QED CORRECTIONS TO EXCLUSIVE LEPTONIC B DECAYS

Cluster of Excellence PRISMA+

Mainz Institute for Theoretical Physics

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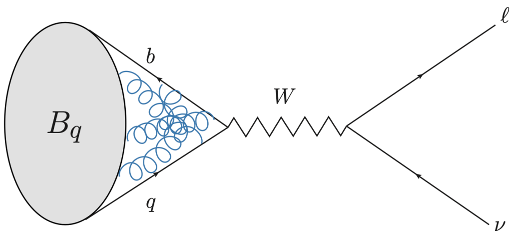
Based on arXiv:2212.14430 [hep-ph] and work in progress [hep-ph]
In collab. w/ C. Cornella, M. König and M. Neubert (JGU Mainz)

Motivations

- Why leptonic B decays ? $\mathscr{B}(B^- \to \ell^- \bar{\nu}_\ell) = \tau_B G_F^2 f_B^2 |V_{ub}|^2 \frac{m_B m_\ell^2}{8\pi} \left(1 - \frac{m_\ell^2}{m_B^2}\right)^2$
  - Direct determination of the CKM element $|V_{ub}|$
  - Chirality-suppressed in the SM $\to$ powerful probe of (pseudo) scalar new physics
  - Testing flavor universality in charged current : Belle II will measure the $\ell = \tau, \mu$ channels at $5 - 6\,\%$ [Belle II Physics Book]. FCC-ee prospects are promising.

- Why QED corrections are needed?
  - Pure hadronic effects are simple and well-understood:

$$\langle 0 | \bar{q}\gamma^\mu \gamma_5 b | B_q(p)\rangle = i f_{B_q} p^\mu$$



and f_{B_q} is known with $\mathcal{O}(1\%)$ precision : $f_{B_q} = 189.4 \pm 1.4$ Mev [FNAL/MILC 1712.09262]

  - In the exclusive channel, with strong cuts on additional soft radiations, QED corrections can be sizable and compete with QCD uncertainties $\to$ need a precise estimation

A multi-scale process

- Focusing on $\ell = \mu$ case, including QED effects introduce new scales (both static and dynamic) to which $B^- \to \mu^- \bar{\nu}_\mu$ is sensitive :

m_W

m_B

$\mu_{hc} = \sqrt{\Lambda_{QCD} m_B}$

$m_\mu \sim \Lambda_{QCD}$

$\mu_{us} = E_0$

$\mu_{usc} \sim E_0 \frac{m_\mu}{m_B}$

Static scales dictated by the energy cut on the additional final radiation

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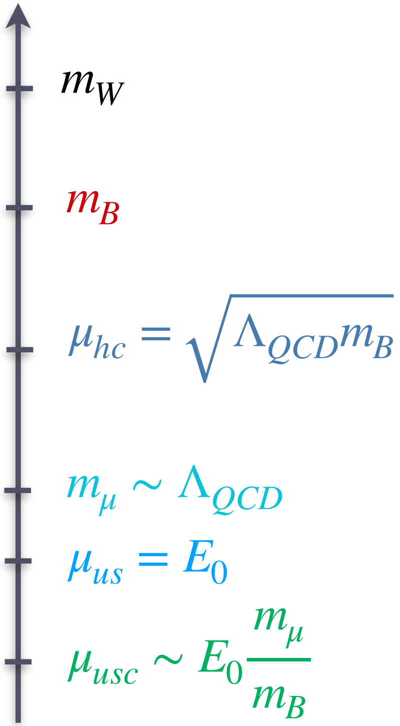
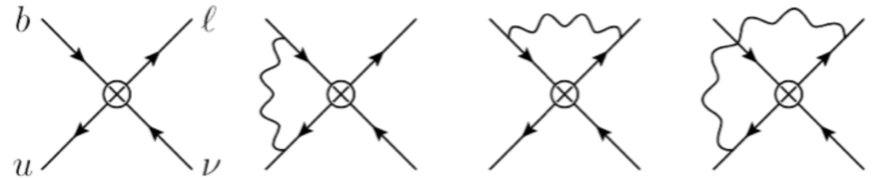
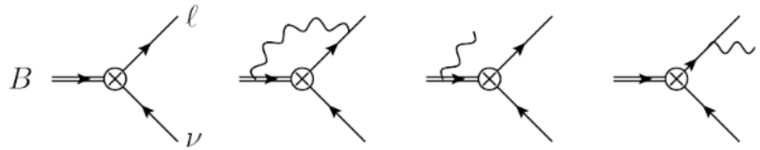




B-meson described as a superposition of Fock-states: $|\bar{u}b\rangle \oplus |\bar{u}bg\rangle \oplus \cdots$

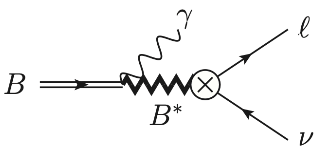
[Beneke et al(2019),JHEP10232]

B-meson described as a point-like pseudo-scalar boson…



[Dai et al (*2022*), Phys.Rev.D 105 3]

…including its first vector excited state as off-shell intermediate propagator !



[Becirevic et al (0907.1845)]

The plan : running down with the scale !

Turning a multi-scale problem to a product of single scale objects by :

- Identifying the appropriate effective description at each scale.
- Performing a step-by-step matching between each EFT.
- Deriving a factorization theorem to break this multi-scale problem into a convolution of single-scale objects.
- Using the renormalisation group to evaluate each object at its natural scale and run it to a common scale to resum logarithms.

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In this talk we are going to embark on an EFT journey :

- Discussing the (re)factorization of the virtual amplitude in the partonic picture.
- Constructing the proper EFT valid at low energy to include the real emissions.
- Deriving the factorization formula for the decay rate.

Fermi theory $\rightarrow$ HQET $\otimes$ SCET$_I$ $\quad \mu \sim m_B$

Power counting: $\lambda = \frac{m_\ell}{m_B} \sim \frac{\Lambda_{QCD}}{m_B}$

Relevant scalings : $p^\mu = (\bar{n} \cdot p, n \cdot p, p_\perp)$

$p \sim (1, \lambda^2, \lambda)$ **« collinear »,**

$p^2 \sim m_\ell^2 \sim \mathcal{O}(\lambda^2)$

$\rightarrow$ given by the lepton virtuality

$p \sim (1, \lambda, \sqrt{\lambda})$ « hard-collinear »,

$p^2 \sim \Lambda_{QCD} m_B \sim \mathcal{O}(\lambda)$

$\rightarrow$ soft and collinear quark X-talk

$p \sim (\lambda, \lambda, \lambda)$ « soft »,

$p^2 \sim \Lambda_{QCD}^2 \sim \mathcal{O}(\lambda^2)$

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Fermi theory $\to$ HQET $\otimes$ SCET$_I$ $\quad \mu \sim m_B$

- The b quark is described by a soft HQET field :

$$b(x) \to e^{-im_b(v\cdot x)}\left(1+\mathcal{O}(\sqrt{\lambda})\right) h_v(x)$$

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- QED corrections $\rightarrow$ hard-collinear momentum exchange between partons & lepton.

