Theoretical issues related to rare kaon decays

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GDR-Intensity Annual Workshop, Cabourg, Nov. 6-8, 2024

OUTLINE

- Introduction
- On short-distance singularities
- $K \to \pi \ell^+ \ell^-$ in the large-N_c limit
- Conclusion

Introduction

Radiative kaon decay modes mediated by neutral-current flavour transitions

(at short distances: W boxes, Z and γ penguins)

- $K \to \gamma^{(*)}\gamma^{(*)}$ e.g. $K_{S,L} \to \gamma\gamma$, $K_{S,L} \to \gamma\ell^+\ell^-$, $K_{S,L} \to \ell^+\ell^-$, $K_{S,L} \to \ell_1^+\ell_1^-\ell_2^+\ell_2^-$
- $K \to \pi\gamma^*$ e.g. $K^\pm \to \pi^\pm\ell^+\ell^-$, $K_{S,L} \to \pi^0\ell^+\ell^-$
- $K \to \pi\gamma^{(*)}\gamma^{(*)}$ e.g. $K_L \to \pi^0\ell^+\ell^-$
- $K \to \gamma\gamma\gamma^{(*)}$
- $\cdots$ V. Cirigliano, G. Ecker, H. Neufeld, A. Pich, J. Portoles, Rev. Mod. Phys. **84**, 399 (2012).

$$\gamma^* \to \ell^+\ell^- \qquad \ell = e, \mu$$

Decay rates naturally highly suppressed S. L. Glashow, J. Iliopoulos and L. Maiani, Phys. Rev. D **2**, 1285 (1970)

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predictions? $\longrightarrow$ determination of the low-energy constants

Most strategies implemented in the strong sector cannot easily be transposed to the weak sector

Need to take a step back

strong sector: ChPT $\longrightarrow$ three-flavour QCD

weak sector: ChPT $\longrightarrow$ three-flavour QCD "augmented" by six four-fermion operators

$$\mathcal{L}^{\rm B}_{\rm QCD} - \frac{G_{\rm F}}{\sqrt{2}} V_{us} V_{ud} \sum_{I=1}^{6} C^{\rm B}_I Q^{\rm B}_I(x) \to \mathcal{L}_{\rm QCD} - \frac{G_{\rm F}}{\sqrt{2}} V_{us} V_{ud} \sum_{I=1}^{6} C_I(\nu) Q_I(x;\nu)$$

$$Q_1 = (\bar{s}^i u_j)_{V-A} (\bar{u}^j d_i)_{V-A} \qquad Q_2 = (\bar{s}^i u_i)_{V-A} (\bar{u}^j d_j)_{V-A}$$

$$Q_3 = (\bar{s}^i d_i)_{V-A} \sum_{q=u}^{s} (\bar{q}^j q_j)_{V-A} \qquad Q_4 = (\bar{s}^i d_j)_{V-A} \sum_{q=u}^{s} (\bar{q}^j q_i)_{V-A}$$

$$Q_5 = (\bar{s}^i d_i)_{V-A} \sum_{q=u}^{s} (\bar{q}^j q_j)_{V+A} \qquad Q_6 (\bar{s}^i d_j)_{V-A} \sum_{q=u}^{s} (\bar{q}^j q_i)_{V+A}$$

$$C_I = z_I + \tau y_I \qquad \tau = -\frac{V_{td} V^*_{ts}}{V_{ud} V_{us}} = -\frac{\lambda_t}{V_{ud} V_{us}} = (1.47 - i0.64) \cdot 10^{-3} \qquad y_1 = y_2 = 0$$

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$$\mathcal{L}^{\mathrm{B}}_{\mathrm{QCD}} - \frac{G_{\mathrm{F}}}{\sqrt{2}} V_{us} V_{ud} \sum_{I=1}^{6} C_I^{\mathrm{B}} Q_I^{\mathrm{B}}(x) \to \mathcal{L}_{\mathrm{QCD}} - \frac{G_{\mathrm{F}}}{\sqrt{2}} V_{us} V_{ud} \sum_{I=1}^{6} C_I(\nu) Q_I(x;\nu)$$

SM only required to provide the list of appropriate operators and the values of their "couplings" (Wilson coefficients) at some scale $\nu_0 \sim 1$ GeV (include RG-improved resummation of LL and NLL in pQCD)

problem shifted to the evaluation of the matrix elements of the Q_I's

problem simplifies in the 't Hooft limit $N_c \to \infty$, $N_c \alpha_s = \mathrm{cst}$

