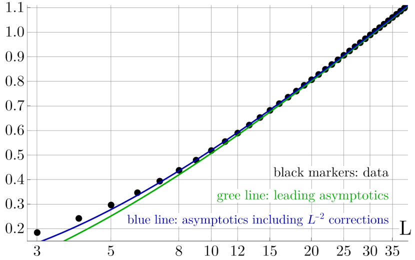
Large-loop asymptotics and N-dependence

- Use exact enumeration and asymptotics in 0-dimensional theory [Balduf and Thürigen 2024].
- E.g. asymptotics $p_L \sim \frac{1}{\Gamma\left(\frac{N+4}{2}\right)}$ implies zeros at $N \in \{-4, -6, \ldots\}$.
- large-L-expansion is factorially divergen, large-N-expansion is not. How that? Consider *average order* of polynomial at fixed L,

$$\langle k \rangle_L := \frac{1}{p_L\big|_{N=1}} \sum_{k=0}^{\infty} k\left[N^k\right] p_L(N) = \frac{\frac{\partial}{\partial N} p_L(N)}{p_L(N)}\bigg|_{N=1} \sim \frac{1}{2}\ln(L) + \frac{1}{2}\gamma_E - \frac{25}{12} + \frac{3}{2}\ln(2) + \frac{11}{8}\frac{1}{L} + \ldots.$$

The (maximum) degree of p_L in N grows like $\frac{2}{3}L$. But $\langle k \rangle_L$ grows only logarithmically.





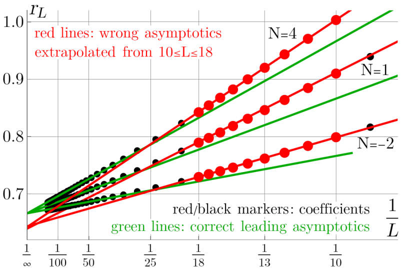
At 30 loops, mean order is $1 \ll 20$

$\Rightarrow$ polynomials are heavily dominated by lowest-order summands N^0, N^1.

Asymptotics of the beta function

UNIVERSITY OF OXFORD
Mathematical Institute

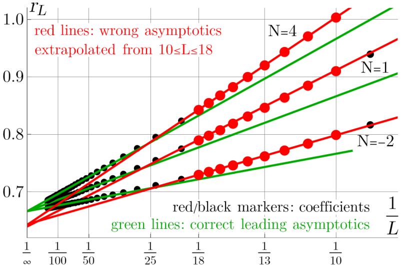
- Coefficient β_L grows factorially with L. Ratio $r_L := \beta_{L+1}/(L \cdot \beta_L)$ should have finite limit, and power series expansion in $\frac{1}{L}$.
- In 0-dimensional theory, known exactly [Balduf and Thürigen 2024]: $r_L \sim \frac{2}{3} + \frac{N+5}{3}\frac{1}{L} - \frac{3N^2-4N-80}{24}\frac{1}{L^2} + \ldots$. Observe that values for $L \leq 18$ suggest wrong asymptotics.

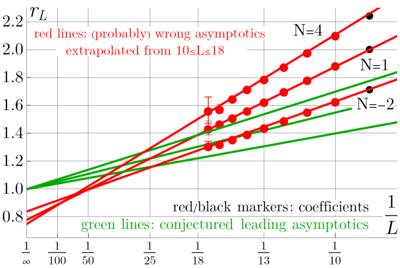


Asymptotics of the beta function

University of Oxford Mathematical Institute

- Coefficient β_L grows factorially with L. Ratio $r_L := \beta_{L+1}/(L \cdot \beta_L)$ should have finite limit, and power series expansion in $\frac{1}{L}$.
- In 0-dimensional theory, known exactly [Balduf and Thürigen 2024]: $r_L \sim \frac{2}{3} + \frac{N+5}{3}\frac{1}{L} - \frac{3N^2-4N-80}{24}\frac{1}{L^2} + \ldots$. Observe that values for $L \leq 18$ suggest wrong asymptotics.
- In 4-dimensional theory, instanton computation [McKane 2019] yields $r_L \sim 1 + \frac{10+N}{2}\frac{1}{L} + \ldots$. Only know data for $L \leq 18$. Does not match expected asymptotics.





$\Rightarrow$ Even at 18 loops, we do not observe the leading asymptotic growth rate.