$\rightarrow$ need a SCET$_I$ subleading power description for the different modes of the spectator and the lepton :

$$q(x) \rightarrow \left(1+\frac{1}{i\bar{n}\cdot D_s}(i\,\not{D}_\perp)\frac{\not{\bar{n}}}{2}\right)\xi_C^{(q)}(x)+\left(1+\frac{1}{i\bar{n}\cdot D_s}eQ_q\not{A}_{C\perp}\frac{\not{\bar{n}}}{2}\right)q_s(x)$$

$$\ell(x) \rightarrow \left(1+\frac{1}{i\bar{n}\cdot D_s}(i\,\not{D}_\perp+m_\ell)\frac{\not{\bar{n}}}{2}\right)\xi_C^{(\ell)}(x)+\left(1+\frac{1}{i\bar{n}\cdot D_s}eQ_q\not{A}_{C\perp}\frac{\not{\bar{n}}}{2}\right)\ell_s(x)$$

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$p^2 \sim \Lambda_{QCD}^2 \sim \mathcal{O}(\lambda^2)$

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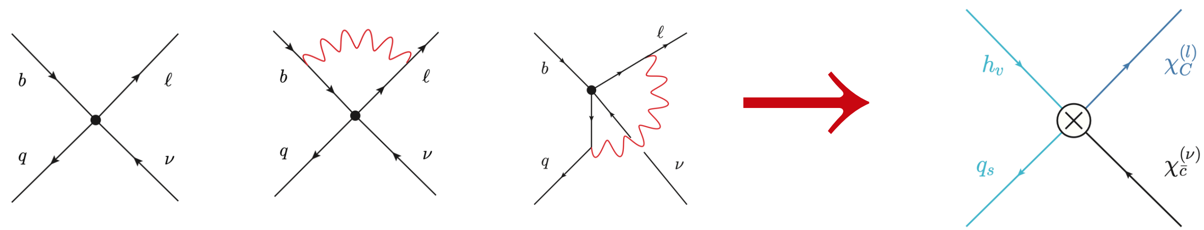
SCET$_I$ operators

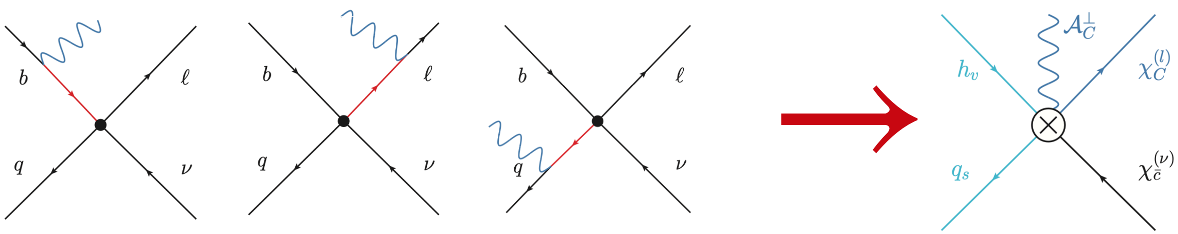
We build our SCET$_I$ basis with the following power counting :

$h_v, q_s \sim \mathcal{O}(\lambda^{3/2})$ $\chi_C \sim \mathcal{O}(\lambda^{1/2})$ $\mathcal{A}_C^\perp \sim \mathcal{O}(\lambda^{1/2})$ $\chi_c \sim \mathcal{O}(\lambda)$ $\mathcal{A}_c^\perp \sim \mathcal{O}(\lambda)$ $\chi_{\bar{c}} \sim \mathcal{O}(\lambda)$

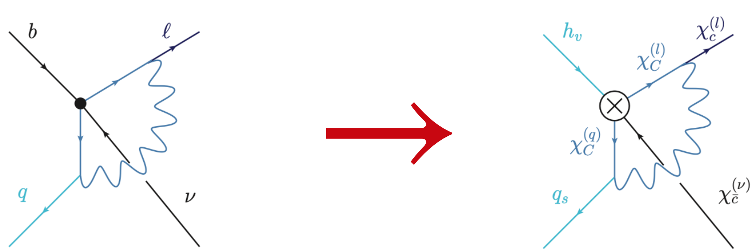
At $\mathcal{O}(\alpha)$, three classes of 4-fermions operators are relevant:



$[O_i^A]$ $(\bar{q}_s \ldots h_v)(\bar{\chi}_C \ldots \chi_{\bar{c}})$



$[O_i^B]$ $(\bar{q}_s \ldots h_v)(\bar{\chi}_C{}^{(\ell)} \ldots \chi_{\bar{c}})\mathcal{A}_C^\perp$



$[O_i^C]$ $(\bar{\chi}_C{}^{(q)} \ldots h_v)(\bar{\chi}_C{}^{(\ell)} \ldots \chi_{\bar{c}})$

$\text{SCET}_I \rightarrow \text{SCET}_{II}$ $\quad \mu \sim \mu_{hc} \sim \sqrt{\Lambda_{QCD} m_B}$

- At $\mu \sim \mu_{hc}$, we lower the virtuality removing hard-collinear modes $\rightarrow$ pure SCET_{II} construction where collinear and soft carry the same virtuality :

$$p_c \sim (1, \lambda^2, \lambda), \qquad p_s \sim (\lambda, \lambda, \lambda), \qquad p_c^2 \sim p_s^2 \sim \mathcal{O}(\lambda^2)$$

- Integrating out hard-collinear propagators introduces non-localities even in soft product :

