Hadron structure with 3DPartons

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A "constructive" introduction

- Consider a generic **bi-local** quark operator (correlation between two space-time points):

$$\mathcal{O} = \overline{\psi}\,(b)\,\Gamma\psi\,(0)$$

- Γ is generic Dirac structure, *i.e.* a linear combination of $\{\mathbb{I}, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^{\mu\nu}\}$.

$$\Gamma = A\mathbb{I} + B\gamma^5 + C_\mu\gamma^\mu + D_\mu\gamma^5\gamma^\mu + E_{\mu\nu}\sigma^{\mu\nu}$$

- In order to give $\mathcal{O}$ a physical meaning, we need to make it gauge invariant.
    - Introduce the parallel-transport operator W (often called **Wilson line** in this context):

$$W(y,x) = \mathcal{P}\exp\left[-igt^a \int_y^x dz^\mu A^a_\mu(z)\right]$$

- The gauge invariant version of $\mathcal{O}$ is then:

$$\mathcal{O} = \overline{\psi}\,(b)\,\Gamma\,W(b,0)\psi\,(0)$$

- Now consider the case in which $\mathcal{O}$ is **highly boosted** along $-z$ (as if it was involved in a high-energy collision): this frame is called Breit (or infinite-momentum) frame.
- Working in the Breit frame has two main important consequences:
    - $b_z \simeq -cb_t$, therefore in light-cone coordinates $b \simeq (0, b^-, \mathbf{b}_T)$.
    - The coefficients $\{A, B, C_\mu, D_\mu, E_{\mu\nu}\}$ get enhanced, unchanged, or suppressed:
        - C_+, D_+, E_{+i} enhanced (twist 2), $A, B, C_i, D_i, E_{ij}, E_{+-}$ unchanged (twist 3), C_-, D_-, E_{-i} suppressed (twist 4).

A "constructive" introduction

- A particularly interesting operator is the "unpolarised" one:

$$\mathcal{O} = \overline{\psi}(b)\,\gamma^+\,W(b,0)\psi(0)\Big|_{b^+=0}$$

(in fact, also the others are interesting but I will focus on this one.)

- To connect this operator to an observable we need to take a matrix element.
- We bracket it between two, generally different hadronic states:

$$\mathcal{M} = \langle H'(p',\lambda')|\overline{\psi}(b)\,\gamma^+\,W(b,0)\psi(0)\,|H(p,\lambda)\rangle\Big|_{b^+=0}$$

- Finally, it is usually phenomenologically more relevant to study the **momentum** behaviour of any such matrix element. We thus take its Fourier transform:

$$\Phi = \int db^- d^2\mathbf{b}_T\, e^{ib^-k^+ - i\mathbf{b}_T\cdot\mathbf{k}_T}\,\langle H'(p',\lambda')|\overline{\psi}(b)\,\gamma^+\,W(b,0)\psi(0)\,|H(p,\lambda)\rangle\Big|_{b^+=0}$$

- This is a (sketchy) definition of **generalised transverse-momentum dependent** (GTMD) correlator.
- GTMDs can be regarded as "**mother distributions**" (*cit.* Meißner, Metz, Schlegel [*JHEP* 08 (2009) 056]).
- They encode "the most general one-body information of partons, corresponding to the full one-quark density matrix in momentum space" (*cit.* Lorcé, Parquini, Vanderhaeghen [*JHEP* 05 (2011) 041])).

Further readings: Ji [*Phys.Rev.Lett.* 91 (2003) 062001], Belitsky, Ji, Yuan [*Phys.Rev.D* 69 (2004) 074014], Belitsky, Radyushkin [*Phys.Rept.* 418 (2005) 1-387]

The general picture

- All relevant hadronic distributions in high-energy physics can be made descend from **GTMDs**. Defining:

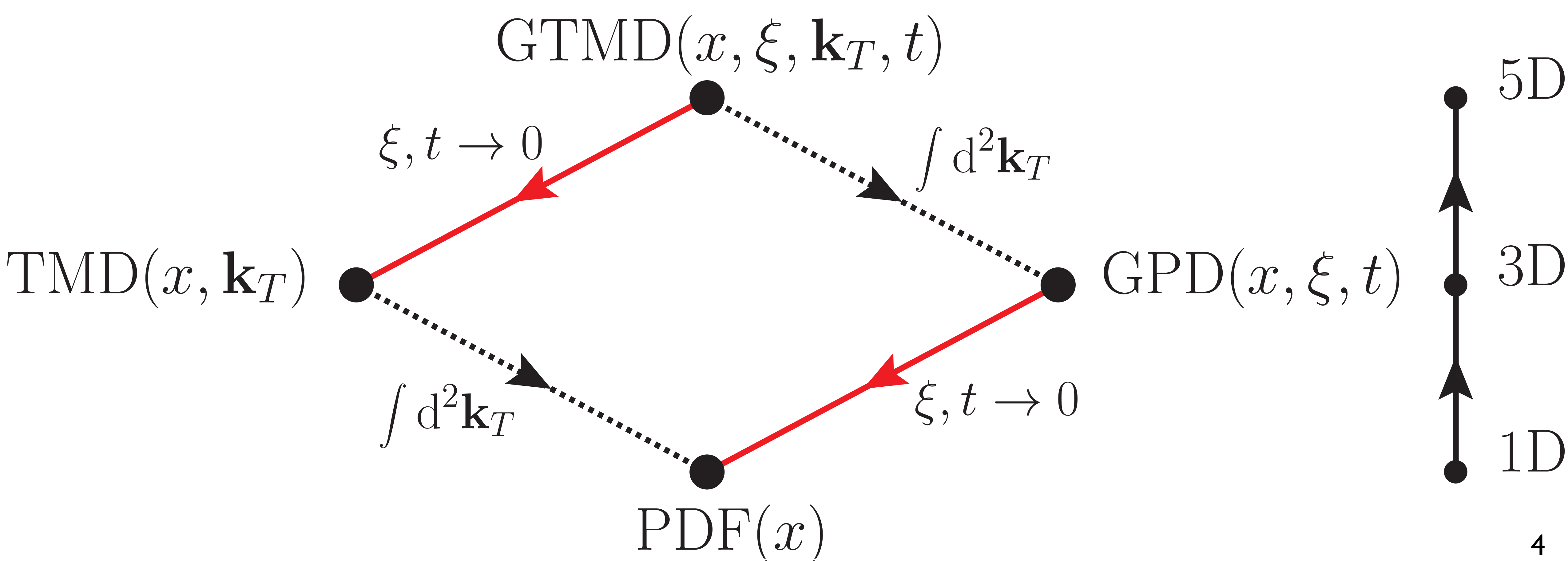
$$P = \frac{p+p'}{2} \qquad \Delta \equiv p - p'$$

p momentum of the incoming hadron
p' momentum of the outgoing hadron
k momentum of the parton

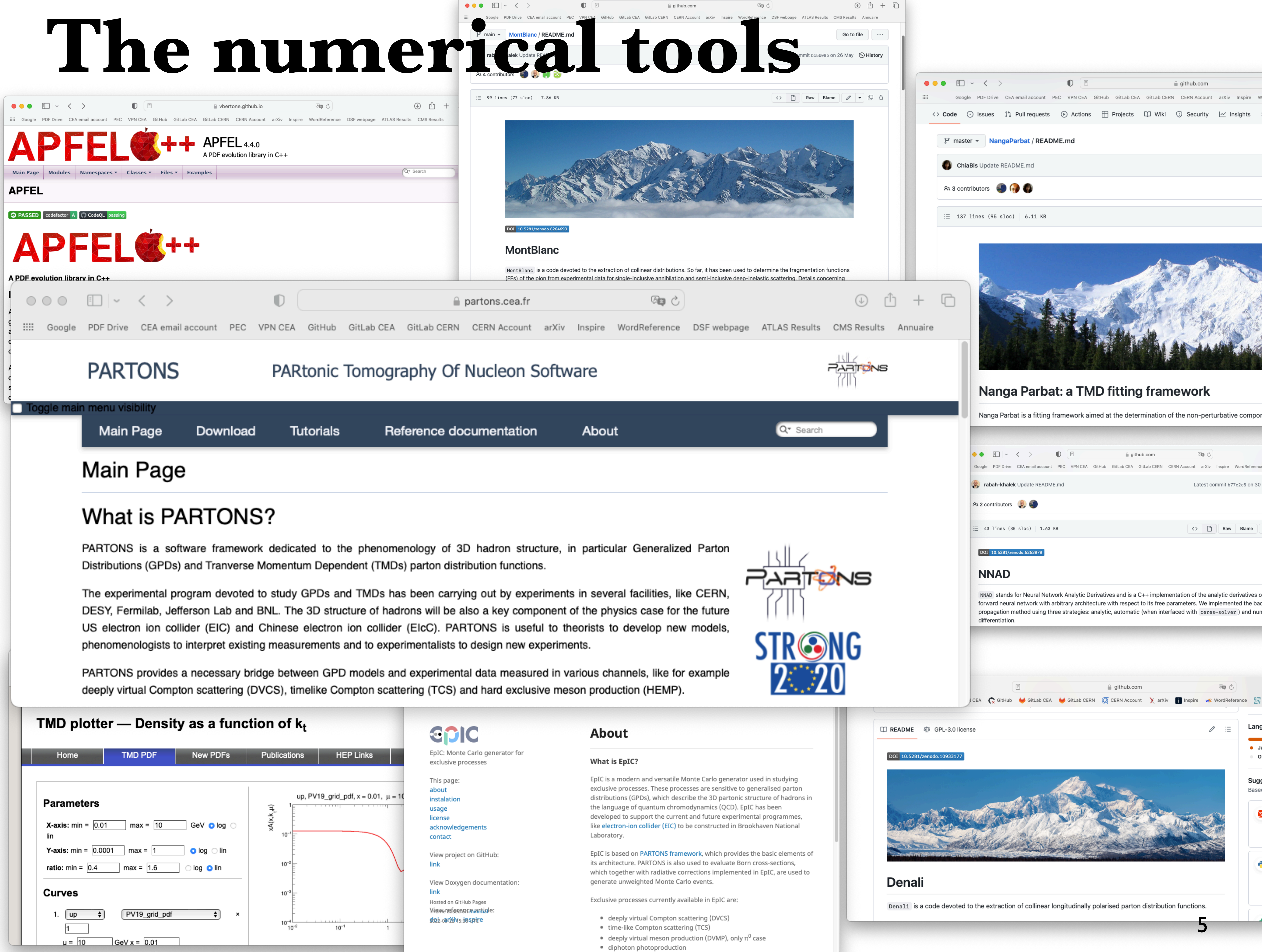
- A common set of kinematic variables used to parameterise GTMDs is:

$$k^+ \equiv xP^+ \qquad \Delta^+ \equiv -2\xi P^+ \qquad t = \Delta^2 \qquad \mathbf{k}_T$$

- A (partial) genealogy of GTMDs looks like this:

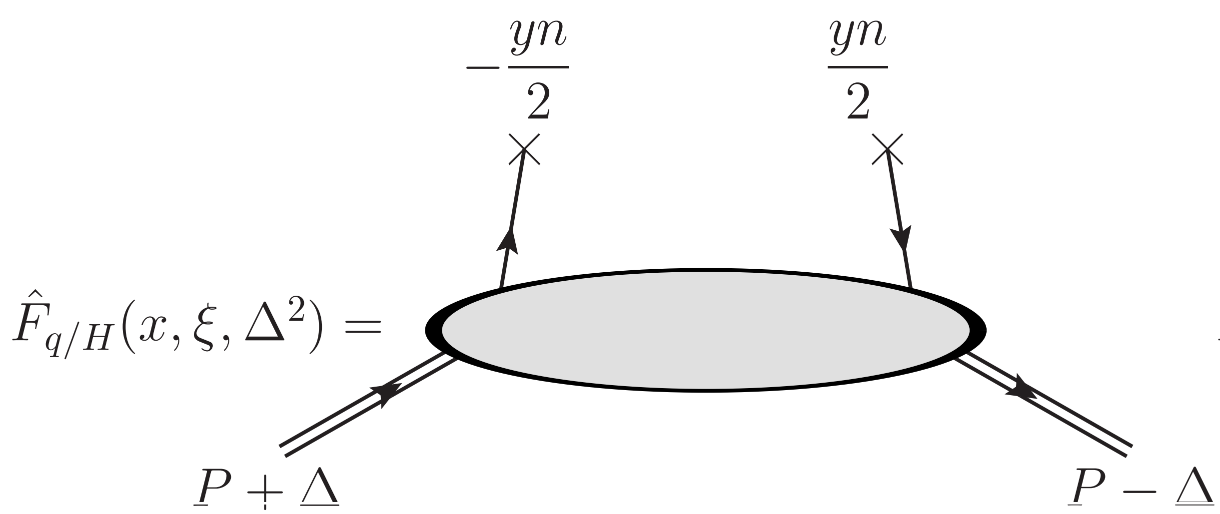


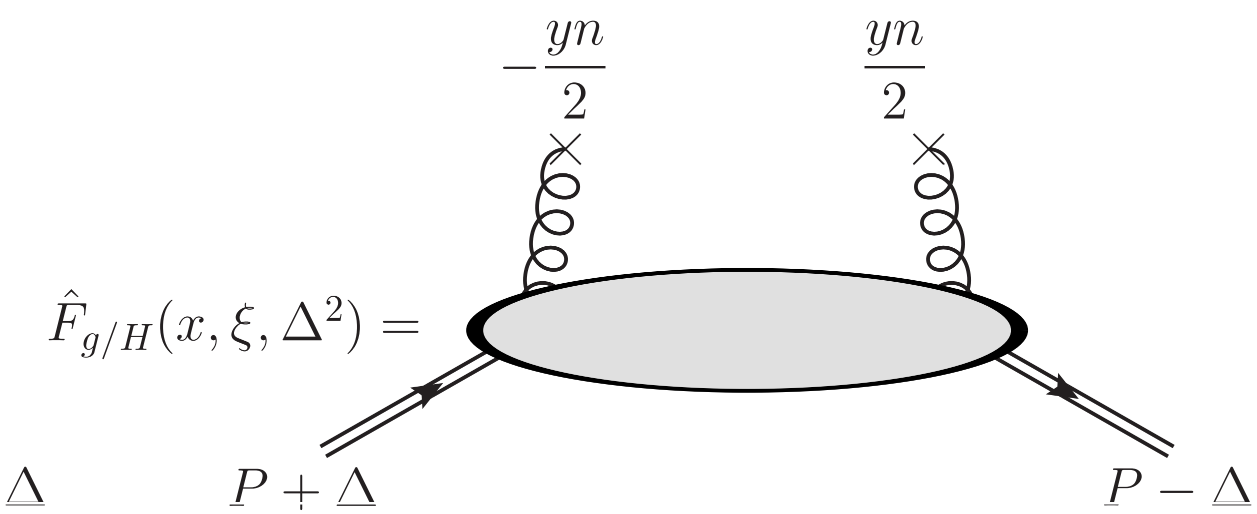
The numerical tools



GPD evolution

- GPDs admit and operator definition that in **light-cone gauge** ($n \cdot A = 0$) reads:





- UV divergences arise that need to be renormalised leading to evolution equations:

$$\frac{dF^{\pm}(x,\xi,\mu)}{d\ln\mu^2} = \int_x^{\infty} \frac{dy}{y} \mathcal{P}^{\pm}\left(y, \frac{\xi}{x}\right) F^{\pm}\left(\frac{x}{y}, \xi, \mu\right)$$

$$\mathcal{P}^{\pm}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{\pm}(y,k) + \theta(\kappa-1)\mathcal{P}_2^{\pm}(y,k)$$

$$F^{-} = F_{q/H} - F_{\overline{q}/H}$$

$$F^{+} = \begin{pmatrix} \sum_{q=1}^{n_f} F_{q/H} + F_{\overline{q}/H} \\ F_{g/H} \end{pmatrix}$$

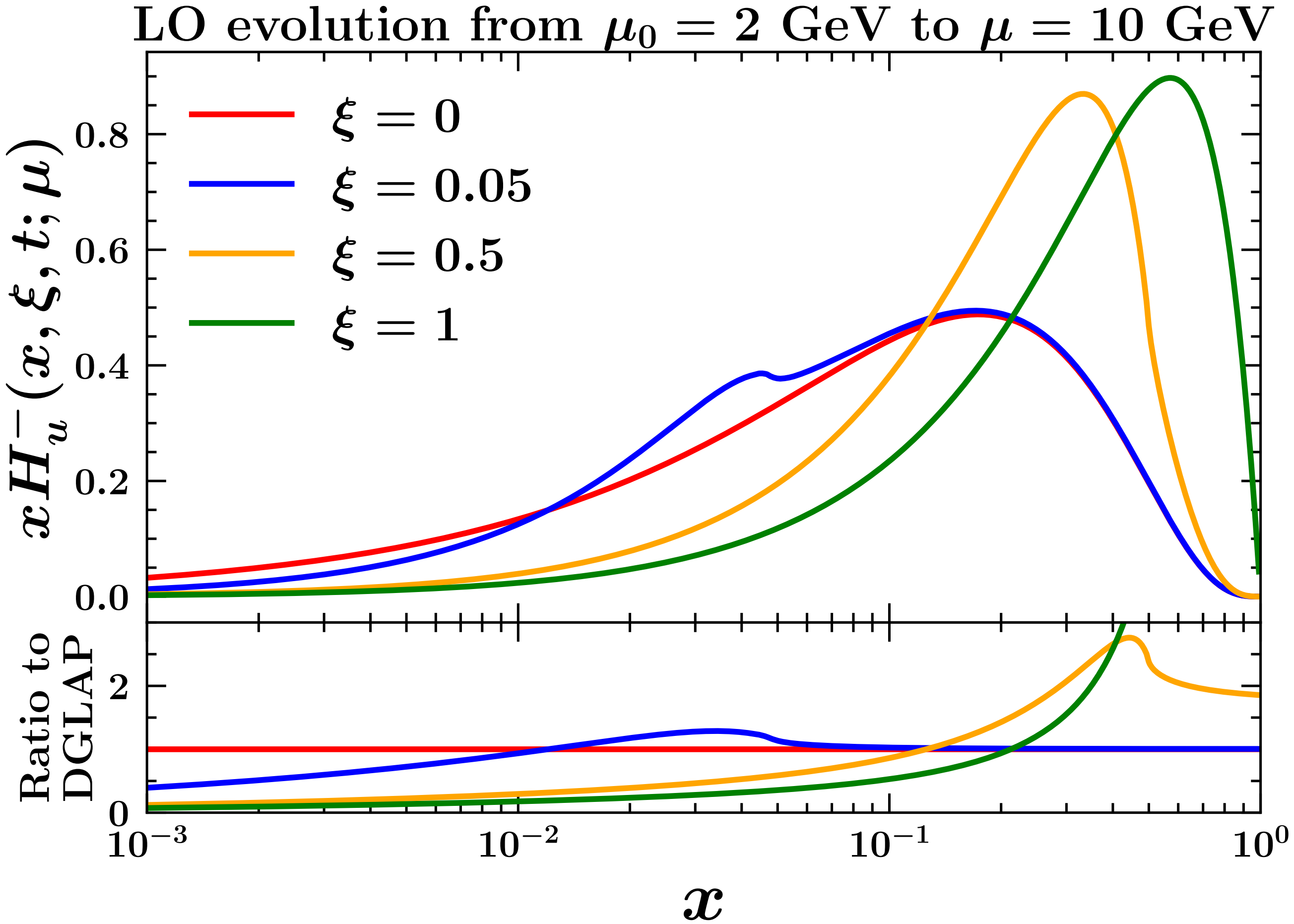
- In [Eur.Phys.J.C 82 (2022) 10, 888] and [Phys.Rev.D 109 (2024) 3, 034023] we have (re)computed the one-loop evolution kernels $\mathscr{P}^{\pm}$ for all of the twist-2 GPDs:

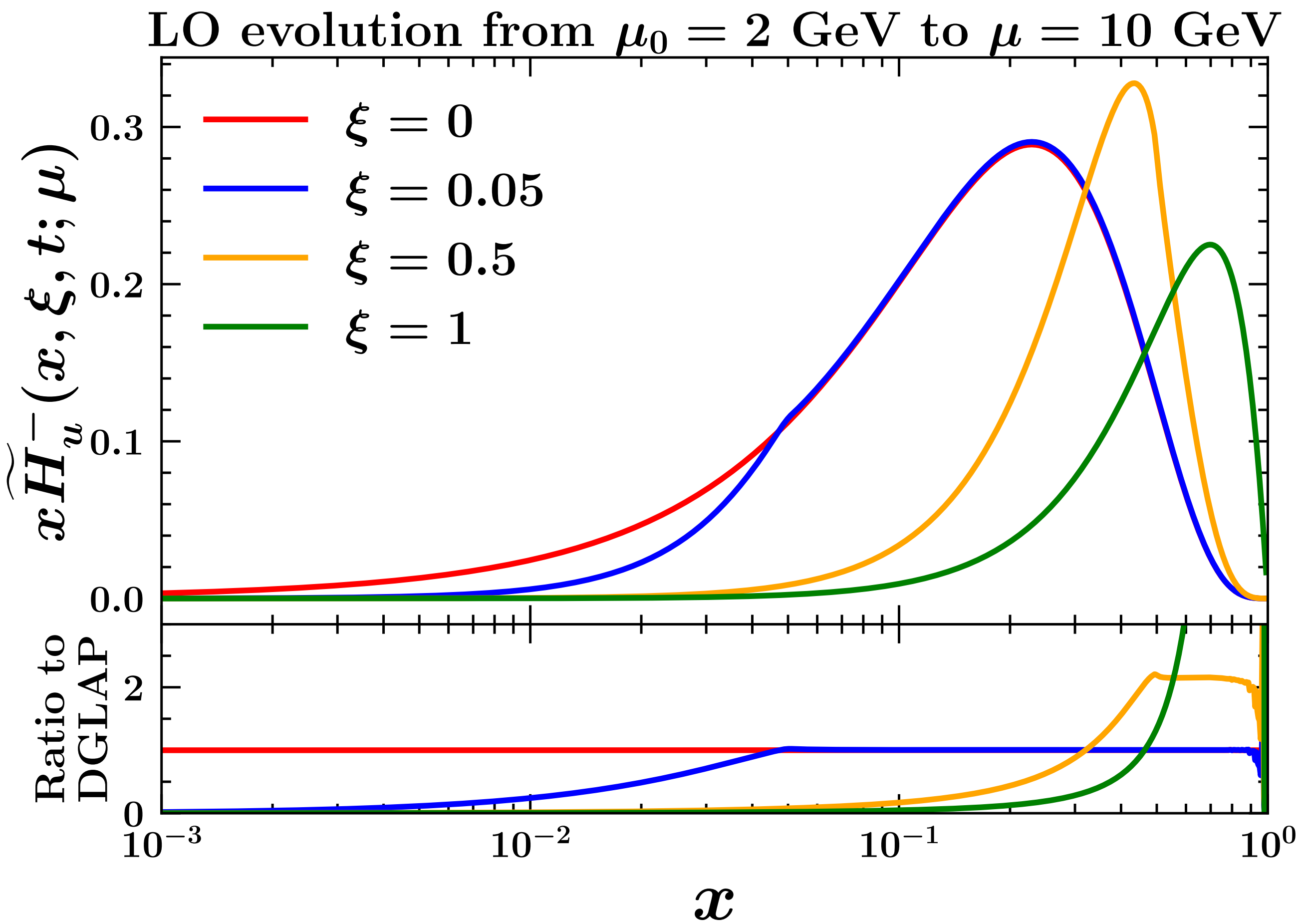
e.g.

$$\begin{cases} \mathcal{P}_1^{-,[0]}(y,\kappa) &= 2C_F\left\{\left(\frac{2}{1-y}\right)_+ - \frac{1+y}{1-\kappa^2y^2} + \delta(1-y)\left[\frac{3}{2} - \ln\left(|1-\kappa^2|\right)\right]\right\}, \\ \mathcal{P}_2^{-,[0]}(y,\kappa) &= 2C_F\left[\frac{1+(1+\kappa)y+(1+\kappa-\kappa^2)y^2}{(1+y)(1-\kappa^2y^2)} - \left(\frac{1}{1-y}\right)_{++}\right], \end{cases}$$

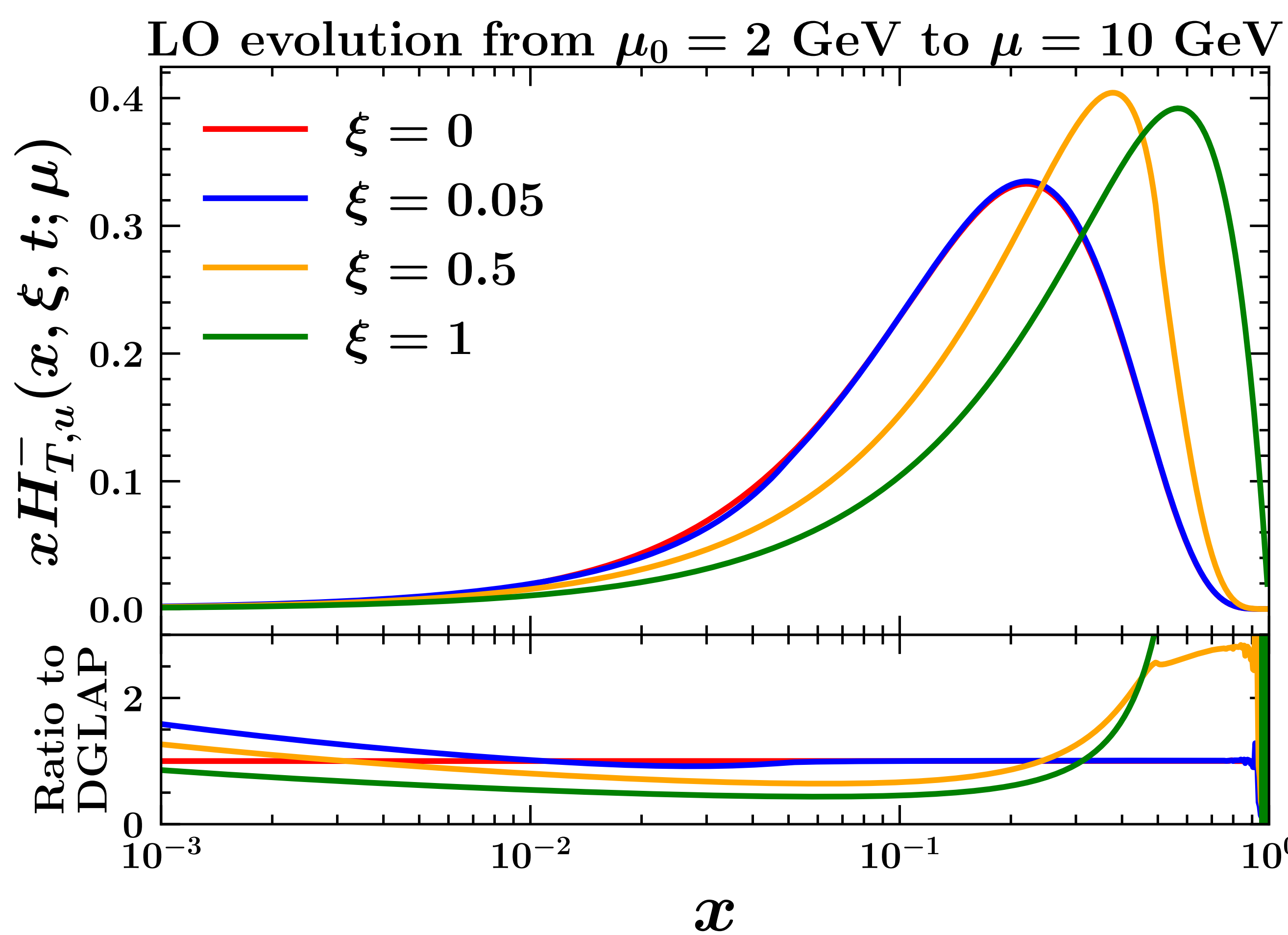
- …and provided a public implementation in **PARTONS** through **APFEL++**.

GPD evolution

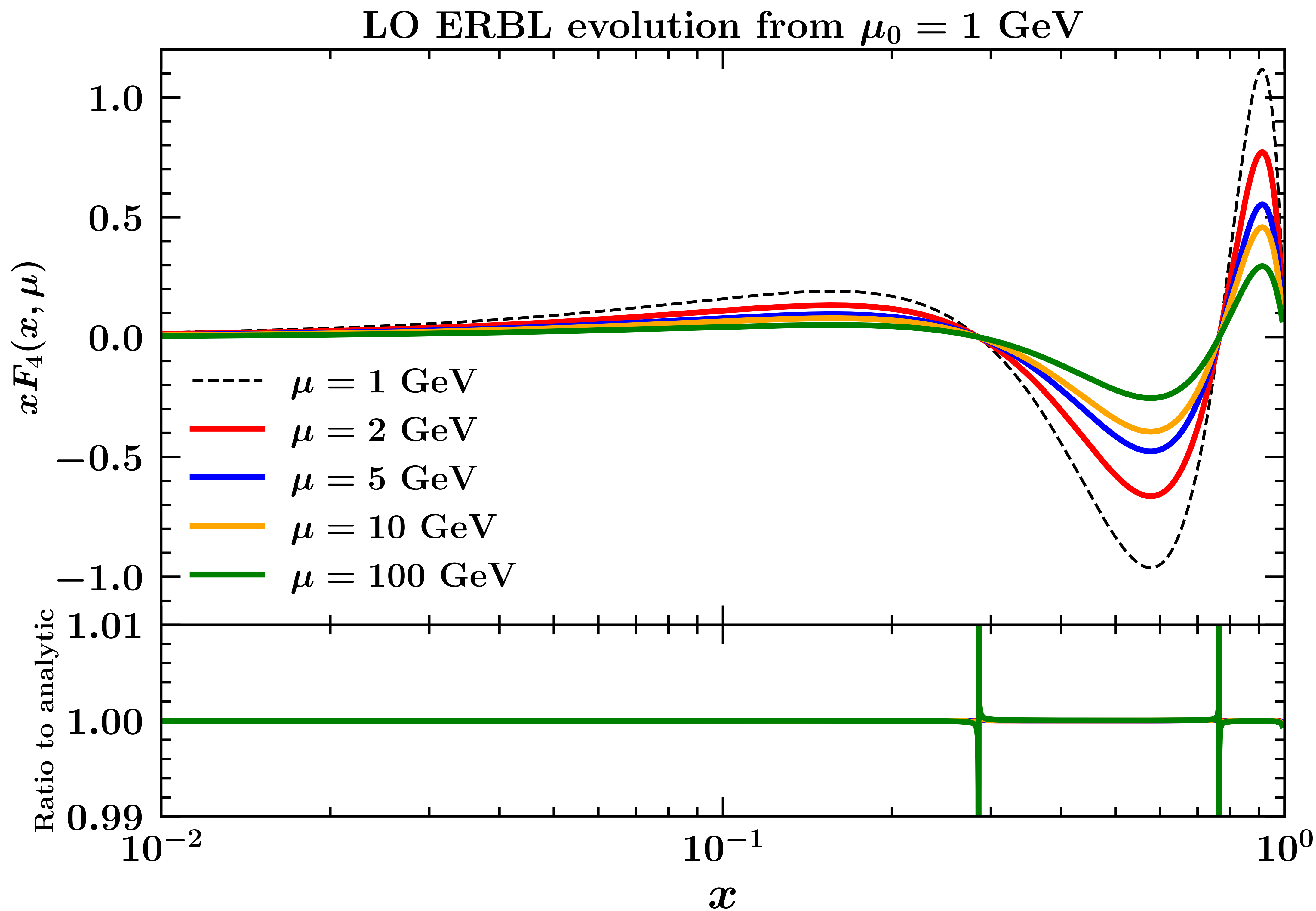




- **DGLAP limit** reproduced within 10^{-5} relative accuracy.
- GPD evolution may significantly deviate from DGLAP evolution.
- The evolution generates a cusp at $x = \xi$ but the distribution remains **continuous** at this point.

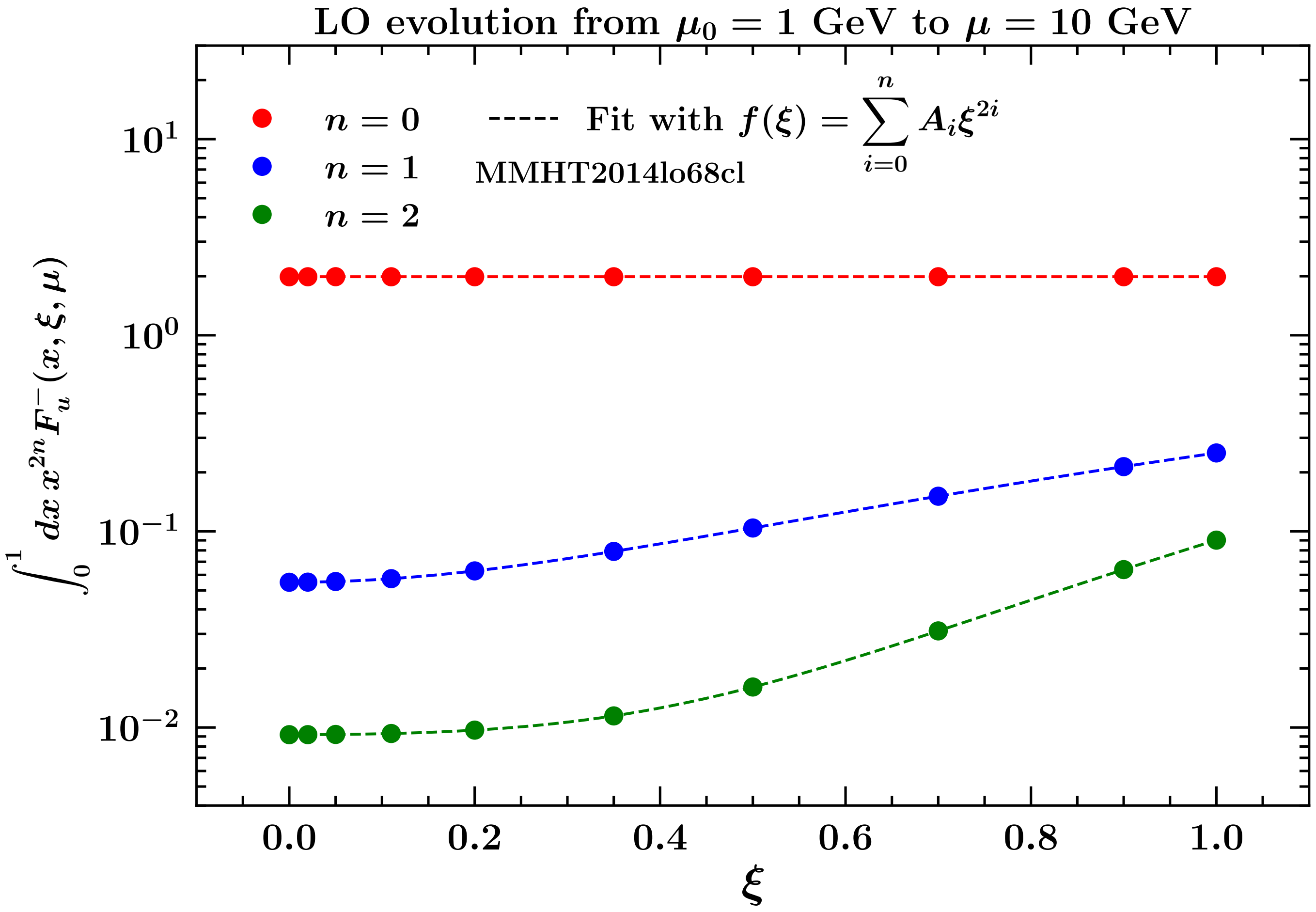


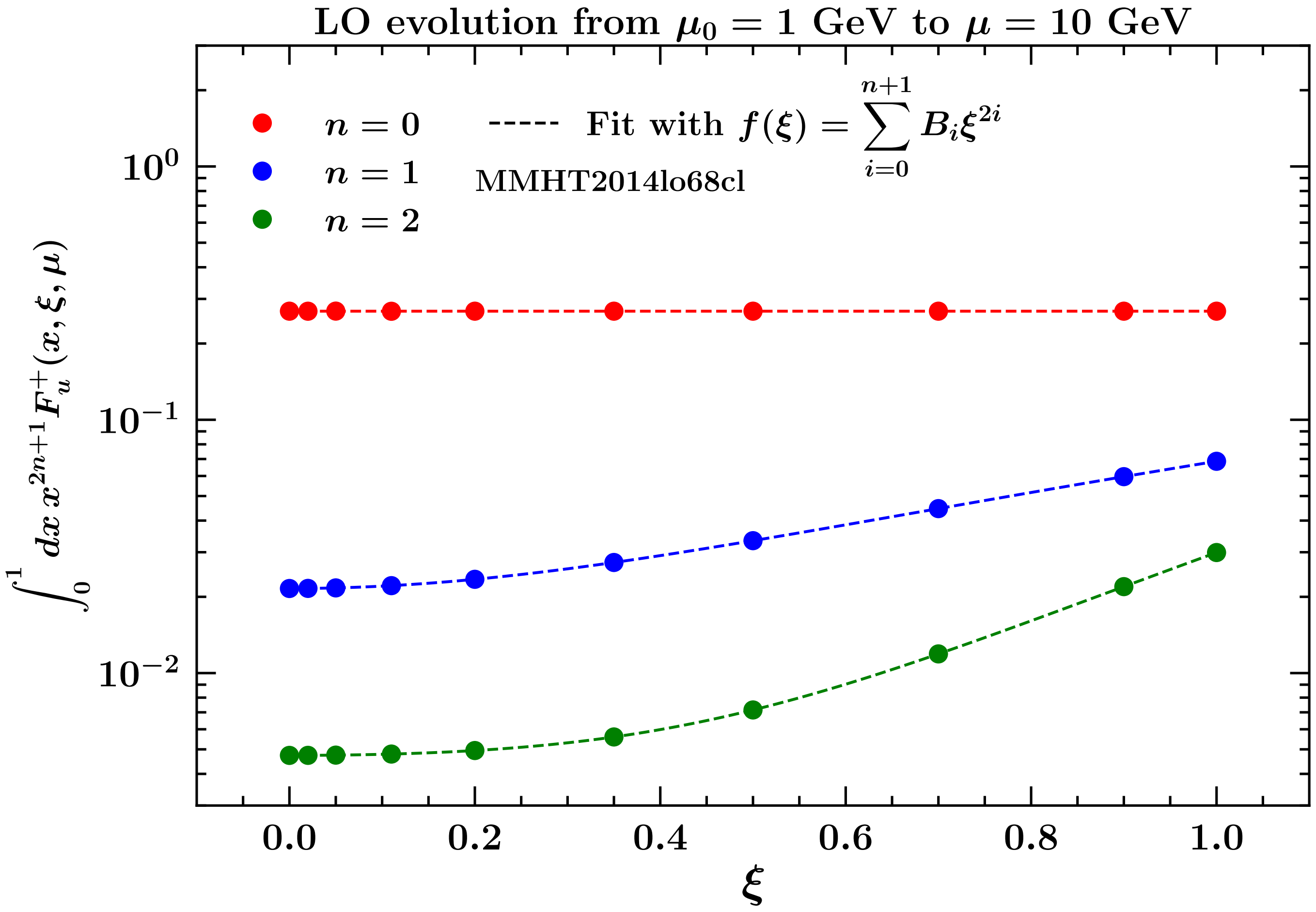
GPD evolution



- **ERBL limit** reproduced within less than 10^{-5} relative accuracy,
- Same accuracy for **higher-degree** Gegenbauer polynomials.

