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Hamiltonian normal forms for the post-Newtonian binary problem

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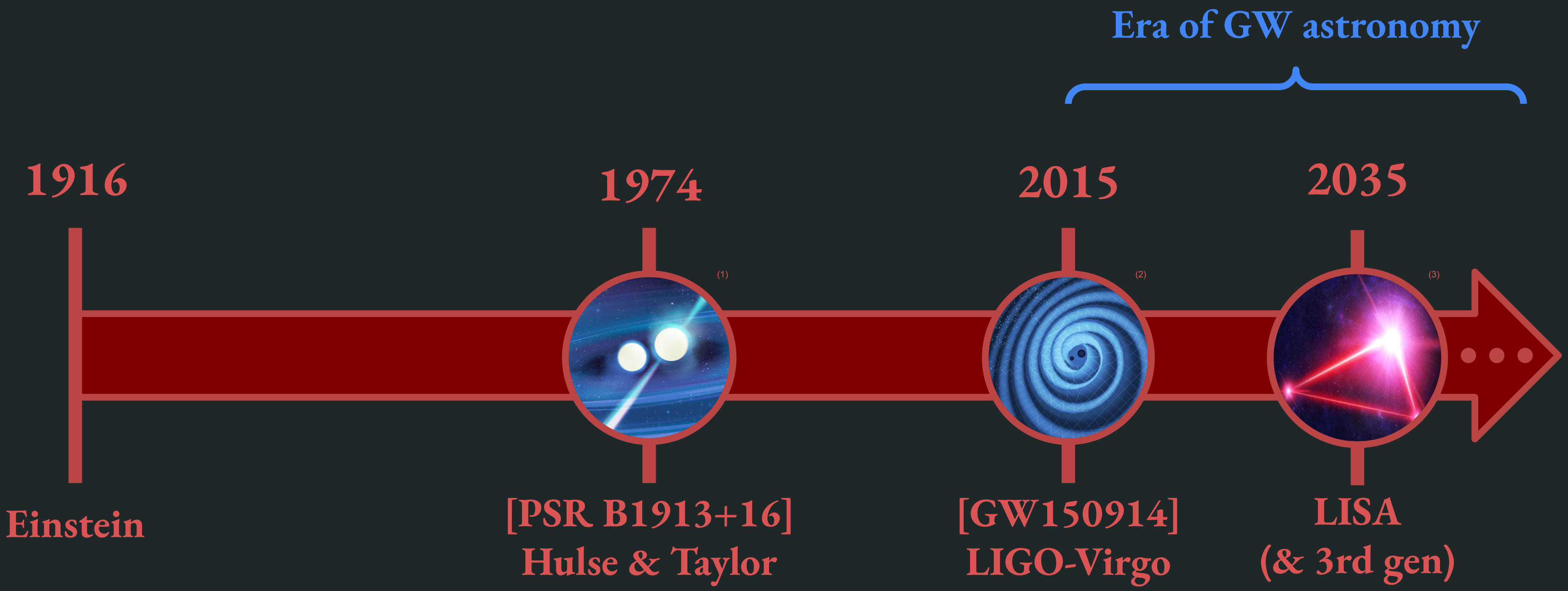
Supervised by C. Le Poncin-Lafitte & A. Bourgoin

