Functional determinants and lifetime of the Standard Model

Katarina Trailović

based on *Phys.Rev.Lett.* 134 (2025) 1, 011601, [arxiv: 2406.05180]
and *JHEP* 05 (2025) 100, [arXiv: 2502.15878]
in collaboration with P. Baratella, M. Nemevšek, Y. Shoji, L. Ubaldi

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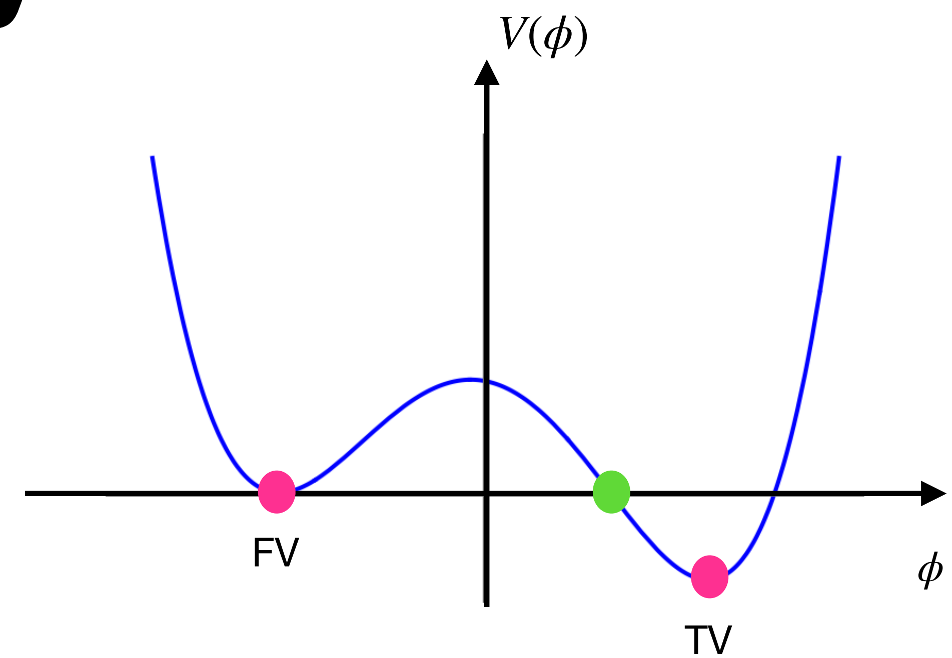
False Vacuum Decay

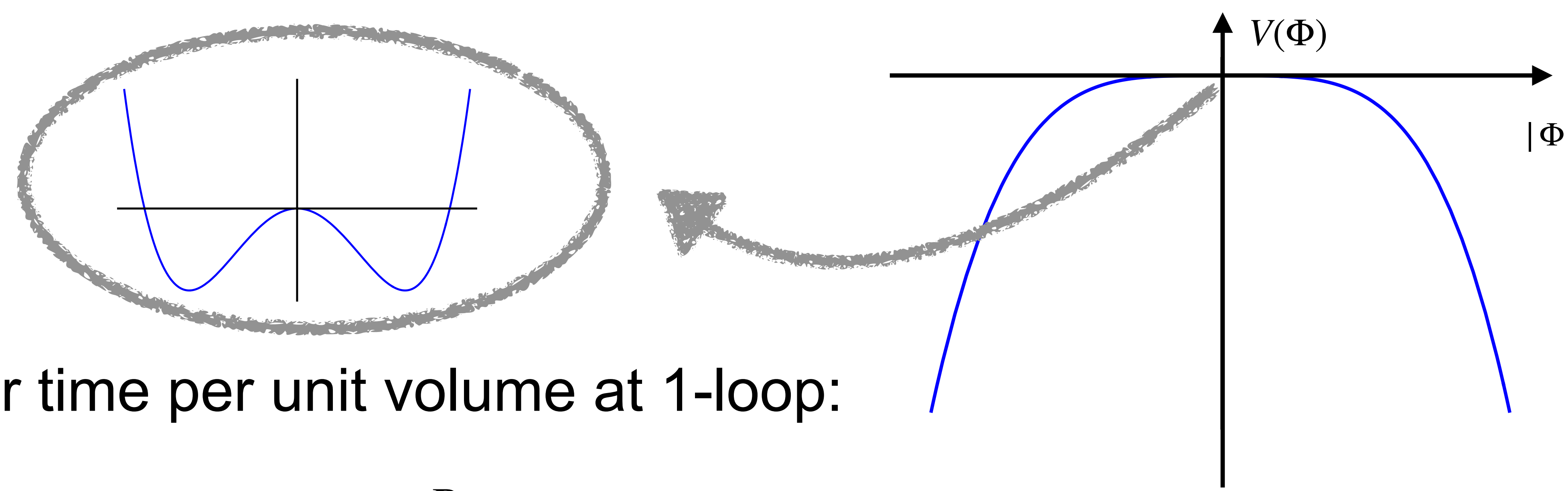


- Metastable state decays via tunnelling to energetically lower true vacuum
- Described by bounce solution
- Example: EW vacuum metastable:

$\lambda < 0$

$$8\pi^2 \frac{d\lambda}{d \log \mu} \simeq -3y_t^4 + 6y_t^2 \lambda$$

$$V(\Phi) = -m^2 |\Phi|^2 + \lambda |\Phi|^4 \simeq \lambda |\Phi|^4$$



- Decay rate per time per unit volume at 1-loop:

$$\gamma = \mathscr{A} e^{-B} \qquad B = S[\bar{\varphi}] - S[\varphi_{FV}]$$

False Vacuum Decay

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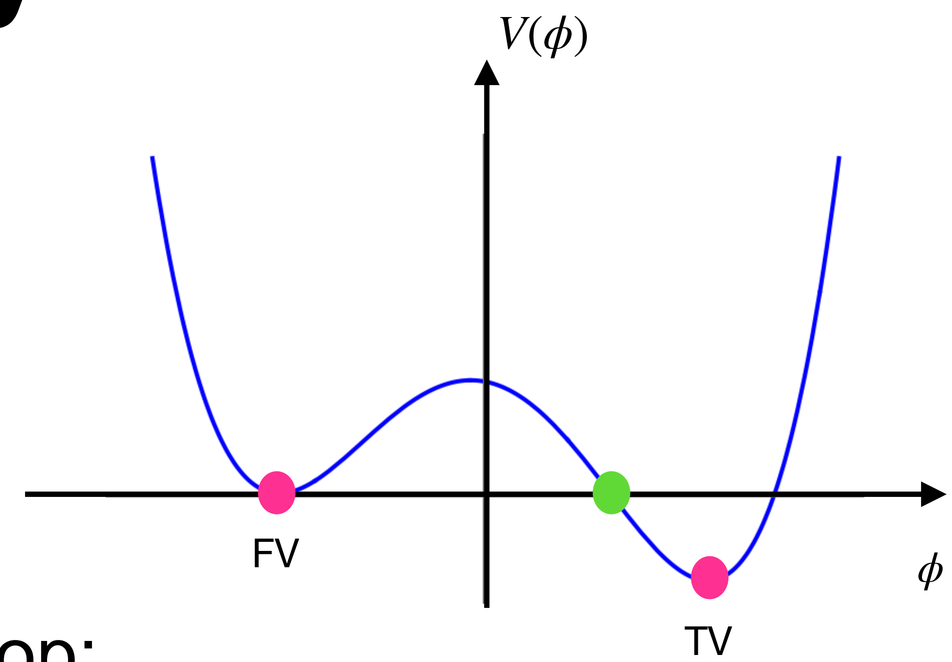
- Described by bounce solution
- Decay rate per time per unit volume at 1-loop:



$$\gamma = e^{-S}\left(\frac{S}{2\pi}\right)^2 \mathrm{Im}\sqrt{\left.\frac{\det\hat{S}''}{\det' S''}\right|_{\phi}}\;\left.\frac{\det S''}{\det\hat{S}''}\right|_{\psi} V_G\, J_G \sqrt{\left.\frac{\det\hat{S}''}{\det' S''}\right|_{A,a}}\;\left.\frac{\det S''}{\det\hat{S}''}\right|_{c\bar{c}}$$

tree level

fluctuation operators at bounce/FV

group volume

zero mode removal

Jacobians for zero modes

Regularisation of Determinant

* fluctuation operators S'' rotationally invariant → can be split into total angular momentum eigenbasis

$$\frac{\det S''}{\det \hat{S}''} = \prod_{\nu=1}^{\infty} \left(\frac{\det S''_\nu}{\det \hat{S}''_\nu} \right)^{d_\nu}$$

* UV divergence of determinant encoded in singular behaviour at large ν

* Regularisation procedure: isolate UV divergent one-loop diagrams
[Baacke, Lavrelashvili, '03]

coordinate space
+ multipole expansion

momentum space
+ dim reg

$$\ln \frac{\det S''}{\det \hat{S}''} = \sum_\nu \left(d_\nu \ln \frac{\det S''_\nu}{\det \hat{S}''_\nu} - \text{subtraction}_\nu \right) + \left(\text{addback}\right)_{\text{regularised}}$$

Scalar Fluctuations

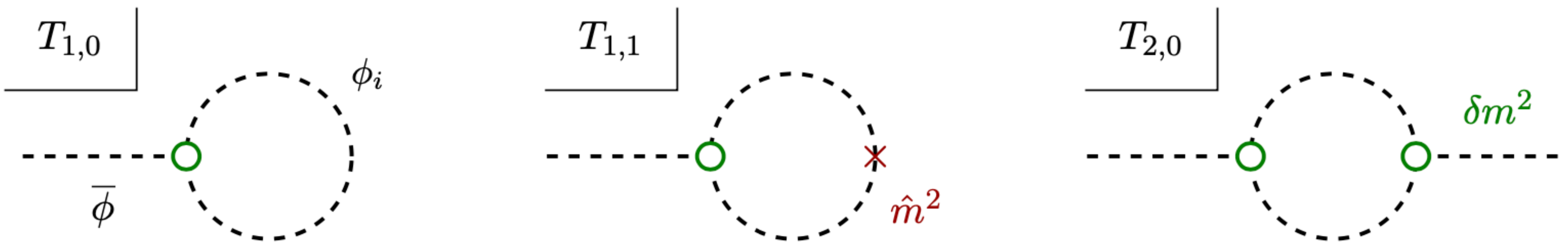
"field dependent mass" $m^2(\rho) = \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}|_{\bar\phi}$

$$S''[\bar\phi] = -\partial^2 + m^2(\rho)$$

$\hat S'' \equiv S''[\phi_{\rm FV}]$

✳ Euclidean action:

$$S[\phi] = \int d^4x \left(\frac{1}{2}(\partial_\mu\phi)^2 + V(\phi)\right) \simeq S[\bar\phi] + \underbrace{S'[\bar\phi]}_{=0}\,\delta\phi + \frac{1}{2}\delta\phi S''[\bar\phi]\delta\phi$$