G. 't Hooft, Nucl. Phys. B **72**, 461 (1974); E. Witten, Nucl. Phys. B **160**, 57 (1979)

On short-distance singularities

The description of the amplitudes for radiative decays will now involve hadronic matrix elements of

$$T\{j_\mu(x)Q_I(0;\nu)\} \quad T\{j_\mu(x)j_\nu(y)Q_I(0;\nu)\} \quad j_\mu \equiv \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s$$

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These time-ordered products are singular at short distances!

$$\lim_{x\to 0} T\{j_\rho(x)Q_1(0;\nu)\} \sim -\frac{1}{18\pi^4}[\bar{s}\gamma_\mu(1-\gamma_5)d](0)\left(\delta_\rho^\mu \Box - \partial_\rho\partial^\mu\right)\frac{1}{(x^2)^2} + \cdots$$

G. Isidori, G. Martinelli, P. Turchetti, Phys. Lett. B **633**, 75 (2006)

G. D'Ambrosio, D. Greynat, M. Knecht, JHEP **02**, 049 (2019); Phys. Lett. B **797**, 134891 (2019)

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Does not concern process where ALL photons are real
e.g. $K \to \gamma\gamma$, $K \to \pi\gamma\gamma$,...

Will be present as soon as there is at least ONE virtual photon
e.g. $K \to \gamma\ell^+\ell^-$, $K \to \pi\ell^+\ell^-$, $K \to \ell^+\ell^-$,...

Means that other (process-dependent) operators are involved, to be identified

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How do these SD singularities manifest themselves in concrete examples?

$K \to \pi\gamma^*$ at $N_c \to \infty$

Structure of the amplitudes

$$\mathcal{A}(K^+ \to \pi^+ \ell^+ \ell^-) = e^2 \bar{\mathrm{u}}(p_{\ell^-})\gamma^\rho \mathrm{v}(p_{\ell^+}) \times \frac{1}{s}\left[(k-p)_\rho (M_K^2 - M_\pi^2) - s(k+p)_\rho\right] \times \frac{\mathcal{W}_+(s)}{16\pi^2 M_K^2}$$

$$\mathcal{A}(K_S \to \pi^0 \ell^+ \ell^-) = e^2 \bar{\mathrm{u}}(p_{\ell^-})\gamma^\rho \mathrm{v}(p_{\ell^+}) \times \frac{1}{s}\left[(k-p)_\rho (M_K^2 - M_\pi^2) - s(k+p)_\rho\right] \times \frac{\mathcal{W}_S(s)}{16\pi^2 M_K^2}$$

Have been studied in ChPT

$$\mathcal{W}_{+,S} = G_F M_K^2 \left[a_{+,S} + b_{+,S}\frac{s}{M_K^2} + \mathcal{V}^{\pi\pi}_{+,S}(s; \alpha_{+,S}, \beta_{+,S}) \right]$$

G. Ecker, A. Pich, E. de Rafael, Nucl. Phys. B **291**, 692 (1987)

G. D'Ambrosio, G. Ecker, G. Isidori, J. Portolés, JHEP **08**, 004 (1998) [arXiv:hep-ph/9808289 [hep-ph]]

$\alpha_{+,S}$ and $\beta_{+,S}$ can be extracted from data on $K \to \pi\pi\pi$ decays

G. D'Ambrosio, M. K. and S. Neshatpour, Phys. Lett. B **835**, 137594 (2022) [arXiv:2209.02143 [hep-ph]]

$a_{+,S}$ ($b_{+,S}$) $\longrightarrow$ unknown LECs at NLO (NNLO)

In practice, the contributions from pion loops to the decay rates are marginal

$$a_{+,S}, b_{+,S} \sim \mathcal{O}(N_c) \qquad \mathcal{V}^{\pi\pi}_{+,S}(s) \sim \mathcal{O}(N_c^0)$$

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$$\mathcal{A}(K_L \to \pi^0\ell^+\ell^-) = \mathcal{A}(K_L \to \pi^0\ell^+\ell^-)|_{\mathrm{mix}} + \mathcal{A}(K_L \to \pi^0\ell^+\ell^-)|_{\mathrm{dir}} + \mathcal{A}(K_L \to \pi^0\ell^+\ell^-)|_{\mathrm{CPC}}$$

$$\mathcal{A}(K_L \to \pi^0\ell^+\ell^-)|_{\mathrm{mix}} = \bar{\epsilon}\mathcal{A}(K_S \to \pi^0\ell^+\ell^-)$$