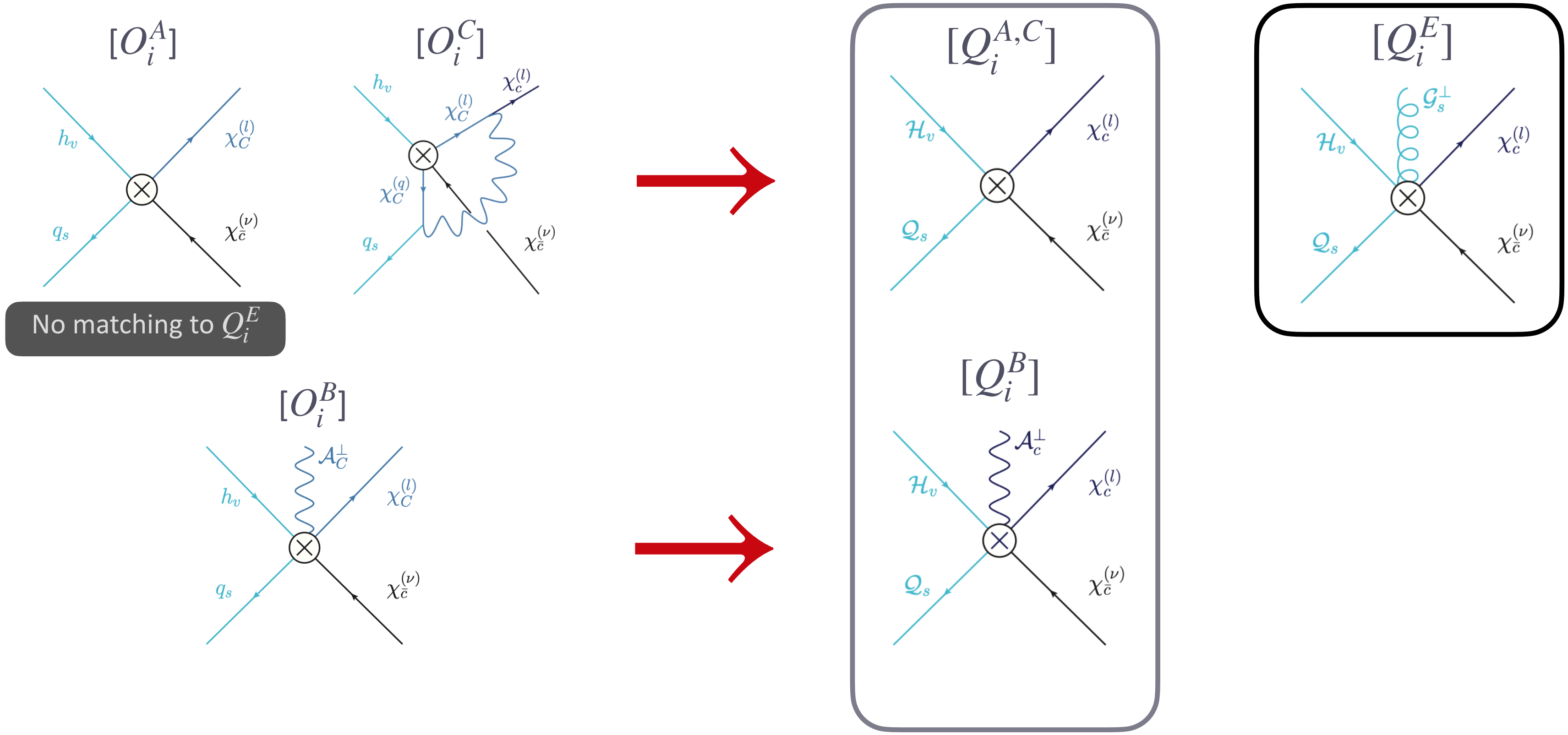
$$\frac{1}{n \cdot \partial} q_s, \left(\frac{1}{n \cdot \partial} \mathcal{G}_s^{\perp}\right)\left(\frac{1}{n \cdot \partial} q_s\right), \cdots$$

$\rightarrow$ Contains more fields but are of the same order !

- Contrary to the $B_s \rightarrow \mu^+\mu^-$ case, those structure-dependent contributions to $B^- \rightarrow \mu^-\bar{\nu}_\mu$ carry the same suppression as the tree level result and do not overcome the chiral suppression.
[Beneke, Bobeth, Szafron 2017, 2019]

SCET$_{II}$ basis

- Starting from our SCET$_I$ operators, we can construct our SCET$_{II}$ basis :



$$|B\rangle = \quad |\bar{u}b\rangle \quad \oplus \quad |\bar{u}bg\rangle \quad \oplus \cdots$$

SCET$_{II}$ basis

- SCET$_{II}$ operators contributing at $\mathcal{O}(\alpha)$:

$$\mathcal{H}_v = Y_n^{(q)\dagger} Y_n^{(\ell)\dagger} h_v \quad , \quad \mathcal{G}_s^\mu = Y_n^\dagger (i D_s^{(G)\mu} Y_n)$$

$$\mathcal{Q}_s = Y_n^{(q)\dagger} q_s \quad , \quad \mathcal{Q}_s^{(+)} = \frac{\not{\bar n}\not{n}}{4}\mathcal{Q}_s \quad , \quad \mathcal{Q}_s^{(-)} = \frac{\not{n}\not{\bar n}}{4}\mathcal{Q}_s$$

$$\omega = n \cdot p_q \, , \omega_g = n \cdot p_g$$

« Local operators »

[A]

$$Q_1^A(\Lambda) = \frac{m_\ell}{(\bar n \cdot \mathcal{P}_c)} \left(\bar{\mathcal{Q}}_s^{(-)} \theta(\Lambda + i\bar n \cdot \overleftarrow{D}_s) \not{n} P_L \mathcal{H}_v \right) \left(\bar\chi_c^{(\ell)} P_L \chi_{\bar c}^{(\nu)} \right)$$

$$Q_2^A = \frac{m_\ell \, \bar n \cdot v}{(\bar n \cdot \mathcal{P}_c)} \left(\bar{\mathcal{Q}}_s P_R \mathcal{H}_v \right) \left(\bar\chi_c^{(\ell)} P_L \chi_{\bar c}^{(\nu)} \right)$$

[B]

$$Q_1^B(x) = \frac{1}{(\bar n \cdot \mathcal{P}_c)} \left(\bar{\mathcal{Q}}_s \not{n} P_L \mathcal{H}_v \right) \left(\bar\chi_c^{(\ell)} \not{\mathcal{A}}_{c[x]}^\perp P_L \chi_{\bar c}^{(\nu)} \right)$$

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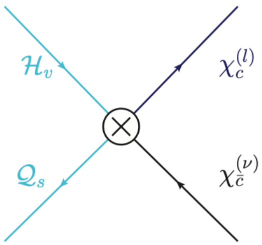

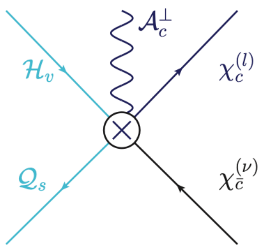

«Non-local operators »

[C]

$$Q_1^C(\omega) = \frac{m_\ell}{(\bar n \cdot \mathcal{P}_c)} \left(\bar{\mathcal{Q}}_{s[\omega]}^{(-)} \not{n} P_L \mathcal{H}_v \right) \left(\bar\chi_c^{(\ell)} P_L \chi_{\bar c}^{(\nu)} \right)$$