CENTRE NATIONAL D'ÉTUDES SPATIALES

École Doctorale d'Astronomie & Astrophysique d'île-de-France

Context: post-Newtonian methods



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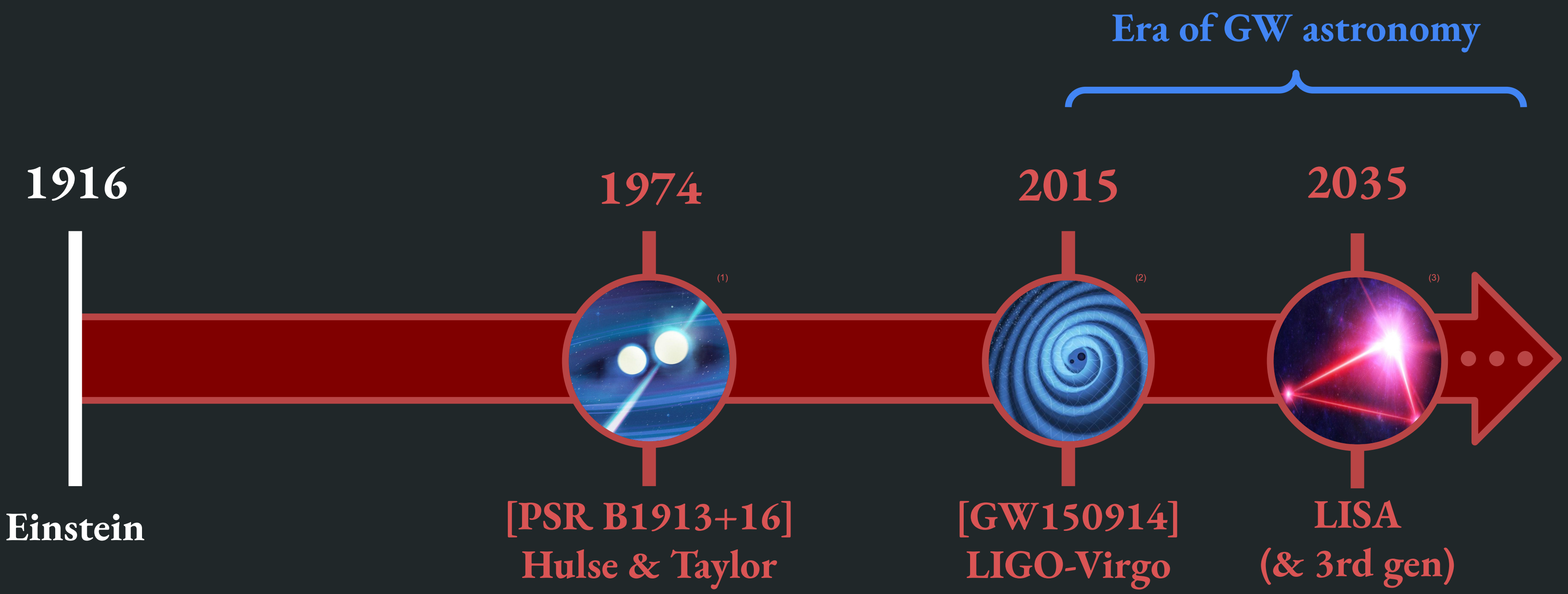

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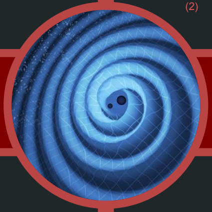
Era of GW astronomy

1916 — Einstein

1974 — [PSR B1913+16] Hulse & Taylor


(1)

2015 — [GW150914] LIGO-Virgo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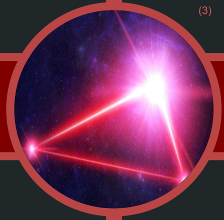
2035 — LISA (& 3rd gen)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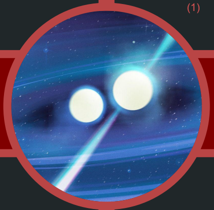

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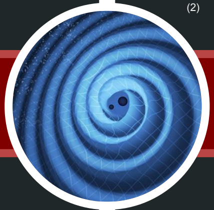
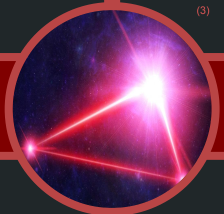


2035 (3) — LISA (& 3rd gen)