Conclusion

UNIVERSITY OF OXFORD
Mathematical Institute

1. Considered *Feynman periods* in ϕ_4^4 theory, subdivergence-free vertex diagrams. Random graphs $\mathcal{G}_{n,4}$ are a decent model.
2. Number of graphs grows factorially, perturbation series diverges.
3. Periods in ϕ_4^4-theory have a fairly smooth distribution, with outliers.
4. Period value is correlated with many properties of the graph.
5. Correlations can be exploited for importance sampling, increases speed 1000-fold $\Rightarrow$ Numerical effort to compute primitive beta function is much lower than expected.
6. Asymptotics of number of graphs, including $O(N)$-symmetry, can be computed exactly.
7. Even for number of primitive graphs alone (=setting all periods to unity), need $L \gg 20$ to see leading asymptotics. Similar (or worse) for 4-dimensional primitive beta function.
8. N-dependence "converges faster" than absolute value (e.g. zeros at negative integer). "Asymptotic regime" depends on the quantity in question, can be $L = 5$ or $L \gg 50$.

Big picture: We're moving towards "big data perturbation theory", exploiting statistics, correlations, and asymptotics of billions of graphs. To determine $L = 16$ primitive beta function to within 400ppm, less than 0.01% of Feynman integrals have actually been computed.

References I

Balduf, Paul-Hermann and Kimia Shaban (2024). "Predicting Feynman Periods in ϕ^4-theory". In: *Journal of High Energy Physics* 2024.11, p. 38. DOI: 10.1007/JHEP11(2024)038. arXiv: 2403.16217 [hep-th].

Balduf, Paul-Hermann and Johannes Thürigen (2024). *Primitive Asymptotics in ϕ^4 Vector Theory*. DOI: 10.48550/arXiv.2412.08617. arXiv: 2412.08617 [hep-th]. Pre-published.

Bender, Edward A and E. Rodney Canfield (1978). "The Asymptotic Number of Labeled Graphs with given Degree Sequences". In: *Journal of Combinatorial Theory, Series A* 24.3, pp. 296–307. DOI: 10.1016/0097-3165(78)90059-6.

Bollobás, Béla (1980). "A Probabilistic Proof of an Asymptotic Formula for the Number of Labelled Regular Graphs". In: *European Journal of Combinatorics* 1.4, pp. 311–316. DOI: 10.1016/S0195-6698(80)80030-8.

— (1982). "The Asymptotic Number of Unlabelled Regular Graphs". In: *Journal of the London Mathematical Society* s2-26.2, pp. 201–206. DOI: 10.1112/jlms/s2-26.2.201.

Borinsky, Michael (2017). "Renormalized Asymptotic Enumeration of Feynman Diagrams". In: *Annals of Physics* 385, pp. 95–135. DOI: 10.1016/j.aop.2017.07.009.

— (2023). "Tropical Monte Carlo Quadrature for Feynman Integrals". In: *Annales de l'Institut Henri Poincaré D* 10.4, pp. 635–685. DOI: 10.4171/AIHPD/158. arXiv: 2008.12310.

Borinsky, Michael and Andrea Favorito (2025). *Feynman Integrals at Large Loop Order and the Log-Γ Distribution*. DOI: 10.48550/arXiv.2503.07803. arXiv: 2503.07803 [hep-th]. arXiv: 2503.07803. Pre-published.

Borinsky, Michael, Henrik J. Munch, and Felix Tellander (2023). "Tropical Feynman Integration in the Minkowski Regime". In: *Computer Physics Communications* 292, p. 108874. DOI: 10.1016/j.cpc.2023.108874.

Broadhurst, D. J. and D. Kreimer (1995). "Knots and Numbers in ϕ^4 Theory to 7 Loops and Beyond". In: *International Journal of Modern Physics C* 06.04, pp. 519–524. DOI: 10.1142/S012918319500037X. arXiv: hep-ph/9504352.

References II

Callan, Curtis G. (1970). "Broken Scale Invariance in Scalar Field Theory". In: *Physical Review D* 2.8, pp. 1541–1547. DOI: 10.1103/PhysRevD.2.1541.

Cvitanović, Predrag, B. Lautrup, and Robert B. Pearson (1978). "Number and Weights of Feynman Diagrams". In: *Physical Review D* 18.6, p. 1939. DOI: 10.1103/PhysRevD.18.1939.

Hastings, W. K. (1970). "Monte Carlo Sampling Methods Using Markov Chains and Their Applications". In: *Biometrika* 57.1, pp. 97–109. DOI: 10.1093/biomet/57.1.97.