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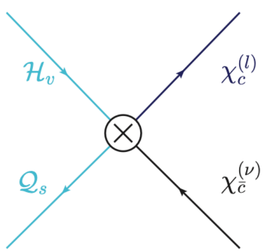
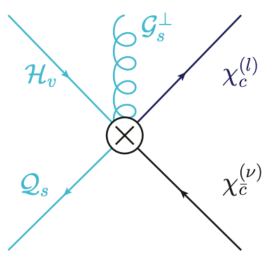
[E]

$$Q_2^E(\omega, \omega_g) = \frac{m_\ell \, \bar n \cdot v}{\omega (\bar n \cdot \mathcal{P}_c)} \left(\bar{\mathcal{Q}}_{s[\omega]} \mathcal{G}_{s[\omega_g]}^{\perp\mu} P_R \mathcal{H}_v \right) \left(\bar\chi_c^{(\ell)} P_L \chi_{\bar c}^{(\nu)} \right)$$





Refactorization

- The matrix element of C-type SCET I operators involve an endpoint-divergent integral→ the theory fails to properly separate hard-collinear modes with low energy (hard-collinear fraction $y \sim \mathcal{O}(\lambda)$) from soft modes.
- This region can be identified with the energetic limit of the soft region → we rearrange terms between the two (RBS scheme) [Liu, Neubert 2019; Liu, Mecaj, Neubert, Wang 2020; Liu, Neubert, Schnubel, Wang 2022; Beneke et al. 2022]

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$$H_1^C(y)\, J_{O_1^C \to Q_1^C}(y, \omega, \Lambda) = H_1^C(y)\, J^{bare}_{O_1^C \to Q_1^C}(y, \omega)$$

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$0 < \eta = \Lambda/m_B \leq 1$: arbitrary scale

$$H_1^C(y)\, J_{O_1^C \to Q_1^C}(y,\omega,\Lambda) = H_1^C(y)\, J^{bare}_{O_1^C \to Q_1^C}(y,\omega) - \theta(\eta - y)[\![H_1^C(y)]\!][\![J_{O_1^C \to Q_1^C}]\!](y,\omega)$$

Singular for $y \to 0$ $\quad \sim 1 + \frac{\alpha_s}{4\pi}\ln(y) \quad \sim y^{-1-\epsilon}$

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Conditions : $[\![H_1^C(y)]\!] = \left(H_1^A + \frac{1}{2} H_2^A\right) S_1^{C'}(\omega')$; $[\![J_{O_1^C \to Q_1^C}]\!](y, \omega) = (\bar{n} \cdot \mathscr{P}_C)\, S_1^{C''}(\omega', \omega)$

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Soft function $\qquad \langle Q_1^A \rangle = \left\langle \frac{m_\ell}{(\bar{n} \cdot \mathscr{P}_c)} \left(\bar{\mathcal{Q}}_s^{(-)}\ \not{n} P_L \mathscr{H}_v \right) \left(\bar{\chi}_c^{(\ell)} P_L \chi_{\bar{c}}^{(\nu)} \right) \right\rangle$

[Cornella, König, Neubert 2023]

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[Cornella, König, Neubert 2023]

New hadronic parameter !

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[Cornella, König, Neubert 2023]

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Factorization formula (virtual corrections)

- Taking SCET_{II} matrix elements, we obtain a factorisation theorem for the virtual corrections :

$$\mathscr{A}_{\text{virt}} = \langle \ell \nu | \mathscr{L}_{\text{SCET II} \otimes \text{HQET}} | B \rangle$$
$$= \sum_{i}^{26} H_i(\{y\}) \otimes_y \sum_{j}^{12} J_{i\to j}(\{y\}, \{\omega\}, \{x\}) \otimes_{x,\omega} \langle Q_j(\{x\}, \{\omega\}) \rangle$$

- SCET_{II} operators factorizes into non perturbative soft hadronic contributions and purely leptonic perturbative collinear functions :

$$\langle Q_i(\{x\}, \{\omega\}) \rangle = S_i(\{\omega\}) \underbrace{\left\langle B \left| Y_n^{(\ell)\dagger} \frac{\varphi_v}{\sqrt{2m_B}} \right| 0 \right\rangle}_{R^{(\ell,B)}} K_i(\{x\}) \frac{m_\ell}{m_B} \langle \ell \nu | \bar{h}_n P_L \nu_{\bar{c}} | 0 \rangle$$

- They define matching coefficients to the low energy theory hadronic and leptonic currents.

Low-energy theory : bHLET ⊗ HSET $\mu < \Lambda_{QCD} \sim m_\mu$

- Below $\mu \sim \Lambda_{QCD}$, quarks hadronize and the meson can be described by a charged scalar of mass m_B. We can describe it with a heavy scalar (HS) :

$$\Phi_B(x) \to \frac{e^{-im_B(v\cdot x)}}{\sqrt{2m_B}}\varphi_v(x)$$

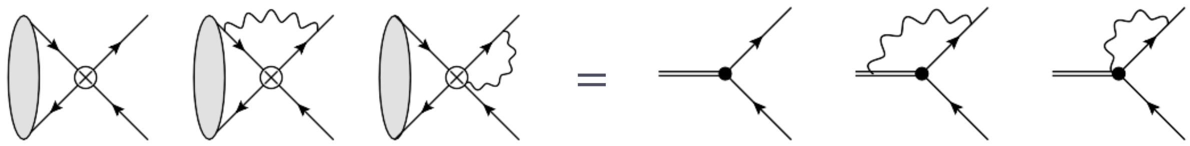
- Since $\Lambda_{\text{QCD}} \sim m_\mu$, also the muon becomes infinitely heavy. We describe it as a boosted heavy lepton field (bHL) :

$$\chi_c^{(\ell)}(x) = e^{-im_\ell(v_\ell\cdot x)}h_n(x)$$

[Fleming et al (hep-ph/0703207); Beneke et al (2305.06401)]

- We match the resulting low energy EFT by taking the hadronic matrix element :

$$\langle \ell\nu | \mathscr{L}_{\text{SCET}_{II}\otimes\text{HQET}} | B\rangle = \langle \ell\nu | \mathscr{L}_{\text{bHLET}\otimes\text{HSET}} | B\rangle$$



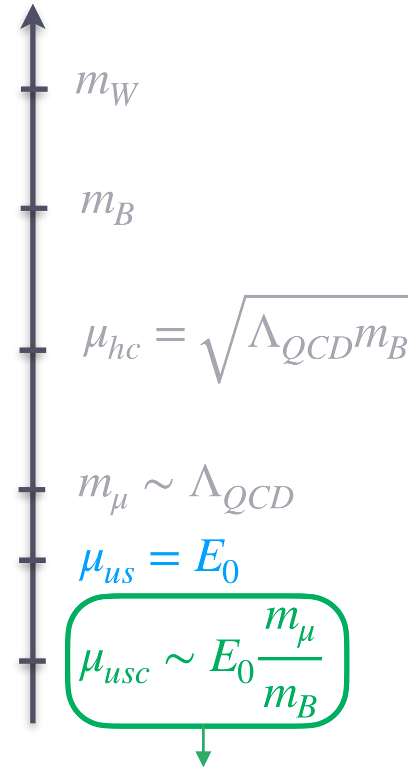
A theory of Wilson lines ?