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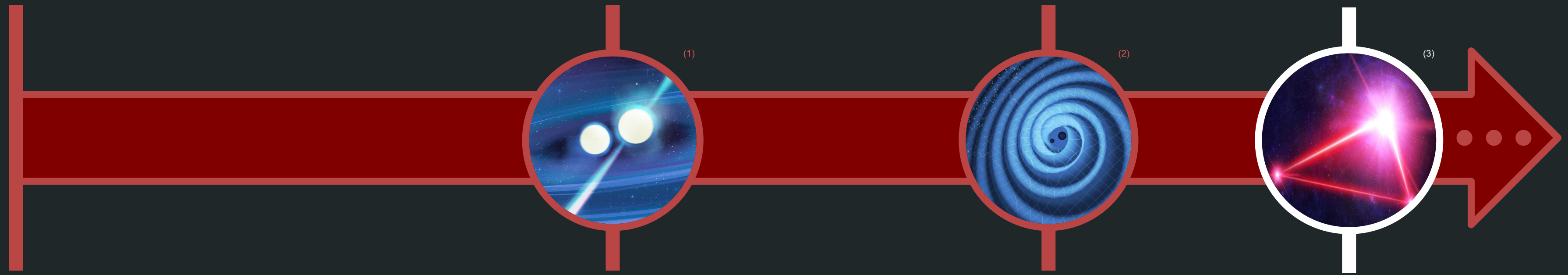

Einstein

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Laser Interferometer Space Antenna (LISA)

★ Space-based GW observatory

★ All-sky survey of millihertz GWs

★ 2.5 Mkm triangular constellation, heliocentric orbit, 20° Earth-trailing

Cosmic Vision	L3 Mission
Launch	~2035
Rocket	Ariane 6
Lifetime	4 yr (+6)
Cost:	>$1.5B

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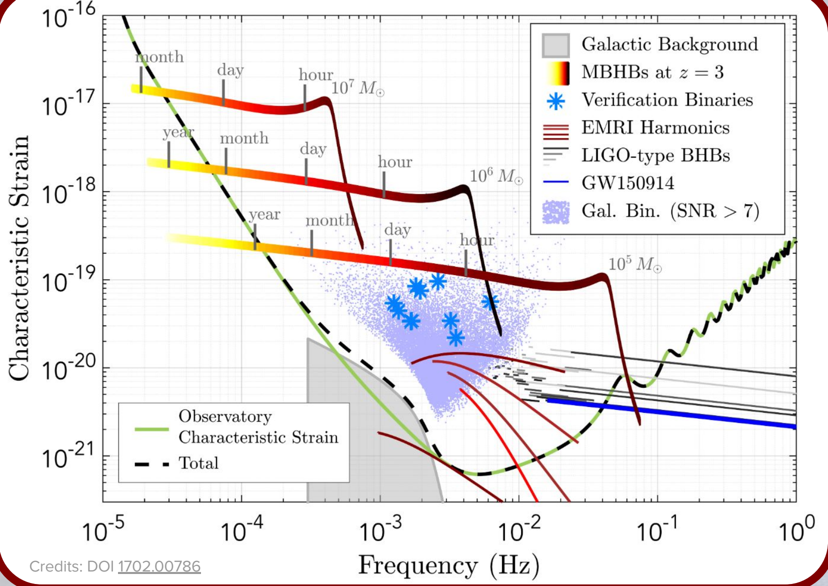

Credits: DOI 1702.00786

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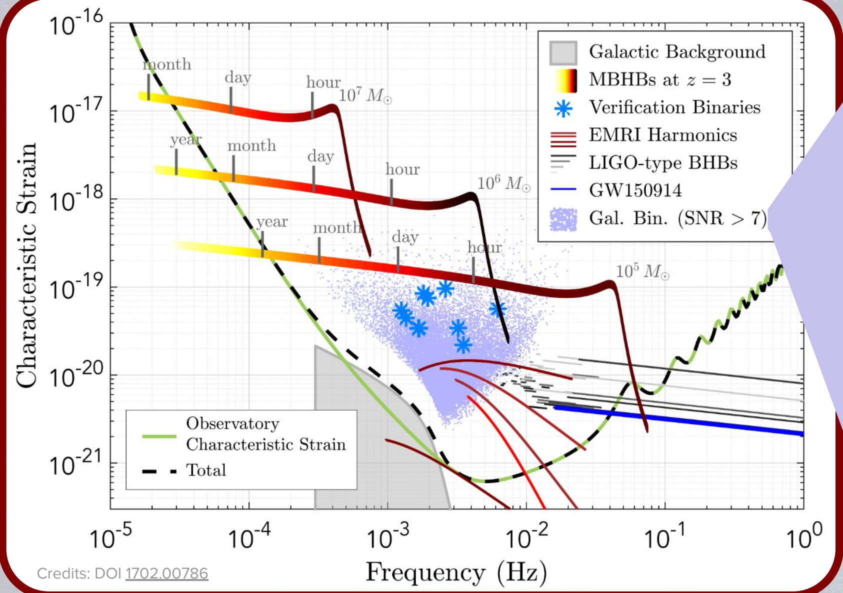