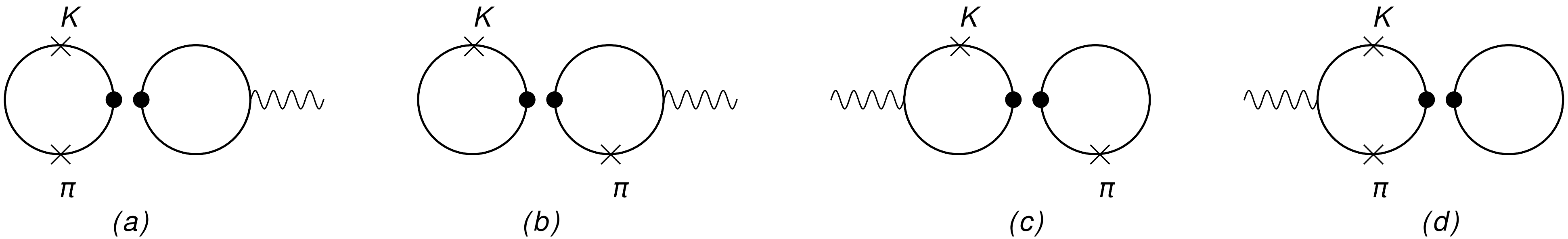
$$|K_S\rangle \simeq |K_1^0\rangle = \frac{|K^0\rangle - |\overline{K}^0\rangle}{\sqrt{2}} \qquad |K_L\rangle \simeq |K_2^0\rangle + \bar{\epsilon}|K_1^0\rangle \qquad |K_2^0\rangle = \frac{|K^0\rangle + |\overline{K}^0\rangle}{\sqrt{2}}$$

$$\bar{\epsilon} \sim \frac{1+i}{\sqrt{2}}|\epsilon|, \ |\epsilon| = 2.228 \cdot 10^{-3}.$$

$$\mathcal{A}(K_L \to \pi^0\ell^+\ell^-)|_{\mathrm{CPC}} = \mathcal{A}(K_2^0 \to \pi^0\gamma^*\gamma^* \to \pi^0\ell^+\ell^-) \sim \mathcal{O}(\alpha^2 G_{\mathrm{F}})$$

When $N_c \to \infty$, the four-quark operators factorize

$\longrightarrow$ hadronic matrix elements of bilinear quark operators



Involves a certain number of form factors related to the matrix elements

$$\langle \pi^0(p)|[\bar{s}\gamma_\mu d](0)|K_S(k)\rangle = -[(k+p)_\mu f_+(s) + (k-p)_\mu f_-(s)]$$

$$(\Gamma^\pi_{VP})_\rho(q,p) = i\int d^4x\, e^{iq\cdot x}\langle \pi^+(p)|T\{\frac{1}{2}[\bar{u}\gamma_\rho u - \bar{d}\gamma_\rho d](x)[\bar{u}i\gamma_5 d](0)\}|0\rangle$$
$$(\Gamma^K_{VP})_\rho(q,k) \equiv i\int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{u}\gamma_\rho u](x)[\bar{s}i\gamma_5 u](0)\}|K^+(k)\rangle$$
$$(\tilde{\Gamma}^K_{VP})_\rho(q,k) \equiv i\int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{s}\gamma_\rho s](x)[\bar{s}i\gamma_5 u](0)\}|K^+(k)\rangle$$

AND

$$i\int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{u}\gamma_\mu u](x)[\bar{u}\gamma_\nu u](0)\}|0\rangle_{\overline{\mathrm{MS}}} = (q_\mu q_\nu - q^2\eta_{\mu\nu})\Pi_{\overline{\mathrm{MS}}}(q^2;\nu)$$

arXiv:2409.08568 [hep-ph]