- We can decouple the interactions of the B and the muon with ultrasoft and ultrasoft-collinear photons :

$$Y_n^{(\ell)\dagger}(x_-)\varphi_v(x) = Y_n^{(\ell)\dagger}(x_-)Y_v^{(B)}(x)C_{\bar{n}}^{(B)}(x_+)\,\varphi_v^0(x) \qquad h_n(x) = C_{v_\ell}^{(\ell)}(x)\,h_n^0(x)$$

$$Y_r^{(i)}(x) = \mathscr{P}\exp\left\{ieQ_i\int_{-\infty}^0 ds\, r\cdot A_{us}(x+sr)\right\} \qquad C_r^{(i)}(x) = \mathscr{P}\exp\left\{ieQ_i\int_{-\infty}^0 ds\, r\cdot A_{usc}(x+sr)\right\}$$

- Real corrections can be expressed as matrix elements of these Wilson lines, convoluted with the measurement function implementing the experimental radiation veto



ultra-soft photon to the muon boosted in the B frame

$$S(E_0,\mu) = \int_0^\infty d\omega_s \int_0^\infty d\omega_{sc}\,\theta\left(\frac{E_0}{2} - \omega_{us} - \omega_{usc}\right) W_{us}(\omega_{us},\mu)W_{usc}(\omega_{usc},\mu)$$

Factorization formula…

- Using this framework, we can write the factorization formula for $B^- \to \mu^- \bar{\nu}_\mu$ at the level of the decay rate.

- Real emissions are factorized at the level of the decay rate . Ultrasoft/ultrasoft-collinear logarithms can be resummed in Laplace space ($\sim \mathcal{O}(10\%)$ corrections).

- The virtual corrections already factorize at the amplitude level and appear as an effective Yukawa coupling containing the hard/hard-collinear logarithms (which could be resummed but lead to a $\sim \mathcal{O}(1\%)$ correction).

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$$y_B(\mu) = C_L(\mu) \otimes H(\mu) \otimes J(\mu) \otimes K(\mu) \otimes S(\mu)$$

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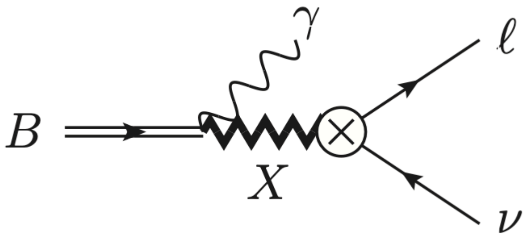
$$S(E_0,\mu) = W_{us}(E_0,\mu) \otimes W_{usc}(E_0,\mu)$$

...incomplete !

- Below Λ_{QCD} , the initial B meson can emit a photon and transition into an excited state X :

[Becirevic et al (0907.1845)]



$$\frac{1}{m_B E_0 - \delta_X^2}$$

- For $E_0 \lesssim \Lambda_{QCD}$, only the B^* needs to be included (heavy-spin symmetry), contributions from higher states being power-suppressed $\rightarrow$ Pole-dominance approximation

$$\delta_X^2 = \frac{m_X^2 - m_B^2}{2m_B} \lesssim E_0 \qquad \delta_{B^*}^2 \sim \mathcal{O}(\lambda^2) \qquad \delta_{B_1'}^2, \delta_{B_2^*}^2 \sim \mathcal{O}(\lambda)$$

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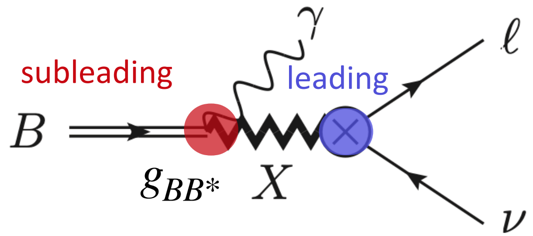

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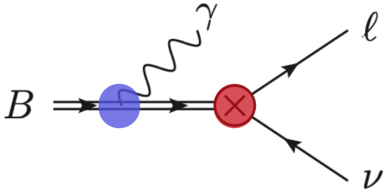
- The $B \rightarrow B^*\gamma$ interaction carries the same power suppression as the effective Yukawa.



- g_{BB^*} is unobserved, has to be estimated by a mixture of QCD sum rules, quark models and lattice QCD.

$HH\chi Pt$ (Heavy Hadron Chiral Perturbation theory)

- Using approximate heavy spin symmetry, the B and B^* can be put into a superfield:

$$H = \frac{1+\not{v}}{2}(\not{\phi}_v - \varphi_v \gamma_5) \quad \text{where} \quad \begin{aligned} H &\to S H \quad \text{under} \quad SU(2)_v \\ \overline{H} &\to \overline{H} S^{-1} \end{aligned}$$

[Wise (PhysRevD.45.R2188), Boyd et al. (9402340),…]

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$$\bar{\Lambda} = m_B - m_b \sim \mathcal{O}(\Lambda_{QCD})$$

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- $SU(2)_v$ breaking effects arise via the chromomagnetic/magnetic operator

$$\frac{g}{4m_B}(\bar{h}_v \sigma^{\mu\nu} h_v) G_{\mu\nu} \quad \text{giving rise to the spurion :} \quad \frac{\Sigma_s^{\mu\nu}}{m_B} = \frac{1+\slashed{v}}{2} \frac{\sigma_s^{\mu\nu}}{m_B} \frac{1+\slashed{v}}{2}$$

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- Subleading $B \to B^*\gamma$ and $B^* \to B^*\gamma$ interactions at $\mathcal{O}(\zeta)$ are described by a single $SU(2)_v$ conserving operator :

$$c_{B^*B^*\gamma} = c_{BB^*\gamma} = c_2 = g_{BB^*}\bar{\Lambda}$$

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$$= \varphi_v^\dagger(iv\cdot\partial)\varphi_v - \rho_{v\mu}^\dagger(iv\cdot\partial)\rho_v^\mu + \delta_{B*}\rho_{v\mu}^\dagger\rho_v^\mu$$

$$+ \frac{ec_{B^*B\gamma}}{2\bar{\Lambda}}\left(\varphi_v v^\mu \rho_v^{\dagger\nu}\tilde{F}_{\mu\nu}^{us} + \text{h.c.}\right) + \frac{iec_{B^*B^*\gamma}}{2\bar{\Lambda}}\rho_v^\mu\rho_v^{\dagger\nu}F_{\mu\nu}^{us}$$

with : $\quad c_{B^*B^*\gamma} = c_{BB^*\gamma} = c_2 = g_{BB^*}\bar{\Lambda} \quad ; \quad \delta_{B^*} = -16c_1\dfrac{\bar{\Lambda}^2}{m_B^2}$