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Galactic binaries

- ★ Continuous, quasi-monochromatic GW sources, stable over > Myrs
- ★ Observation over 10^4–10^7 orbits
- ★ Simultaneous monitoring of tens of thousands of resolvable sources
- ★ Well-described by PN dynamics

Detector sensitive to a variety of physical effects, from eccentricity to spins, magnetic interactions and dissipation.

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(3)

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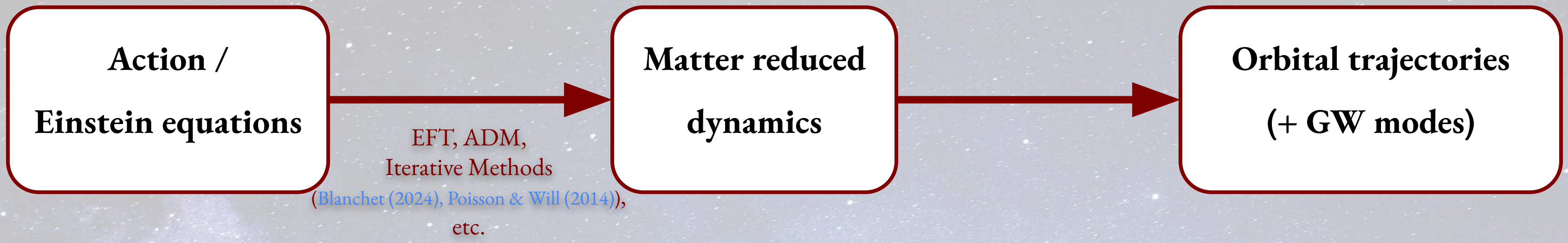
Post-Newtonian procedure



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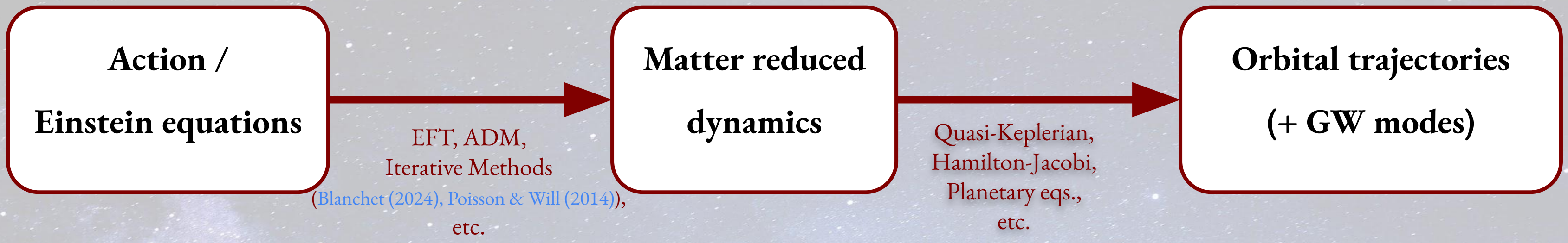

Post-Newtonian procedure

Action / Einstein equations → **Matter reduced dynamics** → **Orbital trajectories (+ GW modes)**

Quasi-Keplerian,
Hamilton-Jacobi,
Planetary eqs.,
etc.

Many of these methods are not available for general systems, or provide only 'secular' (orbit-averaged) solutions. However, the fast-timescale (periodic) behaviour may lead to additional GW harmonics.

Desired properties:

- ✓ Secular & periodic contributions;
- ✓ Valid in eccentric conditions;
- ✓ Can accommodate dissipative effects, spins, and non-gravitational perturbations;

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$\Longrightarrow$ Lie series approaches?