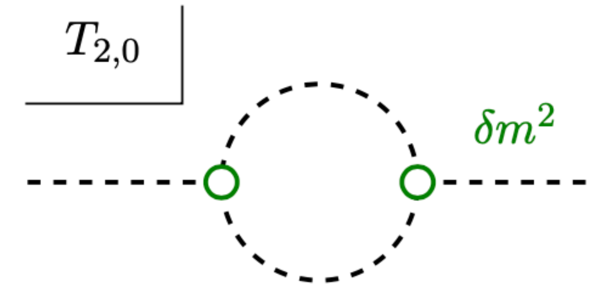


$\delta m^2 \equiv m^2 - \hat m^2$

$$\ln\frac{\det S''}{\det \hat S''} = \mathrm{tr}\ln\left(1 + \frac{1}{-\partial^2 + \hat m^2}\delta m^2\right) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{tr}\left(\frac{1}{-\partial^2+\hat m^2}\delta m^2\right)^n$$

$$= \underbrace{\mathrm{tr}\frac{1}{-\partial^2}\delta m^2}_{T_{1,0}} - \underbrace{\mathrm{tr}\frac{1}{-\partial^2}\hat m^2\frac{1}{-\partial^2}\delta m^2}_{T_{1,1}} - \frac{1}{2}\underbrace{\mathrm{tr}\frac{1}{-\partial^2}\delta m^2\frac{1}{-\partial^2}\delta m^2}_{T_{2,0}} + O\left((\delta m^2)^3, \hat m^4\right)$$

Regularisation of Determinant



✳ Problem in momentum space part: integrals of form

$$\int \mathrm{d}^D q\, \tilde{\phi}^2(q)\tilde{\phi}^2(-q) \int \frac{\mathrm{d}^D k}{(k^2-m^2)((k+q)^2-m^2)}$$

Fourier transform of bounce → typically complicated function

→integral cannot be solved analytically

✳ Simpler recipe with **single integrals in position space**

→ for scalar fluctuations: using ζ-function regularisation

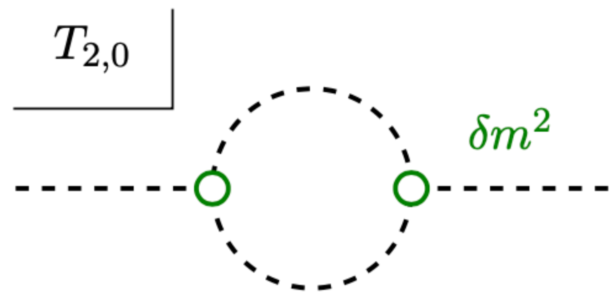
[Dunne, Kirsten, '06]

$$-\frac{1}{8}\int_0^\infty d\rho\ \rho^3 \delta m^4 \left[\ln\frac{\mu\rho}{2}+\gamma+1\right]$$

→ much simpler form!

→ but: derivation complicated & not easily generalisable to fermion and gauge boson fluctuations

Feyman Diagram Approach



- position space:

$d_\nu = \nu^2$

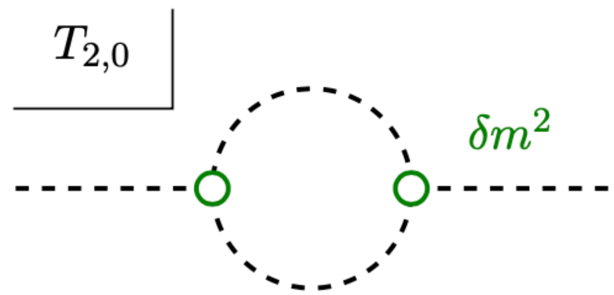
$$T_{2,0} = \sum_\nu \frac{d_\nu}{4\nu^3} \int_\rho \rho^3 \left(\delta m^2\right)^2 - \frac{1}{4} \int_\rho \rho^3 \left(\delta m^2\right)^2 + \frac{1}{2} \int_\rho \rho^3 \delta m^2 \int_0^1 \mathrm{d}x \frac{x^3}{1-x^2} \left(\delta m^2(x\rho) - \delta m^2\right)$$

- momentum space:

$$T_{2,0}^{\mathrm{DR}} = \int_k \widetilde{\delta m^2}(k) \widetilde{\delta m^2}(-k) \mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{q^2(k+q)^2} = \frac{1}{16\pi^2} \int_k \widetilde{\delta m^2}(k) \widetilde{\delta m^2}(-k) \left(\frac{2}{\epsilon} + 2 + \ln \frac{\tilde{\mu}^2}{k^2} \right)$$

