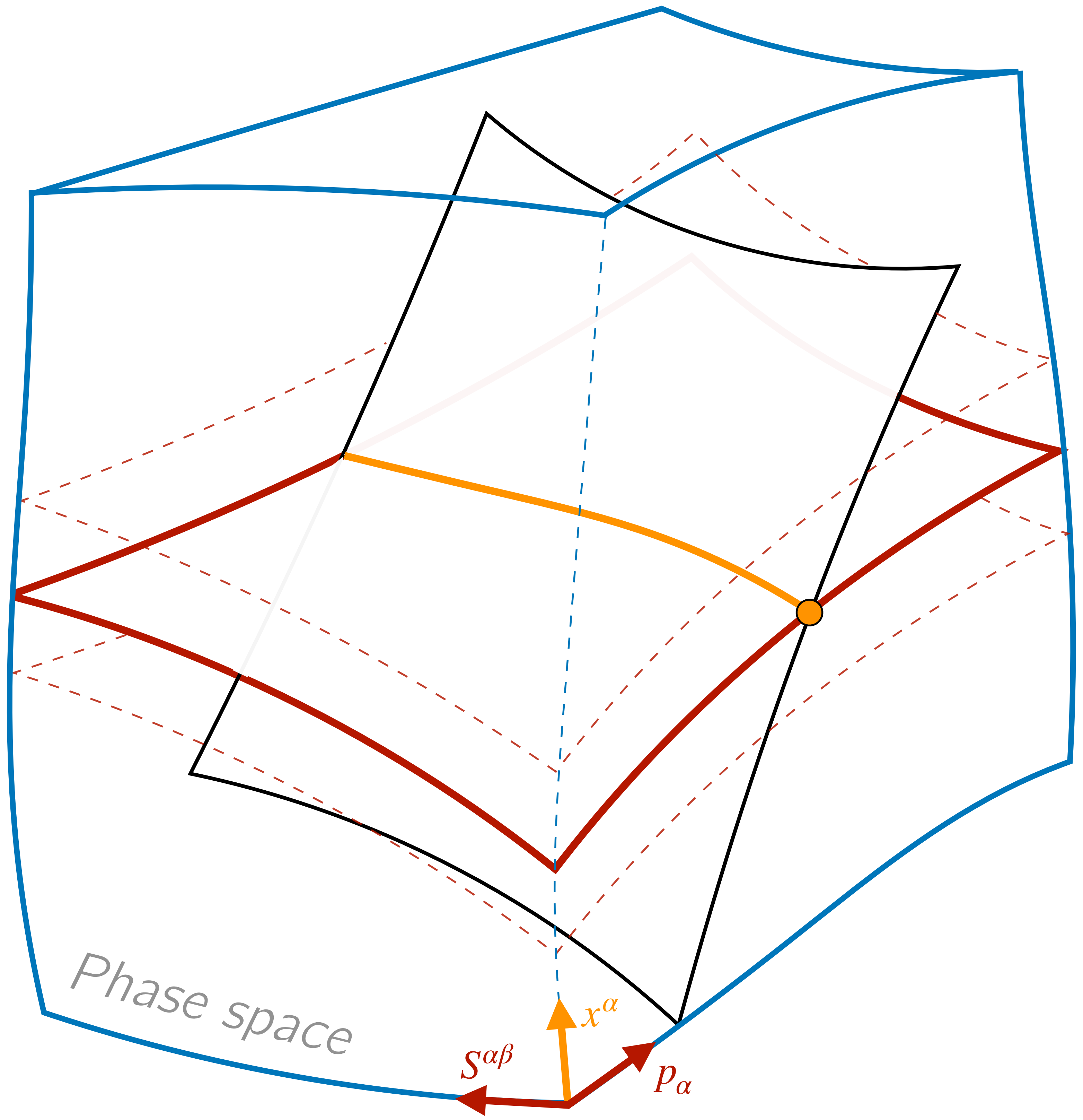
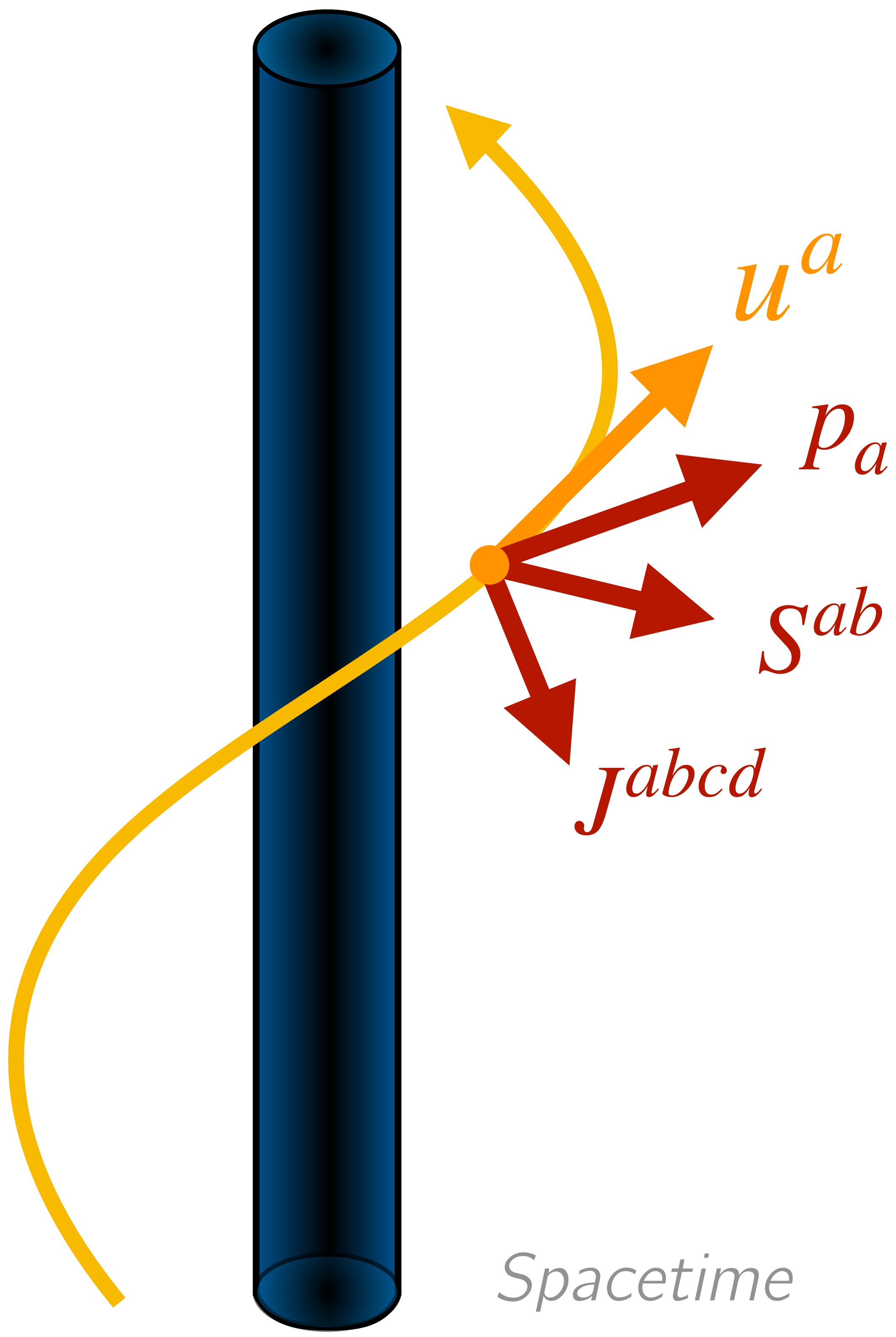
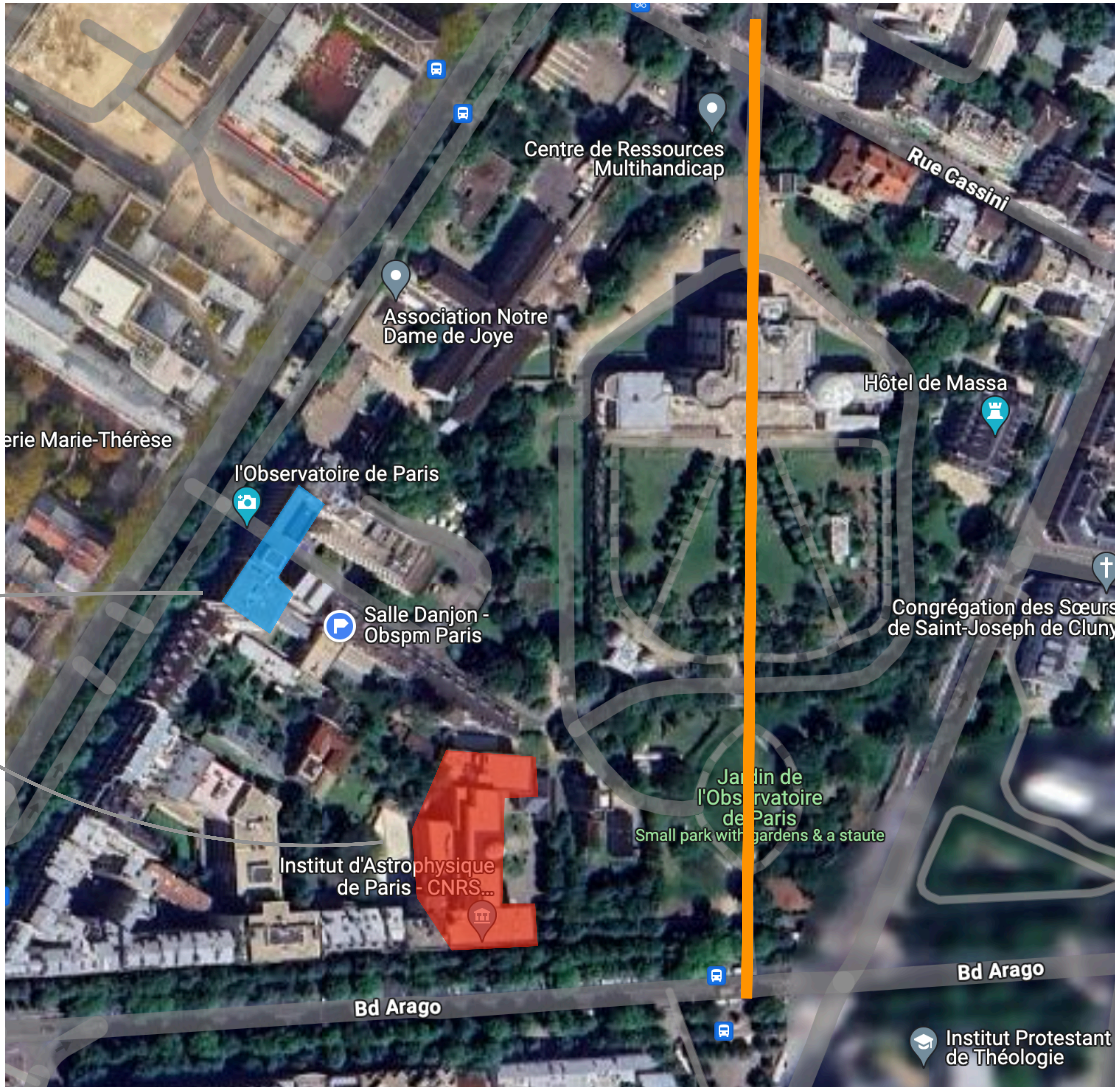
Integrable dynamics of *extended test bodies* around *rotating black holes*





This work's aim:



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with help from
Matthews, Stein, Fejoz,
Hughes-Drummond,
Scheopner-Vines,...