In GR, employed only at 1PN (Biscani and Carloni 2012).

Lie series framework

Hamiltonian formalism

Initial assumptions:

$\|\boldsymbol{r}\|$, $\|\boldsymbol{p}\|$ and $\boldsymbol{p}\cdot\boldsymbol{r}$

- Rotational invariance → e.g. spinless pure gravity
- Time independence → conservative sector
- Local → ADM at < 4PN

Example: non-spinning autonomous local matter-only Hamiltonian (in the CM frame):

$$\mathcal{H}(\boldsymbol{r},\boldsymbol{p}) = \mathcal{H}_0(\boldsymbol{r},\boldsymbol{p}) + \varepsilon\,\mathcal{H}_1(\boldsymbol{r},\boldsymbol{p}) + \varepsilon^2\mathcal{H}_2(\boldsymbol{r},\boldsymbol{p}) + \dots$$

- Leading term describes Newtonian gravity

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Original system

$$\mathcal{H} = \sum_{\ell=0}^{K} \varepsilon^\ell \mathcal{H}_\ell$$

↓

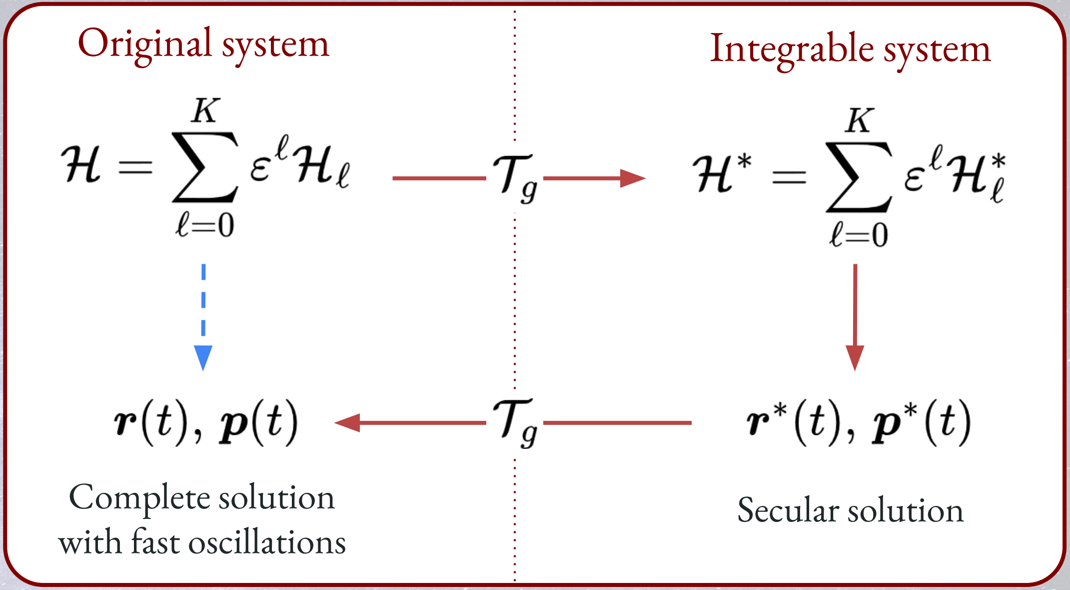
$$\boldsymbol{r}(t),\ \boldsymbol{p}(t)$$

Complete solution with fast oscillations

Lie series framework

Objectives: Determine a near-identity canonical transformation $\mathcal{T}_g$ under which the Hamiltonian becomes integrable (up to order K):

$$\mathcal{H}^* = \mathcal{T}_g(\mathcal{H}), \qquad \boldsymbol{r}^* = \mathcal{T}_g^{-1}(\boldsymbol{r}), \qquad \boldsymbol{p}^* = \mathcal{T}_g^{-1}(\boldsymbol{p})$$



