A holographic analysis of the pion

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based on: PRD 111, 034024 (2025), arXiv:2501.00526 in collaboration with Jeff Forshaw (University of Manchester, UK)

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The pion

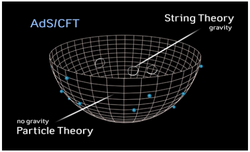
- Lightest hadron in Nature: much lighter than other hadrons
- Simultaneously a bound state and pseudo-Goldstone boson in QCD: probes both confinement and chiral symmetry breaking
- Gell-Mann-Rennes-Oakes (GMOR, 1968) relation

$$f_\pi^2 M_\pi^2 = 2m_q|\langle \bar{q}q \rangle| + \mathcal{O}(m_q^2)$$

- Precise data on non-perturbative observables: mass, decay constant, charge radius, radiative decay width, EM elastic form factor at low Q^2

	M_π [MeV]	f_π [MeV]	r_π [fm]	$\Gamma_{\pi\to\gamma\gamma}$ [eV]
PDG	139	130.2 ± 1.7	0.659 ± 0.004	7.82 ± 0.23

Holographic duality (G. 't Hooft, L. Susskind, J. Maldacena: 1990s)



https://rationalisingtheuniverse.org/2018/11/27/ads-cft-a-deep-duality/

- String theory in higher dimensional anti de Sitter (curved) spacetime has same information content as conformal QFT at the (flat) spacetime boundary : AdS/QFT
- Weak-Strong coupling duality
- Gravity dual for QCD (not conformal) unknown

Light-front QCD and quark model

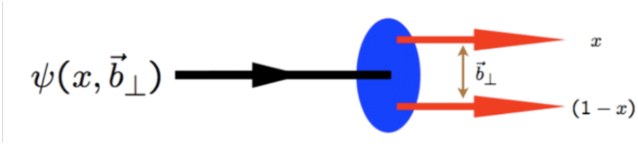


Figure: Stan Brodsky

Connects to Quark Model

$$\left(\frac{-\nabla^2}{x(1-x)} + \frac{m_q^2}{x} + \frac{m_{\bar{q}}^2}{1-x} + U(x, \mathbf{b}) \right) \Psi(x, \mathbf{b}) = M^2 \Psi(x, \mathbf{b})$$

Light-front QCD: factorization

Separation of transverse and longitudinal d.o.f

$$\Psi(x,\mathbf{b}) = \frac{\phi(\zeta)}{\sqrt{2\pi\zeta}} e^{iL\varphi} X(x) \qquad \zeta = \sqrt{x(1-x)}\mathbf{b} \qquad M^2 = M_\perp^2 + M_\parallel^2$$

Transverse dynamics: underlying conformal symmetry

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + \frac{4L^2-1}{4\zeta^2} + U_\perp\right]\phi(\zeta) = M_\perp^2\phi(\zeta) \qquad \int_0^\infty \mathrm{d}\zeta\, |\phi(\zeta)|^2 = 1$$

Longitudinal dynamics: captures conformal symmetry breaking

$$\boxed{\left[\frac{m_q^2}{x(1-x)} + U_\parallel\right] X(x) = M_\parallel^2 X(x)} \qquad \int_0^1 \frac{\mathrm{d}x}{x(1-x)} |X(x)|^2 = 1$$

Holographic mapping

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + \frac{4L^2-1}{4\zeta^2} + U_\perp\right]\phi(\zeta) = M_\perp^2\phi(\zeta) \qquad \int_0^\infty \mathrm{d}\zeta\, |\phi(\zeta)|^2 = 1$$

maps onto EOM of freely propagating spin-J string modes in warped AdS_5 if

- $\zeta = z_5$
- $L^2 = (\mu_5 R)^2 + (2-J)^2$

where

$$U_\perp(\zeta) = \frac{1}{2}\varphi''(z_5) + \frac{1}{4}\varphi'(z_5)^2 + \frac{2J-3}{2z_5}\varphi'(z_5)$$

with $\varphi(z_5)$ is a dilaton field in AdS_5.