Plan

I. Geodesics

1. geodesic motion
2. hamiltonian formulation
3. integrable systems

II. Adding spin

1. linear-in-spin motion
2. hamiltonian formulation
3. integrability in Kerr

III. Quadrupoles

1. quadratic-in-spin motion
2. hamiltonian formulation
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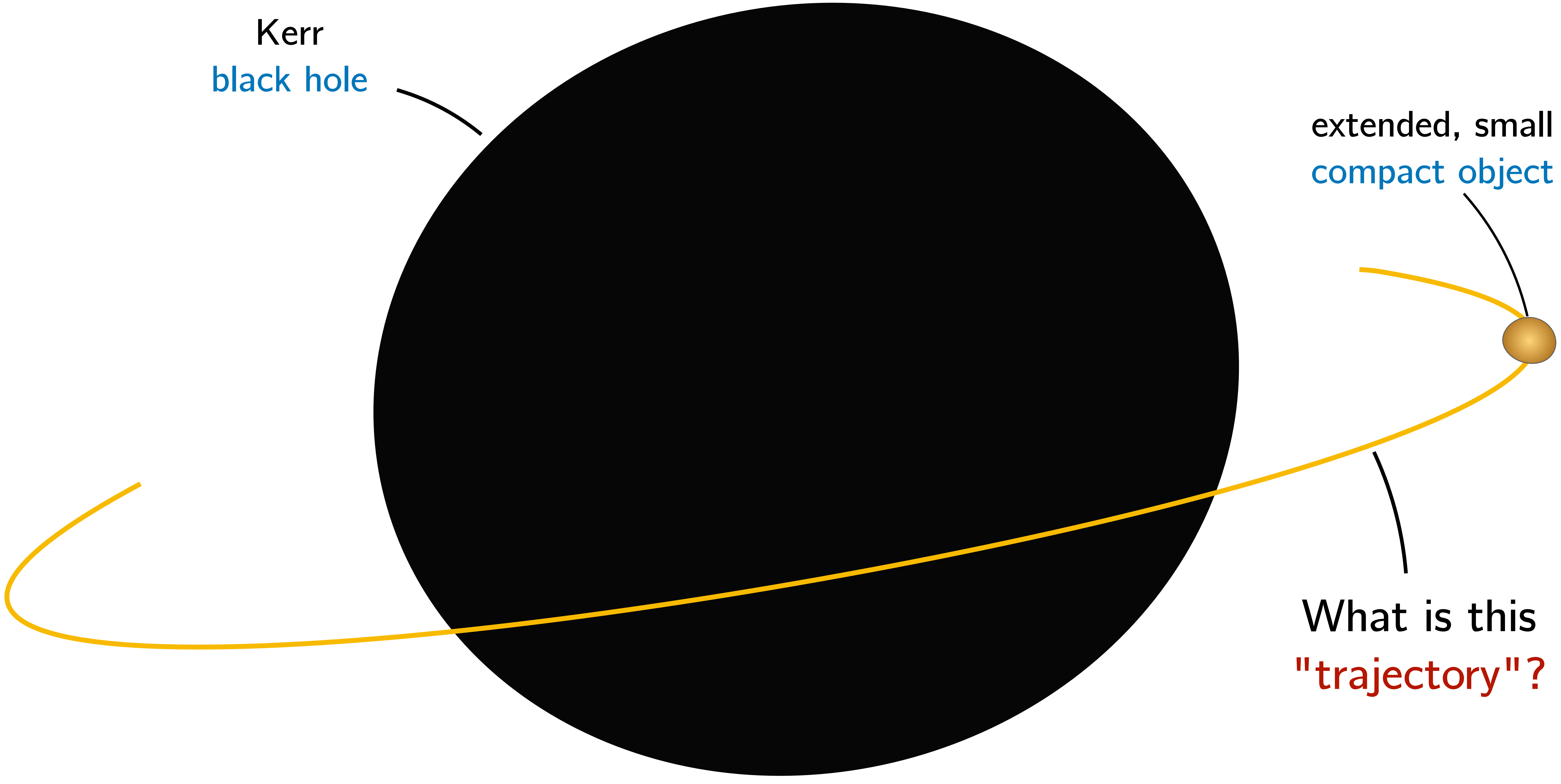
1. Geodesic motion around black holes



⇒ Relativistic mechanics

Die formale Grundlage der allgemeinen Relativitätstheorie.

Von A. Einstein.

1914

No field equations, but...

"From a mathematical point of view, the motion of a test point corresponds to a geodesic curve on the four-dimensional manifold"

$$u^\alpha \nabla_\alpha u^\beta = 0$$

Duale Sechservektoren. Ist ferner $(F^{\mu\nu})$ ein antisymmetrischer Tensor (zweiten Ranges), so können wir zu ihm einen zweiten antisymmetrischen Tensor $F^{\mu\nu *}$ bilden nach der Gleichung

$$F^{\mu\nu *} = \frac{1}{2} \sum_{\alpha\beta} G^{\mu\nu}_{\alpha\beta} F^{\alpha\beta} . \tag{24}$$

Man nennt $F^{\mu\nu *}$ den zu $F^{\mu\nu}$ »dualen« kontravarianten Sechservektor. Umgekehrt ist $F^{\mu\nu}$ zu $F^{\mu\nu *}$ dual. Denn multipliziert man (24) mit $G^{\sigma\tau}_{\mu\nu}$, und summiert über μ und ν, so erhält man

$$\frac{1}{2} \sum G^{\sigma\tau}_{\mu\nu} F^{\mu\nu *} = \frac{1}{4} \sum_{\alpha\beta\mu\nu} G^{\sigma\tau}_{\mu\nu} G^{\mu\nu}_{\alpha\beta} F^{\alpha\beta} ;$$

da aber nach (22)

$$\sum_{\mu\nu} G^{\sigma\tau}_{\mu\nu} G^{\mu\nu}_{\alpha\beta} = \sum_{\mu\nu\lambda\varkappa\lambda'\varkappa'} \sqrt{g}\, \delta_{\mu\nu\lambda\varkappa}\, g^{\lambda\sigma} g^{\varkappa\tau} \frac{1}{\sqrt{g}}\, \delta_{\mu\nu\lambda'\varkappa'}\, g_{\lambda'\alpha}\, g_{\varkappa'\beta} = 2(\delta^{\sigma}_{\alpha}\delta^{\tau}_{\beta} - \delta^{\sigma}_{\beta}\delta^{\tau}_{\alpha}) ,$$

ist[1], so ergibt sich

$$\frac{1}{4} \sum_{\alpha\beta\mu\nu} G^{\sigma\tau}_{\mu\nu} G^{\mu\nu}_{\alpha\beta} F_{\alpha\beta} = \frac{1}{2} (F^{\sigma\tau} - F^{\tau\sigma}) = F^{\sigma\tau} ,$$

woraus die Behauptung folgt.

Ganz Entsprechendes gilt für kovariante Sechservektoren. Man beweist ferner leicht, daß Sechservektoren, welche zwei dualen reziprok sind, selbst dual sind.

§ 7. Geodätische Linie bzw. Gleichungen der Punktbewegung.

In § 2 ist bereits dargelegt, daß die Bewegung eines materiellen Punktes im Gravitationsfelde nach der Gliederung

$$\delta\left\{\int ds\right\} = 0 \tag{1}$$

vor sich geht. Der Bewegung eines Punktes entspricht also vom mathematischen Standpunkte eine geodätische Linse in unserer vierdimensionalen Mannigfaltigkeit. Wir wollen der Vollständigkeit halber die

[1] Die zweite dieser Umformungen beruht darauf, daß $\delta_{\mu\nu\lambda\varkappa}$ nur dann nicht verschwindet, wenn alle Indizes verschieden sind. Es bleiben deshalb nur die beiden Möglichkeiten $(\lambda = \lambda', \varkappa = \varkappa')$ und $(\lambda = \varkappa', \varkappa = \lambda')$; mit Rücksicht darauf ergibt sich zunächst durch Summation über μ und ν der Ausdruck

$$2 \sum_{\lambda\varkappa} \left\{ g^{\lambda\sigma} g^{\varkappa\tau} g^{\lambda\alpha} g^{\varkappa\beta} - g^{\lambda\sigma} g^{\varkappa\tau} g^{\lambda\beta} g^{\varkappa\alpha} \right\} ,$$

wobei die Summe zunächst nur über solche Indexkombinationen $(\lambda\varkappa)$ zu erstrecken ist, für welche $\lambda \neq \varkappa$. Da aber die Klammer für $\lambda = \varkappa$ ohnehin verschwindet, so kann die Summe über alle Kombinationen erstreckt werden. Mit Rücksicht auf (10) ergibt sich hieraus der im Text angegebene Ausdruck.

wird auch in der verallgemeinerten Relativitätstheorie der Fall sein. Schließen wir den letzteren Fall $(ds = 0)$ von der Betrachtung aus, so können wir als Parameter λ die auf der geodätischen Linie gemessene »Bogenlänge« s wählen. Dann geht die Gleichung der geodätischen Linie über in

$$\sum_{\mu} g_{\sigma\mu} \frac{d^2 x_\mu}{ds^2} + \sum_{\mu\nu} \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} = 0 , \tag{23a}$$

wobei nach Christoffel die Abkürzung

$$\begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix} = \frac{1}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \right) \tag{24}$$

eingeführt ist, welcher Ausdruck bezüglich der Indizes μ und ν symmetrisch ist. Endlich multipliziert man (23a) mit $g^{\sigma\tau}$ und summiert über σ. Mit Rücksicht auf (10) und bei Benutzung des bekannten Christoffelschen Symbols

$$\begin{Bmatrix} \mu\nu \\ \tau \end{Bmatrix} = \sum_{\sigma} g^{\sigma\tau} \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix} \tag{24a}$$

erhält man dann an Stelle von (23a)