GPD evolution





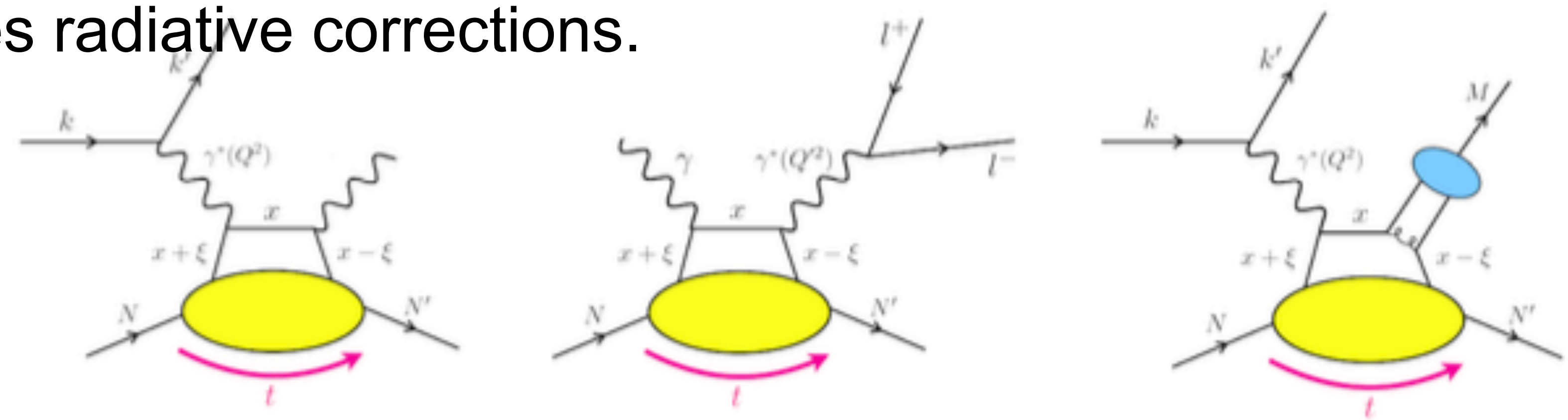
- **First moment** for both singlet and non-singlet is **constant** in ξ:
  - this was expected and the expectation is very nicely fulfilled.
- **Second and third moments** follow the expected power laws:
  - including odd-power terms in the fit gives coefficients very close to zero.

EpIC generator

[*Eur.Phys.J.C* 82 (2022) 9, 819]

EpIC

- EpIC: an event generator for exclusive reactions
- EpIC uses the PARTONS framework [B. Berthou et al., Eur.Phys.J. C78 (2018)]: takes advantage of
  - multiple GPD models that already exist
  - flexibility for adding new models [see H. Moutarde's talk]
- Multiple channels: DVCS, TCS, DVMP (pseudoscalar mesons)
- Includes radiative corrections.



- Written in C++
- XML interface for automated tasks
- Open-source [https://pawelsznajder.github.io/epic]

Kemal Tezgin (BNL) | EpIC | 26 October 2022 | 2/22

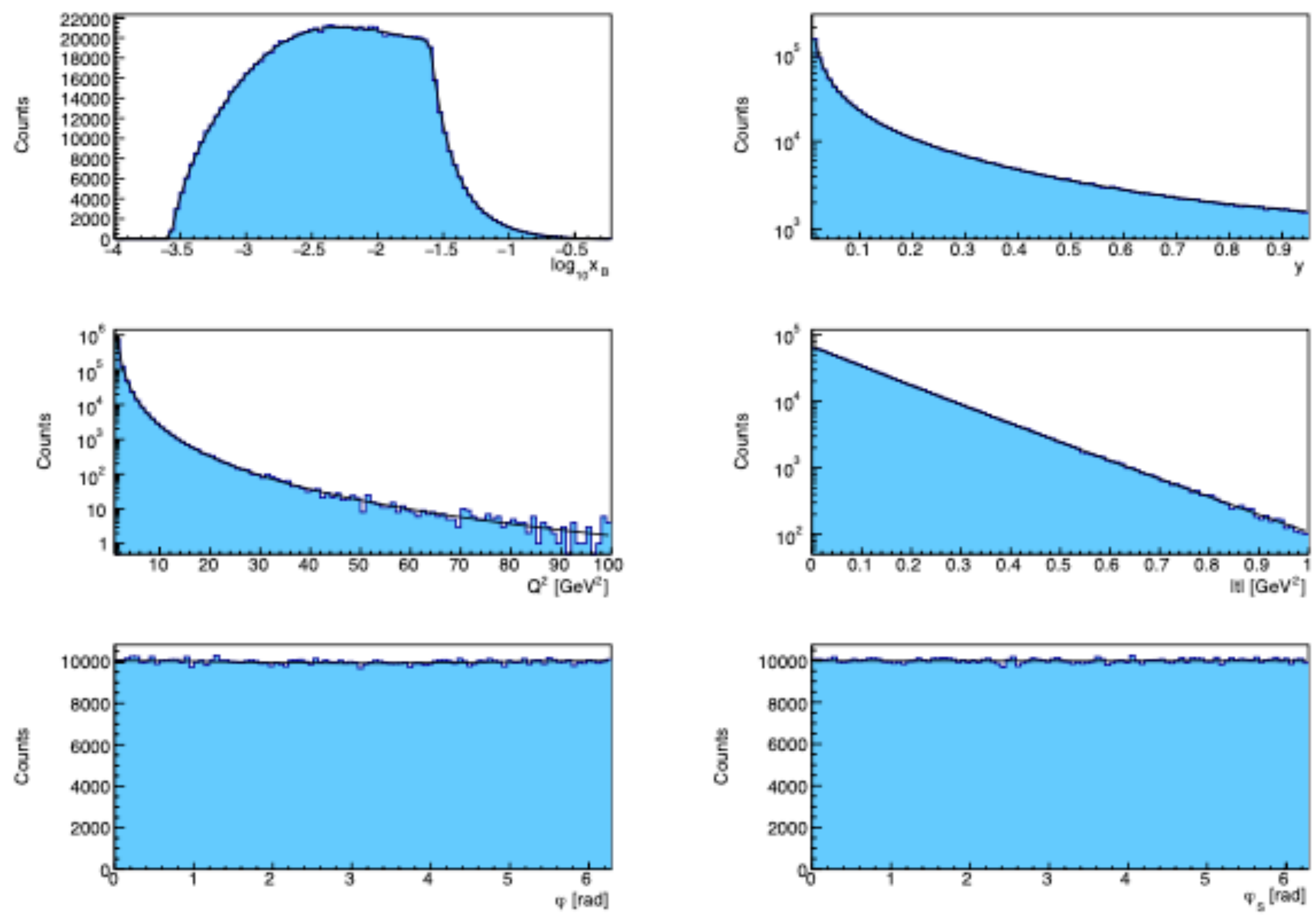
Credit: Kemal Tezgin

EpIC generator

[*Eur.Phys.J.C* 82 (2022) 9, 819]

EpIC – DVCS

Unpolarized target, $E_e = 10$ GeV, $E_p = 100$ GeV (DVCSProcessBMJ12 & GK GPDs)

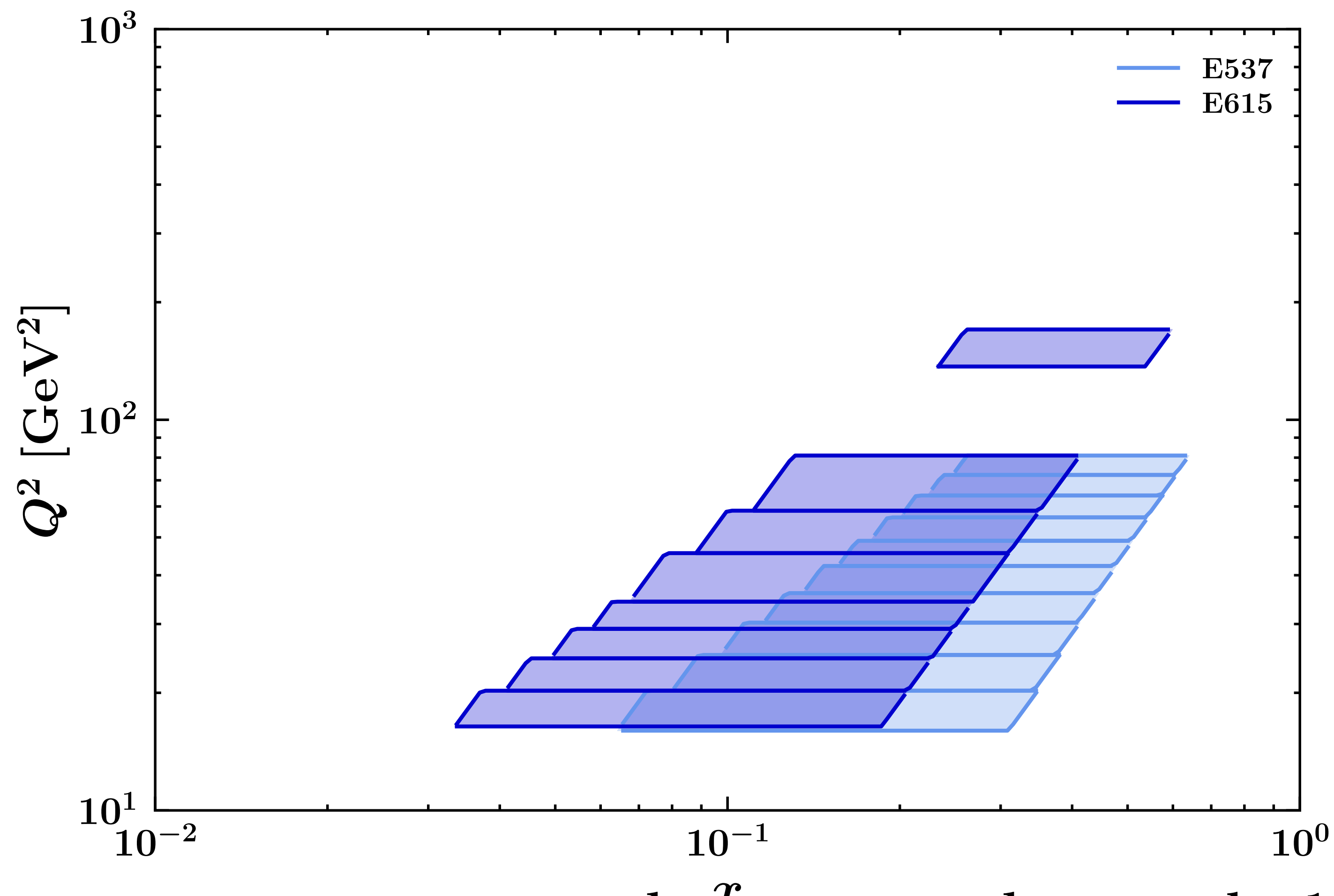


$0.0001 \leq x_B \leq 0.6$, $0.01 \leq y \leq 0.95$, $1 \leq Q^2 \leq 100$ GeV^2, $0 \leq |t| \leq 1$ GeV^2

Credit: Kemal Tezgin

An extraction of pion TMDs

- **NangaParbat** used to extract pion TMDs and published the result in [*Phys.Rev.D* 107 (2023)].

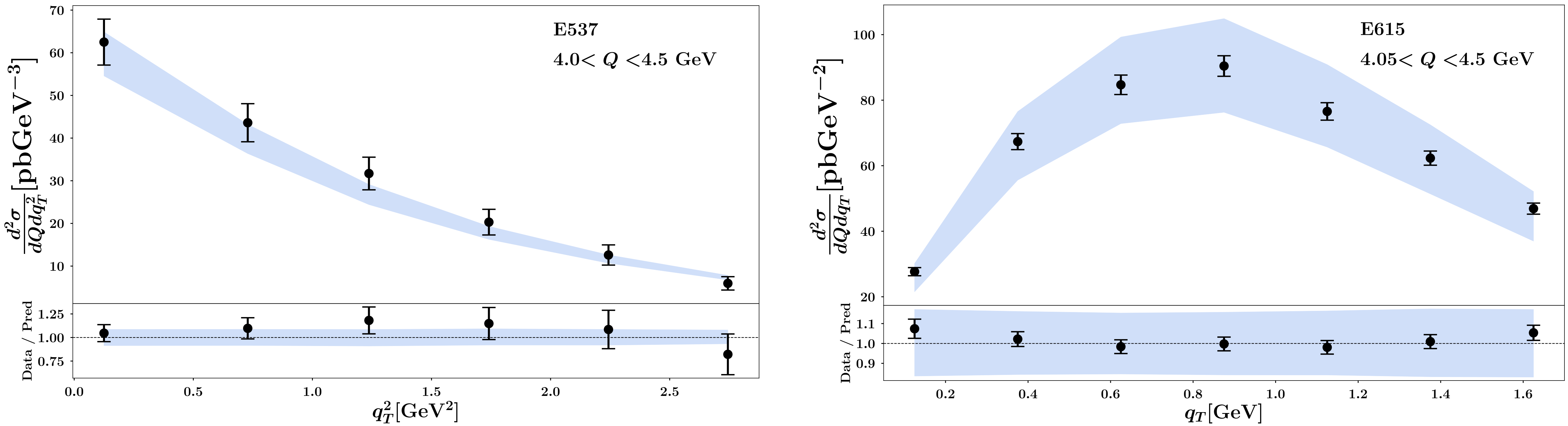


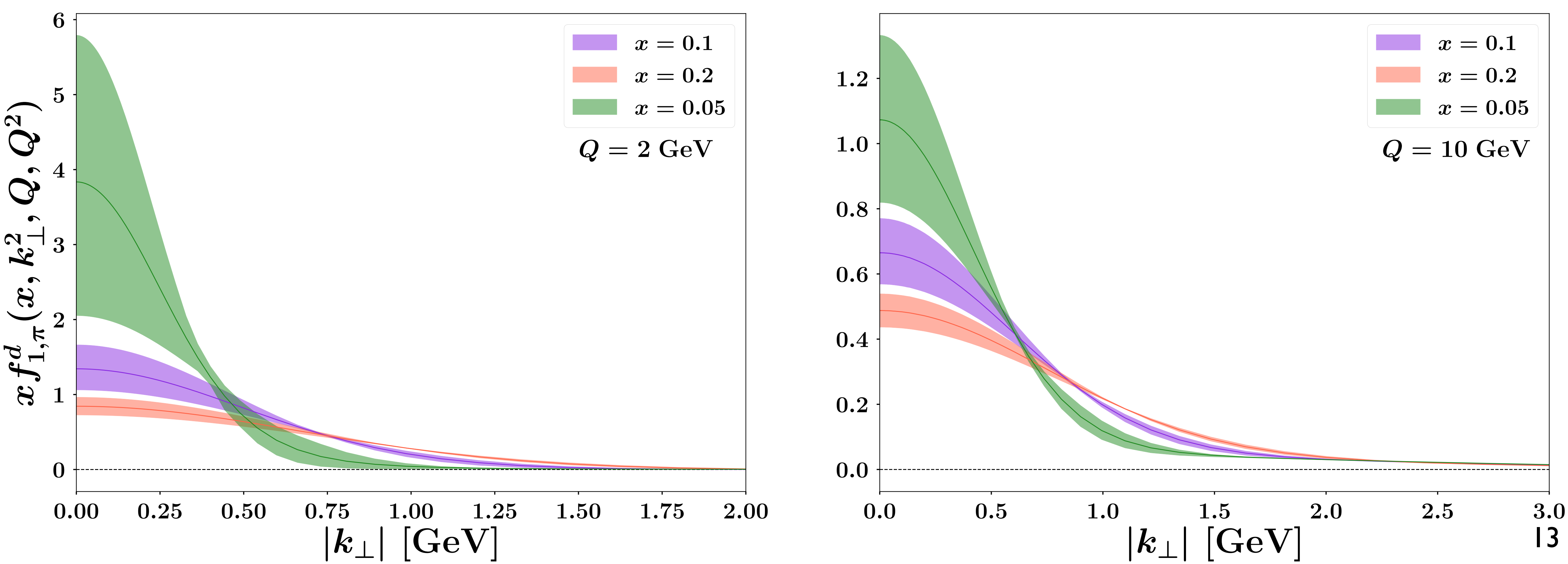
- Moderate data coverage as compared to proton data: only 138 data points.
    - Old fixed target data sets from FNAL.
    - Can the **EIC** help with it?
- Simple functional form for $f_{\rm NP}$:

$$f^{\pi}_{\rm NP}(x, b_T, Q) = g_{1\pi}(x)\, e^{-g_{1C}(x)\frac{b_T^2}{4}} \left(\frac{Q}{Q_0^2}\right)^{\frac{g_K(b_T)}{2}} \qquad g_{1\pi}(x) = N_{1\pi}\frac{x^{\sigma_\pi}(1-x)^{\alpha_\pi^2}}{\hat{x}^{\sigma_\pi}(1-\hat{x})^{\alpha_\pi^2}}$$

- Only 3 free parameters.

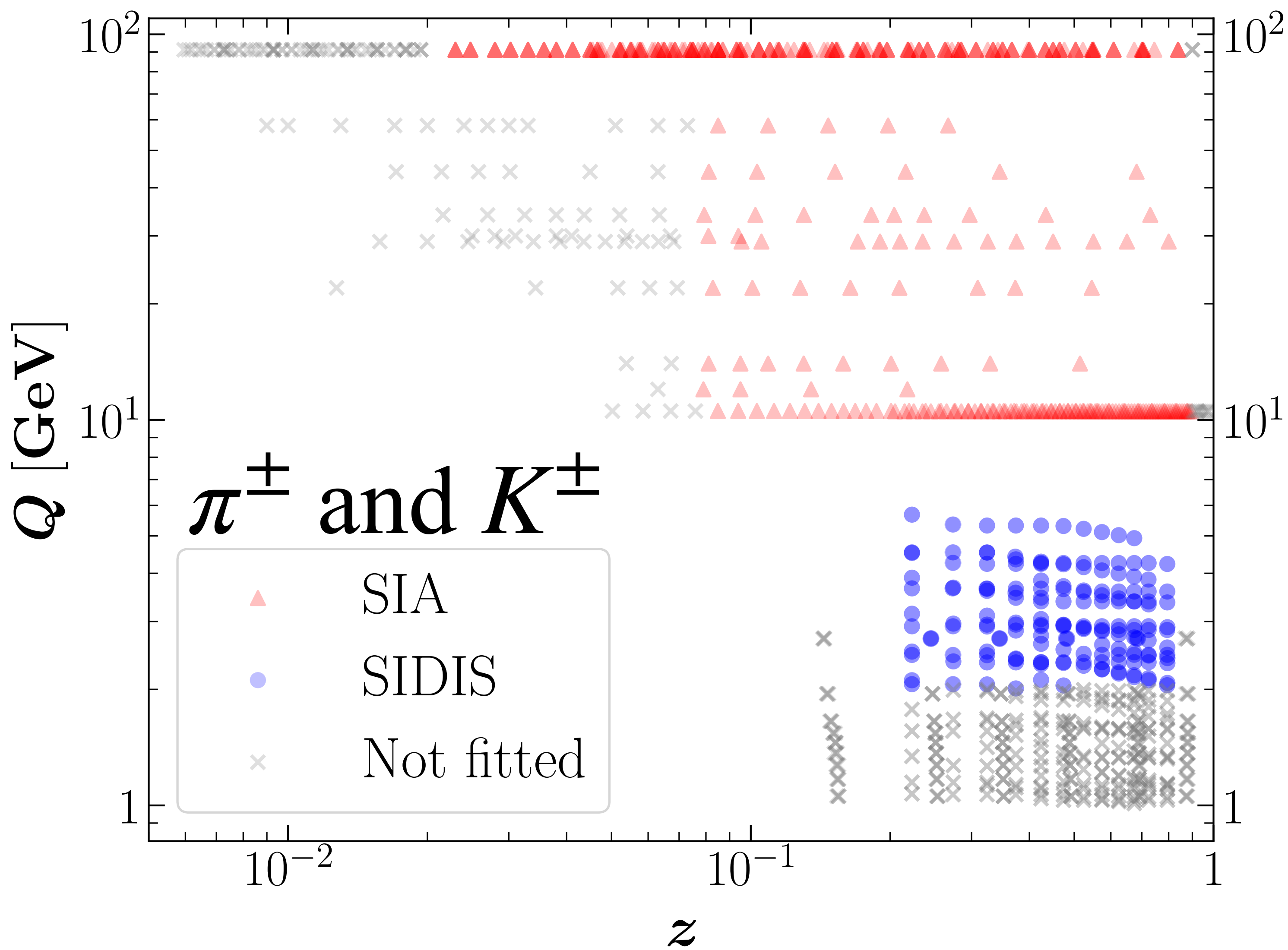
An extraction of pion TMDs





Light-hadron FFs at NNLO

- In [Phys.Lett.B 834 (2022) 137456] we have extracted $\pi^\pm$ and $K^\pm$ fragmentation functions (FFs) from a broad set of single-inclusive e^+e^- annihilation (SIA) and SIDIS data at NNLO accuracy (the first ever FF sets made public at this order).