$D = 4 - \epsilon$

Feyman Diagram Approach



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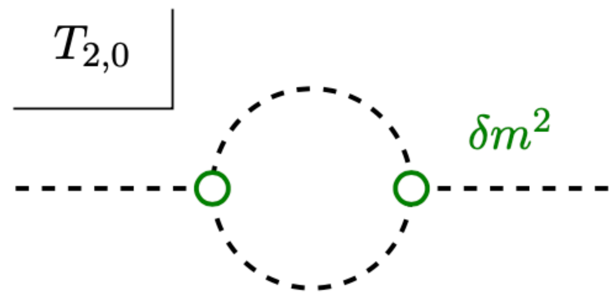
$$\int_k \tilde{f}(\left|k\right|)^2 \ln k^2 = -4\pi^2 \int_\rho \rho^3 f(\rho) \left(f(\rho) \left(\gamma_E - 1 + \ln \frac{\rho}{2} \right) + 2 \int_0^1 \mathrm{d}x\, x^3 \frac{f(x\rho) - f(\rho)}{1 - x^2} \right)$$

Feyman Diagram Approach



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$$= \frac{1}{8} \int_\rho \rho^3 \left(\delta m^2\right)^2 \left(\frac{2}{\epsilon} + 2\gamma_E + 2 \ln \frac{\tilde{\mu}\rho}{2} \right) + \frac{1}{2} \int_\rho \rho^3 \delta m^2 \int_0^1 \mathrm{d}x \frac{x^3}{1-x^2} \left(\delta m^2(x\rho) - \delta m^2\right)$$

Final Simple Formulas

[Baratella, Nemevšek, Shoji, K.T., Ubaldi, arxiv: 2502.15878]

- scalar fluctuations: [Dunne, Kirsten, '06]

$$\ln \frac{\det' S''}{\det \hat{S}''}\Bigg|_{\phi} = \sum_{\nu=1}^{\infty} \left(d_\nu^\phi \ln R_\nu^\phi - \frac{\nu}{2}\int_\rho \rho\, \mathrm{tr}\delta m^2 + \frac{1}{8\nu}\int_\rho \rho^3 \mathrm{tr}\delta m^4 \right) - \frac{1}{8}\int_\rho \rho^3 \mathrm{tr}\delta m^4 \left(\frac{1}{\epsilon} + 1 + \gamma_E + \ln\frac{\tilde{\mu}\rho}{2} \right)$$

- fermion fluctuations:

$$\ln \frac{\det S''}{\det \hat{S}''}\Bigg|_{\psi} = \sum_{\nu=1}^{\infty} \left(d_\nu^\psi \ln R_\nu^\psi - \left(\nu + \tfrac{1}{2}\right)\int_\rho \rho\, \mathrm{tr}\delta m_\psi^2 + \frac{1}{4\nu}\int_\rho \rho^3 \mathrm{tr}\left(\delta m_\psi^4 + \dot{m}_\psi{}^2 \right) \right)$$
$$-\frac{1}{4}\int_\rho \rho^3 \mathrm{tr}\left(\delta m_\psi^4 + \dot{m}_\psi{}^2 \right)\left(\frac{1}{\epsilon} + 1 + \gamma_E + \ln\frac{\tilde{\mu}\rho}{2} \right)$$

- gauge boson fluctuations:

$$\ln \frac{\det' S''}{\det \hat{S}''}\Bigg|_{A,a}^{\xi=1} = \sum_{\nu=1}^{\infty} \left(d_\nu^\phi \ln R_\nu^{(Da)} + 2\, d_\nu^T \ln R_\nu^{(T)} - \frac{1}{2}\int_\rho \rho \left(\nu\, \delta m_a^2 + 2(2\nu+1)\,\delta m_A^2 \right) + \frac{1}{8\nu}\int_\rho \rho^3 \left(\delta m_a^4 + 4\,\delta m_A^4 + 8\,\dot{m}_A^2 \right) \right)$$
$$+\frac{1}{2}\int_\rho \rho\, \delta m_A^2 + \frac{1}{8}\int_\rho \rho^3 \delta m_A^4 - \frac{1}{8}\int_\rho \rho^3 \left(\frac{1}{\epsilon} + 1 + \gamma_E + \ln\frac{\tilde{\mu}\rho}{2} \right)\left(\delta m_a^4 + 4\,\delta m_A^4 + 8\,\dot{m}_A^2 \right)$$

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✳ gauge boson fluctuations:

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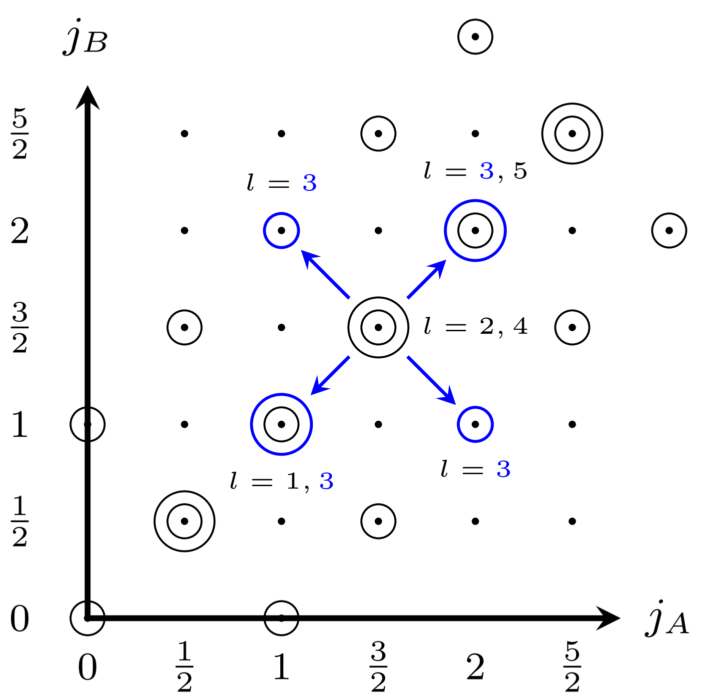
$$+\frac{1}{2}\int_\rho \rho\, \delta m_A^2 + \frac{1}{8}\int_\rho \rho^3 \delta m_A^4 - \frac{1}{8}\int_\rho \rho^3\left(\frac{1}{\epsilon} + 1 + \gamma_E + \ln\frac{\tilde{\mu}\rho}{2}\right)\left(\delta m_a^4 + 4\,\delta m_A^4 + 8\,\dot{m}_A^2\right)$$

Standard Model Lifetime

[Isidori, Ridolfi, Stumia, '01; Andreassen, Frost, Schwartz, '17; Chigusa, Moroi, Shoji, '17& '18]

* $SO(4) \simeq SU(2)_A \otimes SU(2)_B$ representation of gauge bosons:

$$\bigoplus_{l=0}^{\infty} (\tfrac{1}{2},\tfrac{1}{2}) \otimes (\tfrac{l}{2},\tfrac{l}{2}) = (0,0)_1 \oplus \bigoplus_{j=1}^{\infty}\left((\tfrac{j}{2},\tfrac{j}{2})_{j-1} \oplus (\tfrac{j}{2},\tfrac{j}{2})_{j+1} \oplus (\tfrac{j-1}{2},\tfrac{j+1}{2})_j \oplus (\tfrac{j+1}{2},\tfrac{j-1}{2})_j \right)$$



transverse modes → degeneracy factor:

$$d_j^T \equiv \text{dim}\left(\frac{j \pm 1}{2}, \frac{j \mp 1}{2}\right) = j(j+2)$$

[Baratella, Nemevšek, Shoji, K.T., Ubaldi, arxiv: 2406.05180]

$$J_A^2 \,|\, j_A, m_A\rangle = j_A(j_A+1)\,|\, j_A, m_A\rangle$$

$$J_{A,3}\,|\, j_A, m_A\rangle = m_A\,|\, j_A, m_A\rangle, \quad -j_A \le m_A \le j_A$$