Hepp, Klaus (1966). "Proof of the Bogoliubov-Parasiuk Theorem on Renormalization". In: *Communications in Mathematical Physics* 2.1, pp. 301–326. DOI: 10.1007/BF01773358.

Hu, Simone et al. (2022). "Further Investigations into the Graph Theory of ϕ^4-Periods and the c_2 Invariant". In: *Annales de l'Institut Henri Poincaré D* 9.3, pp. 473–524. DOI: 10.4171/AIHPD/123. arXiv: 1812.08751 [hep-th]. arXiv: 1812.08751.

Kontsevich, Maxim and Don Zagier (2001). "Periods". In: *Mathematics Unlimited - 2001 and Beyond*. Springer, pp. 771–808. URL: http://preprints.ihes.fr/M01/M01-22.ps.gz.

Kreimer, Dirk (1998). "On the Hopf Algebra Structure of Perturbative Quantum Field Theories". In: *Adv. Theor. Math. Phys.* 2, pp. 303–334. DOI: 10.4310/ATMP.1998.v2.n2.a4. arXiv: q-alg/9707029 [q-alg].

Martin, Pierre (1977). "Enumérations eulériennes dans les multigraphes et invariants de Tutte-Grothendieck". thesis. Institut National Polytechnique de Grenoble - INPG ; Université Joseph-Fourier - Grenoble I. URL: https://tel.archives-ouvertes.fr/tel-00287330.

McKane, A. J., D. J. Wallace, and O. F. de Alcantara Bonfim (1984). "Non-Perturbative Renormalisation Using Dimensional Regularisation: Applications to the ε Expansion". In: *Journal of Physics A: Mathematical and General* 17.9, pp. 1861–1876. DOI: 10.1088/0305-4470/17/9/021.

McKane, Alan J. (2019). "Perturbation Expansions at Large Order: Results for Scalar Field Theories Revisited". In: *Journal of Physics A: Mathematical and Theoretical* 52.5, p. 055401. DOI: 10.1088/1751-8121/aaf768. arXiv: 1807.00656 [hep-th].

References III

McKay, B. D. and N. C. Wormald (1984). "Automorphisms of Random Graphs with Specified Vertices". In: *Combinatorica* 4.4, pp. 325–338. DOI: 10.1007/BF02579144.

McKay, Brendan D., Nicholas C. Wormald, and Beata Wysocka (2004). "Short Cycles in Random Regular Graphs". In: *The Electronic Journal of Combinatorics* 11.1, R66. DOI: 10.37236/1819.

Metropolis, Nicholas et al. (1953). "Equation of State Calculations by Fast Computing Machines". In: *The Journal of Chemical Physics* 21.6, pp. 1087–1092. DOI: 10.1063/1.1699114.

Panzer, Erik (2022). "Hepp's Bound for Feynman Graphs and Matroids". In: *Annales de l'Institut Henri Poincaré D* 10.1, pp. 31–119. DOI: 10.4171/aihpd/126. arXiv: 1908.09820 [math-ph]. arXiv: 1908.09820.

Panzer, Erik and Karen Yeats (2023). *Feynman Symmetries of the Martin and c_2 Invariants of Regular Graphs*. DOI: 10.48550/arXiv.2304.05299. arXiv: 2304.05299 [hep-th, physics:math-ph]. arXiv: 2304.05299. Pre-published.

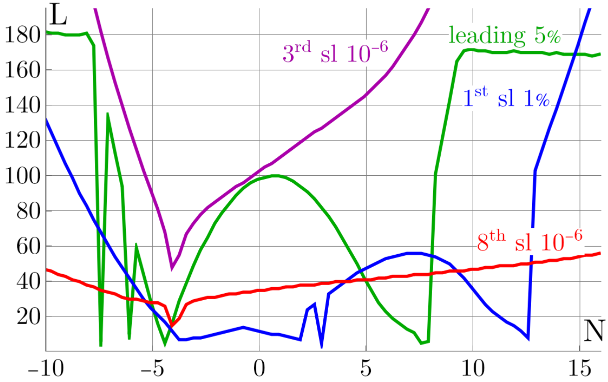
Schnetz, Oliver (2010). "Quantum Periods: A Census of ϕ^4-Transcendentals". In: *Communications in Number Theory and Physics* 4.1, pp. 1–47. DOI: 10.4310/CNTP.2010.v4.n1.a1. arXiv: 0801.2856 [hep-th].