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$$q^\rho(\Gamma^K_{VP})_\rho(q,k) = -\sqrt{2}F_K\frac{M_K^2}{m_s+\hat{m}}, \qquad q^\rho(\tilde{\Gamma}^K_{VP})_\rho(q,k) = +\sqrt{2}F_K\frac{M_K^2}{m_s+\hat{m}}.$$

$$\begin{aligned}(m_s+\hat{m})(\Gamma^K_{VP})_\rho(q,k) \quad = \quad & \sqrt{2}F_K M_K^2\frac{(2k-q)_\rho}{(q-k)^2-M_K^2}F_u^K(q^2) + \sqrt{2}F_K M_K^2\frac{F_u^K(q^2)-1}{q^2}q_\rho \\ & +\sqrt{2}[q^2k_\rho-(q\cdot k)q_\rho]\mathcal{P}^K(q^2,(q-k)^2)\end{aligned}$$

$$\begin{aligned}(m_s+\hat{m})(\tilde{\Gamma}^K_{VP})_\rho(q,k) \quad = \quad & \sqrt{2}F_K M_K^2\frac{(2k-q)_\rho}{(q-k)^2-M_K^2}F_s^K(q^2) + \sqrt{2}F_K M_K^2\frac{F_s^K(q^2)+1}{q^2}q_\rho \\ & +\sqrt{2}[q^2k_\rho-(q\cdot k)q_\rho]\tilde{\mathcal{P}}^K(q^2,(q-k)^2)\end{aligned}$$

$$\langle K^+(k')|[\bar{u}\gamma_\rho u](0)|K^+(k)\rangle = (k'+k)_\rho F_u^K(q^2), \quad \langle K^+(k')|[\bar{s}\gamma_\rho s](0)|K^+(k)\rangle = (k'+k)_\rho F_s^K(q^2)$$

arXiv:2409.08568 [hep-ph]

$$i \int d^d x \, e^{iq\cdot x} \langle 0 | T\{[\bar{u}\gamma_\mu u](x)[\bar{u}\gamma_\nu u](0)\}|0\rangle = (q_\mu q_\nu - q^2 \eta_{\mu\nu}) \left[\Pi_{\overline{\rm MS}}(q^2;\nu) + \mathcal{O}\left(\frac{1}{d-4}\right)\right]$$

The divergence is absorbed by the “bare” Gilman-Wise term

$$\mathcal{L}^{\rm B}_{\rm QCD} - \frac{G_{\rm F}}{\sqrt{2}} V_{us} V_{ud} \Big[\sum_{I=1}^{6} C^{\rm B}_I Q^{\rm B}_I(x) + C^{\rm B}_{7V} Q_{7V} \Big] \to \mathcal{L}_{\rm QCD} - \frac{G_{\rm F}}{\sqrt{2}} V_{us} V_{ud} \Big[\sum_{I=1}^{6} C_I(\nu) Q_I(x;\nu) + C_{7V}(\nu) Q_{7V} \Big]$$

$$Q_{7V} = (\bar{\ell}\gamma_\mu \ell)[\bar{s}\gamma^\mu(1-\gamma_5)d] \qquad C_{7V}(\nu) = z_{7V}(\nu) + \tau y_{7V}(\nu)$$

F. J. Gilman and M. B. Wise, Phys. Rev. D **20**, 2392 (1979); Phys. Rev. D **21**, 3150 (1980)

$$\frac{\mathcal{W}_+(s)}{16\pi^2 M_K^2} = +\frac{G_F}{\sqrt{2}} V_{us} V_{ud} \times \Bigg\{ f_+(s) \times \left[\frac{2}{3} \Pi_{\overline{\rm MS}}(s;\nu)({\rm Re}\, C_1 - {\rm Re}\, C_4) + \frac{{\rm Re}\, C_{7V}(\nu)}{4\pi\alpha} \right]$$

$$+({\rm Re}\, C_2 + {\rm Re}\, C_4) \times \Bigg[- F_K \mathcal{P}^\pi(s, M_K^2) - \frac{2}{3} F_\pi \mathcal{P}^K(s, M_\pi^2) + \frac{1}{3} F_\pi \tilde{\mathcal{P}}(s, M_\pi^2)$$

$$- \frac{2}{3} \frac{F_K F_\pi}{M_K^2 - M_\pi^2} \left(3 M_\pi^2 \frac{F_V^\pi(s) - 1}{s} - 2 M_K^2 \frac{F_u^K(s) - 1}{s} + M_K^2 \frac{F_s^K(s) + 1}{s} \right) \Bigg] + \cdots \Bigg\}$$