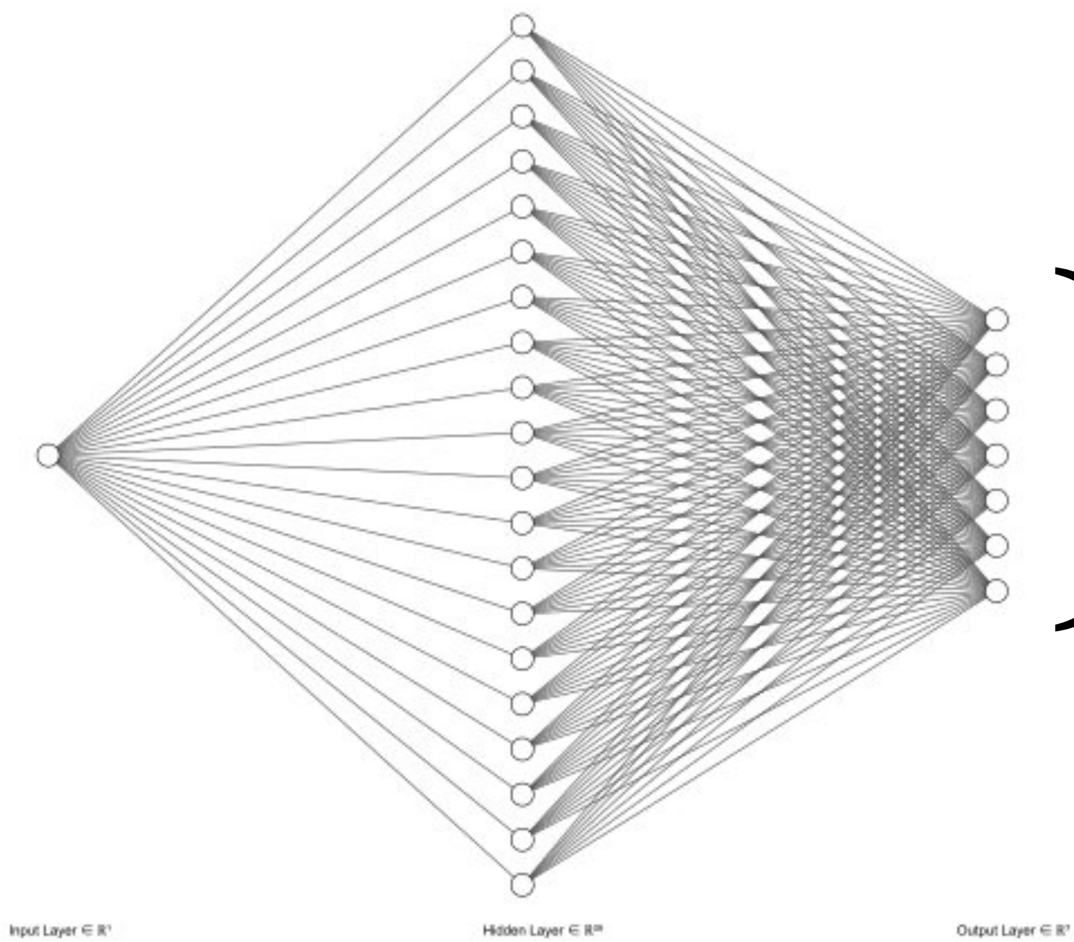


- Around 700 data points for **SIA and SIDIS** for both pions and kaons.
- **No *pp* data**: NNLO corrections not known yet.
- **Wide coverage** in z and Q.
- Extraction performed using the **MontBlanc** framework.

Light-hadron FFs at NNLO

- **All** fitted FFs are parameterised using a **single** NN:
  - architecture 1-20-7 (187 free parameters).

$z \longrightarrow$



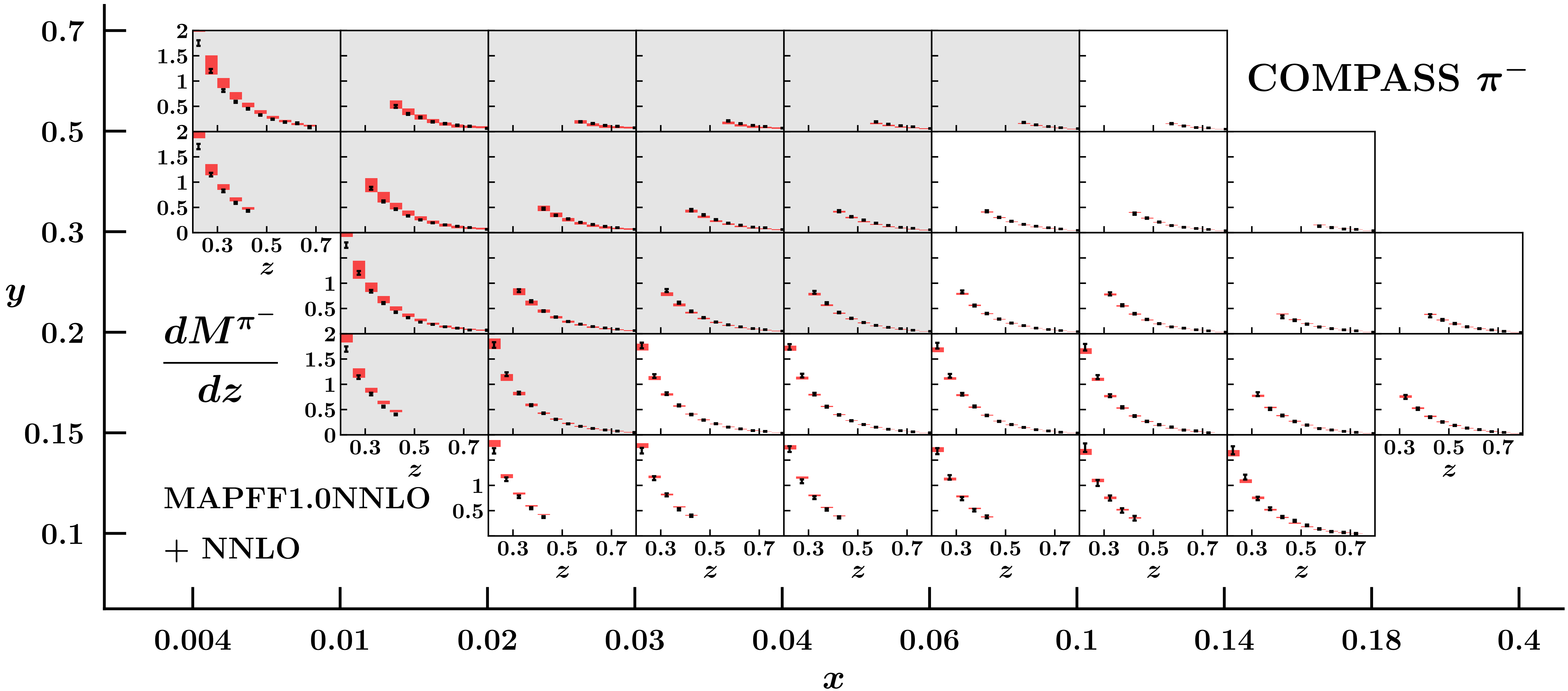
1) $D_u^{\pi^+}(z, Q_0)$
2) $D_{\bar{d}}^{\pi^+}(z, Q_0)$
3) $D_d^{\pi^+}(z, Q_0)$
4) $D_s^{\pi^+}(z, Q_0) = D_{\bar{s}}^{\pi^+}(z, Q_0)$
5) $D_c^{\pi^+}(z, Q_0) = D_{\bar{c}}^{\pi^+}(z, Q_0)$
6) $D_b^{\pi^+}(z, Q_0) = D_{\bar{b}}^{\pi^+}(z, Q_0)$
7) $D_g^{\pi^+}(z, Q_0)$

- Exploit the ability to compute the **analytic derivatives** of any NN w.r.t. its free parameters using the **NNAD** library. [R. Abdul Khalek, V. Bertone, arXiv:2005.07039]

- This enormously simplifies the task of the minimiser in that the gradient of the χ^2 can be computed analytically (as opposed to numerical or automatic derivatives).

- Monte Carlo method to propagate uncertainties.

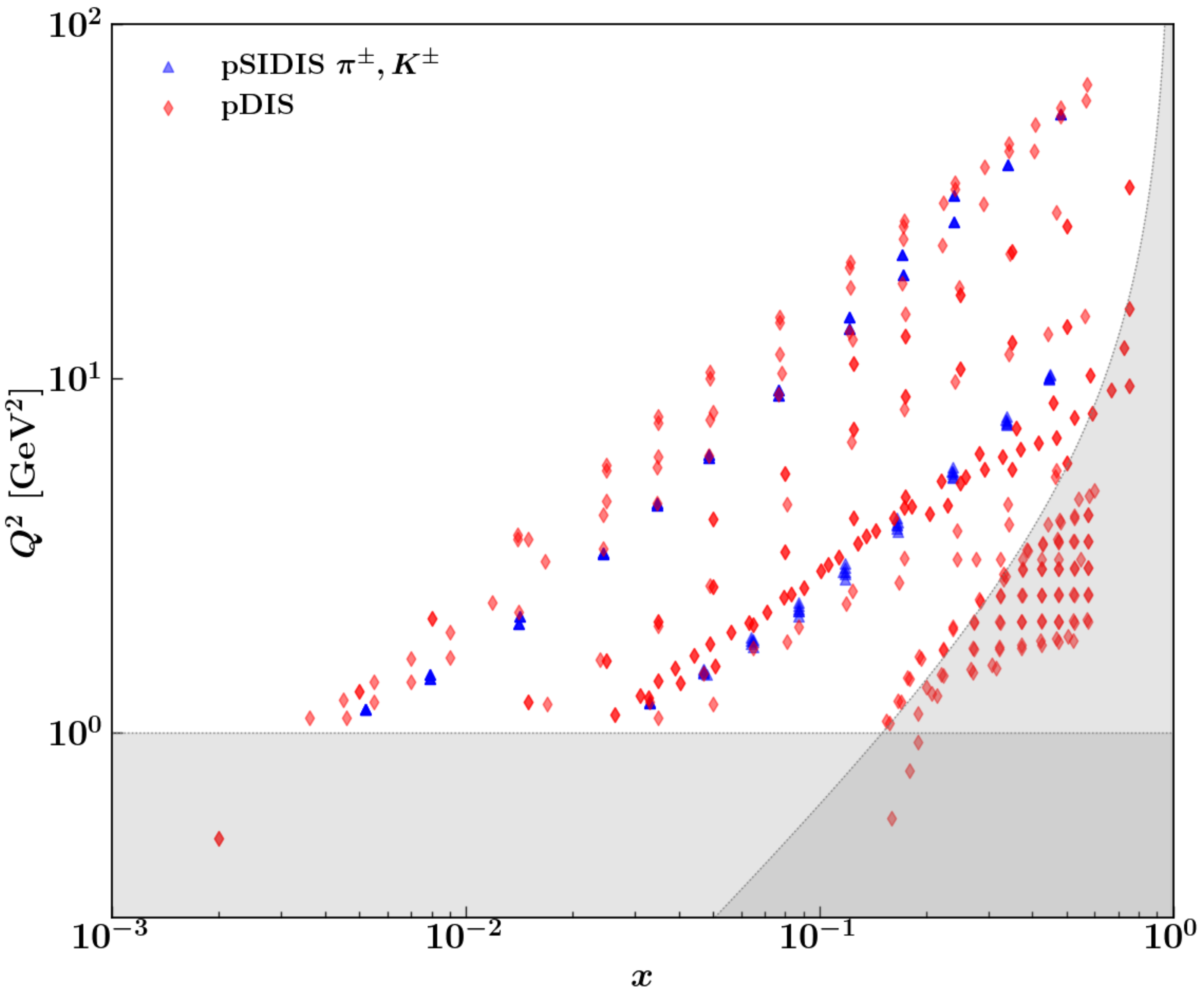
Light-hadron FFs at NNLO



- Good description of the data included in the fit:
  - even for bins that are not included because of kinematic cuts.

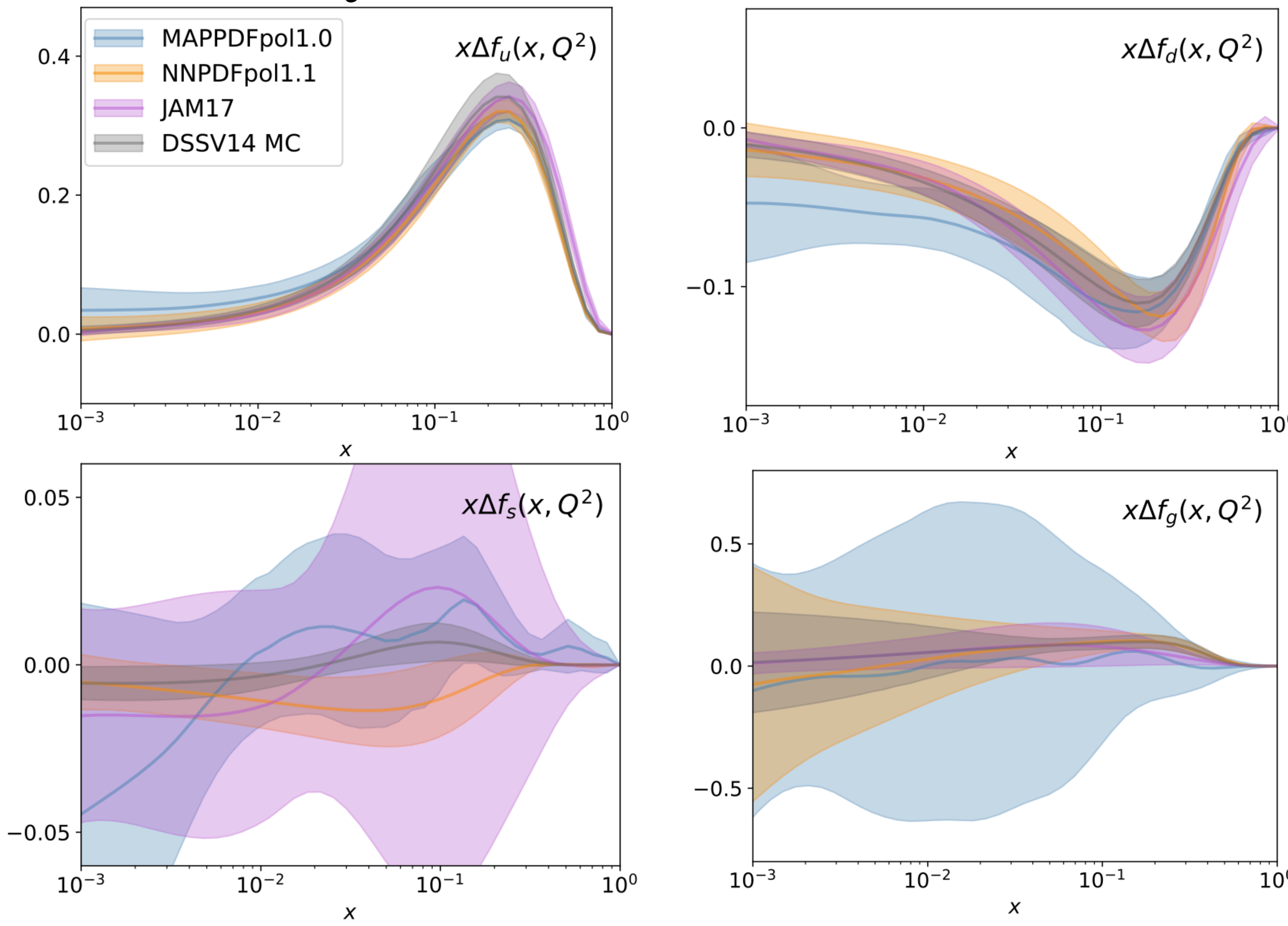
Helicity PDFs at NNLO

- A similar technology was used in [arXiv:2404.04712] to extract longitudinally polarised PDFs at NNLO accuracy



- Around 360 data points for **DIS and SIDIS** for both pions and kaons.
- **No *pp* data**: NNLO corrections not known yet.
- **Wide coverage** in x and Q.
- Extraction performed using the **Denali** framework.

Helicity PDFs at NNLO



Reconstructing GTMDs

- In [*Eur.Phys.J.C* 82 (2022) 10, 941] GTMD matching functions computed at one-loop accuracy.
- The unpolarised GTMD correlator can be decomposed as: Meißner, Metz, Schlegel [*JHEP* 08 (2009) 056]

$$\mathcal{F}_{i/H} = \frac{1}{2M}\overline{u}(P_{\rm out})\left[F^i_{1,1} + \frac{i\sigma^{\mathbf{k}_T n}}{n\cdot P}F^i_{1,2} + \frac{i\sigma^{\mathbf{\Delta}_T n}}{n\cdot P}F^i_{1,3} + \frac{i\sigma^{\mathbf{k}_T\mathbf{\Delta}_T}}{M^2}F^i_{1,4}\right]u(P_{\rm in})$$

- Each function $F^i_{1,l}$ is complex and can be decomposed into a real and an imaginary part:

$$F^i_{1,l} = F^{i,e}_{1,l} + iF^{i,o}_{1,l} \qquad\qquad F^{i,e}_{1,l}, F^{i,o}_{1,l} \in \mathbb{R}$$

- $F^{i,e}_{1,1}$ for $b_T \simeq 0$ and $Q \simeq 1/b_T$ is related to the GPDs H_j and E_j as follows:

$$F^{i,e}_{1,1}(x,\xi,b_T,t,Q) \underset{b_T\simeq 0}{=} \mathcal{C}_{i/j}(x,\xi,b_T,Q) \underset{x}{\otimes} \left[(1-\xi^2)H_j(x,\xi,t,Q) - \xi^2 E_j(x,\xi,t,Q)\right]$$

- Moreover, the forward limit of $F^{i,e}_{1,1}$ is the unpolarised TMD $f_{1,i}$:

$$\lim_{\xi,t\to 0} F^{i,e}_{1,1}(x,\xi,b_T,t,\mu,\zeta) = f_{1,i}(x,b_T,\mu,\zeta)$$

- As for TMDs, the value of $F^{i,e}_{1,1}$ for any values of b_T and Q is achieved by introducing a non-perturbative function ($f_{\rm NP}$) and solving appropriate evolution equations:
    - $f_{\rm NP}$ is (mostly) the same as that of TMDs,
    - also the evolution equations closely follow those for TMDs and can thus be solved analogously.

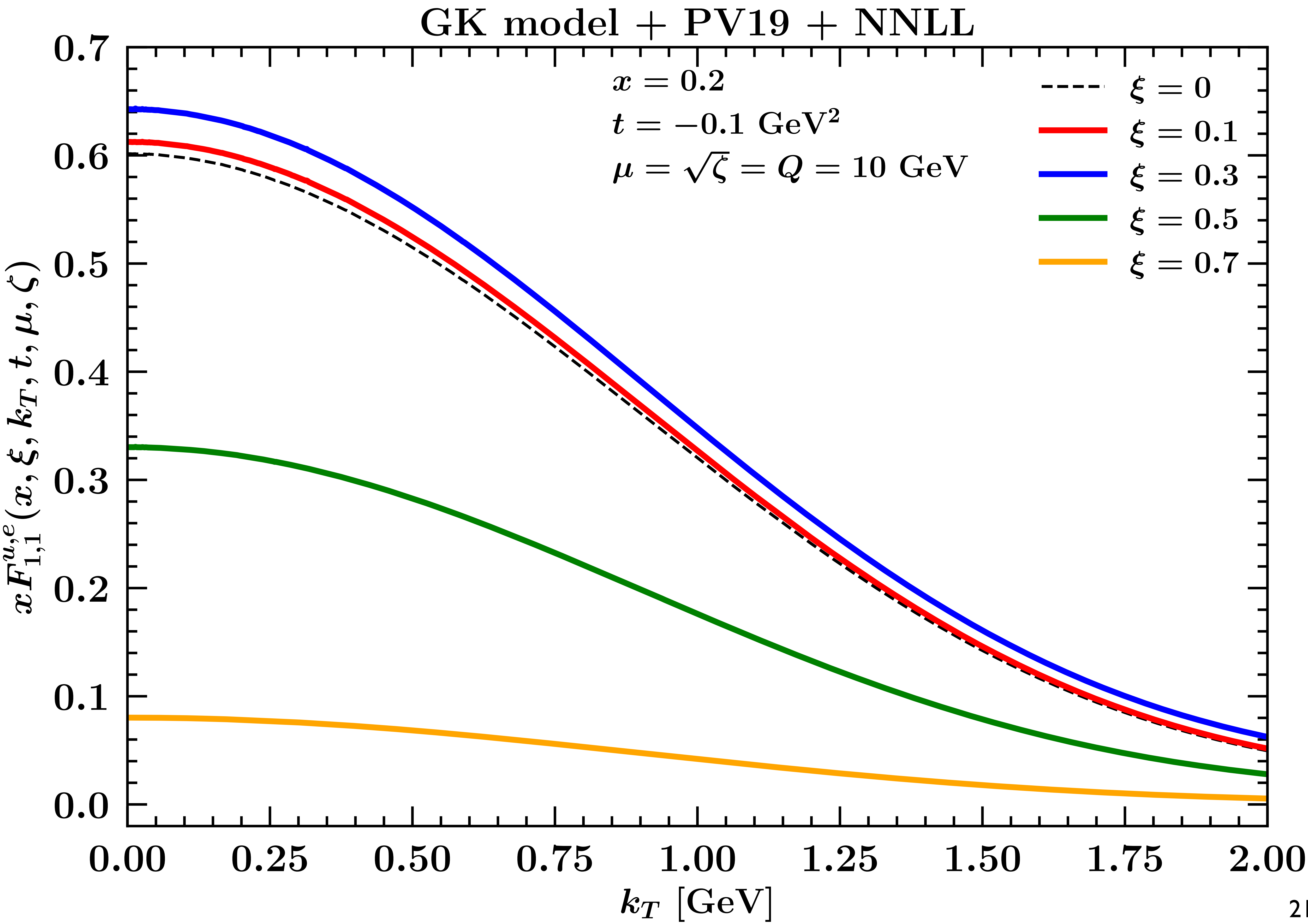
Reconstructing GTMDs

- A numerical code to compute $F_{1,1}^{i,e}$ as briefly described above is public at:

  https://github.com/vbertone/GTMDMatching

- and is based on a combination VA2 public codes.
- **PARTONS** for the handling of GPDs:
  - the Goloskokov-Kroll (GK) model for the GPDs H_j and E_j has been used.
- **NangaParbat** for the handling of TMDs:
  - the PV19 [*JHEP* 07 (2020) 117] determination of $f_{\rm NP}$ along with the b_* function.
- **APFEL++** is used for:
  - the numerical computation of the **convolutions**,
  - the **collinear evolution of GPDs**,
  - the computation of the **Sudakov form factor**,
  - the **inverse Fourier transform**.