Review: S. J. Brodsky, G. F. de Téramond, H. G. Dosch, J. Erlich, Phys. Rept. 584 (2015) 1-105

Uniqueness of a quadratic dilaton

- A quadratic dilaton, $\varphi = \kappa^2 z_5^2$, gives

$$U_\perp(\zeta) = \kappa^4 \zeta^2 + 2\kappa^2 (J-1)$$

- (Only) harmonic potential preserves the underlying conformal invariance

S. J. Brodsky, G. F. de Téramond, H. G. Dosch, Phys. Lett. B729 (2014) 3-8

The pion in light-front holography

- LFH Predicts

$$M_\perp^2(n_\perp, J, L) = 4\kappa^2 \left(n_\perp + \frac{J+L}{2} \right)$$

- Massless pion: $M_\pi = M_\perp(0,0,0) = 0$
- Gaussian pion wavefunction:

$$\Psi_\pi(x, \mathbf{b}) = \frac{\kappa}{\sqrt{\pi}} \sqrt{x(1-x)} \exp\left\{ \left(\frac{-\kappa^2 x(1-x) b_\perp^2}{2} \right) \right\}$$

- Need longitudinal dynamics to generate pion mass consistent with GMOR relation ($M_\pi^2 \propto m_q$)

Connecting to 't Hooft $(1+1)$-dim, large N_c QCD

$$\left[\frac{m_q^2}{x(1-x)} + U_\parallel\right] X(x) = M_\pi^2 X(x) \qquad \int_0^1 \frac{\mathrm{d}x}{x(1-x)} |X(x)|^2 = 1$$

$$\chi(x) = \frac{X(x)}{\sqrt{x(1-x)}}$$

$$\left[\frac{m_q^2}{x(1-x)} + V_\parallel\right] \chi(x) = M_\pi^2 \chi(x)$$

$$V_\parallel = \frac{1}{\sqrt{x(1-x)}} U_\parallel \sqrt{x(1-x)} \qquad \int_0^1 \mathrm{d}x \, |\chi(x)|^2 = 1$$

This is the 't Hooft Equation if

$$V_\parallel = -g^2 \mathcal{P} \int_0^1 \mathrm{d}y \frac{\chi(y) - \chi(x)}{(x-y)^2}$$

Chabysheva & Hiller, Annals Phys. 337, 143 (2013)

Li & Vary model for $V_{\|}$

1. 't Hooft Nucl. Phys. B75, 461 (1974)

$$\frac{m_q^2 - g^2}{x(1-x)}\chi(x) - g^2\mathcal{P}\int_0^1 \mathrm{d}y \frac{\chi(y)}{(x-y)^2} = M_\pi^2\chi(x)$$

2. Li & Vary Phys. Lett. B 758, 118 (2016)

$$\frac{m_q^2}{x(1-x)}\chi(x) - \sigma^2\partial_x(x(1-x)\partial_x)\chi(x) = M_\pi^2\chi(x)$$

- Very different analytical forms
- Yet completely degenerate in describing pion data with $\chi(x) \sim (x(1-x))^\beta$, $\beta = m_q/\sigma$ or $\beta \sim m_q/g$

2021-23: Ahmady et al., Li & Vary, Brodsky & de Téramond, Weller & Miller

An underlying link in AdS_3

(D. Vegh, 2301.07154 [hep-th])

- Ansatz

$$\left(\frac{m_q^2 - g^2}{x(1-x)}\right)\chi(x) - g^2\mathcal{P}\int_0^1 \mathrm{d}y\,\frac{\chi(y)}{(x-y)^2} - \sigma^2\partial_x(x(1-x)\partial_x)\chi(x) = M_\pi^2\chi(x)$$

- Holographic dictionary

$$\boxed{g^2 = m_q^2 + \sigma^2/4 \qquad g_s = g^2/(4\sigma^2) \qquad \mu^2 = M_\pi^2/(4\sigma^2)}$$

- Vegh's string equation in pure AdS_3

$$-\mathcal{P}\int_0^1 \mathrm{d}y\,\frac{\chi(y)}{(x-y)^2} - 4g_s\sqrt{x(1-x)}\partial_x^2\sqrt{x(1-x)}\chi(x) = \mu^2\chi(x)$$

g_s: dimensionless string coupling

Observables

After integrating transverse d.o.f

$$M_\pi^2 = \int_0^1 \mathrm{d}x\, \chi^*(x) \left[\frac{m_q^2}{x\bar{x}} + V_\parallel\right] \chi(x)$$

$$f_\pi = \frac{\sqrt{6}}{\pi}\kappa \int_0^1 \mathrm{d}x \sqrt{x\bar{x}}\chi(x) \qquad r_\pi^2 = \frac{3}{2\kappa^2}\int_0^1 \mathrm{d}x \frac{(1-x)}{x}|\chi(x)|^2$$

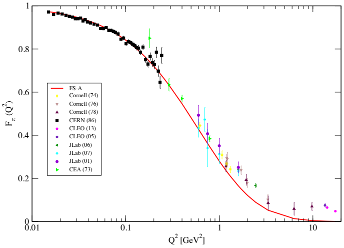
$$F_\pi(Q^2) = \int_0^1 \mathrm{d}x\, |\chi(x)|^2 \, \exp\left(-\frac{\bar{x}}{x}\frac{Q^2}{4\kappa^2}\right)$$