$$\frac{\mathcal{W}_S(s)}{16\pi^2 M_K^2} = -\frac{G_F}{\sqrt{2}} V_{us} V_{ud} \times \Bigg\{ f_+(s) \times \left[\frac{2}{3} \Pi_{\overline{\rm MS}}(s;\nu)({\rm Re}\, C_1 - {\rm Re}\, C_4) + \frac{{\rm Re}\, C_{7V}(\nu)}{4\pi\alpha} \right]$$

$$+({\rm Re}\, C_1 - {\rm Re}\, C_4) \times \Bigg[\frac{2}{3} \frac{F_\pi F_K M_K^2}{M_K^2 - M_\pi^2} \frac{F_u^K(s) + F_s^K(s)}{s}$$

$$- \frac{F_\pi}{3} \mathcal{P}^K(s, M_\pi^2) - \frac{F_\pi}{3} \mathcal{P}^K(s, M_\pi^2) \Bigg] + \cdots \Bigg\}$$

- $|C_I(\nu)| \ll |C_{1,2}(\nu)|$ at $\nu \sim 1\ \mathrm{GeV}$ for $I = 3, 4, 5, 6$
- Q_2 does not contribute to $\mathcal{W}_S(s)$ in the large-N_c limit
- In the large-N_c limit

$$\nu \frac{d}{d\nu} \frac{C_{7V}(\nu)}{4\pi\alpha} = -\frac{2}{3} [C_1 - C_4] \times \nu \frac{d}{d\nu} \Pi_{\overline{\rm MS}}(q^2;\nu)$$

[can be checked explicitly at order $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$]

Lowest-meson-dominance approximation to the large-N_c limit:

one resonance per channel (pseudo-scalar channel is saturated by the kaon state)

S. Peris, M. Perrottet, E. de Rafael, JHEP 05, 011 (1998)

$$F^{\pi}_{V;\,\mathrm{VMD}}(s) = \frac{M_\rho^2}{M_\rho^2 - s}, \quad F^{K}_{u;\,\mathrm{VMD}}(s) = \frac{M_\rho^2}{M_\rho^2 - s}, \quad F^{K}_{s;\,\mathrm{VMD}}(s) = \frac{M_\phi^2}{s - M_\phi^2}. \quad f_+(s) \sim \frac{1}{M_{K^*}^2 - s}$$

$\mathcal{P}^\pi(q^2,(q+p)^2), \mathcal{P}^K(q^2,(q-k)^2), \tilde{\mathcal{P}}^K(q^2,(q-k)^2)$ are described by poles due to radial excitations of the pion or kaon, e.g. $\pi(1300)$

$$\mathcal{P}^\pi(q^2, M_K^2) \simeq 0, \quad \mathcal{P}^K(q^2, M_\pi^2) \simeq 0, \quad \tilde{\mathcal{P}}^K(q^2, M_\pi^2) \simeq 0$$

This simple picture with a single resonance does not work in the case of $\Pi_{\overline{\rm MS}}(s;\nu)$

$$\Pi_{\overline{\rm MS}}(s;\nu) = \frac{2f_\rho^2 M_\rho^2}{M_\rho^2 - s} + \cdots$$

The asymptotic logarithmic behaviour must involve an *infinite* number of resonances

$$\Pi_{\overline{\rm MS}}(s;\nu) = \frac{2f_\rho^2 M_\rho^2}{M_\rho^2 - s} + \frac{1}{4\pi^2}\frac{N_c}{3}\left[\frac{5}{3} - \ln(M^2/\nu^2) - \psi\left(3 - \frac{s}{M^2}\right)\right]$$

$$\psi\left(3 - \frac{s}{M^2}\right) = -\gamma_E + \frac{3}{2} + \sum_{n\geq 1}\frac{1}{n+2}\,\frac{s}{s - M_n^2} \qquad M_n = \sqrt{n+2}M$$

$$\psi\left(3 - \frac{s}{M^2}\right) \sim \ln(-s/M^2) - \frac{5}{2}\frac{M^2}{s} + \cdots \qquad [s \to -\infty]$$

In QCD, the Adler function $s\partial\Pi_{\overline{\rm MS}}(s;\nu)/\partial s$ does not have a term $\propto 1/s$ in its asymtotic expansion (in the chiral limit)

S. Peris, M. Perrottet, E. de Rafael, JHEP 05, 011 (1998)

$$\longrightarrow\ M^2 = \frac{16\pi^2}{5}\frac{3}{N_c}f_\rho^2 M_\rho^2 \qquad M \sim 0.9\ \text{GeV} \qquad M_1 \sim 1.5\ \text{GeV} \sim M_{\rho(1450)}$$