Reconstructing GTMDs

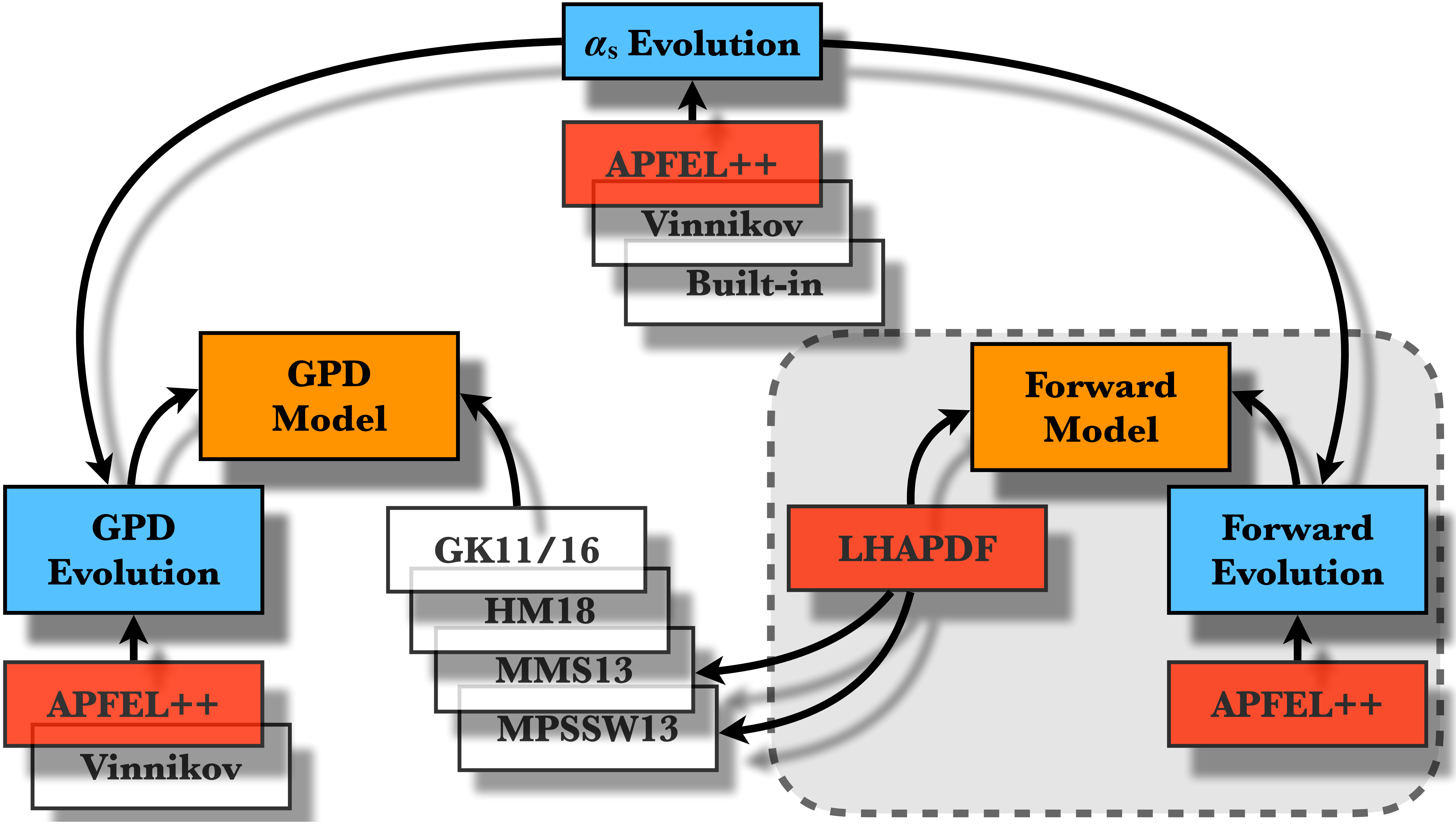


Conclusions

- Over the past few years 3DPartons has produced a great deal of results:
  - many relevant physics results,
  - much numerical and open-source infrastructure has been developed,
  - existing codes are being interfaced to create a seamless framework,
  - most of the involved codes are written in C++, that guarantees performance, modularity, and maintainability.
  - Python wrappers have also been developed.
  - These developments are having a tangible impact on different experimental physics programmes:
    - the preparatory work in view of the EIC,
    - the determination of the W mass at the LHC,
    - physics at JLab,
    - …

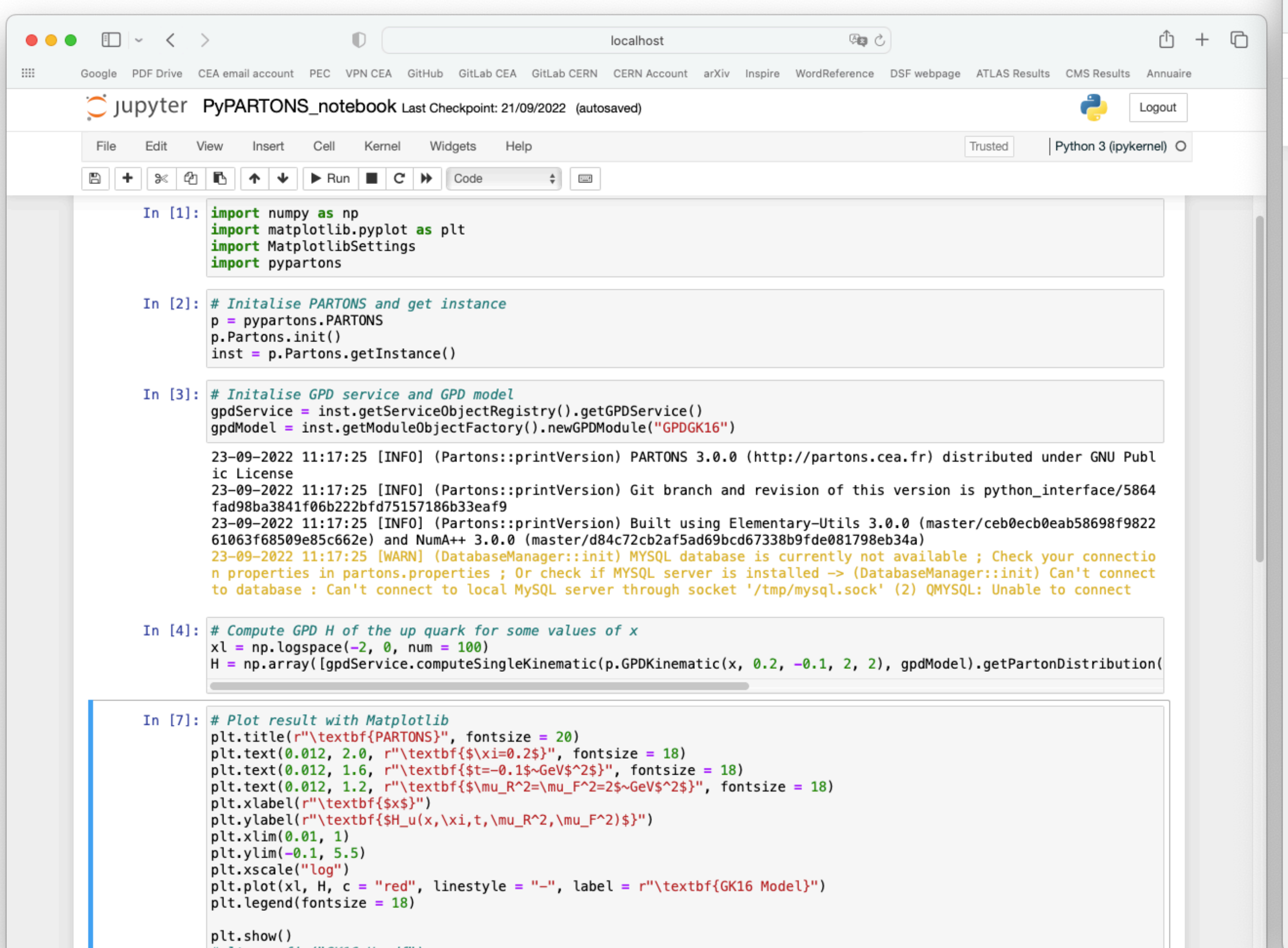
Back up

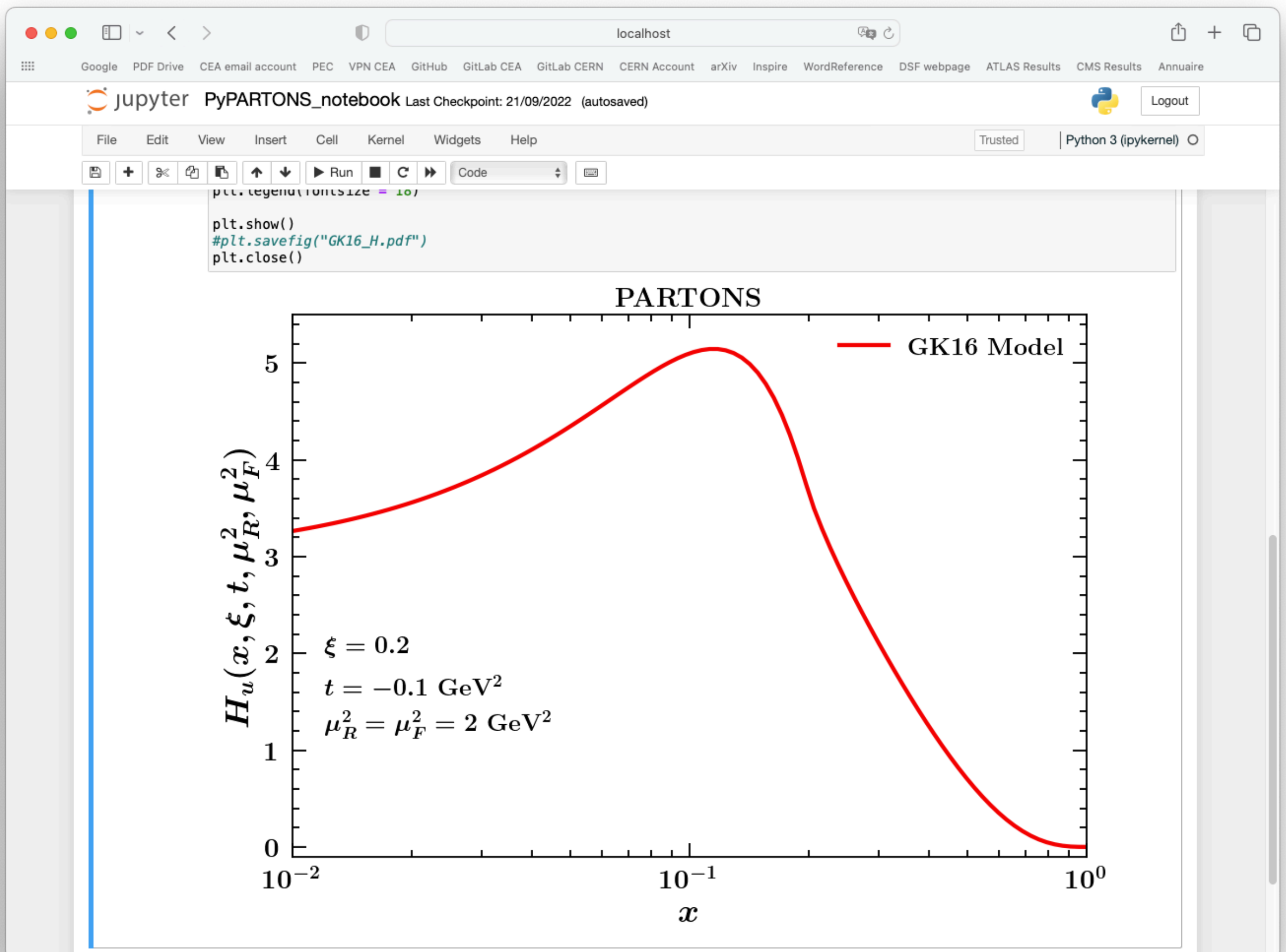
GPD modeling in PARTONS



A python interface to PARTONS

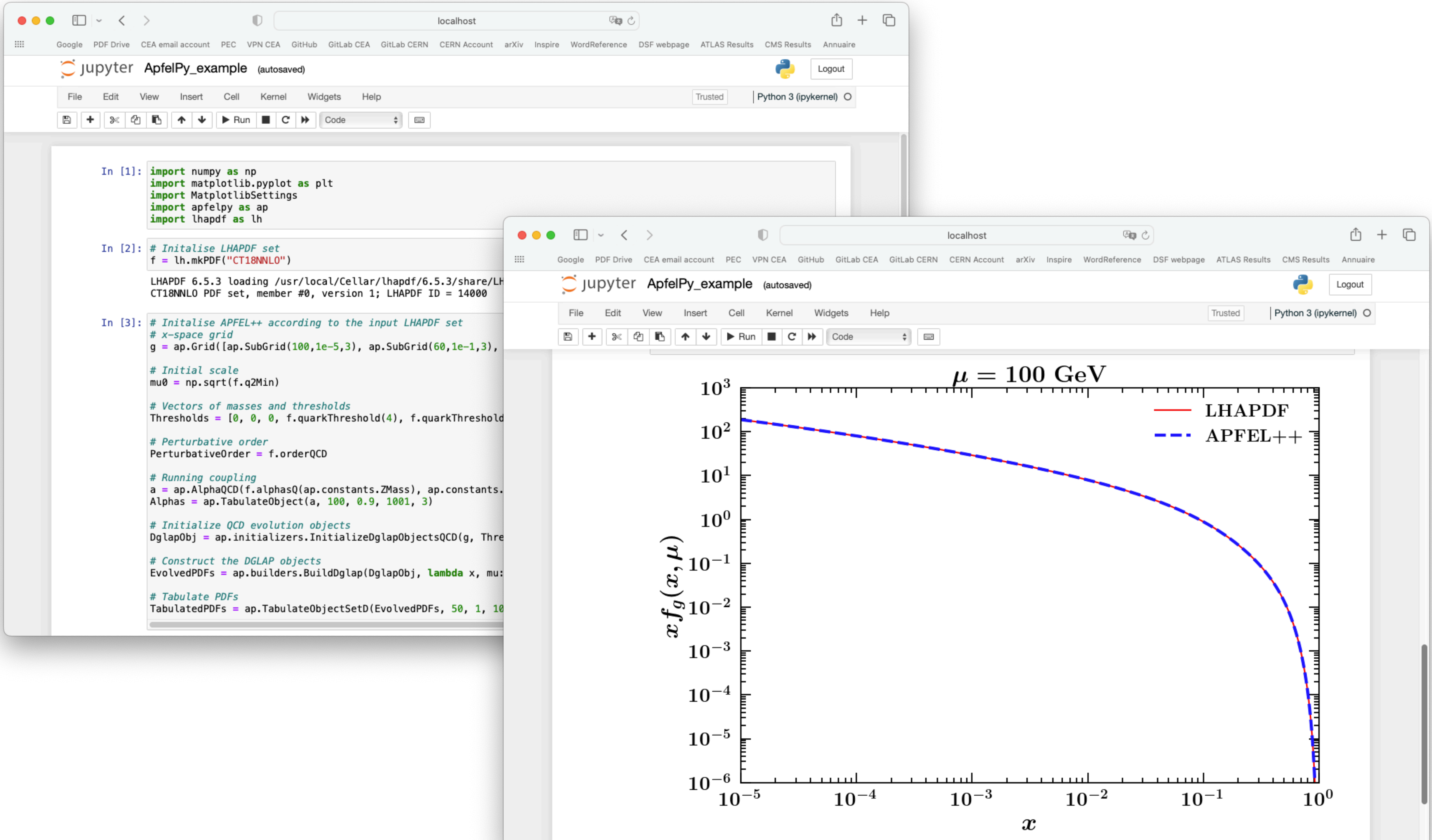
- Developed using **pybind11**, a C++11 compliant library:
  - similar to Boost.python but much lighter (simpler to install locally),
  - generated at compilation level *if* pybind11 is found
  - the developer writes the interface with direct control on the functionalities to be exposed,
  - the module gets installed locally allowing for *interoperability* with other python packages,
  - facilitates **usability** and thus **dissemination**.





A python interface to APFEL++

- Same strategy as for PARTONS



The deconvolution problem

- Given an arbitrarily large and precise set of data for the Compton form factor (CFF) $\mathscr{H}$ and the convolution formula:

$$\mathcal{H}(\xi, Q^2) = \int_{-1}^{1} \frac{dx}{\xi} T\left(\frac{x}{\xi}, \frac{Q^2}{\mu}, \alpha_s(\mu^2)\right) H(x, \xi, \mu^2) \equiv [T \otimes H](\xi, Q)$$

- where *T* is perturbatively known to some fixed order and *H* is a GPD, can we uniquely extract *H*? This often goes under the name of *deconvolution problem*.
- Since the dependence on *x* of the GPD is integrated over, one may superficially expect that it is *not* possible to extract the GPD *H* uniquely.
- While this argument has long been advocated and proven to tree level, it was also believed that evolution effects may provide a handle on *H*.
- Indeed, a general answer to this question (valid to any perturbative order and scale) requires considering evolution effects provided by the solution of:

$$\frac{dH(x, \xi, \mu^2)}{d \ln \mu^2} = [P \otimes H](x, \xi, \mu^2)$$

- Clearly, the evolution entangles x, ξ, and μ^2 and may potentially allow one to find a unique solution to the deconvolution problem.