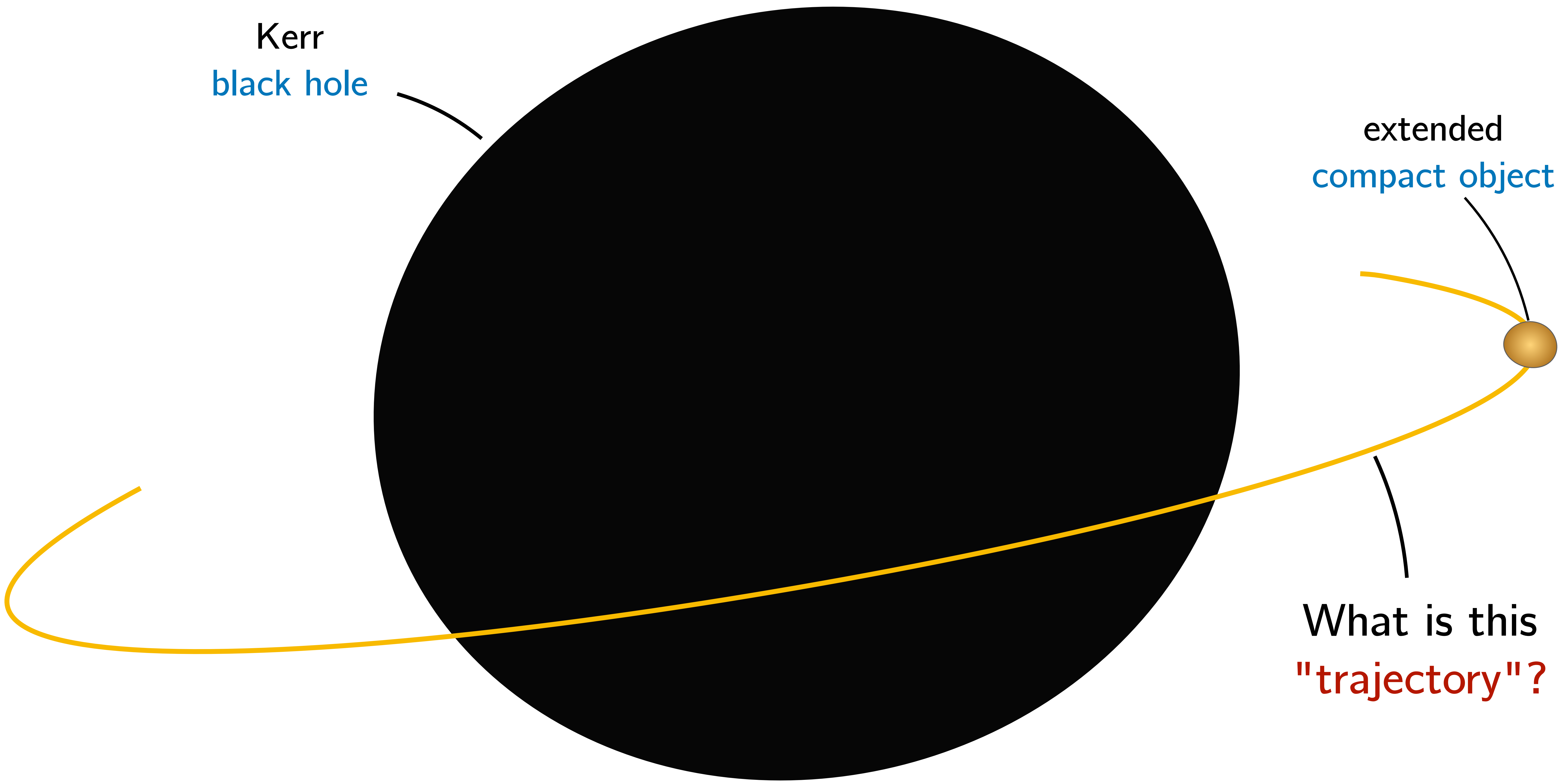
$$\frac{d^2 x_\tau}{ds^2} + \sum_{\mu\nu} \begin{Bmatrix} \mu\nu \\ \tau \end{Bmatrix} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} = 0 . \tag{23b}$$

Dies ist die Gleichung der geodätischen Linie in ihrer übersichtlichsten Form. Sie drückt die zweiten Ableitungen der x_τ nach s durch die ersten Ableitungen aus. Durch Differenzieren von (23b) nach s erhielte man Gleichungen, die auch eine Zurückführung der höheren Differenzialquotionenten bei Koordinaten nach s auf die ersten Ableitungen gestatten; man erhielte so die Koordinaten in Taylorscher Entwicklung nach den Variabeln s. Gleichung (23b) entspricht der Bewegungsgleichung des materiellen Punktes in Minkowskischer Form, indem s die »Eigenzeit« bedeutet.

§ 8. Bildung von Tensoren durch Differentiation.

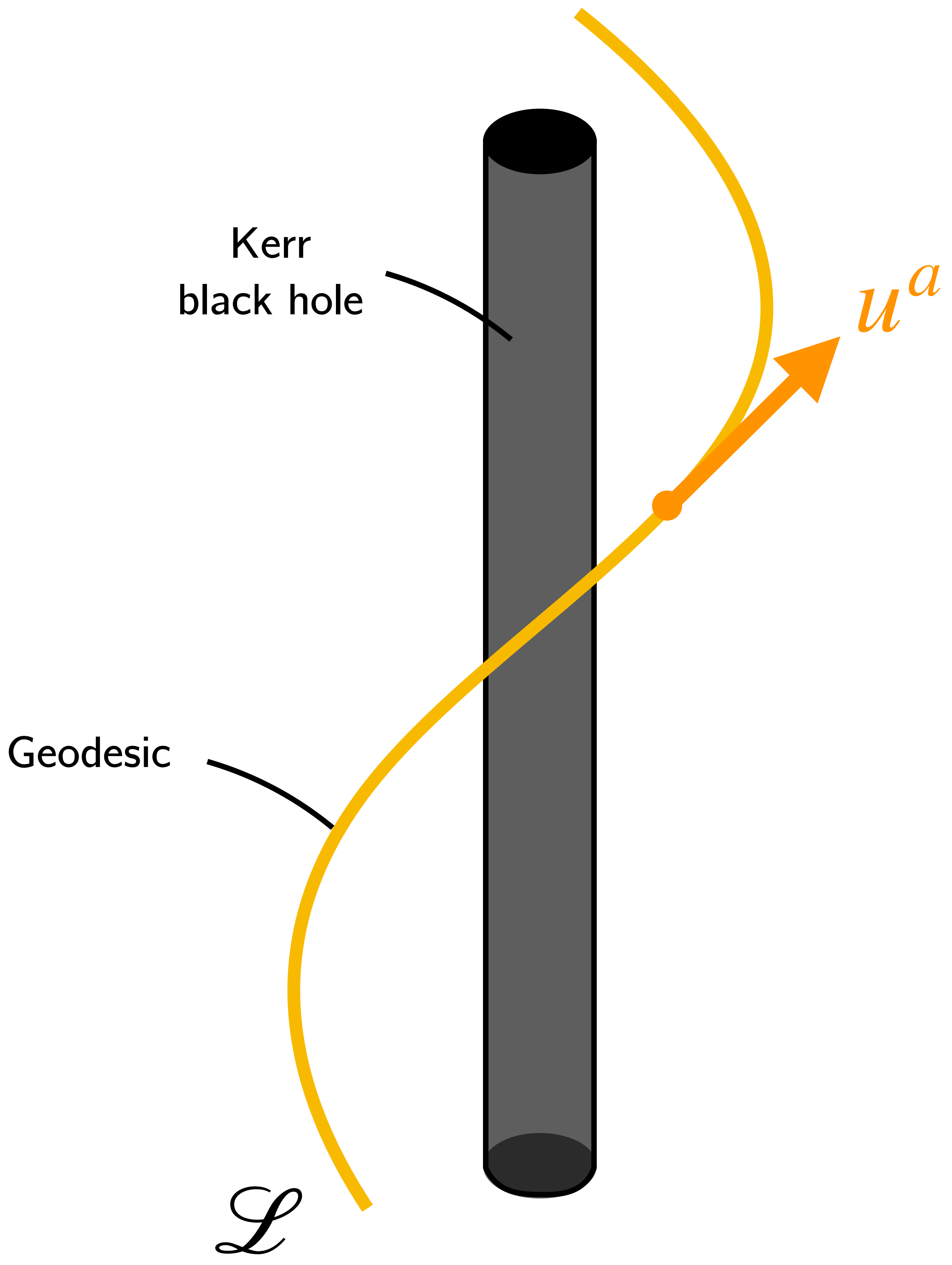
Die fundamentale Bedeutung des Tensorbegriffes beruht bekanntlich darauf, daß die Transformationsgleichungen für die Tensorkomponenten linear und homogen sind. Dies bringt es mit sich, daß die Komponenten eines Tensors bezüglich eines jeden beliebigen Koordinatensystems verschwinden, falls sie bezüglich eines Koordinatensystems verschwinden. Hat man also eine Gruppe von physikalischen Gleichungen in eine Form gebracht, welche das Verschwinden aller Komponenten eines Tensors aussagt, so hat dieses Gleichungssystem

How do things fall around black holes?



⇒ a Kerr geodesic!

Geodesics in spacetime



Kerr BH

- Coordinates $x^\alpha = (t, r, \theta, \phi)$
- Metric coefficients $g_{\alpha\beta}(x)$

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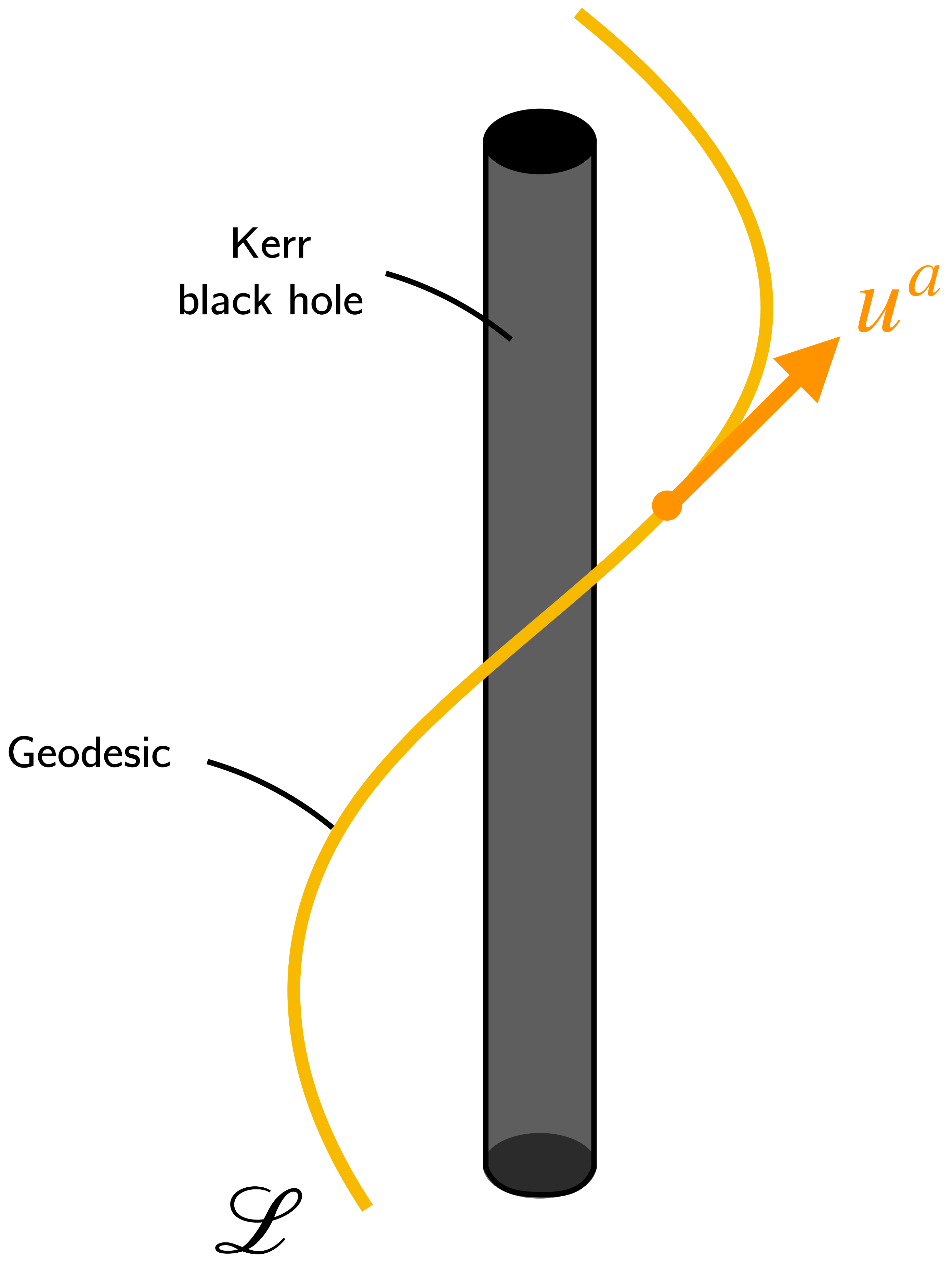