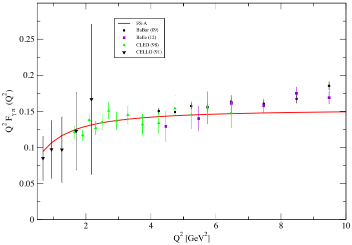
$$Q^2 F_{\pi\gamma}(Q^2) = \frac{2\kappa}{\sqrt{3}\pi}\int_0^1 \mathrm{d}x \sqrt{x\bar{x}}\chi(x) \times \int_0^\infty \mathrm{d}b_\perp (m_q b_\perp) K_1(m_q b_\perp) \exp\left(-\frac{\kappa^2 x\bar{x} b_\perp^2}{2}\right) Q J_1(b_\perp \bar{x} Q)$$

$$\Gamma_{\gamma\gamma} = \frac{\pi}{4}\alpha_{\mathrm{em}}^2 M_\pi^3 |F_{\pi\gamma}(0)|^2$$

Extracting the parameters from data

- Reproduce exactly M_π, f_π and r_π
- Predicts excellent agreement for form factors at low Q^2
- Predicts $\Gamma_{\gamma\gamma} = 7.0$ eV. PDG: 7.82 ± 0.22 eV

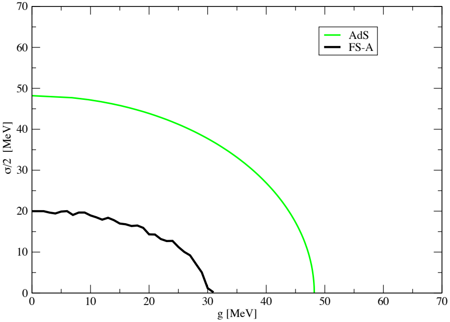




Left: Elastic EM form factor
Right: Transition Form Factor

Correlation between parameters

$$\kappa = 423 \text{ MeV} \; m_q = 48.2 \text{ MeV}$$



- Green: Holographic prediction $g^2 = m_q^2 + \sigma^2/4$
- Qualitative agreement

Similarity transformations

- Eigenvalue invariant under transformation

$$V_\parallel \to (x(1-x))^{n/2}\, V_\parallel \, \frac{1}{(x(1-x))^{n/2}}$$

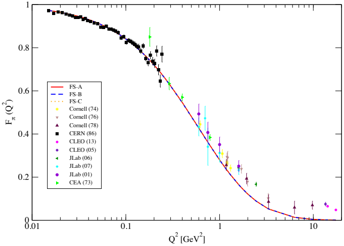
- Eigenfunction transforms as

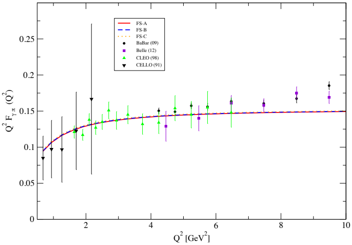
$$\chi(x) \to (x(1-x))^{n/2}\chi(x)$$

- Before: $n = 0$
- Now:
  - $n < 0$: bad chiral limit behaviour
  - $n > 2$: discarded by data
  - Model A: $n = 0$, Model B: $n = 1$, Model C: $n = 2$

Completely degenerate in fitting data

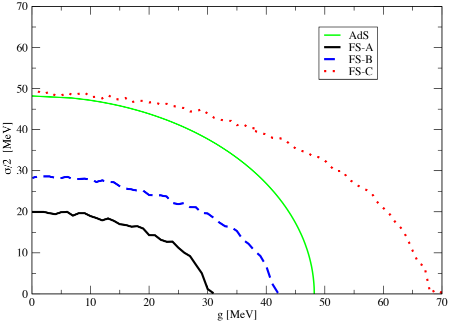
- Reproduce exactly M_π, f_π and r_π
- Predict $\Gamma_{\gamma\gamma} = 7.0, 7.2, 7.4$ eV for Models A, B, C. PDG: 7.82 $\pm$ 0.22 eV
- Predict excellent agreement for form factors at low Q^2





Degeneracy lifted in parameter space

$$\kappa = 423 \text{ MeV} \ m_q = 48.2 \text{ MeV}$$



- Green: Holographic prediction $g^2 = m_q^2 + \sigma^2/4$
- Model A, Model B, Model C
- Model C: quantitative agreement with holography in weak string coupling limit

Concluding remarks

- Intriguing agreement with holographic prediction for Model C
- Deserves further investigation

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