Factorization formula…. now complete !

$$\Gamma = \Gamma_0\Big(\underbrace{|y_B(\mu)|^2}_{\text{Non-radiative}} \underbrace{S(E_0,\mu)}_{\text{Radiative}} + \underbrace{|y_B^*|^2 S_{B*}(E_0)}_{\text{Structure-dependent radiative corrections}} \Big)$$

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$$y_B(m_\ell) = 6.30 \cdot 10^{-8}\,(1 \pm 0.05_{V_{ub}} \pm 0.02_{F_{m_B}} \pm 0.01_{f_B})\,,$$
$$y_{B^*}(m_\ell) = 6.69 \cdot 10^{-8}\,(1 \pm 0.05_{V_{ub}} \pm 0.06_{f_{B^*}})\,.$$

We choose $\Lambda = m_B$

$$F(m_B, m_b) = (1.00 \pm 0.02) \cdot F_{\rm QCD}(m_b)$$

Conservative assumption !

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- We get an estimation of the branching ratio for various values of $E_\gamma^{max} = E_0/2$:

$$\mathscr{B}(B^- \to \mu^- \bar{\nu}_\mu)\Big|_{E_\gamma^{\max}=25\,\text{MeV}} = 4.00 \cdot 10^{-7} \Big(1 \pm 0.09_{V_{ub}} \pm 0.04_{F_{m_B}} \pm 0.01_{f_B}\Big)$$
$$\mathscr{B}(B^- \to \mu^- \bar{\nu}_\mu)\Big|_{E_\gamma^{\max}=100\,\text{MeV}} = 4.16 \cdot 10^{-7} \Big(1 \pm 0.09_{V_{ub}} \pm 0.04_{F_{m_B}} \pm 0.01_{f_B} \pm 0.01_{g_{BB^*}}\Big)$$

Factorization formula…. now complete !

$$\Gamma = \Gamma_0\Big(\underbrace{|\, y_B(\mu)\,|^2}_{\text{Non-radiative}} \underbrace{S(E_0,\mu)}_{\text{Radiative}} + \underbrace{|\, y_B^* \,|^2 S_{B*}(E_0)}_{\text{Structure-dependent radiative corrections}} \Big)$$

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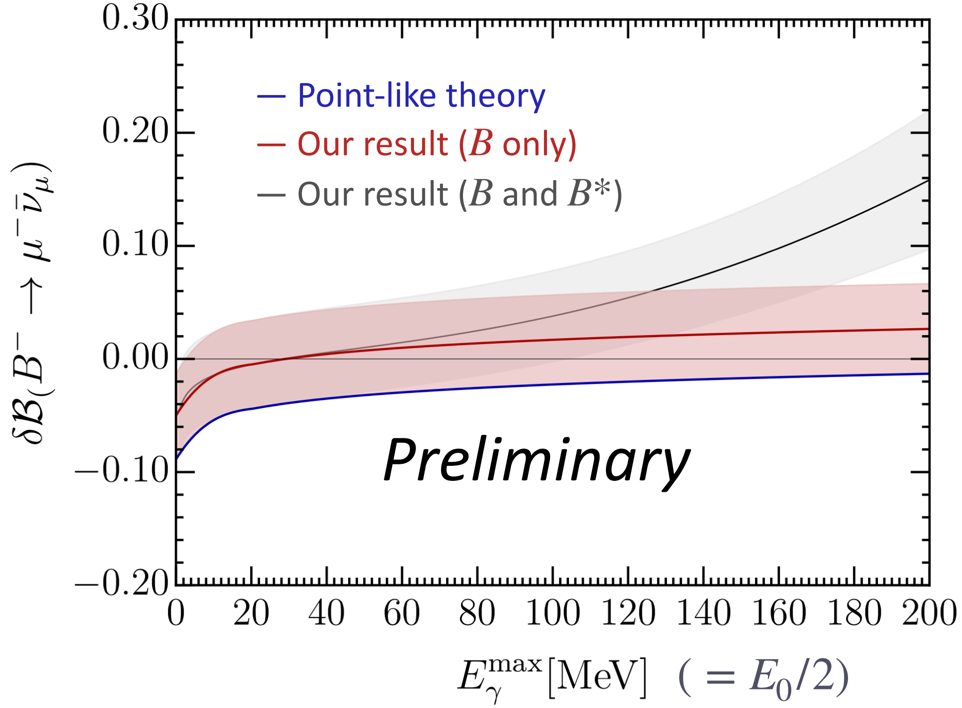
Precision for the extraction of V_{ub}

$$\mathscr{B} = \mathscr{B}_0\left(|y_B(\mu)|^2 S(E_s,\mu) + |y_B^*|^2 S_{B*}(E_S)\right)$$

$$\delta\mathscr{B}(B^- \to \mu^-\bar{\nu}_\mu) \equiv \frac{\mathscr{B}(B^- \to \mu^-\bar{\nu}_\mu)}{\mathscr{B}_0(B^- \to \mu^-\bar{\nu}_\mu)} - 1$$

$$F(m_B, m_b) = (1.00 \pm 0.02) \cdot F_{\text{QCD}}(m_b)$$

Conservative assumption !



$$S(E_0,\mu) \sim \log\frac{E_0}{m_B}\log\frac{m_\ell}{m_B}$$

$$S_{B*}(E_0) \sim E_0^2$$

Conclusions

- The exclusive leptonic decay $B^- \to \mu^- \bar{\nu}_\mu$, is an important channel for test of physics in SM and beyond. Since future experimental accuracy calls for percent precision in the theory prediction, QED corrections are needed.

- On top of large logarithms of lepton mass and photon cuts, we find (single logarithmic) structure-dependent effects in the virtual & real corrections:

  – virtual: hard-collinear photons between the lepton and light spectator quark
  – real: B* contribution; gets more important for a looser cut.