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Small object

- 4-momentum: $p^a := \mu u^a$
- mass: $\mu^2 := -g_{ab}p^a p^b$

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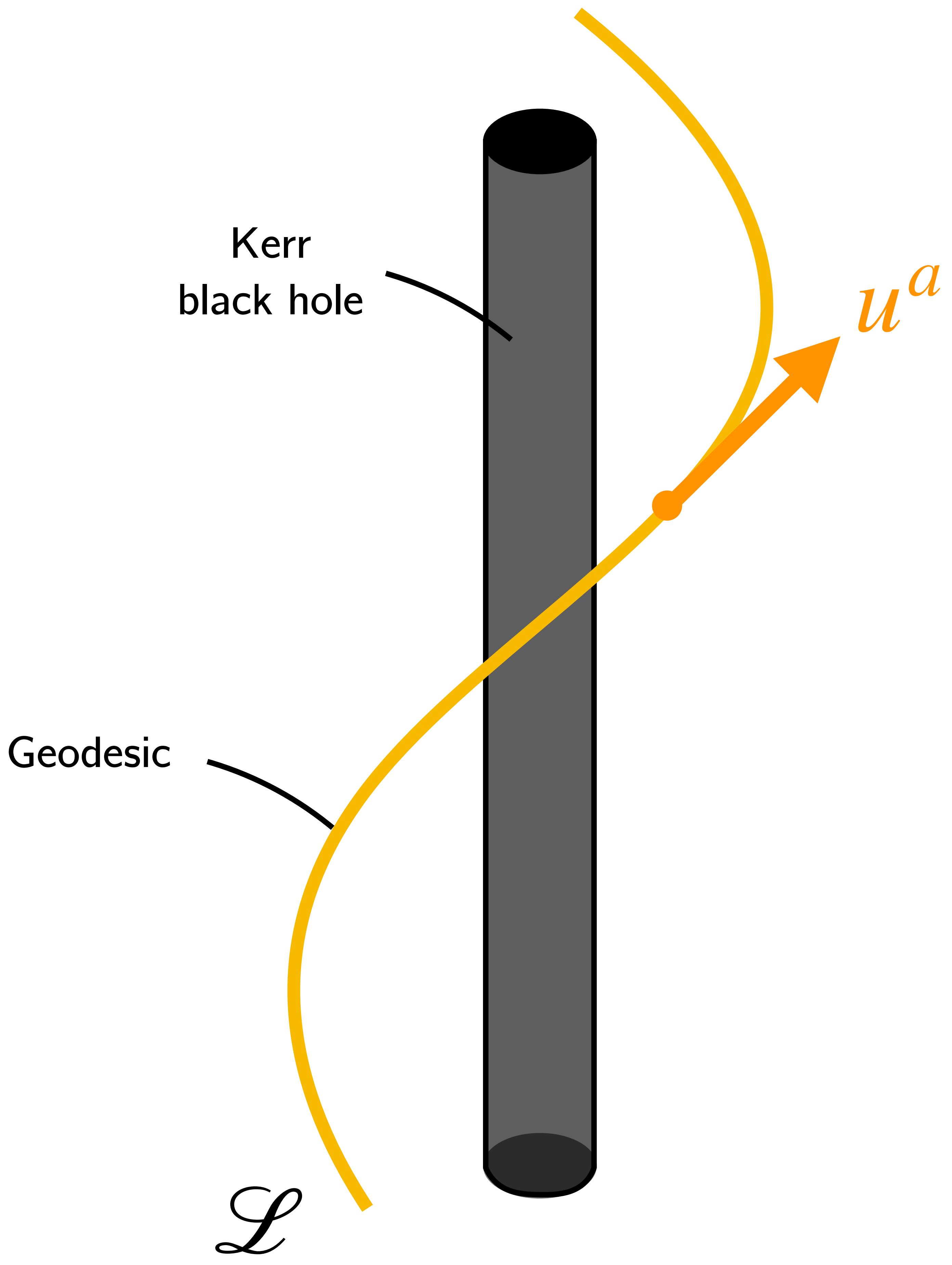

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Motion

- 4-velocity u^a is *parallel-transported* along $\mathscr{L}$
- $\mathscr{L}$'s parametrization: $u^\alpha = \dfrac{dx^\alpha}{d\tau}$
- functions $x^\alpha(\tau)$ solve the geodesic equation

1. Geodesics

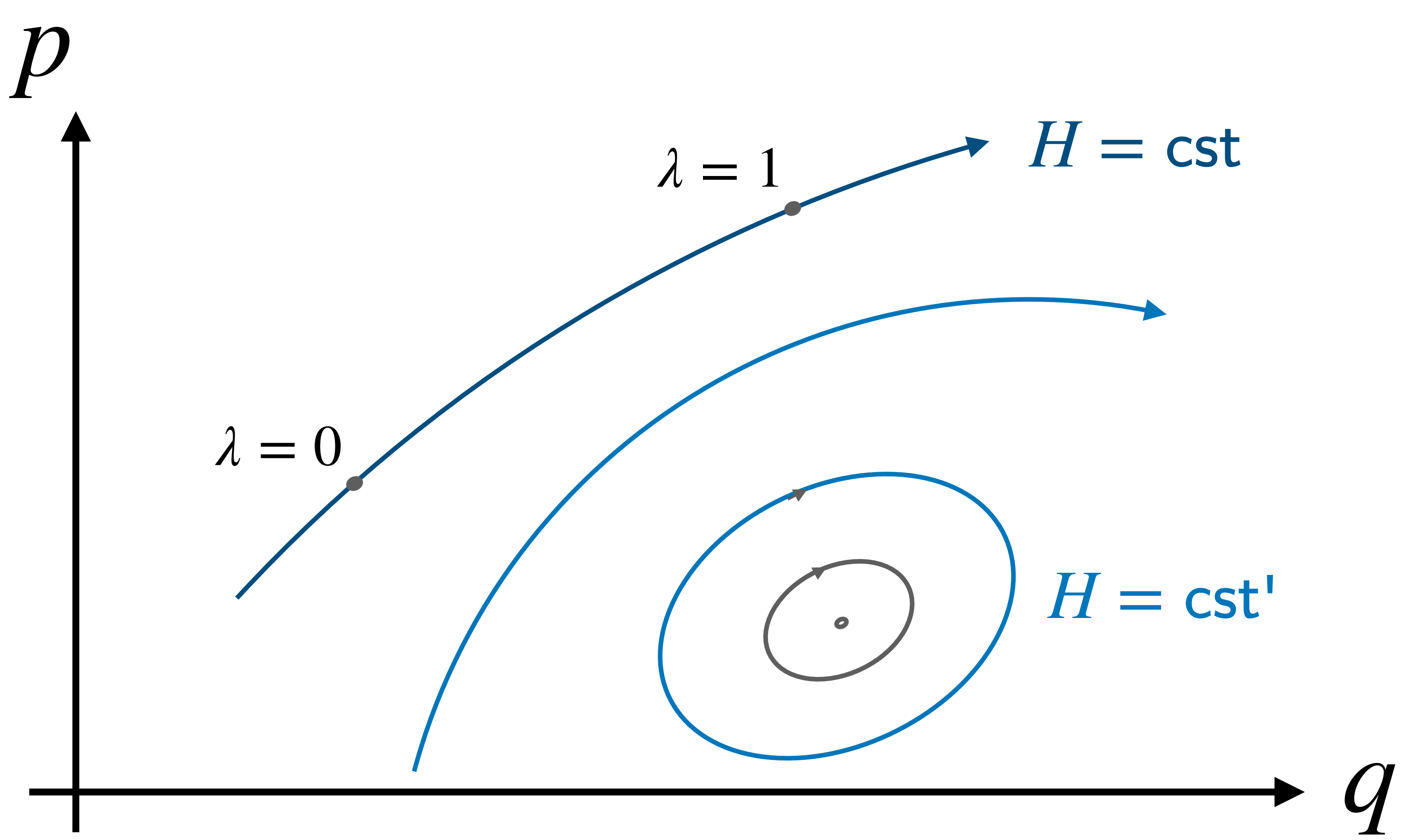
2. Hamiltonian formulation

Geodesics in phase space

in general

Ham. system

- Phase space $\mathscr{M}$
- Poisson brackets $\{,\}$
- Hamiltonian H



Law of motion

$$\frac{dF}{d\lambda} = \{F, H\}$$

+ Leibniz rule

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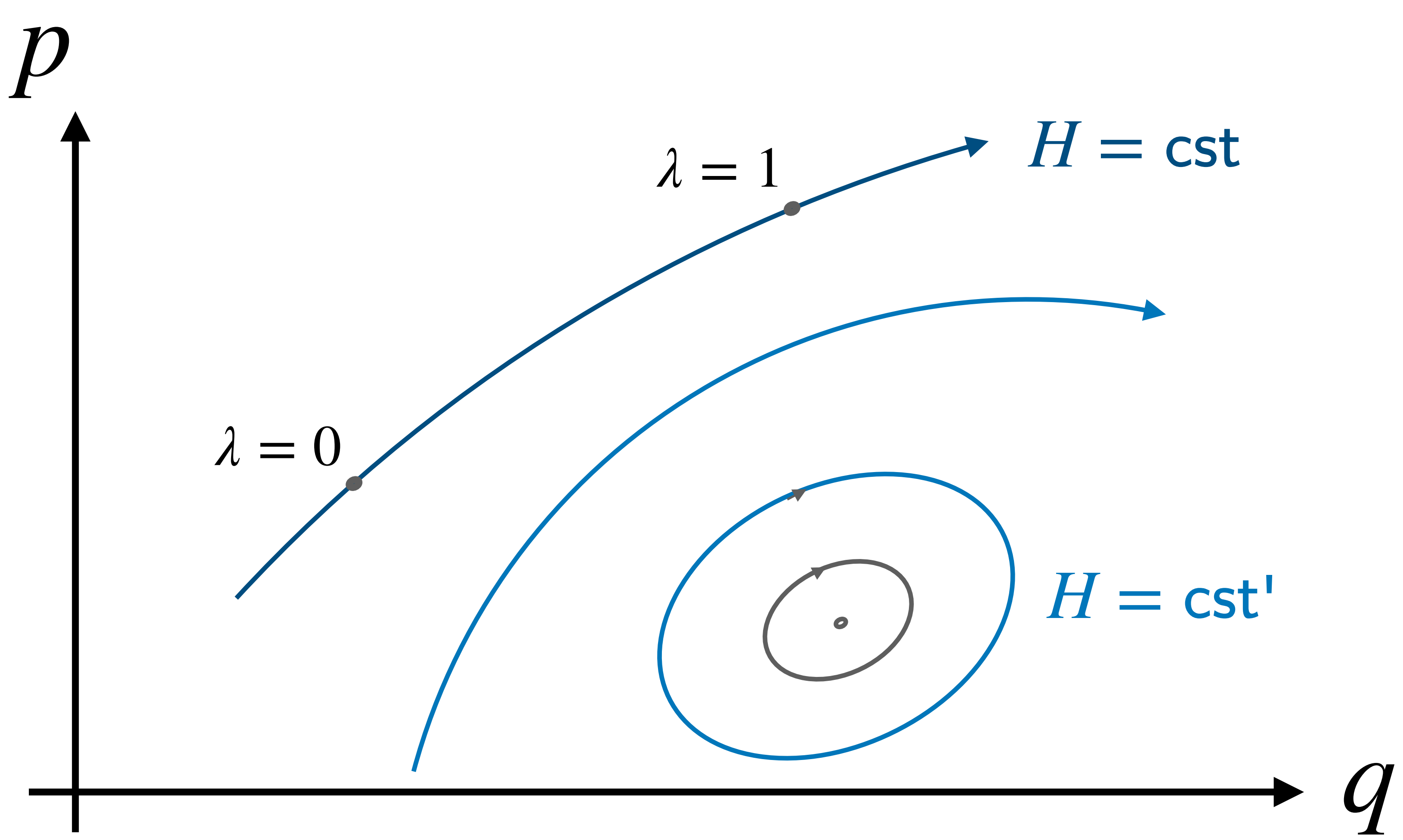