The deconvolution problem

- In [Phys.Rev.D 103 (2021) 11, 114019] we have addressed the deconvolution problem accounting for NLO corrections in T and evolution effects for H.
- The striking result of this paper is that it is possible to identify non-trivial GPDs with *arbitrarily small* imprint on the CFFs: the **shadow GPDs**.
- This is the *explicit proof* that the deconvolution problem has no unique solution.
- The shadow GPD is constructed in double-distribution (DD) space:

$$H_{\text{shadow}}(x,\xi) = \int_{-1}^{1} d\beta \int_{-1+|y|}^{1+|y|} d\alpha\, \delta(x-\beta-\xi\alpha) F_{\text{shadow}}(\alpha,\beta)$$

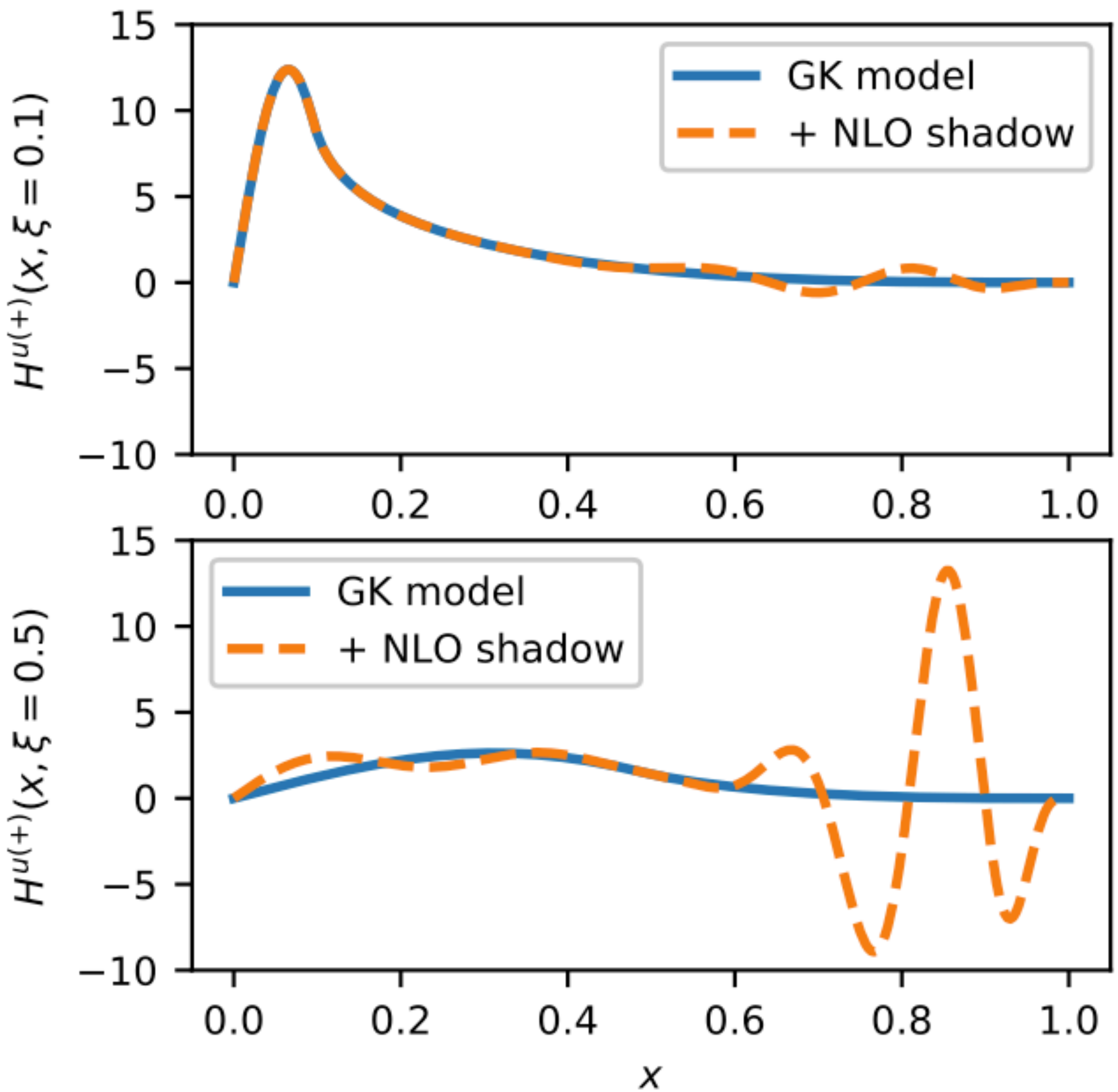
- Accounting for polynomiality of GPDs, we approximate F_{shadow} as:

$$F_{\text{shadow}}(\alpha,\beta) = \sum_{\substack{m \text{ even} \\ n \text{ odd}}}^{m+n\leq N} c_{mn}\alpha^m \beta^n$$

- We then require that the CFF at some scale vanishes as well as $H(x,0) = 0$ in a way that it does not affect the forward limit of the GPD (*i.e.* the PDF).
- The final result is an **underconstrained** problem that admits infinitely many solutions provided that the degree N is large enough.

The deconvolution problem

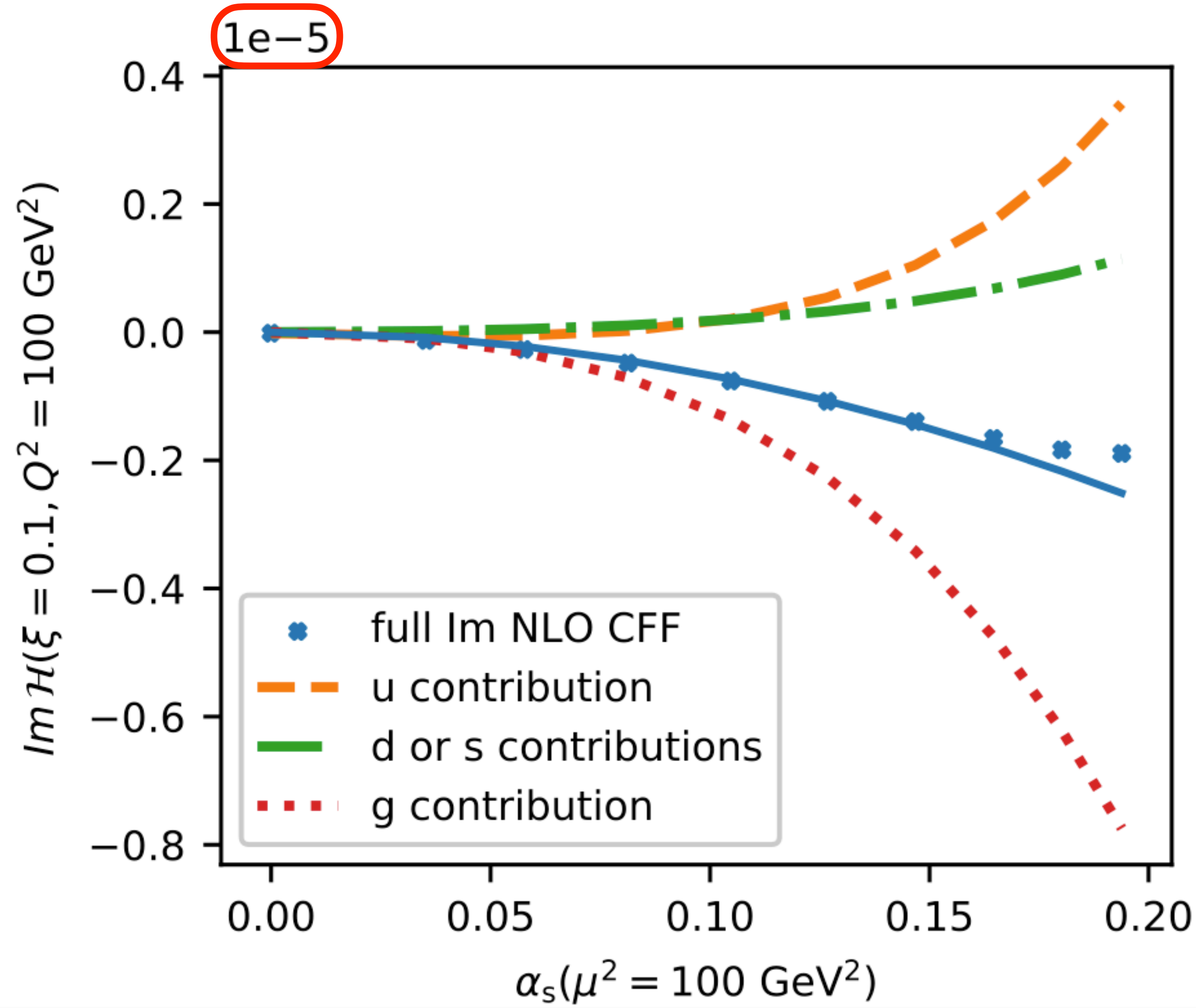
- The effect of a possible shadow GPD on the GK model:



- Numerical results obtained with **PARTONS** interfaced to **APFEL++**, both developed within the VA2 work package.

The deconvolution problem

- NLO CFF generated by a shadow GPD evolved from $\mu_0^2 = 1\ \mathrm{GeV}^2$ to $\mu^2 = 100\ \mathrm{GeV}^2$:
    - it scales quadratically with $\alpha_s(\mu^2 = 100\ \mathrm{GeV}^2)$ as expected,
    - it is $\mathcal{O}(10^{-5})$, *i.e.* negligible w.r.t. a typical physical value.



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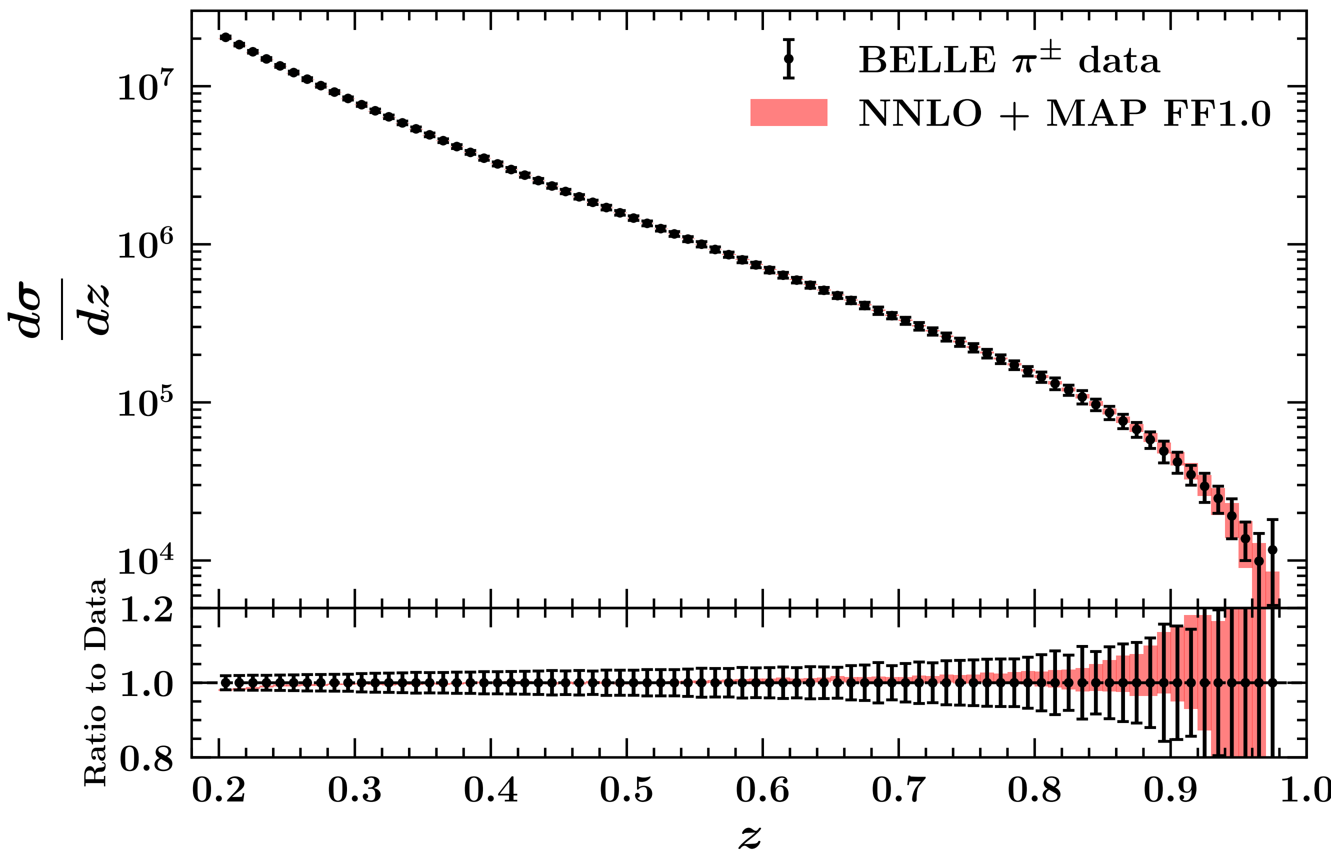
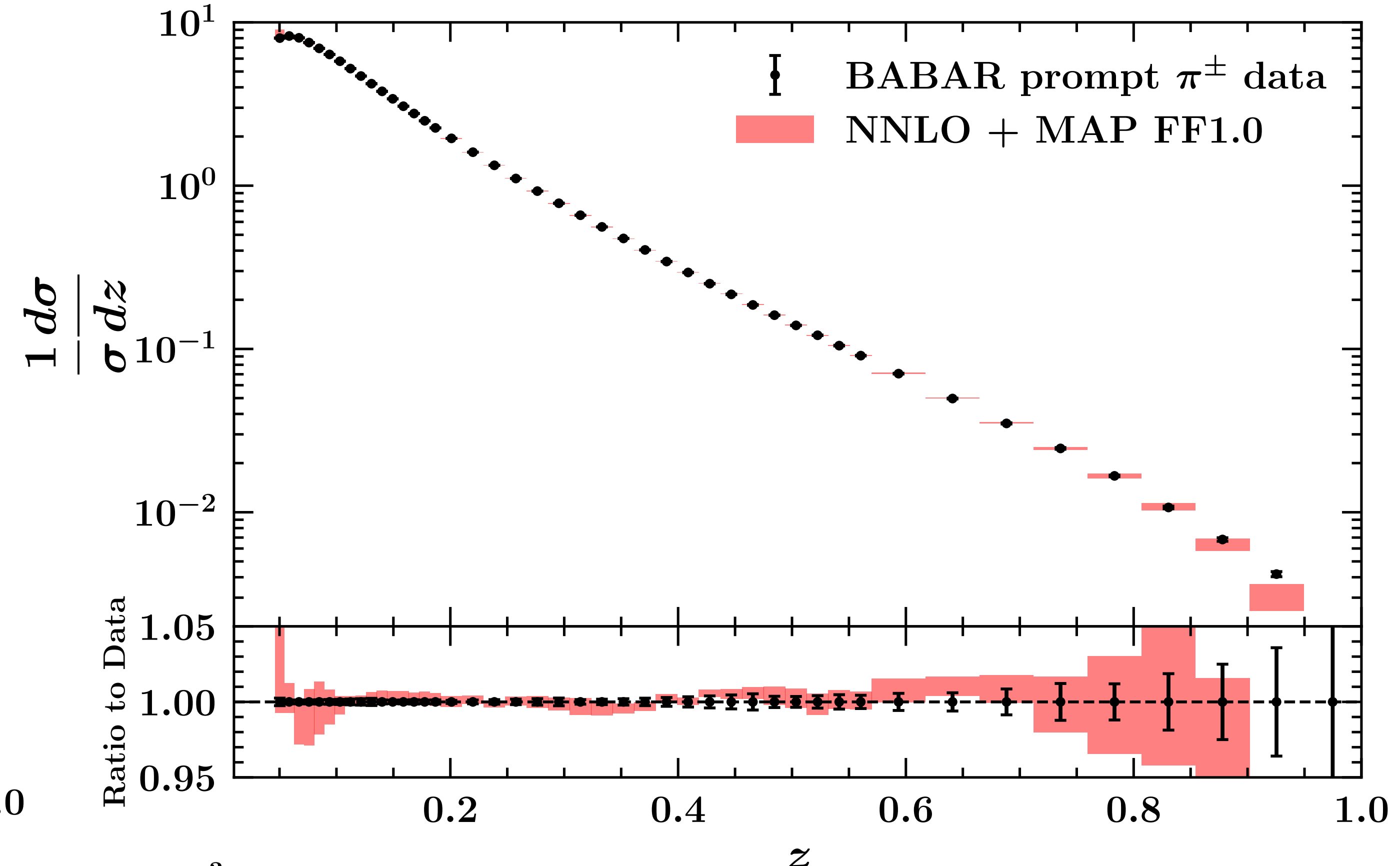
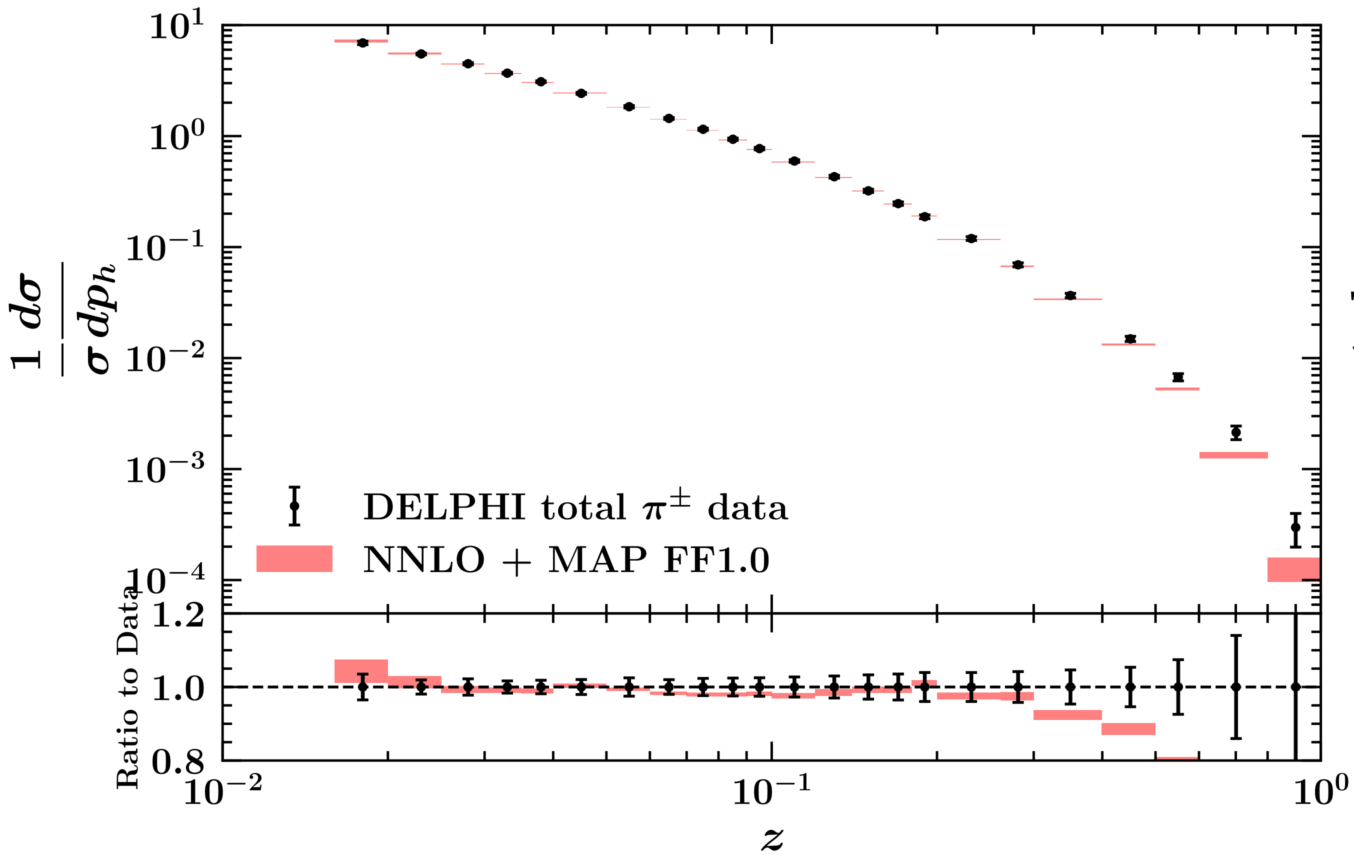
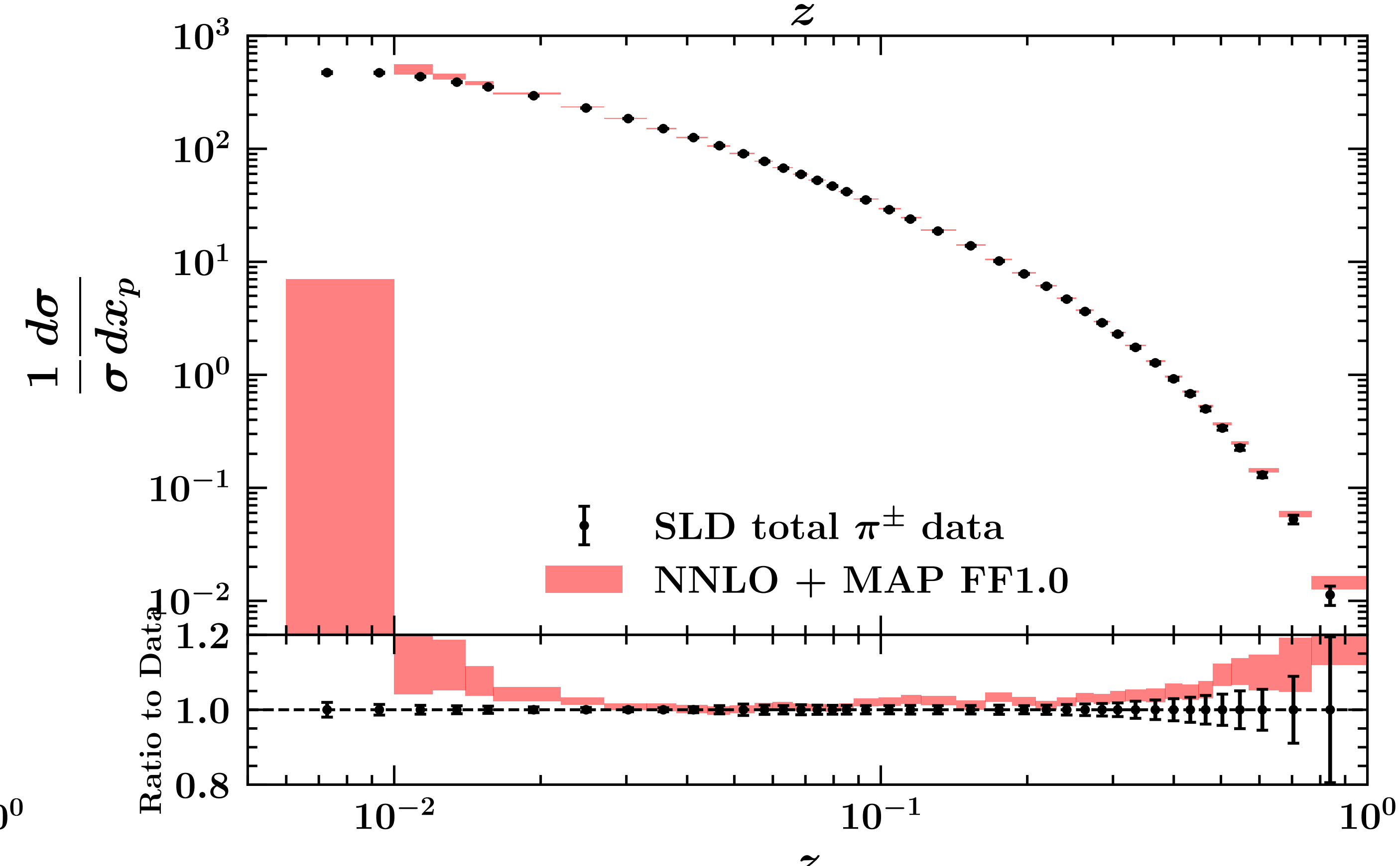
Light-hadron FFs at NNLO