$$\text{dim}(j_A,\, j_B) = (2j_A+1)(2j_B+1)$$

Standard Model Lifetime

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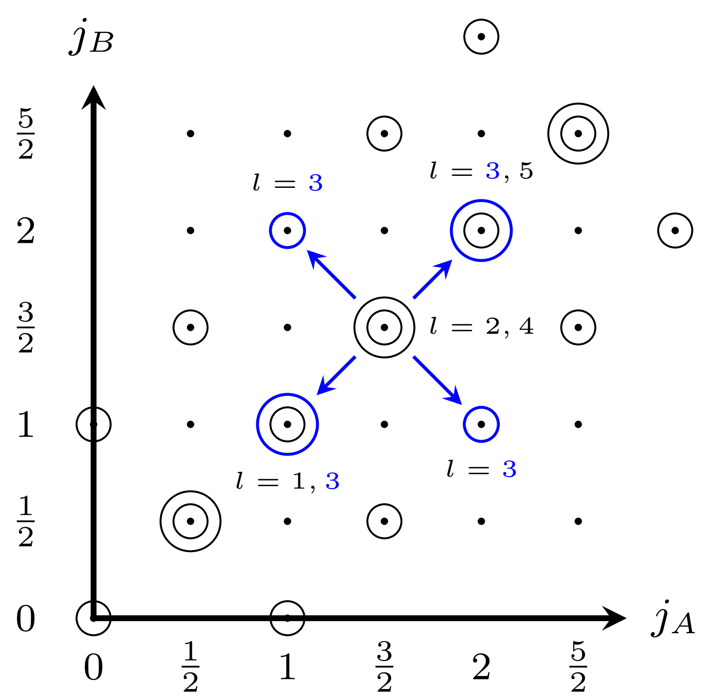
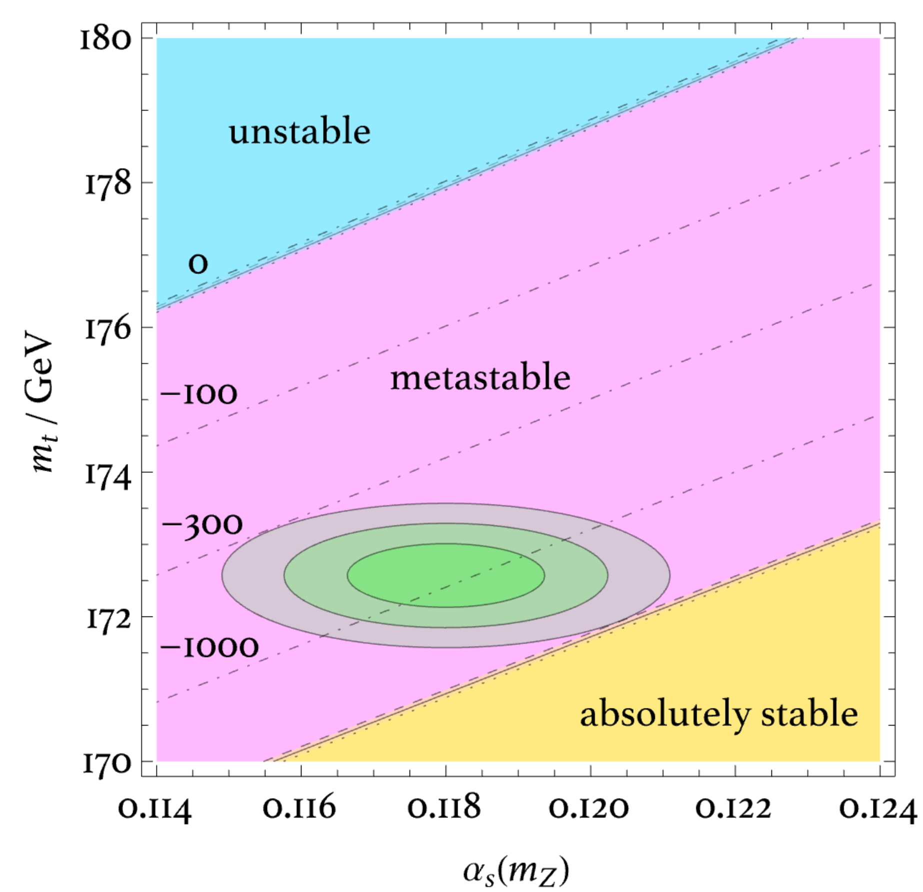


transverse modes → degeneracy factor:

$$d_j^T \equiv \dim\left(\frac{j\pm1}{2},\frac{j\mp1}{2}\right) = j(j+2) \quad \text{corrected}$$

$$= (j+1)^2 \quad \text{previous}$$

$$\frac{\det S''}{\det \hat{S}''} = \prod_{\nu=1}^{\infty} \left(\frac{\det S''_\nu}{\det \hat{S}''_\nu}\right)^{d_\nu}$$



✳ Updated EW decay rate:

[Baratella, Nemevšek, Shoji, K.T., Ubaldi, arxiv: 2406.05180]

$$\log_{10} \frac{\gamma}{\mathrm{Gyr}^{-1}\mathrm{Gpc}^{-3}} = -877 \rightarrow -871^{+35\ +175\ +209}_{-37\ -253\ -330}$$

Summary

* with double expansion and multipole subtraction arrive at simple expressions for regularised determinants
* method extended to scalar, fermion and gauge boson fluctuations
* final prescription only **single ρ-integrals,** bypasses Fourier transform, may thus be numerically more stable and easier to implement
* results general:
  - → generic scalar potential
  - → generic bounce solution
  - → generic number of species of spin 0, 1/2, 1
  - → easily generalise to non-abelian gauge bosons
* corrected degeneracy factor of T-modes in gauge sector, important for any meta-stability model including gauge bosons

Thank you

Scalar Fluctuations

$$\ln \frac{\det S''}{\det \hat{S}''} = \sum_{\nu=1}^{\infty} d_\nu \ln \frac{\det S''_\nu}{\det \hat{S}''_\nu} - \left(T_{1,0} - T_{1,1} - \frac{1}{2} T_{2,0} \right) + \left(T^{\rm DR}_{1,0} - T^{\rm DR}_{1,1} - \frac{1}{2} T^{\rm DR}_{2,0} \right)$$

✳ Representation of scalar field under $SO(4) \simeq SU(2)_A \otimes SU(2)_B$:

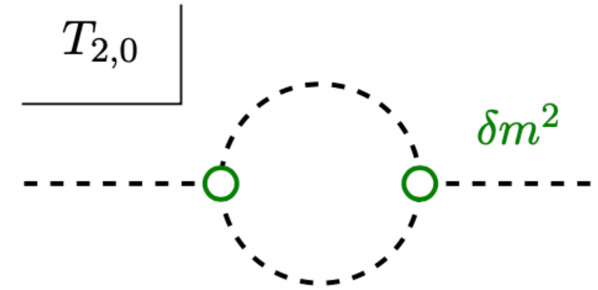
$$\bigoplus_{\nu=1}^{\infty} \left(\frac{\nu-1}{2}, \frac{\nu-1}{2} \right) \longrightarrow (j_A, j_B) \longrightarrow \begin{array}{l} J_A^2 \,|\, j_A, m_A \rangle = j_A(j_A+1) \,|\, j_A, m_A \rangle \\ J_{A,3} \,|\, j_A, m_A \rangle = m_A \,|\, j_A, m_A \rangle, \quad -j_A \le m_A \le j_A \end{array}$$