Law of motion

$$\frac{dF}{d\lambda} = \{F, H\}$$

+ Leibniz rule

for geodesics

Phase space: $\mathscr{M} = \mathbb{R}^4 \times \mathbb{R}^4$

$(x^\alpha, p_\alpha) = (t, r, \theta, \phi, p_t, p_r, p_\theta, p_\phi)$

Poisson brackets: $\{x^\alpha, p_\beta\} = \delta^\alpha_\beta$

canonical (conjugated pairs)

Hamiltonian: $H := \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta$

free-body in curved space(time)

I. Geodesics

3. Integrable systems

Example: Schwarzschild

- Hamiltonian $$H(t, p_t, r, p_r, \theta, p_\theta, \phi, p_\phi) = -\frac{p_t^2}{2f} + \frac{f p_r}{2} + \frac{1}{2r^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right)$$

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- **Integrals of motion**

energy: $E := -p_t$

component of ang. mom.: $J_z := p_\phi$

norm of ang. mom.: $J^2 := p_\theta^2 + p_\phi^2 \csc^2\theta$

mass: $\mu^2 := -2H$

Integrable systems

"number of integrals of motion = number of degrees of freedom"

extra assumptions required

half the dimension of phase space

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- **Integral** of motion: a function $\mathscr{I} : \mathscr{M} \to \mathbb{R}$ such that $\{\mathscr{I}, H\} = 0$
- Pairwise **involution**: integrals of motion satisfy $\{\mathscr{I}_i, \mathscr{I}_j\} = 0$
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Note:

- Integrability → **mathematical** notion (a *physical system* <u>is not</u> *integrable per se*)
- Here: "**Liouville-Arnold**" integrability → for non-deg. "classical" Ham. systems
- **Integrability** involves all **three ingredients**: $\mathscr{M}, \{,\}$ and H.

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In physics, we usually replace "0" by $O(\varepsilon^n)$ (this may not be harmless...)

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Why integrable systems ?

Maths:

- Integrability $\Longrightarrow$ **Liouville-Arnold theorem:** phase space foliated by invariant torii

Kerr astrophysics:

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- Integrability $\Longrightarrow$ **preferred set of canonical coordinates:** action-angle $(\vartheta^i, \mathcal{J}_i)$

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- Integrability $\Longrightarrow$ **well-defined notion of "Hamiltonian" frequency:** $\Omega_i := \partial H/\partial \mathcal{J}_i$
- Integrability $\Longrightarrow$ **well-understood perturbation theory:** KAM/Birkhoff theorems

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- action-angle $\Longrightarrow$ **"first" and "flux-balance" laws** and **analytic solutions**
- frequencies $\Longrightarrow$ well-defined notion of **resonances** (lots of astrophysical phenomena)
- perturbed systems $\Longrightarrow$ adapted to **multi-timescale expansions** and **dissipation** (EMRIs)

Geodesic integrals and Killing fields

Killing field	Definition	Integral for geodesics		
k^a	$\nabla_{(a} k_{b)} = 0$	$k^\alpha p_\alpha$		

Geodesic integrability around **black holes**

Killing field	Definition	Integral for geodesics	**Schwarzschild** geodesics			
			static	axisym.	SO(3)	
k^a	$\nabla_{(a}k_{b)} = 0$	$k^\alpha p_\alpha$	$(\partial_t)^a$	$(\partial_\phi)^a$		
			E	L_z	L^2	

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Killing field	Definition	Integral for geodesics	**Schwarzschild** geodesics			**Kerr** geodesics	
k^a	$\nabla_{(a}k_{b)} = 0$	$k^\alpha p_\alpha$	static $(\partial_t)^a$ E	axisym. $(\partial_\phi)^a$ L_z	SO(3) L^2	static $(\partial_t)^a$ E	axisymmetric $(\partial_\phi)^a$ L_z

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K^{ab}	$\nabla_{(a}K_{bc)} = 0$ + symmetric	$K^{\alpha\beta}p_\alpha p_\beta$					

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$a \to 0$

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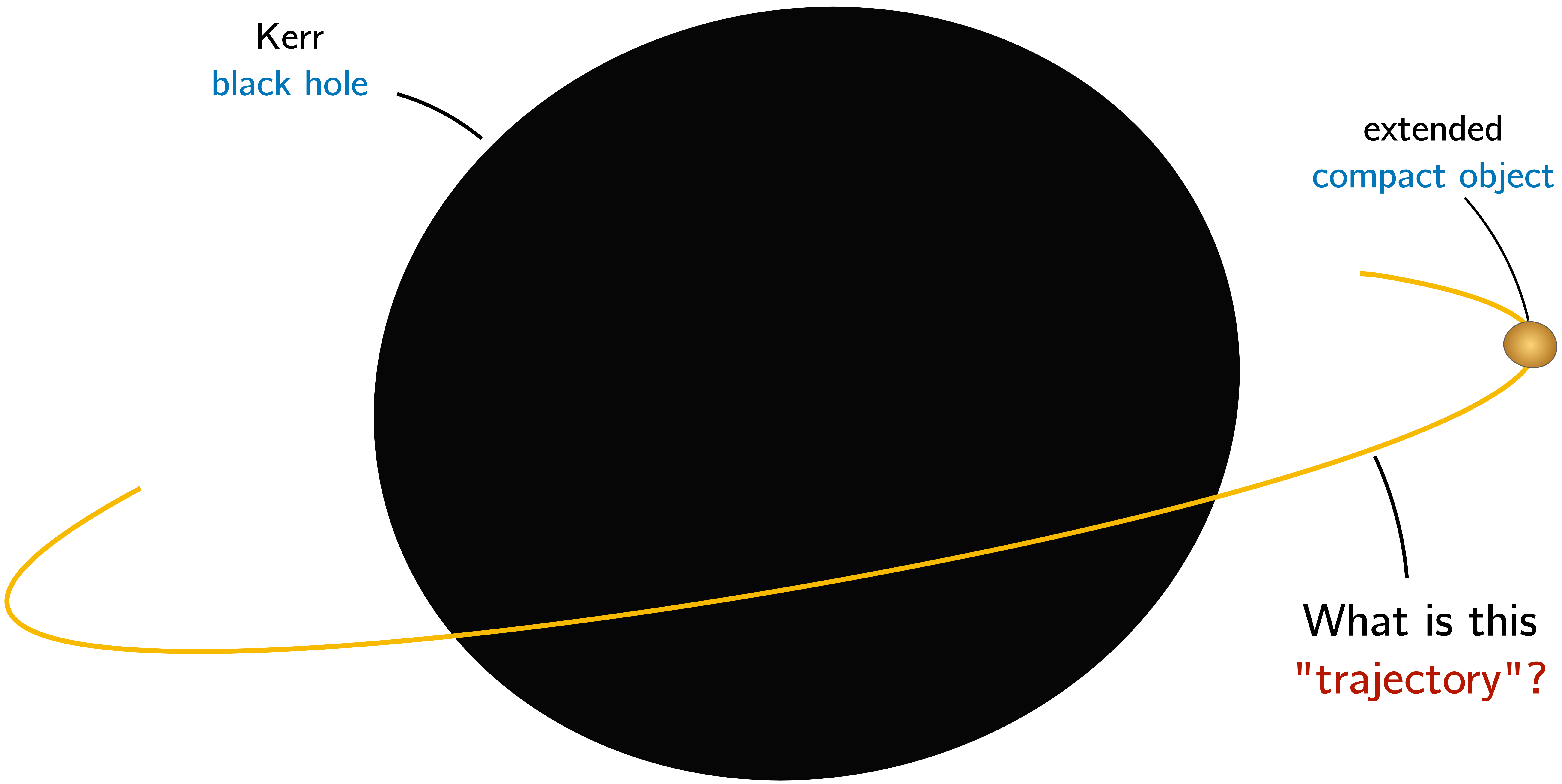
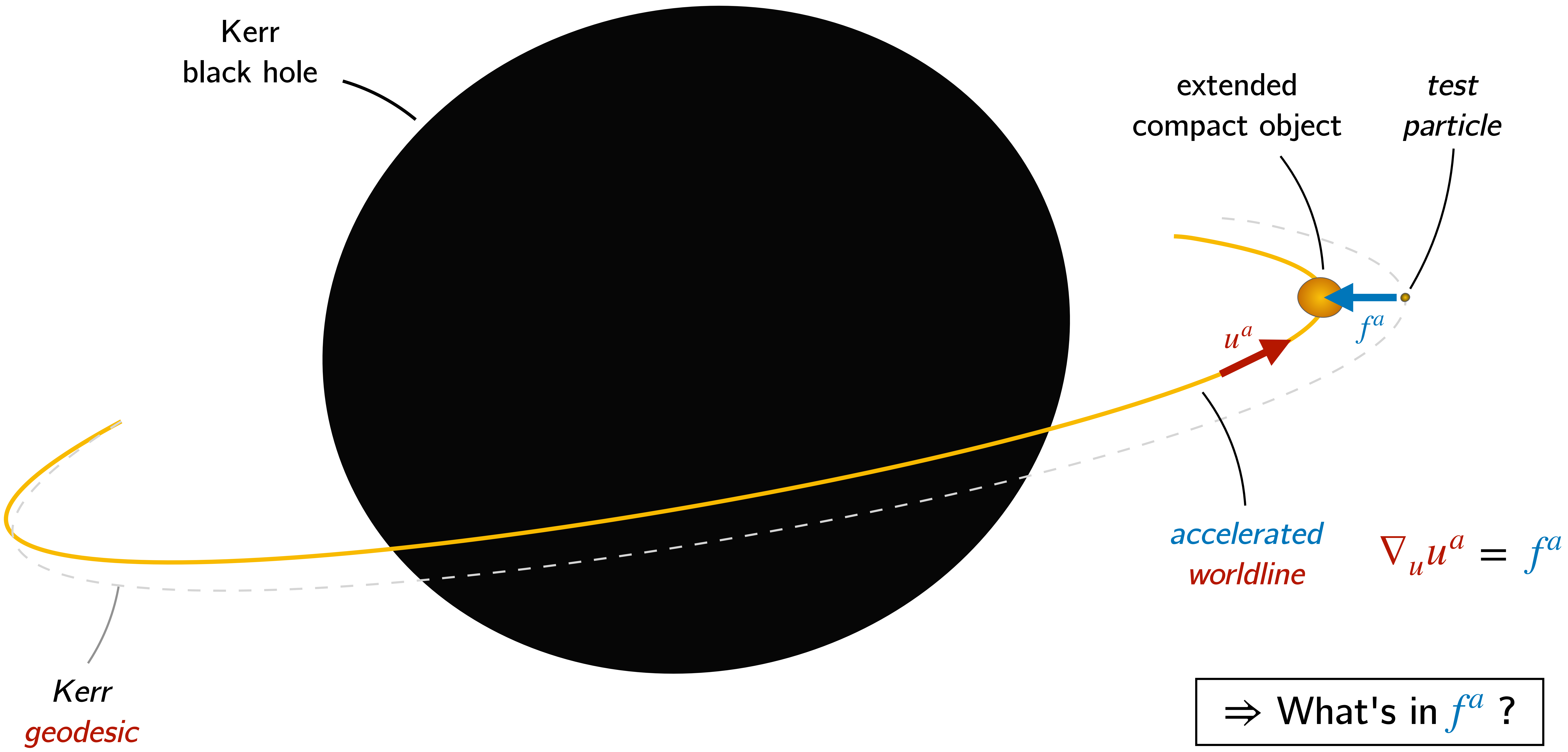


How do things **really** fall around black holes?