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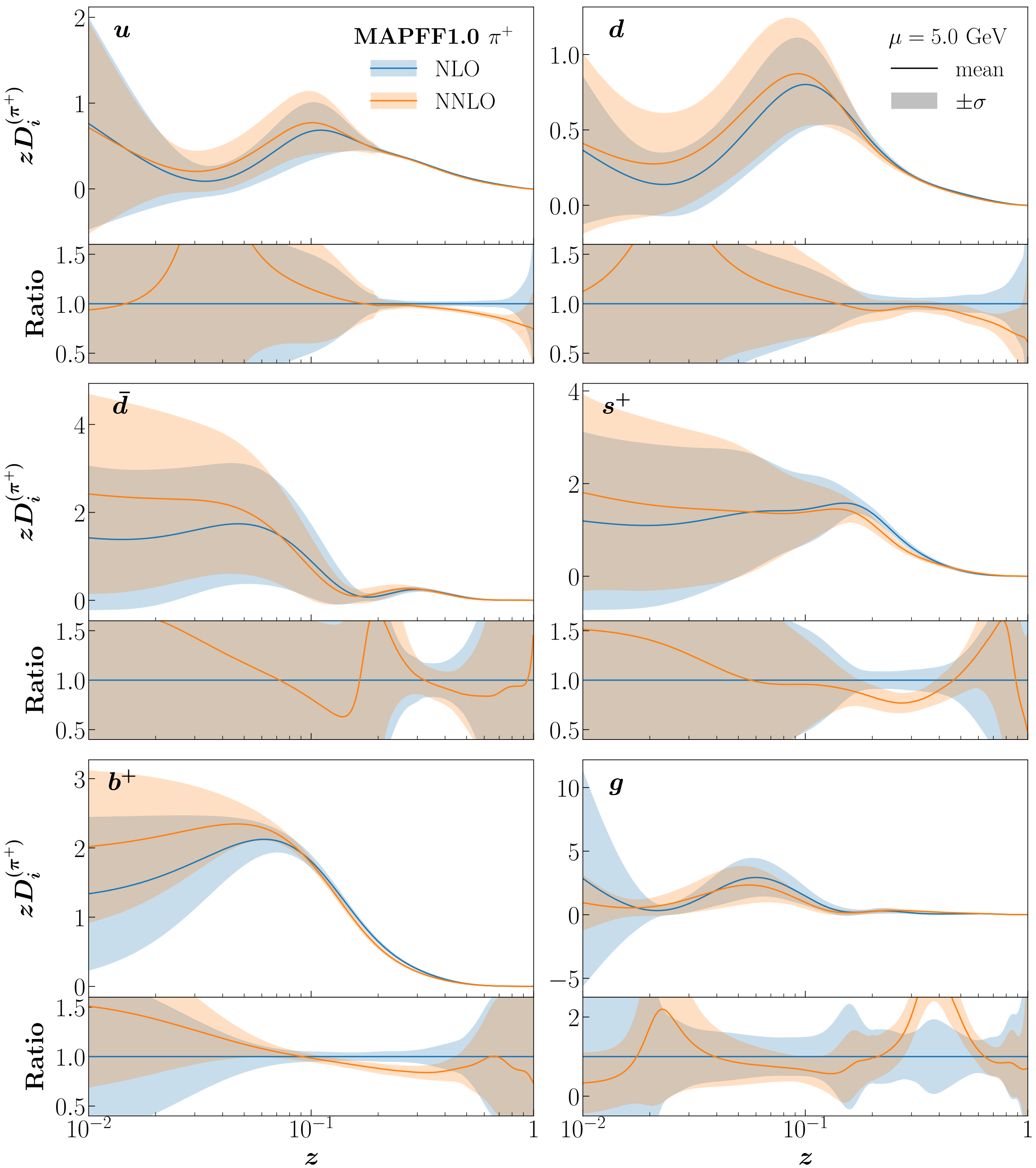
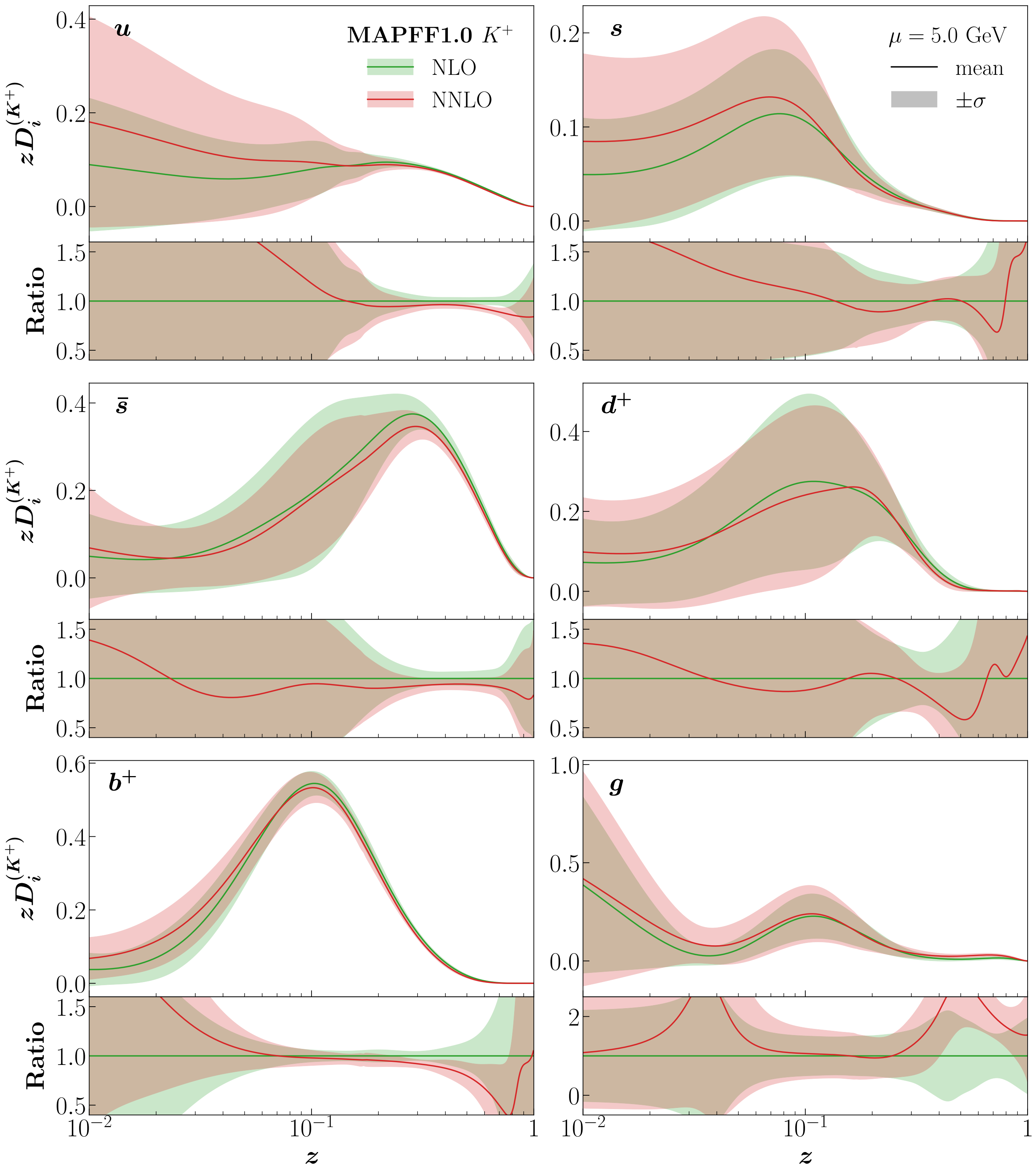
Pions



Kaons



Numerical setup

- The evolution kernels for *unpolarised* evolution that we have recomputed are now implemented in **APFEL++** and available through **PARTONS** allowing for LO GPD evolution in momentum space.

- The remarkable properties of the evolution kernels allowed us to obtain for the first time a stable numerical implementation over the full range $\mathbf{0 \leq \xi \leq 1}$:

  - first numerical check that both the **DGLAP** and **ERBL** limits are recovered,

  - first numerical check of **polynomiality** conservation.

- Numerical tests mostly use the MMHT14 PDF set at LO as an initial-scale set of distributions evolved from 1 to 10 GeV for the first time in the **variable-flavour-number scheme**, *i.e.* accounting for heavy-quark-threshold crossing.

- Tests have also been performed using more realistic GPD models such as the Goloskokov-Kroll model [*Eur.Phys.J.C* 53 (2008) 367-384] based on the Radyushkin double-distribution ansatz [*Phys.Lett.B* 449 (1999) 81-88].

The ERBL limit

- The limit $\xi \to \mathbf{1}$ ($\kappa \to 1/x$) we should reproduce the **ERBL equation**.

- It is well known that in this limit **Gegenbauer polynomials** decouple upon LO evolution, such that:

$$F_{2n}(x,\mu_0) = (1-x^2)C_{2n}^{(3/2)}(x) \quad \Rightarrow \quad F_{2n}(x,\mu) = \exp\left[\frac{V_{2n}^{[0]}}{4\pi}\int_{\mu_0}^{\mu} d\ln\mu^2\alpha_s(\mu)\right] F_{2n}(x,\mu_0)$$

- where the kernels $V_{2n}^{[0]}$ can be read off, for example, from [Brodsky, Lepage, *Phys.Rev.D* 22 (1980) 2157] or [Efremov, Radyushkin, *Phys.Lett.B* 94 (1980) 245-250].

- We have compared this expectation with the numerical results for GPD evolution by setting $\kappa = 1/x$ and using a Gegenbauer polynomial as an initial-scale GPD.

Conformal-space evolution

- In order to check that LO GPD evolution ($\xi \neq 0$) in conformal space is diagonal in a **realistic** case, we have considered the RDDA:

$$H_q(x,\xi,\mu_0) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \delta\left(x-\beta-\xi\alpha\right) q(|\beta|)\pi(\beta,\alpha)$$

with:

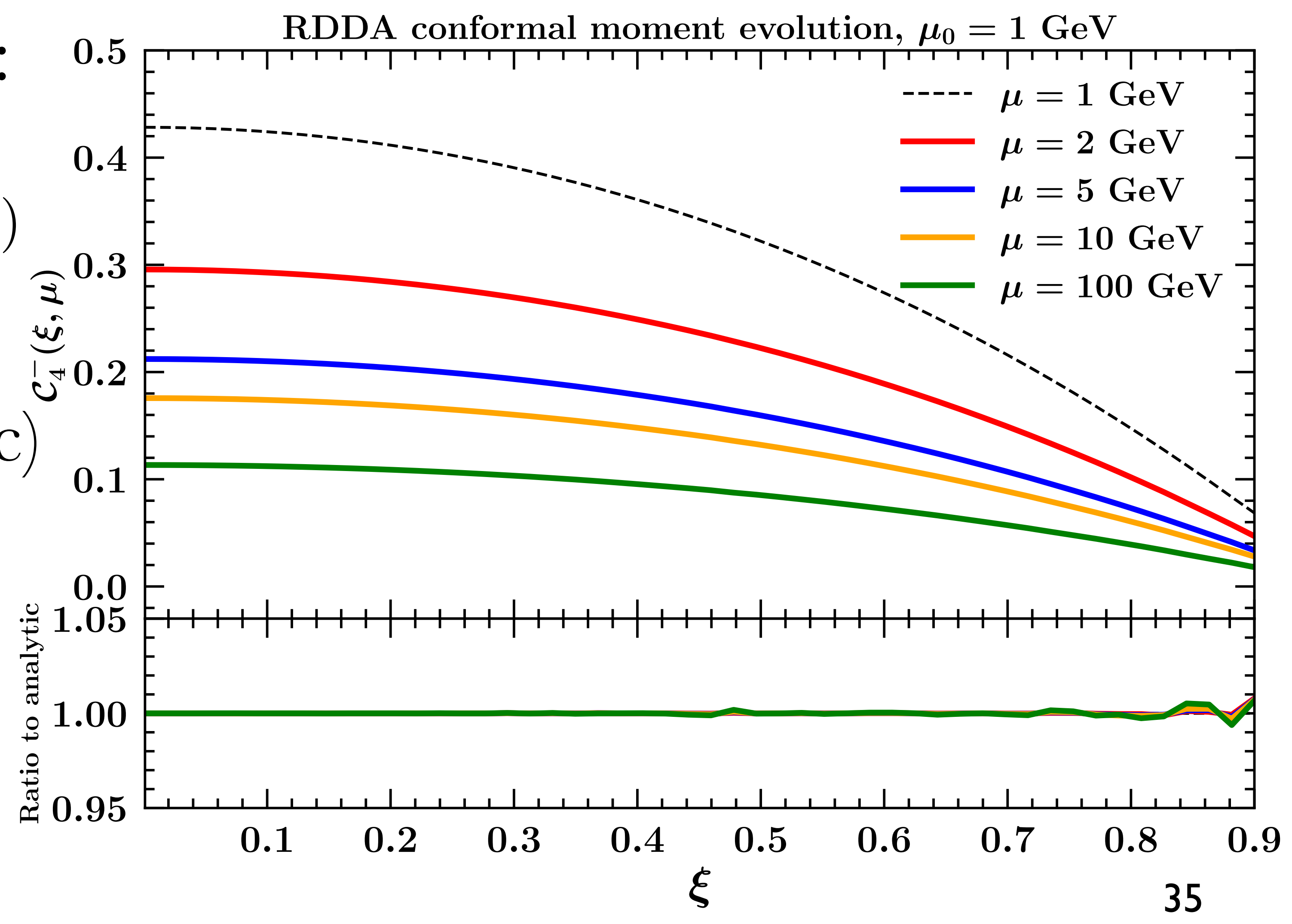
$$q(x) = \frac{35}{32} x^{-1/2}(1-x)^3, \quad \pi(\beta,\alpha) = \frac{3}{4}\frac{\left((1-|\beta|)^2-\alpha^2\right)}{(1-|\beta|)^3}$$

We have evolved the 4th moment:

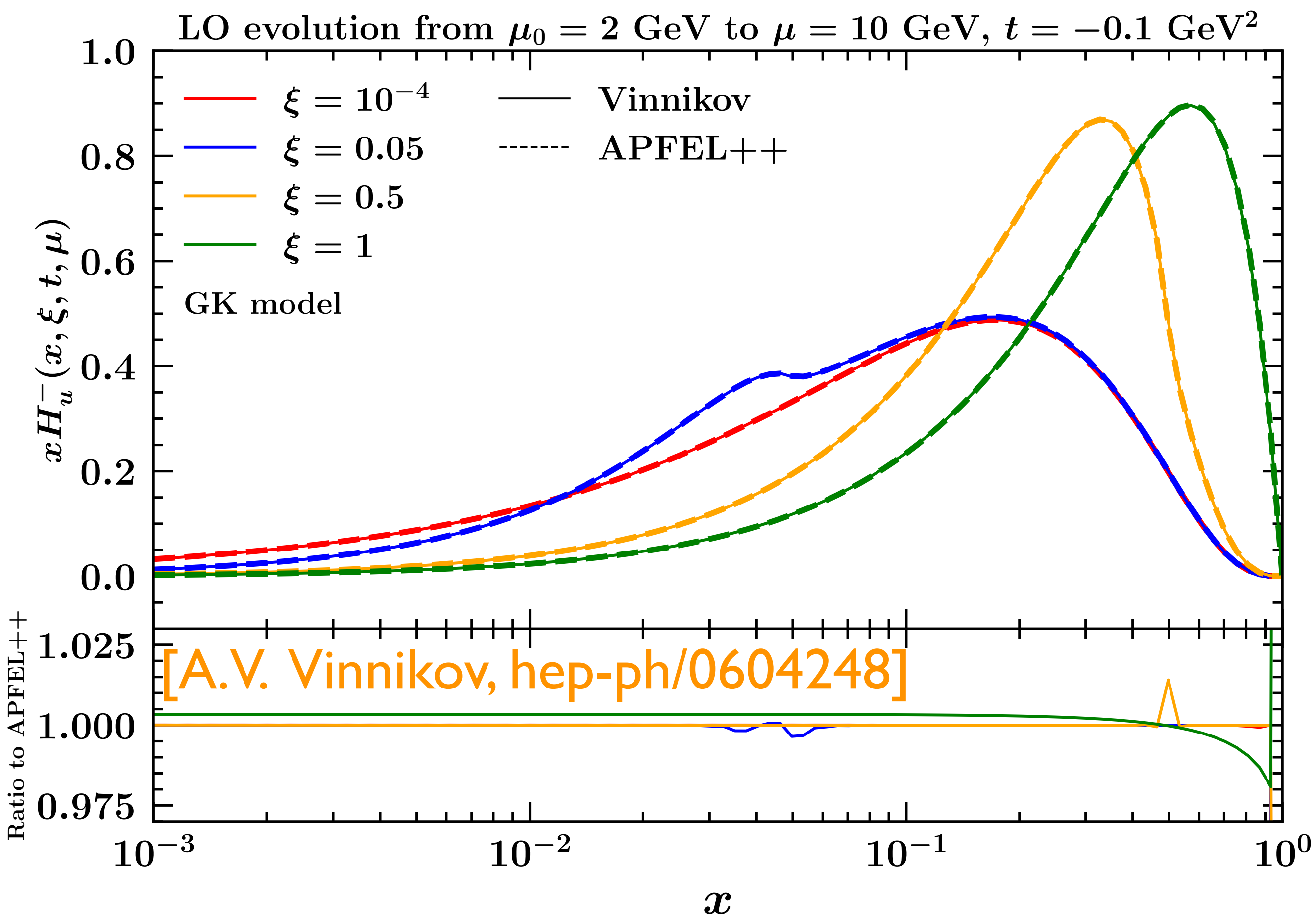
$$\mathcal{C}_4^-(\xi,\mu) = \xi^4 \int_{-1}^{1} dx\, C_4^{(3/2)}\left(\frac{x}{\xi}\right) H_q(x,\xi,\mu)$$