spherical harmonics $Y_{\nu m_A m_B}$ eigenfunctions of $J^2 \equiv 2(J_A^2 + J_B^2),\ J_{A,3}\ J_{B,3}$

$$\Longrightarrow \phi(x) = \sum_{\nu m_A m_B} c_{\nu m_A m_B}(\rho)\ Y_{\nu m_A m_B}(\hat{x})$$

$$S''_\nu = -\partial_\rho^2 - \frac{3}{\rho}\partial_\rho + \frac{\nu^2-1}{\rho^2} + m^2(\rho)$$ with degeneracy $d_\nu = \nu^2$

Scalar Fluctuations



- position space:

$$T_{2,0} = \sum_{\nu m} \int_\rho \rho^3 \int_\lambda \int_{\rho'} \rho'^{\,3} \int_{\lambda'} \langle \rho; \nu m | \frac{1}{-\partial^2} | \lambda; \nu m \rangle \langle \lambda; \nu m | \delta m^2 | \rho'; \nu m \rangle \langle \rho'; \nu m | \frac{1}{-\partial^2} | \lambda'; \nu m \rangle \langle \lambda'; \nu m | \delta m^2 | \rho; \nu m \rangle$$

scalar localised at $x^2 = \rho^2$

$$\langle \rho; \nu m | \rho'; \nu' m' \rangle = \rho^{-3} \delta(\rho - \rho') \delta_{\nu\nu'} \delta_{mm'}$$

$$-\partial^2 | \lambda; \nu m \rangle = \lambda^2 | \lambda; \nu m \rangle$$

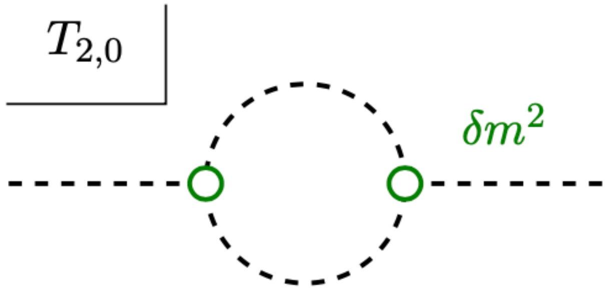
$$\langle \lambda; \nu m | \lambda'; \nu' m' \rangle = \delta(\lambda - \lambda') \delta_{\nu\nu'} \delta_{mm'}$$

$$\langle \rho; \nu m | \lambda; \nu' m' \rangle = \frac{\sqrt{\lambda}}{\rho} J_\nu \left(\lambda \rho \right) \; \delta_{\nu\nu'} \delta_{mm'}$$

$$= \sum_\nu d_\nu \frac{1}{2\nu^2} \int_\rho \rho^3 \delta m^2 \int_0^1 \mathrm{d}x \; x^{2\nu+1} \delta m^2(x\rho)$$

$$= \sum_\nu \frac{d_\nu}{4\nu^3} \int_\rho \rho^3 \left(\delta m^2 \right)^2 - \frac{1}{4} \int_\rho \rho^3 \left(\delta m^2 \right)^2 + \frac{1}{2} \int_\rho \rho^3 \delta m^2 \int_0^1 \mathrm{d}x \frac{x^3}{1 - x^2} \left(\delta m^2(x\rho) - \delta m^2 \right)$$

Scalar Fluctuations



✳ position space:

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$$= \sum_\nu \frac{d_\nu}{4\nu^3} \int_\rho \rho^3 \left(\delta m^2\right)^2 - \frac{1}{4} \int_\rho \rho^3 \left(\delta m^2\right)^2 + \frac{1}{2} \int_\rho \rho^3 \delta m^2 \int_0^1 \mathrm{d}x \frac{x^3}{1-x^2} \left(\delta m^2(x\rho) - \delta m^2\right)$$

✳ momentum space:

$$T_{2,0}^{\rm DR} = \int_x \int_p \int_y \int_q \langle x | \frac{1}{-\partial^2} | q \rangle \langle q | \delta m^2 | y \rangle \langle y | \frac{1}{-\partial^2} | p \rangle \langle p | \delta m^2 | x \rangle = \int_k \widetilde{\delta m^2}(k) \widetilde{\delta m^2}(-k) \mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{q^2 (k+q)^2}$$

$$= \frac{1}{16\pi^2} \int_k \widetilde{\delta m^2}(k) \widetilde{\delta m^2}(-k) \left(\frac{2}{\epsilon} + 2 + \ln \frac{\tilde{\mu}^2}{k^2} \right)$$

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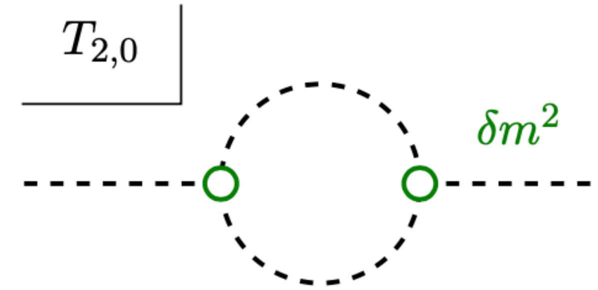
Scalar Fluctuations