$\Rightarrow$ What's in f^a ?

non-GR, hairs, environment, GSF, extended-body effects,...

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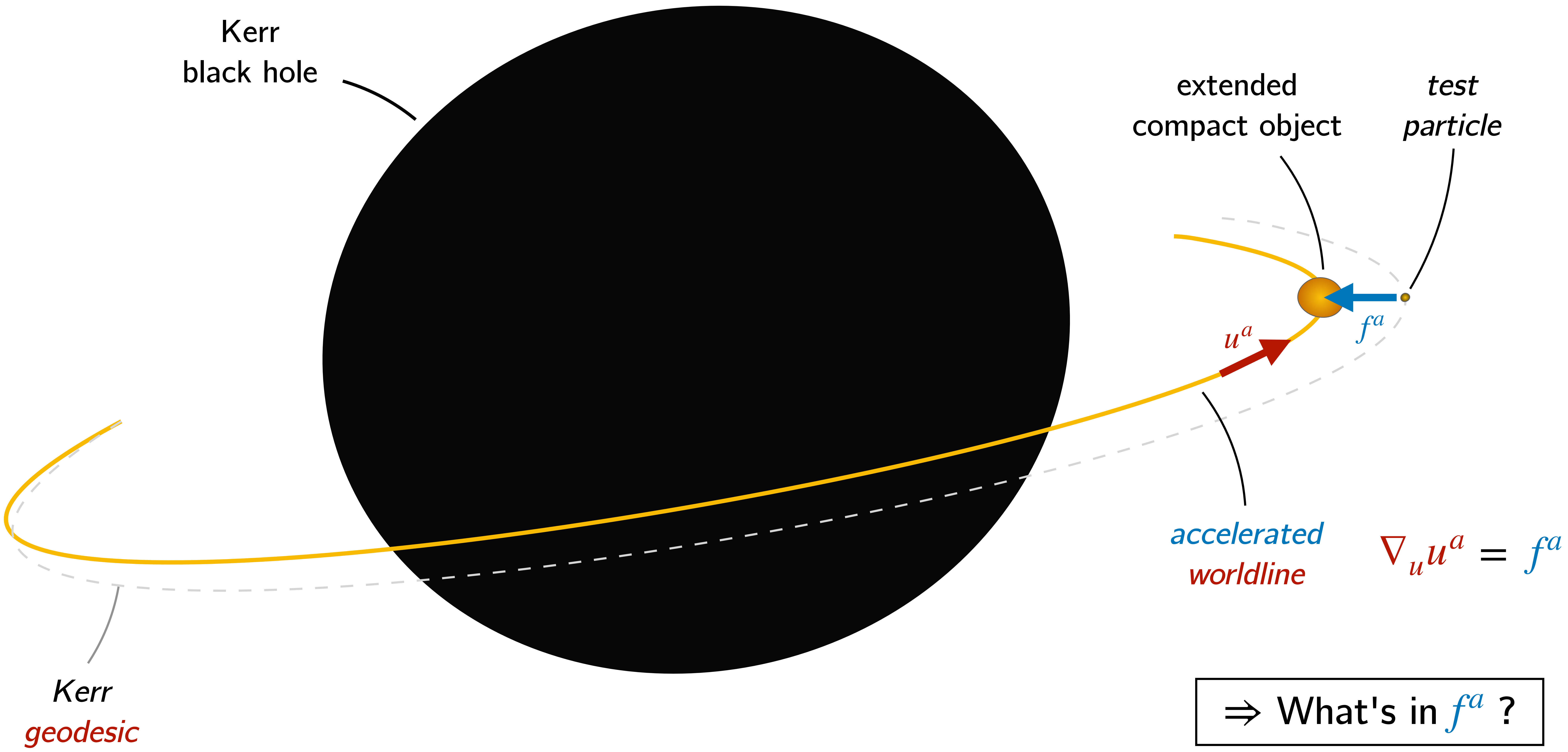

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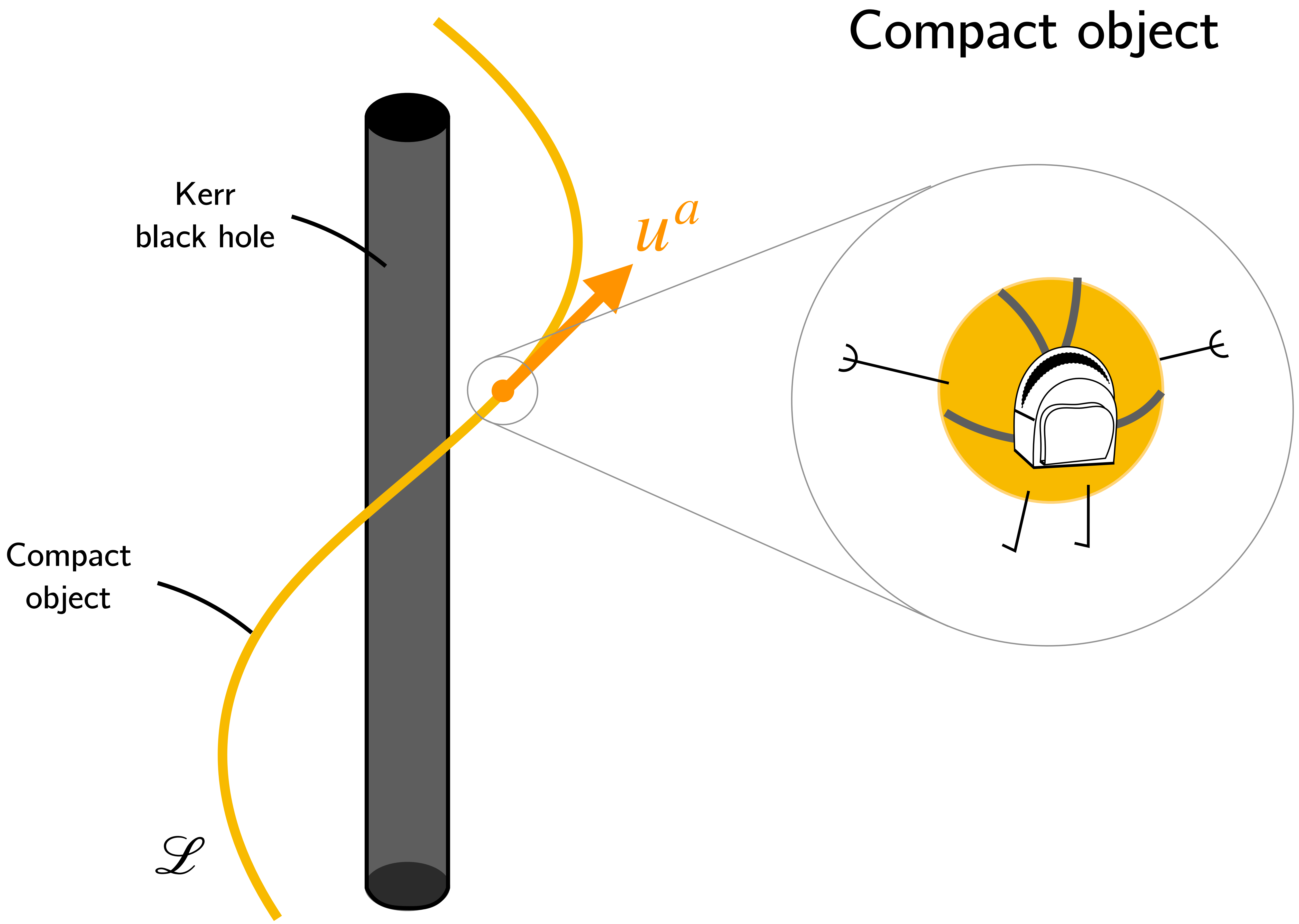
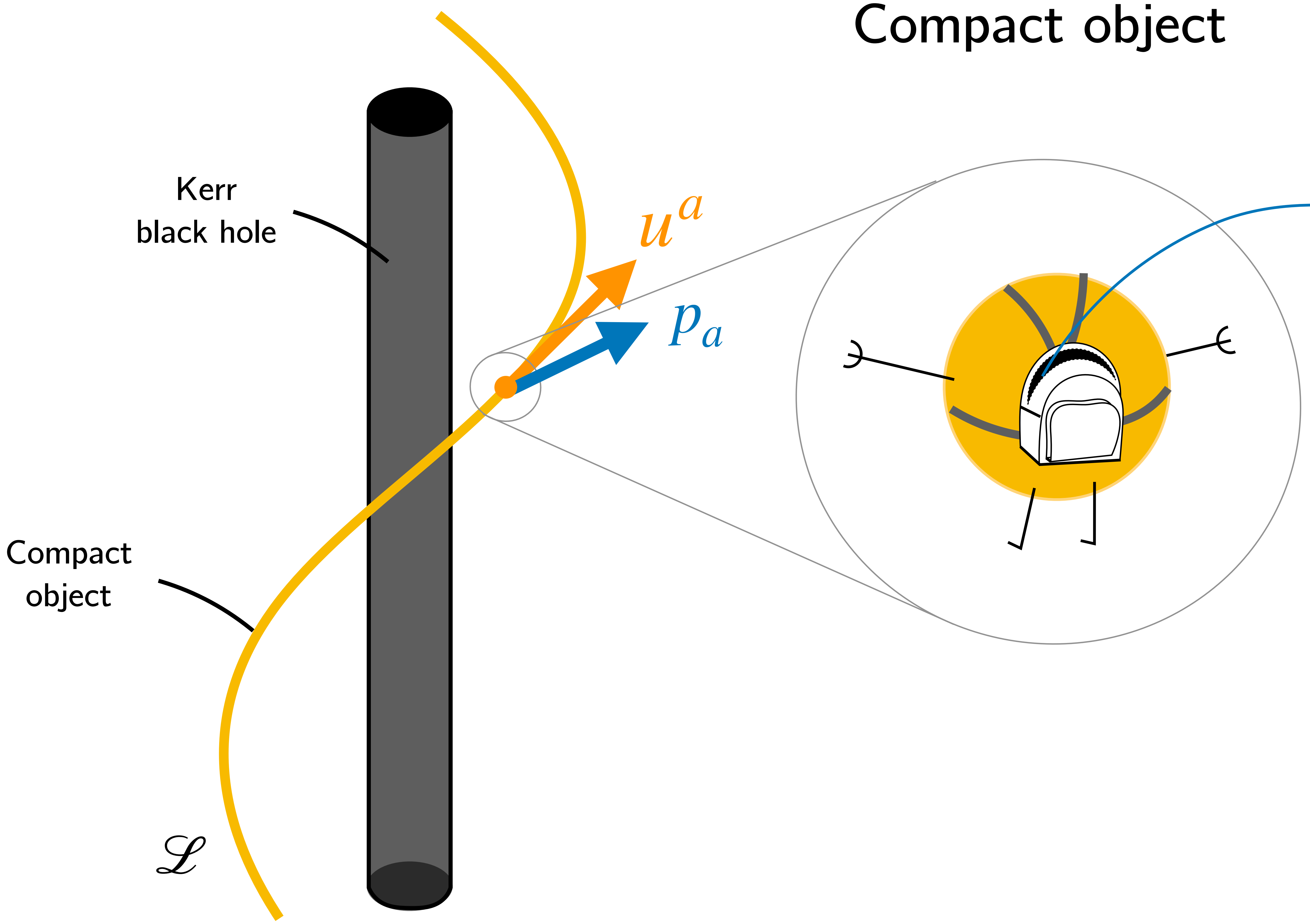
What happens to **Kerr integrability**
for the motion of **spinning objects** ?