Symanzik, Kurt (1970). "Small Distance Behaviour in Field Theory and Power Counting". In: *Communications in Mathematical Physics* 18.3, pp. 227–246. DOI: 10.1007/BF01649434.

Wormald, Nicholas C. (1981). "The Asymptotic Distribution of Short Cycles in Random Regular Graphs". In: *Journal of Combinatorial Theory, Series B* 31.2, pp. 168–182. DOI: 10.1016/S0095-8956(81)80022-6.

Minimum required loop number

- In order to reach a given accuracy with the r^{th}-order asymptotic expansion, how many loops are needed?
- Depends heavily on N.



Importance sampling for periods

- Idea of importance sampling: If we know a function $\bar{\mathcal{P}}$ which approximates the period *and* $\bar{\mathcal{P}}$ is fast to compute, then:
  1. Evaluate $\langle \bar{\mathcal{P}} \rangle$ in a large sample of size $N_s \cdot n$.
  2. Generate a smaller random sample S of n graphs weighted proportional to $\bar{\mathcal{P}}$. Evaluate $\left\langle \frac{\mathcal{P}}{\bar{\mathcal{P}}} \right\rangle_S$ in this sample.
  3. Law of conditional probability:

$$\langle \mathcal{P} \rangle = \underbrace{\left\langle \frac{\mathcal{P}}{\bar{\mathcal{P}}} \right\rangle_S}_{\substack{\text{slow individually,} \\ \text{but small sample}}} \cdot \underbrace{\langle \bar{\mathcal{P}} \rangle}_{\substack{\text{large sample,} \\ \text{but fast individually}}} .$$

- First factor sampling accuracy is limited by $\delta := \sigma\left(\frac{\mathcal{P}}{\bar{\mathcal{P}}}\right)$ (i.e. accuracy of the prediction function).
- Second factor sampling accuracy is limited by feasible N_s, i.e. by speed of the prediction function $\bar{P}$. Scales like $\frac{1}{\sqrt{N_s}} \propto \sqrt{t_a}$, where $t_a \ldots$ approximation time for one graph.
- Use Metropolis-Hastings sampling algorithm [Metropolis et al. 1953; Hastings 1970].

Hepp bound

UNIVERSITY OF OXFORD
Mathematical Institute

- Several combinatorial *invariants* are known that respect the symmetries of $\mathcal{P}$.
- One example: *Hepp bound* $\mathcal{H}(G)$ [Hepp 1966; Panzer 2022]. Arises from "tropicalization" of period integral, replacing Symanzik polynomial ψ_G.

$$\mathcal{H}(G) := \left(\prod_{e \in E_G} \int_0^\infty \mathrm{d}a_e \right) \delta\left(1 - \sum_{e=1}^{|E_G|} a_e \right) \frac{1}{\psi^2_{G,\text{trop}}(\{a_e\})} \in \mathbb{Q}$$

$$\psi_{G,\text{trop}} := \text{maximum monomial of } \psi_G.$$

- Which monomial is maximum depends on the particular values of edge variables $\{a_e\}$.
- The Hepp sector integrals of $\mathcal{H}(G)$ are over monomials $\Rightarrow$ can be done analytically $\Rightarrow$ recursive combinatorial formula for $\mathcal{H}(G)$ without explicit integration.

Martin invariant

UNIVERSITY OF OXFORD
Mathematical Institute

- If G is a decompletion, the *Martin invariant* is the linear coefficient of the Martin polynomial $M(G,N) := \frac{J(G,N-2)}{N-2}$ [Martin 1977; Panzer and Yeats 2023].
- Equals evaluation of $O(N)$-symmetric vector symmetry factor at $N=-2$.
- Replace every edge in G by k parallel edges, compute J and M, obtain "higher" Martin invariant $M^{[k]}$.
- Like Hepp bound, can be computed by explicit combinatorial enumeration.

Martin invariant

- *Martin invariant* $M^{[k]}$ [Panzer and Yeats 2023] is $O(N)$ vector-model symmetry factor (circuit partition polynomial $J(G,N)$) at $N=-2$ for a graph where every edge is replaced by k parallel edges.
- Linear function of $\ln M^{[1]}$ gives $\delta \approx 4\%$, higher $M^{[k]}$ are much better. k^{th}-order polynomial of $M^{[k]}$ can get very accurate when cobined with cuts $\ln(c_j)$, reach $\delta \ll 0.1\%$.
- Like Hepp, requires recurrence over decompositions and caching.