- position space:

$$T_{2,0} = \sum_{\nu m} \int_\rho \rho^3 \int_\lambda \int_{\rho'} \rho'^{\,3} \int_{\lambda'} \langle \rho; \nu m | \frac{1}{-\partial^2} | \lambda; \nu m \rangle \langle \lambda; \nu m | \delta m^2 | \rho'; \nu m \rangle \langle \rho'; \nu m | \frac{1}{-\partial^2} | \lambda'; \nu m \rangle \langle \lambda'; \nu m | \delta m^2 | \rho; \nu m \rangle$$

$$= \sum_\nu \frac{d_\nu}{4\nu^3} \int_\rho \rho^3 \left(\delta m^2\right)^2 - \frac{1}{4} \int_\rho \rho^3 \left(\delta m^2\right)^2 + \frac{1}{2} \int_\rho \rho^3 \delta m^2 \int_0^1 \mathrm{d}x \frac{x^3}{1-x^2} \left(\delta m^2(x\rho) - \delta m^2\right)$$

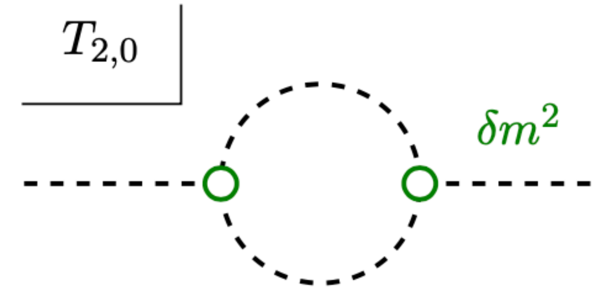
- momentum space:

$$T_{2,0}^{\mathrm{DR}} = \int_x \int_p \int_y \int_q \langle x | \frac{1}{-\partial^2} | q \rangle \langle q | \delta m^2 | y \rangle \langle y | \frac{1}{-\partial^2} | p \rangle \langle p | \delta m^2 | x \rangle = \int_k \widetilde{\delta m^2}(k) \widetilde{\delta m^2}(-k) \mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{q^2 (k+q)^2}$$

$$= \frac{1}{16\pi^2} \int_k \widetilde{\delta m^2}(k) \widetilde{\delta m^2}(-k) \left(\frac{2}{\epsilon} + 2 + \ln \frac{\tilde{\mu}^2}{k^2} \right)$$

$$\int_k \tilde{f}(|k|)^2 \ln k^2 = -4\pi^2 \int_\rho \rho^3 f(\rho) \left(f(\rho) \left(\gamma_E - 1 + \ln \frac{\rho}{2} \right) + 2 \int_0^1 \mathrm{d}x\, x^3 \frac{f(x\rho) - f(\rho)}{1 - x^2} \right)$$

Scalar Fluctuations



✳ position space:

$$T_{2,0} = \sum_{\nu m} \int_\rho \rho^3 \int_\lambda \int_{\rho'} \rho'^{\,3} \int_{\lambda'} \langle \rho; \nu m | \frac{1}{-\partial^2} | \lambda; \nu m \rangle \langle \lambda; \nu m | \delta m^2 | \rho'; \nu m \rangle \langle \rho'; \nu m | \frac{1}{-\partial^2} | \lambda'; \nu m \rangle \langle \lambda'; \nu m | \delta m^2 | \rho; \nu m \rangle$$

$$= \sum_\nu \frac{d_\nu}{4\nu^3} \int_\rho \rho^3 \left(\delta m^2\right)^2 - \frac{1}{4} \int_\rho \rho^3 \left(\delta m^2\right)^2 + \frac{1}{2} \int_\rho \rho^3 \delta m^2 \int_0^1 \mathrm{d}x \frac{x^3}{1-x^2} \left(\delta m^2(x\rho) - \delta m^2\right)$$

✳ momentum space:

$$T_{2,0}^{\mathrm{DR}} = \int_x \int_p \int_y \int_q \langle x | \frac{1}{-\partial^2} | q \rangle \langle q | \delta m^2 | y \rangle \langle y | \frac{1}{-\partial^2} | p \rangle \langle p | \delta m^2 | x \rangle = \int_k \widetilde{\delta m^2}(k) \widetilde{\delta m^2}(-k) \mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{q^2 (k+q)^2}$$

$$= \frac{1}{16\pi^2} \int_k \widetilde{\delta m^2}(k) \widetilde{\delta m^2}(-k) \left(\frac{2}{\epsilon} + 2 + \ln \frac{\tilde{\mu}^2}{k^2} \right)$$

$$= \frac{1}{8} \int_\rho \rho^3 \left(\delta m^2\right)^2 \left(\frac{2}{\epsilon} + 2\gamma_E + 2 \ln \frac{\tilde{\mu}\rho}{2} \right) + \frac{1}{2} \int_\rho \rho^3 \delta m^2 \int_0^1 \mathrm{d}x \frac{x^3}{1-x^2} \left(\delta m^2(x\rho) - \delta m^2\right)$$