1. account for the object's spin
2. describe as a Hamiltonian system
3. find enough integrals of motion

Multipolar description of extended bodies



Multipolar description of extended bodies



Compact object = Worldline + Momenta

linear momentum p_a

translation

Multipolar description of extended bodies

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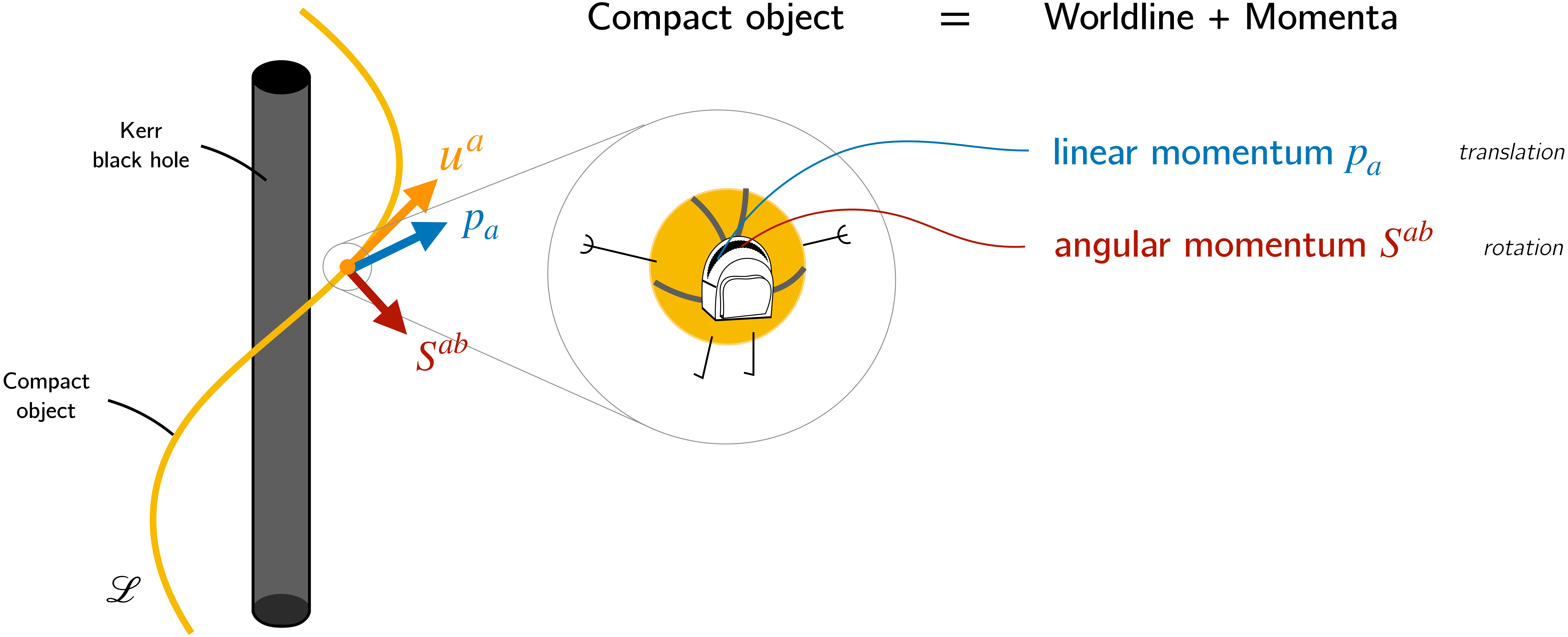

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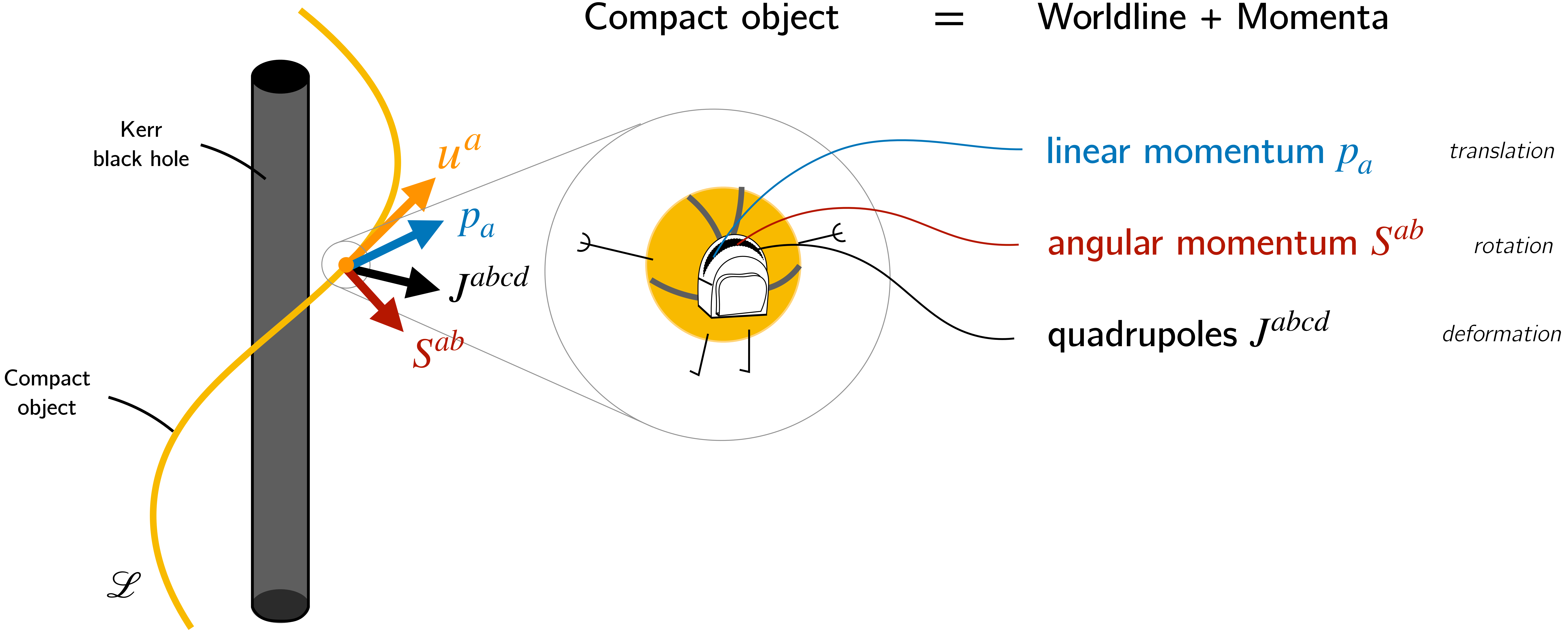

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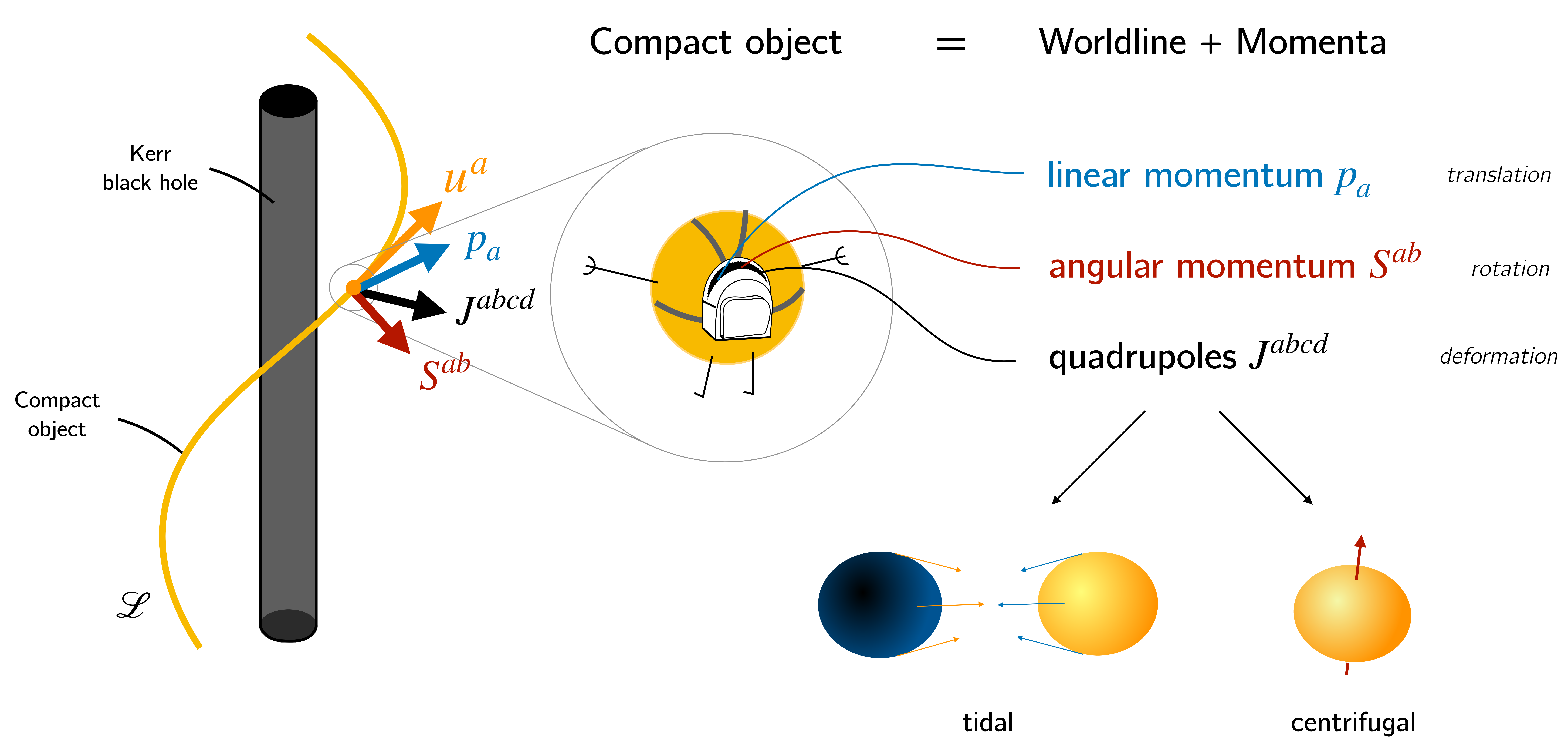
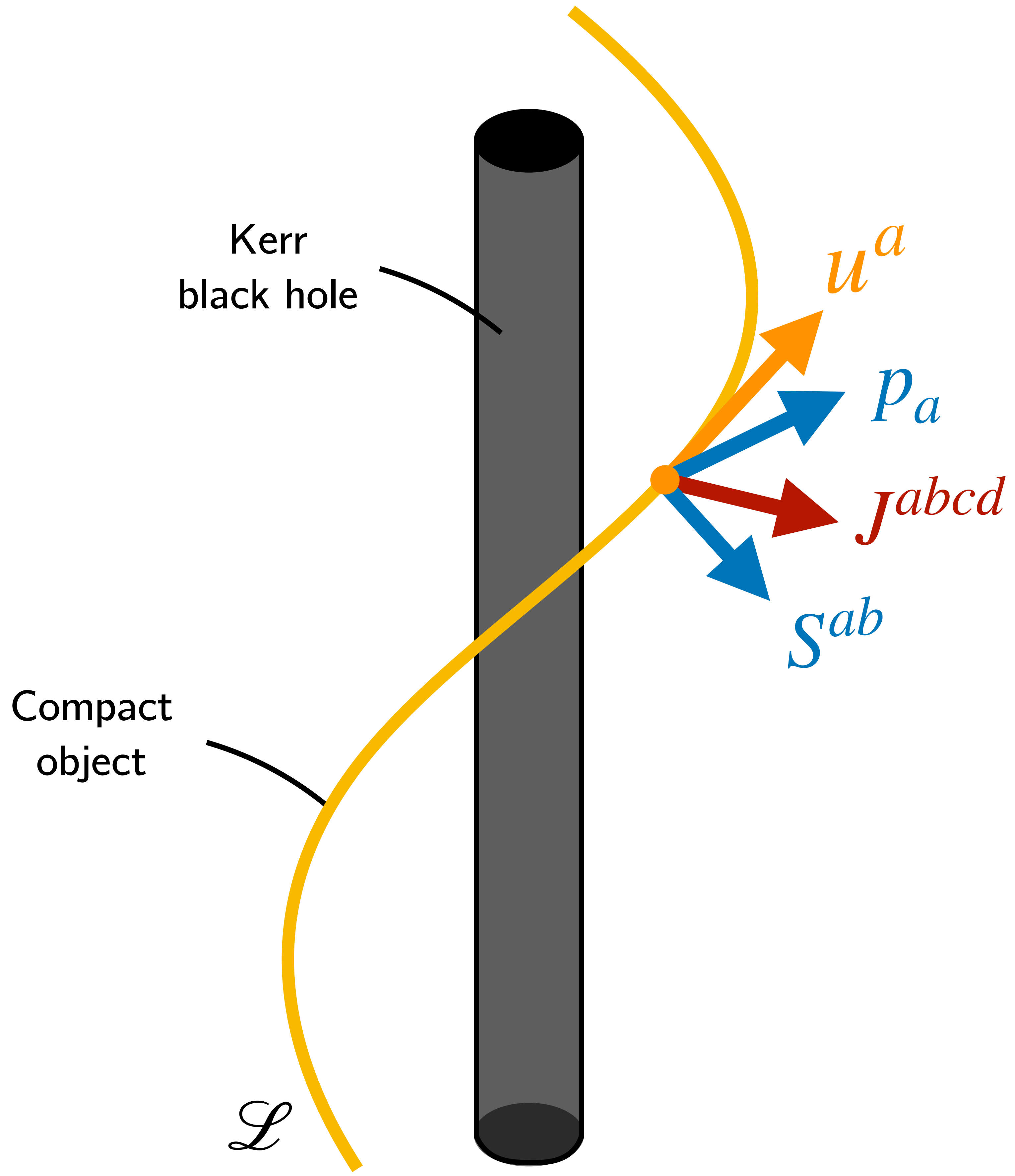