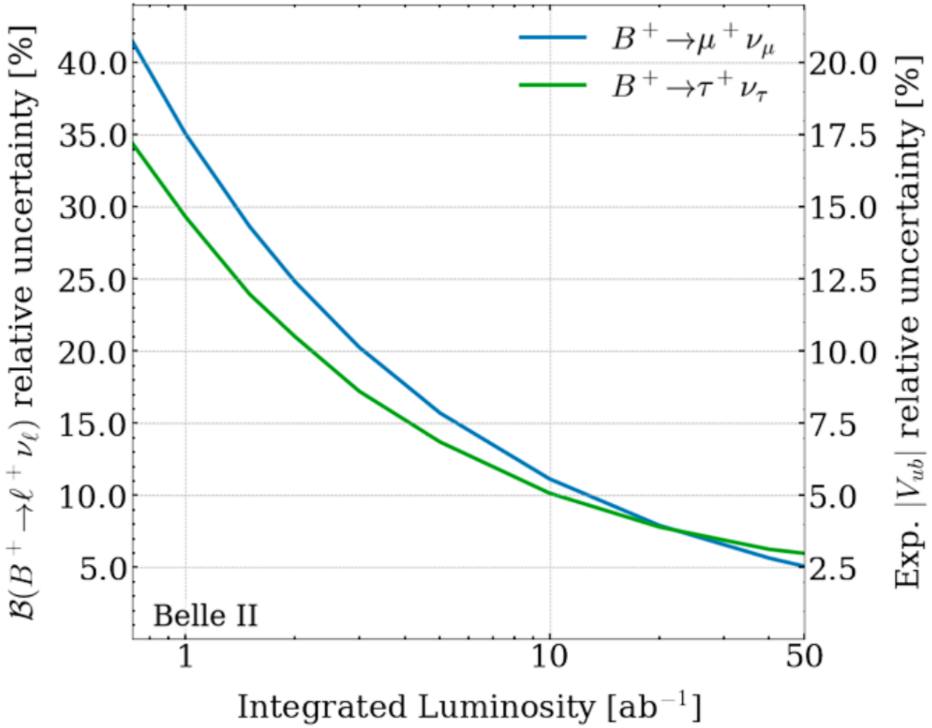
- These corrections introduce new uncertainties: they probe the inner structure of the B meson & introduce new hadronic parameters which need to be estimated (Sum rules, lattice determination ?)

- These EFT methods can be adapted to tackle other leptonic decays (D, D_s ,...) and need to be extended for semi-leptonic channels ($B \to \pi \ell \nu, B \to D \ell \nu$,...), crucial for the extraction of $|V_{ub}|$ and $|V_{cb}|$.

Thanks for listening !

Backup-slides

Belle II projections



[Belle II Physics Book]

Figure 1: Projection of uncertainties on the branching fractions $\mathcal{B}(B^+ \to \mu^+ + \nu_\mu)$ and $\mathcal{B}(B^+ \to \tau^+ + \nu_\tau)$. The corresponding uncertainty on the experimental value of $|V_{ub}|$ is shown on the right-hand vertical axis.

Virtual amplitude at NLO QED

$$\mathscr{A}_{\text{virtual}} = i\sqrt{2}\, G_F\, K_{\text{EW}}(\mu) V_{ub}\, \frac{m_\ell}{m_B}\sqrt{m_B}\, F(\Lambda,\mu)\, \bar{u}(p_\ell) P_L\, v(p_\nu) \sum_j \mathscr{M}_j(\mu)\,,$$

$$\begin{aligned}
\mathscr{M}_{2\text{p}}(\mu) = 1 + \frac{\alpha}{4\pi}\Bigg\{ & Q_b^2\left[-\frac{3}{2}\ln\frac{\mu^2}{m_b^2} - 2\right] \\
& - Q_\ell^2\left[\frac{1}{\epsilon_{\text{IR}}}\left(2 + \ln\frac{m_\ell^2}{m_B^2}\right) - \frac{1}{2}\ln^2\frac{\mu^2}{m_\ell^2} + \frac{3}{2}\ln\frac{\mu^2}{m_\ell^2} - \frac{\pi^2}{12} + 2\right] \\
& - Q_b Q_\ell\left[\frac{1}{2}\ln^2\frac{\mu^2}{m_B^2} - 2\ln\frac{\mu^2}{m_B^2} + \ln\frac{\mu^2}{m_\ell^2} - 1 + \frac{\pi^2}{12}\right] \\
& + Q_\ell Q_u\left[-5 - 2\ln\frac{\mu^2}{m_\ell^2} + \int_0^\infty d\omega\, \phi_-(\omega)\left(\ln\frac{\mu^2}{\omega\sqrt{m_B\Lambda}}\ln\frac{\Lambda^2}{m_B^2} - 2\ln\frac{\mu^2}{\omega\sqrt{m_B\Lambda}} + \ln\frac{\Lambda}{m_B} - 2 - \frac{\pi^2}{3}\right)\right]\Bigg\} \\
& + \frac{C_F\alpha_s}{4\pi}\left[-\frac{3}{2}\ln\frac{\mu^2}{m_b^2} - 2\right],
\end{aligned}$$

$$\mathscr{M}_{3\text{p}}(\mu) = -\frac{\alpha}{\pi} Q_\ell Q_u \left(1 + \ln\frac{\Lambda}{m_B}\right) \times \int_0^\infty d\omega \int_0^\infty d\omega_g\, \frac{\phi_3(\omega,\omega_g)}{\omega_g}\left[\frac{1}{\omega_g}\ln\left(1 + \frac{\omega_g}{\omega}\right) - \frac{1}{\omega + \omega_g}\right].$$

A theory of Wilson lines ?

$$Y_r^{(i)}(x) = \mathscr{P}\exp\left\{ieQ_i\int_{-\infty}^{0} ds\, r\cdot A_{us}(x+sr)\right\} \quad C_r^{(i)}(x) = \mathscr{P}\exp\left\{ieQ_i\int_{-\infty}^{0} ds\, r\cdot A_{usc}(x+sr)\right\}$$

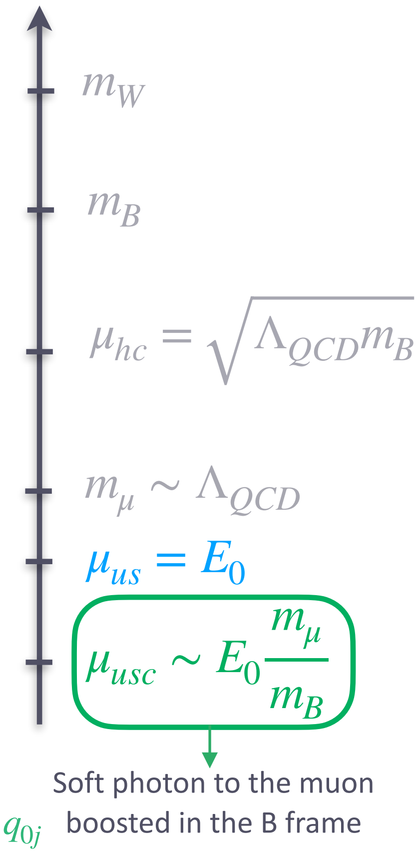
- We can decouple the interactions of the B and the muon with ultrasoft and ultrasoft-collinear photons :

$$Y_n^{(\ell)\dagger}(x_-)\varphi_v(x) = Y_n^{(\ell)\dagger}(x_-)Y_v^{(B)}(x)C_{\bar{n}}^{(B)}(x_+)\,\varphi_v^0(x) \qquad h_n(x) = C_{v_\ell}^{(\ell)}(x)\,h_n^0(x)$$