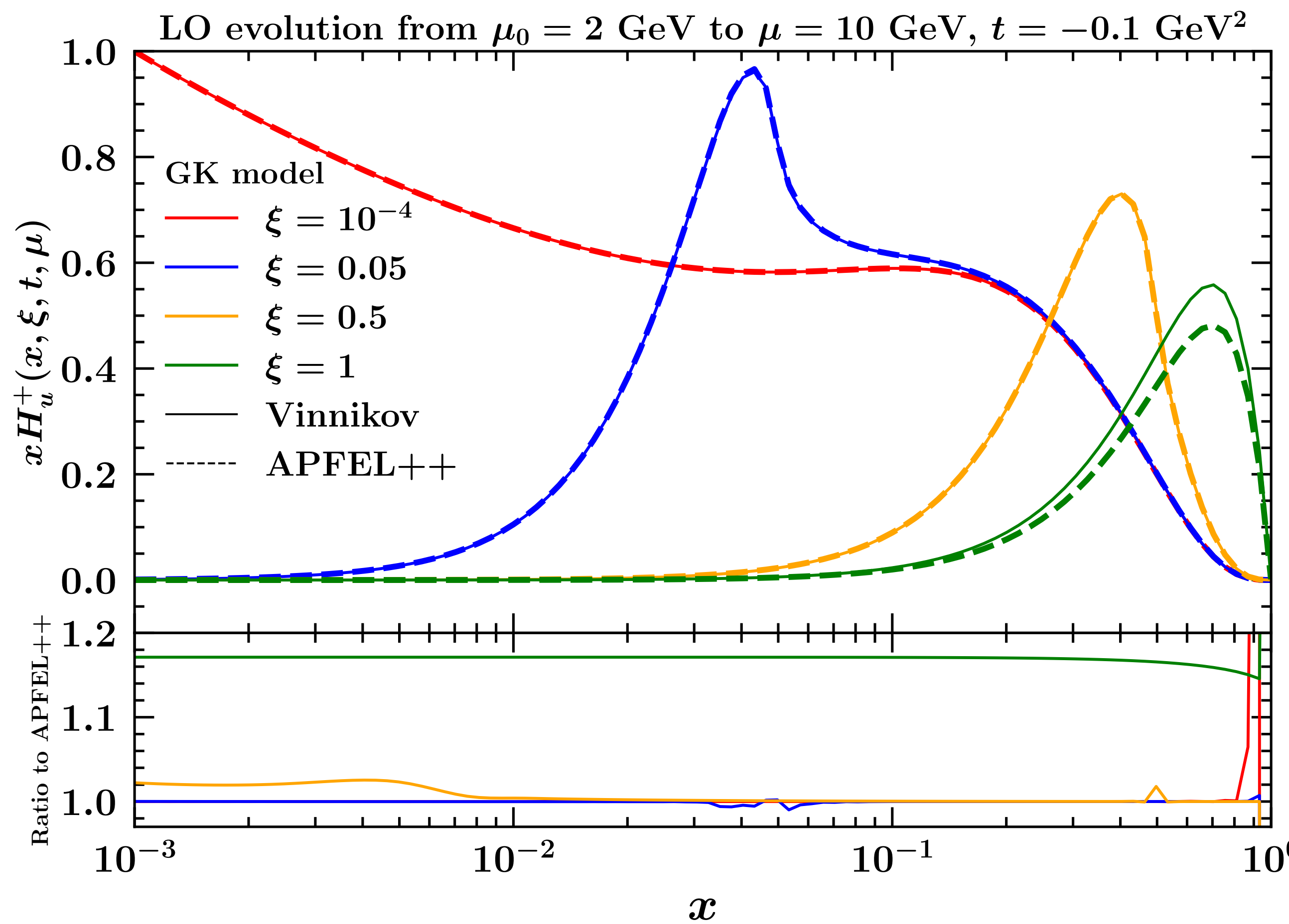
from $\mu_0 = 1$ GeV using the (analytic) conformal-space evolution and the (numerical) momentum-space evolution.

we found excellent agreement.

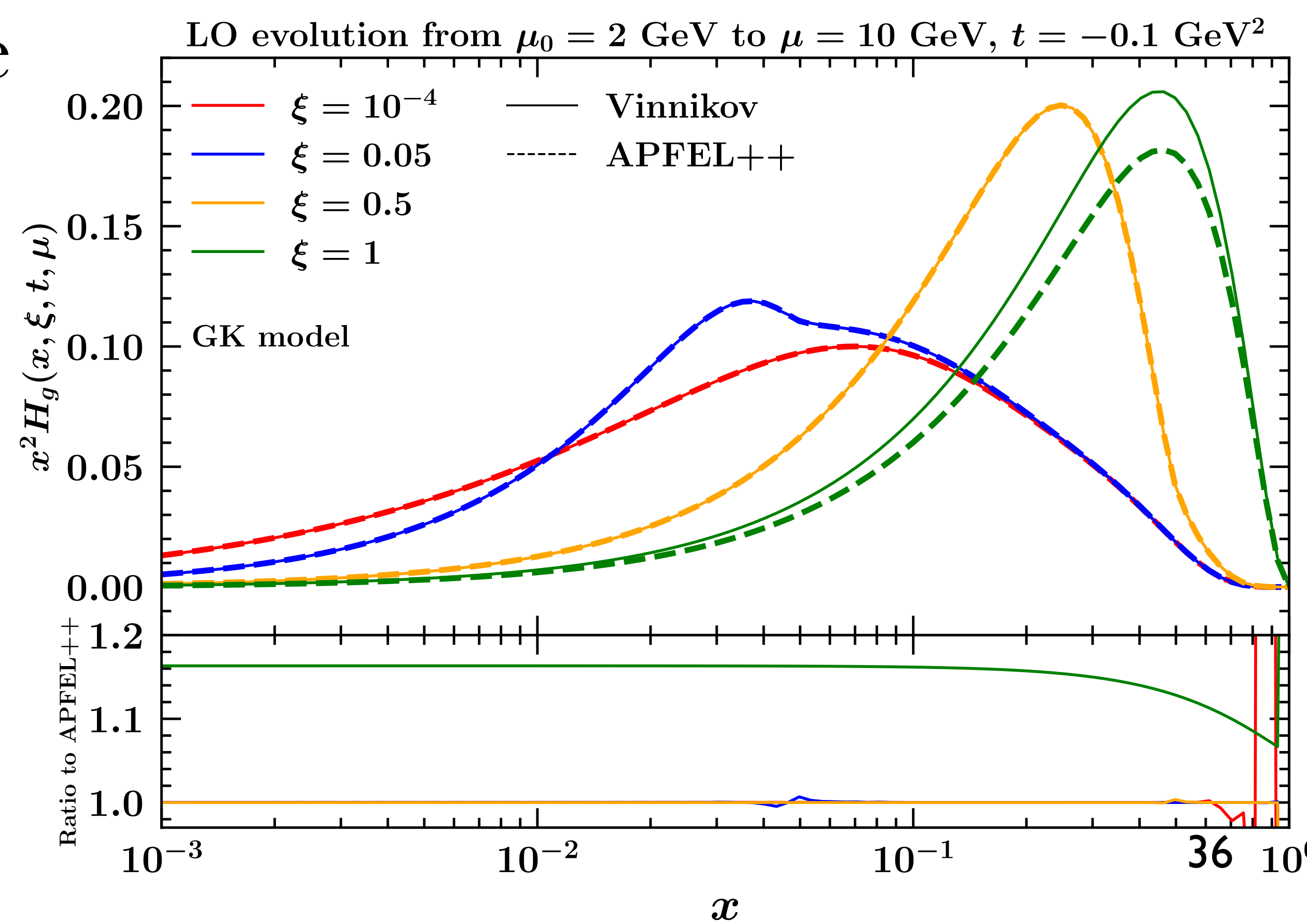


APFEL vs. Vinnikov's code





- **Excellent agreement** between the two code for $\xi \lesssim 0.6$.
- Agreement deteriorates for $\xi \gtrsim 0.6$:
  - discrepancy larger for the singlets (~20%) than for the non-singlet (~1%).
  - possible numerical instabilities of Vinnikov's code?
  - Inability to check the ERBL limit.



Polynomiality

- GPD evolution should preserve **polynomiality**.
  [Xiang-Dong Ji, *J.Phys.G* 24 (1998) 1181-1205] [A.V. Radyushkin, *Phys.Lett.B* 449 (1999) 81-88]
- The following relations for the Mellin moments must hold at **all scales**:

$$\int_0^1 dx\, x^{2n} F_q^-(x,\xi,\mu) = \sum_{k=0}^{n} A_k(\mu)\xi^{2k}$$

$$\int_0^1 dx\, x^{2n+1} F_q^+(x,\xi,\mu) = \sum_{k=0}^{n+1} B_k(\mu)\xi^{2k}$$

- Polynomiality predicts that the first moment ($n = 0$) of the *non-singlet* distribution is **constant** in ξ.

- The coefficient of the ξ^{2n+2} term of the *singlet* (D-term) is absent in our initial conditions and it is *not* generated by evolution, so that also the first moment of the singlet is expected to be **constant** in ξ.

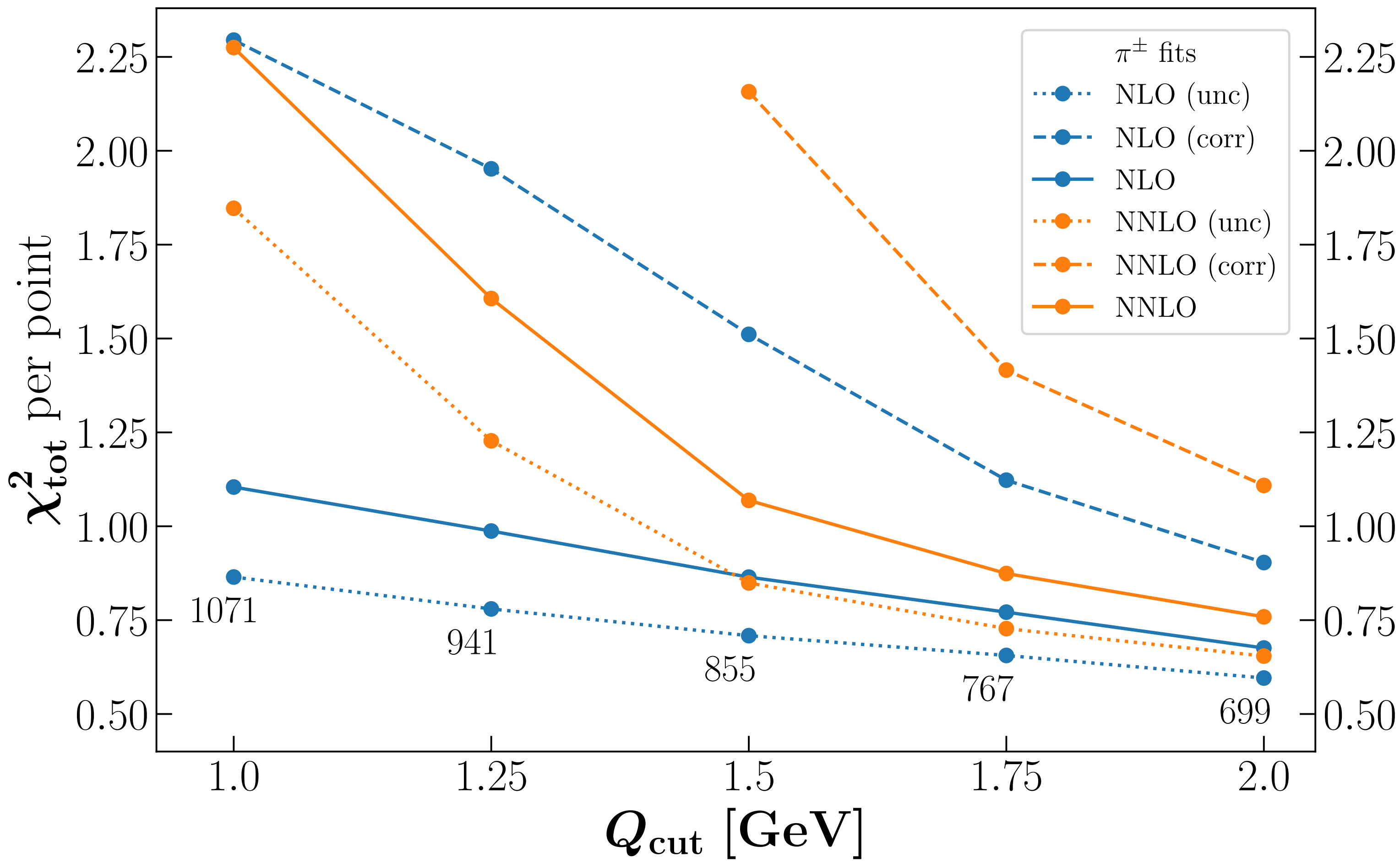
- For the other values of n one can just **fit** the behaviour in ξ and check that it follows the **expected power law**.

FFs at NNLO

NLO vs. NNLO

- While both MAPFF1.0 and BDSS confirm that COMPASS high-Q data is better described by NNLO, it is not clear as yet where NNLO starts doing better than NLO.

Bertone *et al.* [arXiv:2204.10331]



Borsa *et al.* [arXiv:2202.05060]

Experiment	$Q^2 \geq 1.5\,\mathrm{GeV}^2$			$Q^2 \geq 2.0\,\mathrm{GeV}^2$			$Q^2 \geq 2.3\,\mathrm{GeV}^2$			$Q^2 \geq 3.0\,\mathrm{GeV}^2$		
	#data	NLO	NNLO	#data	NLO	NNLO	#data	NLO	NNLO	#data	NLO	NNLO
SIA	288	1.05	0.96	288	0.91	0.87	288	0.90	0.91	288	0.93	0.86
COMPASS	510	0.98	1.14	456	0.91	1.04	446	0.91	0.92	376	0.94	0.93
HERMES	224	2.24	2.27	160	2.40	2.08	128	2.71	2.35	96	2.75	2.26
TOTAL	1022	1.27	1.33	904	1.17	1.17	862	1.17	1.13	760	1.16	1.07

Reconstructing GTMDs

- We can evolve $F_{1,1}^{i,e}$ to *any* scale by solving the evolution equations:
  - $\mathcal{O}(\alpha_s)$ matching functions allow us to reach **NNLL accuracy**. Anomalous dimensions (that coincide with the TMD ones) need to be evaluated accordingly.
- Extrapolation to large $|\mathbf{b}_T|$ is obtained *a la* CSS, *i.e.* by means of a b_* prescription:

$$b_*(b_T) = \frac{b_0}{Q}\left(\frac{1-\exp\left(-\frac{b_T^4 Q^4}{b_0^4}\right)}{1-\exp\left(-\frac{b_T^4}{b_0^4}\right)}\right)^{\frac{1}{4}}$$

- and introducing an appropriate non-perturbative function $f_{\rm NP}$. The final result is:

$$\begin{aligned}F_{1,1}^{i,e}(x,\xi,b_T,t,\mu,\zeta) &= \mathcal{C}_{i/j}(x,\kappa,b_*,\mu_{b_*},\mu_{b_*}^2)\underset{x}{\otimes}\left[(1-\xi^2)H_j(x,\xi,t,\mu_{b_*})-\xi^2E_j(x,\xi,t,\mu_{b_*})\right]\\ &\times\quad R_i\left[(\mu,\zeta)\leftarrow(\mu_{b_*},\mu_{b_*}^2)\right]\\ &\times\quad f_{\rm NP}(x,b_T,(1-\xi^2)\zeta)\end{aligned}$$

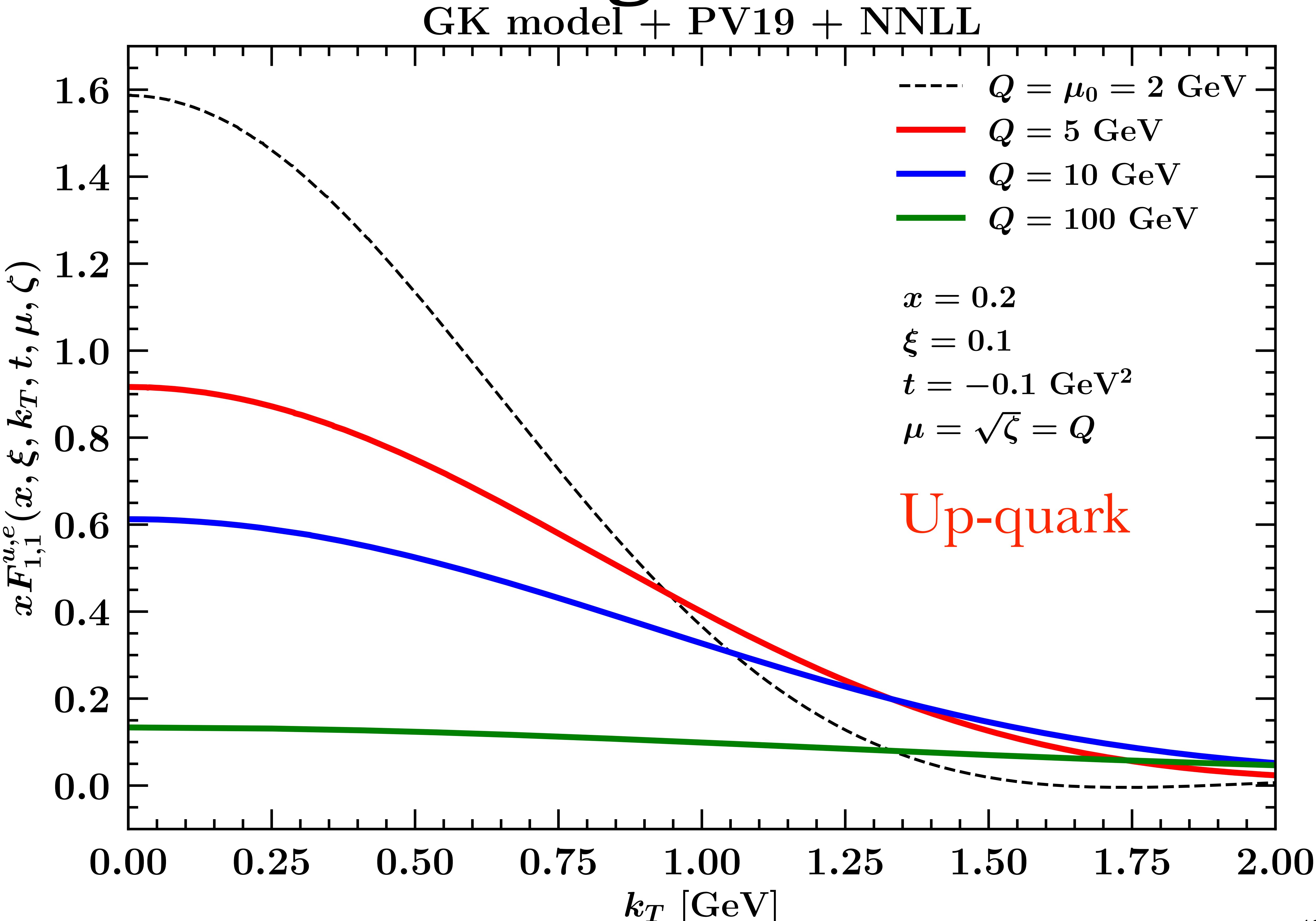
- The evolution operator (or Sudakov form factor) is given by:

$$R_i = \exp\left\{K_i(b_*,\mu_{b_*})\ln\frac{\sqrt{(1-\xi^2)\zeta}}{\mu_{b_*}}+\int_{\mu_{b_*}}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_{F,i}(\alpha_s(\mu'))-\gamma_{K,i}(\alpha_s(\mu'))\ln\frac{\sqrt{(1-\xi^2)\zeta}}{\mu'}\right]\right\}$$

- Finally the GTMDs in $\mathbf{k}_T$ space are obtained by inverse Fourier transform:

$$F_{1,1}^{i,e}(x,\xi,k_T,t,\mu,\zeta) = \frac{1}{2\pi}\int_0^\infty db_T\, b_T J_0(k_T b_T) F_{1,1}^{i,e}(x,\xi,b_T,t,\mu,\zeta)$$

Reconstructing GTMDs



Reconstructing GTMDs