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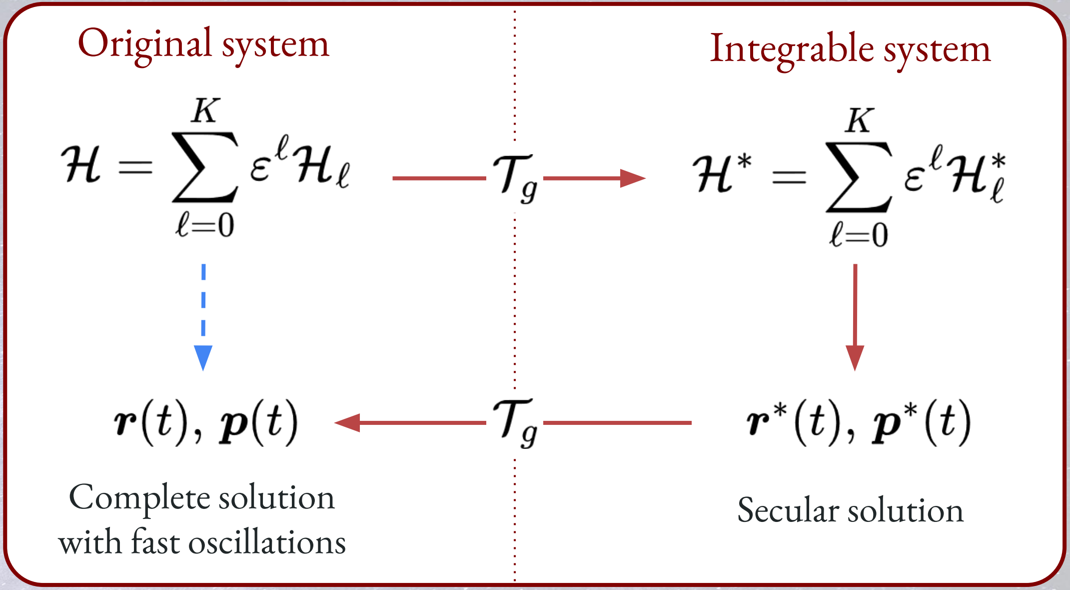
Such transformations form a Lie group generated by a smooth function g = O(ε):

$$\mathcal{T}_g(x) = e^{\{\,\cdot\,,g\}}(x)$$

$$= x + \varepsilon^2\{x,g\} + \frac{\varepsilon^4}{2}\{\{x,g\},g\} + \ldots$$

Expanding the new Hamiltonian in terms of the perturbation parameter and 'g' yields a sequence of equations of the form:

$$\{g_\ell, \mathcal{H}_0\} = F_\ell - \mathcal{H}^*_\ell, \quad \ell = 1,\ldots,K,$$



Solutions

Homologic equations:

- $\{g_\ell, \mathcal{H}_0\} = F_\ell - \mathcal{H}^*_\ell, \quad \ell = 1, 2, \ldots,$

Key steps:

- Computing the integrable Hamiltonian ('Birkhoff normal-form'):

$$\mathcal{H}^*_\ell(\boldsymbol{I}) = \langle F_\ell \rangle := \frac{1}{2\pi} \int_0^{2\pi} F_\ell(\theta, \boldsymbol{I}) \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^{-1}_{\text{kep}} \mathrm{d}\theta$$

- Computing the generator:

$$g_\ell = \int (F_\ell - \mathcal{H}^*_\ell) \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^{-1}_{\text{kep}} \mathrm{d}\theta \quad - \quad \langle \cdot \rangle$$

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Under initial assumptions:

⇒ General parametric solutions;

⇒ Application to 2PN dynamics;

(see Aykroyd et al. [arXiv] 2409.12204)

Further extensions?

- Spins (non-canonical degrees-of-freedom) — care must be taken to deal with resonances;
- Account for time-dependence — e.g. dissipation at 2.5PN (Aykroyd et al. in prep).

Extensions to non-conservative interactions ?

Non-conservative systems are often seen as incompatible with an action principle, needing to be treated at the level of the equations of motion.