Some consequences of phenomenological interest arXiv:2409.08568 [hep-ph]

Branching fractions [with input from A. J. Buras et al., Nucl. Phys. B 423, 349 (1994)]

$$\mathrm{Br}(K_S \to \pi^0 e^+ e^-)|_{m_{ee}>165\ \mathrm{MeV}} = 2.9(1.0)\cdot 10^{-9}$$
$$\mathrm{Br}^{\mathrm{exp}}(K_S \to \pi^0 e^+ e^-)|_{m_{ee}>165\ \mathrm{MeV}} = (3.0^{+1.5}_{-1.2} \pm 0.2)\cdot 10^{-9}$$

J. R. Batley *et al.* [NA48/1], Phys. Lett. B **576**, 43 (2003) [arXiv:hep-ex/0309075 [hep-ex]]

$$\mathrm{Br}(K_S \to \pi^0 \mu^+ \mu^-) = 1.3(0.4)\cdot 10^{-9}$$
$$\mathrm{Br}^{\mathrm{exp}}(K_S \to \pi^0 \mu^+ \mu^-) = (2.9^{+1.5}_{-1.2}(\mathrm{stat}) \pm 0.2(\mathrm{syst}))\cdot 10^{-9}$$

J. R. Batley *et al.* [NA48/1], Phys. Lett. B **599**, 197 (2004) [arXiv:hep-ex/0409011 [hep-ex]]

Some consequences of phenomenological interest

arXiv:2409.08568 [hep-ph]

Direct CPV in $K_L \to \pi^0 \ell^+ \ell^-$

$$\mathrm{Br}(K_L \to \pi^0 \ell^+ \ell^-) = 10^{-12} \left[C_{\mathrm{mix}}^{(\ell)} + C_{\mathrm{int}}^{(\ell)} \frac{\mathrm{Im}\,\lambda_t}{10^{-4}} + C_{\mathrm{dir}}^{(\ell)} \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}} \right)^2 + C_{\gamma^* \gamma^*}^{(\ell)} \right].$$

$$C_{\gamma^* \gamma^*}^{(e)} \sim \mathcal{O}(10^{-2}) \qquad\qquad C_{\gamma^* \gamma^*}^{(\mu)} = 5.2(1.6)$$

J. F. Donoghue, B. R. Holstein, G. Valencia, Phys. Rev. D **35**, 2769 (1987)

G. Ecker, A. Pich, E. de Rafael, Nucl. Phys. B **303**, 665 (1988)

G. Buchalla, G. D'Ambrosio, G. Isidori, Nucl. Phys. B **672**, 387 (2003) [arXiv:hep-ph/0308008 [hep-ph]]

G. Isidori, C. Smith, R. Unterdorfer, Eur. Phys. J. C **36**, 57 (2004) [arXiv:hep-ph/0404127 [hep-ph]]

F. Mescia, C. Smith, S. Trine, JHEP **08**, 088 (2006) [arXiv:hep-ph/0606081 [hep-ph]]

$$C_{\mathrm{int}}^{(e)} = +7.8(2.6)\, \frac{y_{7V}}{\alpha} \qquad\qquad C_{\mathrm{int}}^{(\mu)} = +1.9(0.6)\, \frac{y_{7V}}{\alpha}$$

The interference between direct and indirect CP violation in the branching ratio for $K_L \to \pi^0 \ell^+ \ell^-$ is unambiguously predicted to be constructive in the large-N_c limit of QCD

Summary

ChPT provides the natural framework for the theoretical study of kaon physics...

... but it is difficult to make predictions for radiative decay modes of kaons without at least some knowledge of counterterms

Implement large-N_c approach directly in the underlying theory: QCD with three active flavours extended by an appropriate set of four-fermion operators (SM LEEFT)

Form factors for FCNC-induced transitions exhibit a more complex structure than in the strong sector, necessary to address the issue of QCD short-distance singularities

Large-N_c QCD interesting framework to study these aspects (seldom mentioned, never addressed)

In the case of $K \to \pi\gamma^*$ it even leads to interesting phenomenological results, cf. arXiv:2409.08568

Other radiative decay modes are being studied within this framework: $K \to \gamma^*\gamma^*$, ...