- Real corrections can be expressed as matrix elements of Wilson lines :

$$W_s(\omega_{us},\mu) = \left[\sum_{n_{us}=0}^{\infty}\prod_{i=1}^{n_{us}}\int d\Pi_i(q_i)\right]\left|\langle n_{us}\gamma_{us}(q_i)\,|\,Y_v^{(B)}Y_n^{\dagger(\ell)}\,|\,0\rangle\right|^2\delta(w_{us}-q_0^{us}) \quad , \quad q_0^{us} = \sum_i q_{0i}$$

$$W_{usc}(\omega_{usc},\mu) = \left[\sum_{n_{usc}=0}^{\infty}\prod_{j=1}^{n_{usc}}\int d\Pi_j(q_j)\right]\left|\langle n_{usc}\gamma_{usc}(q_j)\,|\,C_{\bar{n}}^{(B)}C_{v_\ell}^{\dagger(\ell)}\,|\,0\rangle\right|^2\delta(w_{usc}-q_0^{usc}) \quad , \quad q_0^{usc} = \sum_j q_{0j}$$



- These radiative functions are convoluted with the measurement function implementing the experimental radiation veto

$$S(E_0,\mu) = \int_0^\infty d\omega_s\int_0^\infty d\omega_{sc}\,\theta\left(\frac{E_0}{2}-\omega_{us}-\omega_{usc}\right)W_{us}(\omega_{us},\mu)W_{usc}(\omega_{usc},\mu)$$

Generalized decay constant

- The QED corrected soft function of local operators define a generalized decay constant :

$$S_1^A(\Lambda) \equiv \frac{1}{R^{(\ell,B)}} \left\langle 0 \left| \bar{q}_s \theta(\Lambda + i\bar{n}\cdot \overleftarrow{D}_s) \not{n} P_L Y_n^{(l)\dagger} h_v \right| B^- \right\rangle$$

$$= -\frac{i\sqrt{m_B}}{2} F(\Lambda, \mu)$$

$$R^{(\ell,B)} = \langle 0 |\, Y_{v_B}^{(B)} Y_n^{(l)\dagger} \,| 0 \rangle$$

$$Y_r^{(i)}(x) = \mathscr{P} \exp\left\{ ieQ_i \int_{-\infty}^{0} ds\, r \cdot A_s(x + sr) \right\}$$

[Beneke et al (2108.05589)]

- For $\alpha \to 0$, F reduces to the standard HQET decay constant :

$$\sqrt{m_B} f_B \left[1 + C_F \frac{\alpha_s(m_b)}{2\pi} + \mathcal{O}(\alpha_s^2) \right] = F(\Lambda, m_b)\Big|_{\alpha \to 0} = F_{\text{QCD}}(m_b)\,,$$

Generalized decay constant

- The QED corrected soft function of local operators define a generalized decay constant :

$$S_1^A(\Lambda) \equiv \frac{1}{R^{(\ell,B)}}\left\langle 0\left|\bar{q}_s\theta(\Lambda + i\bar{n}\cdot\overleftarrow{D}_s)\not{n}P_L Y_n^{(l)\dagger}h_v\right|B^-\right\rangle$$
$$= -\frac{i\sqrt{m_B}}{2}F(\Lambda,\mu)$$

$$R^{(\ell,B)} = \langle 0|\, Y_{v_B}^{(B)}Y_n^{(l)\dagger}\,|0\rangle$$

$$Y_r^{(i)}(x) = \mathscr{P}\exp\left\{ieQ_i\int_{-\infty}^{0} ds\, r\cdot A_s(x+sr)\right\}$$

[Beneke et al (2108.05589)]

- $F(\Lambda,\mu)$ is an unknown non-perturbative parameter following evolution equations for Λ and μ :

$$F(\Lambda,\mu) = F(m_B,\mu)\left[1 - \frac{\alpha}{4\pi}Q_\ell Q_u \log\frac{\Lambda^2}{m_B^2}\int d\omega\left(\phi_-(\omega)\log\frac{\mu^2}{\omega\sqrt{m_B\Lambda}} + 2\int d\omega_g\frac{\phi_3(\omega,\omega_g)}{\omega_g^2}\log\frac{\omega}{\omega+\omega_g}\right) + \cdots\right]$$

two-particle LCDA　　three-particle LCDA

$$F(m_B,\mu) = F(m_B,m_B)\left[1 + \frac{\alpha}{8\pi}Q_\ell Q_u \log^2\frac{\mu^2}{m_B^2} + \ldots\right]$$

- Need to be computed at a single value of Λ and μ. Lattice determination ? QCD SR estimate ?

Coft Wilson-lines cancellation

- One can also define $F(\Lambda)$ using time-like Wilson-lines :

$$-\frac{i\sqrt{m_B}}{2}F(\Lambda,\mu) \equiv \frac{\left\langle 0 \middle| \bar{q}_s \theta(\Lambda + i\bar{n}\cdot \overleftarrow{D}_s)\not{n} P_L \mathcal{S}^{(l)\dagger}_{v_\ell} h_v \middle| B^- \right\rangle}{\langle 0 |\, \mathcal{S}^{(B)}_v \mathcal{S}^{(l)\dagger}_{v_\ell} \,| 0 \rangle}$$

- Those time-like WL can be splitted between soft and soft-collinear (coft) contributions :

$$\mathcal{S}^{(l)}_v(x) = Y^{(B)}_v(x) C^{(B)}_{\bar{n}}(x_+) = Y^{(B)}_v(x) C^{(b)}_{\bar{n}}(x_+) C^{(q)\dagger}_{\bar{n}}(x_+) \qquad \mathcal{S}^{(l)}_{v_\ell}(x) = Y^{(\ell)}_n(x_-) C^{(\ell)}_{v_\ell}(x)$$

- After soft-collinear decoupling, $h_v \to C^{(b)}_{\bar{n}} h_v \quad , \quad q_s \to C^{(q)}_{\bar{n}} q_s$

contributions from the coft Wilson-lines cancel in the ratio :

$$-\frac{i\sqrt{m_B}}{2}F(\Lambda,\mu) = \frac{\left\langle 0 \middle| \bar{q}_s \theta(\Lambda + i\bar{n}\cdot \overleftarrow{D}_s)\not{n} P_L Y^{(\ell)\dagger}_n h_v \middle| B^- \right\rangle}{\langle 0 |\, Y^{(B)}_v Y^{(\ell)\dagger}_n \,| 0 \rangle}$$