But in fact several formalisms have evolved to address this limitation:

- Explicit time dependence / extended phase-space;
- Rayleigh dissipation function (see e.g. Virga 2015);
- Fractional derivatives;
- Contact Hamiltonian mechanics (e.g. Bravetti, Cruz, and Tapias 2017);
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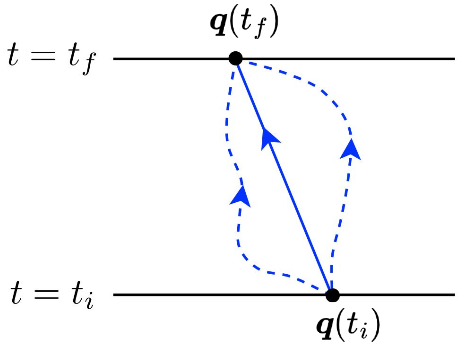

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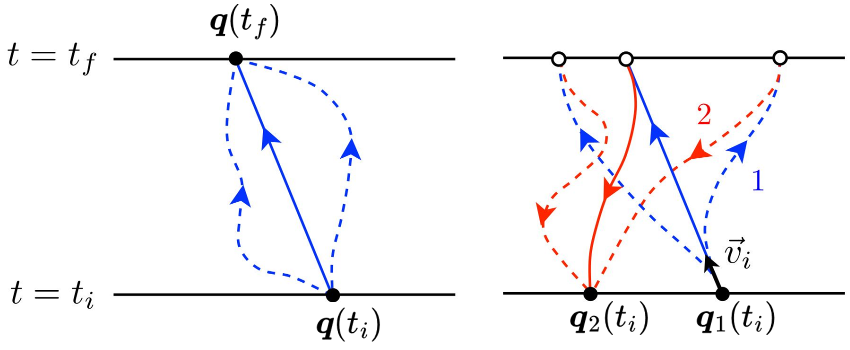

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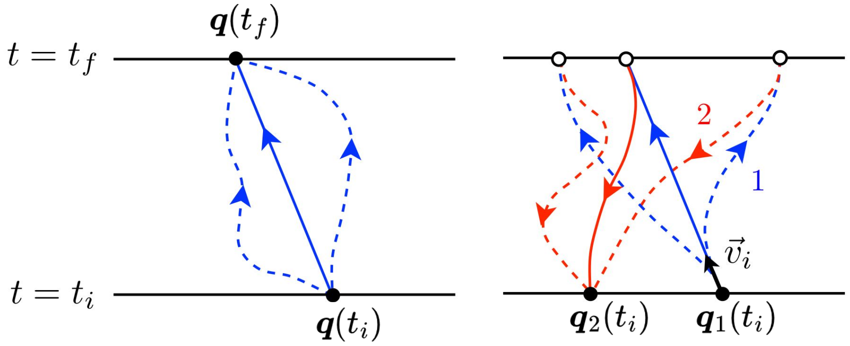

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Non-conservative Lie series (Aykroyd et al. in prep)

Non-conservative Hamiltonian: additional coupling term

- $\mathcal{A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) = \mathcal{H}(\boldsymbol{r}_1, \boldsymbol{p}_1) - \mathcal{H}(\boldsymbol{r}_2, \boldsymbol{p}_2) + \mathcal{K}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2)$

Non-conservative homologic equations:

- $\{g_\ell, \mathcal{A}_0\} = \mathscr{F}_\ell - \mathcal{A}_\ell^*$

will now depend on two fast variables:

$$g_\ell(\theta, \boldsymbol{I}) \to g_\ell(\theta_1, \theta_2, \boldsymbol{I}_1, \boldsymbol{I}_2)$$

$$F_\ell(\theta, \boldsymbol{I}) \to \mathscr{F}_\ell(\theta_1, \theta_2, \boldsymbol{I}_1, \boldsymbol{I}_2)$$

With the appropriate coordinate transformations, the non-conservative homological equation can be reduced to a set of six conservative homological equations:

$$\{g_\ell, \mathcal{A}_0\} = \mathscr{F}_\ell - \mathcal{A}_\ell^*$$

$$\Downarrow$$

$$\{g_{\ell,m}, \mathcal{H}_0\} = \mathscr{F}_{\ell,m} - \mathcal{A}_{\ell,m}^*$$

$$m \in \{1, \ldots, 6\}$$

Can be directly extracted from the eqs. of motion!
(Hamiltonian is not needed)

Thank you!

Questions?