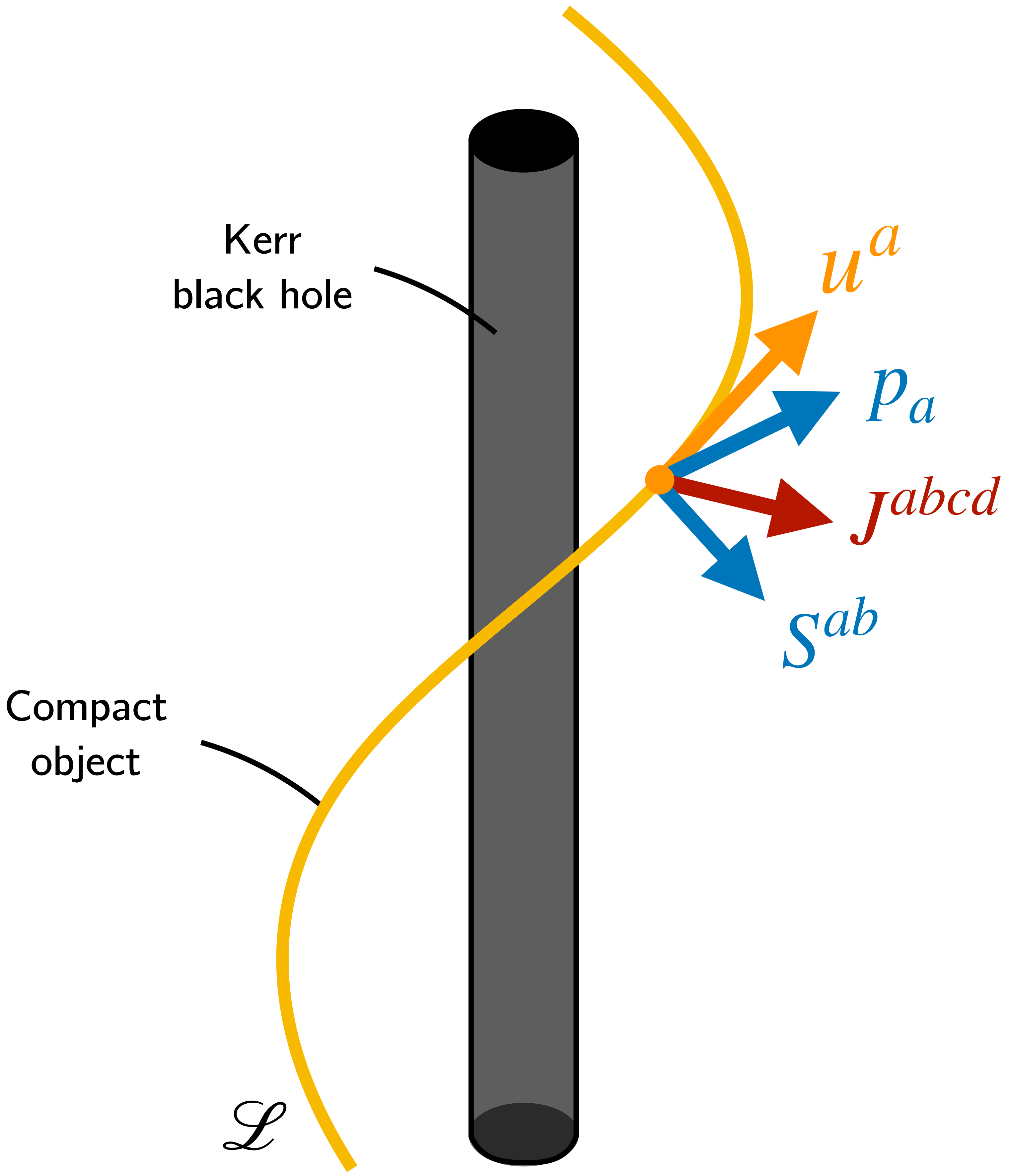
Evolution equations



Dixon-Harte equations

Evolution = Kinematics + Dynamics

Evolution equations

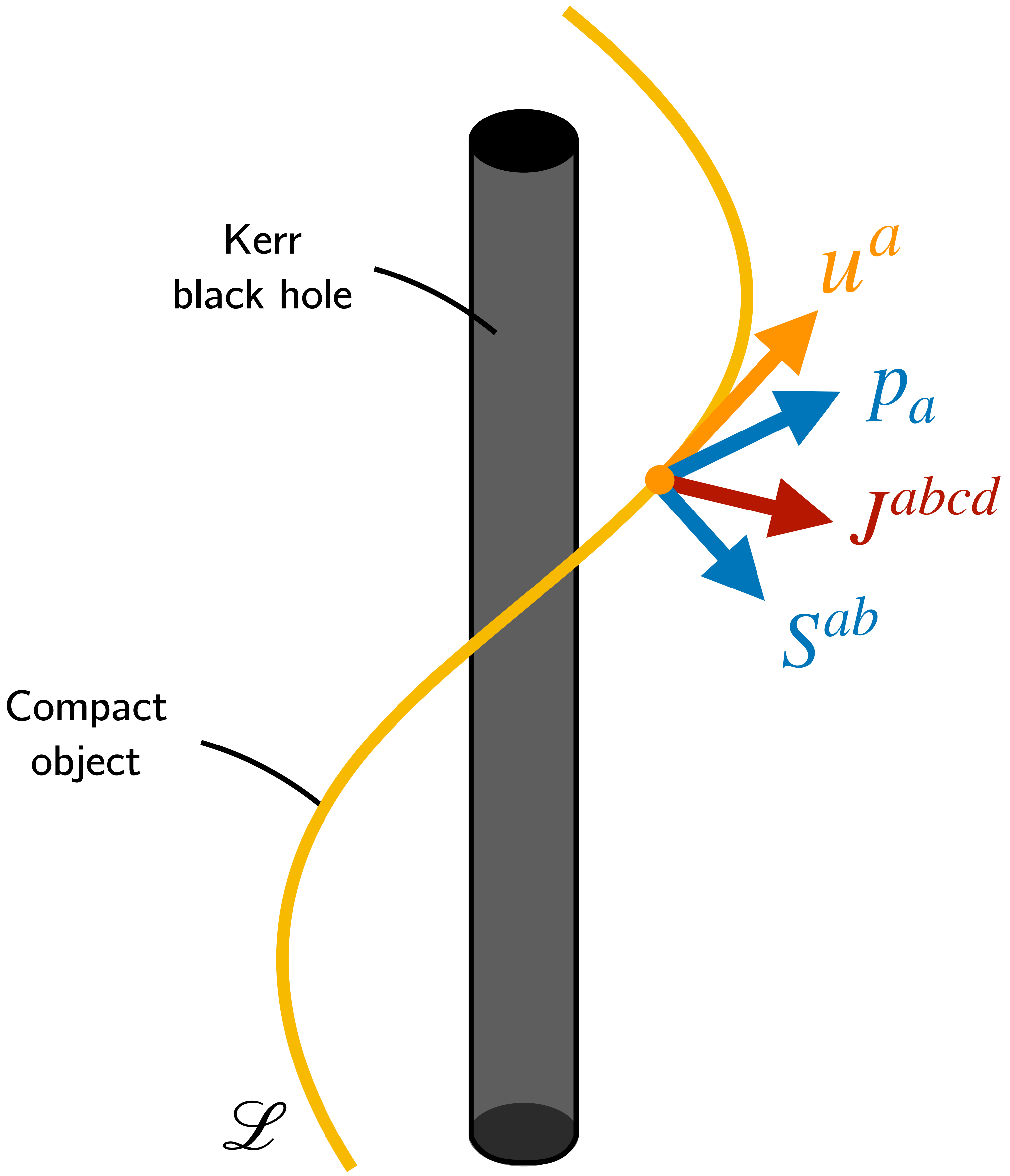


Dixon-Harte equations

Evolution = Kinematics + Dynamics

$$\nabla_u p_a = R_{abcd} S^{bc} u^d + F_a$$

Evolution equations



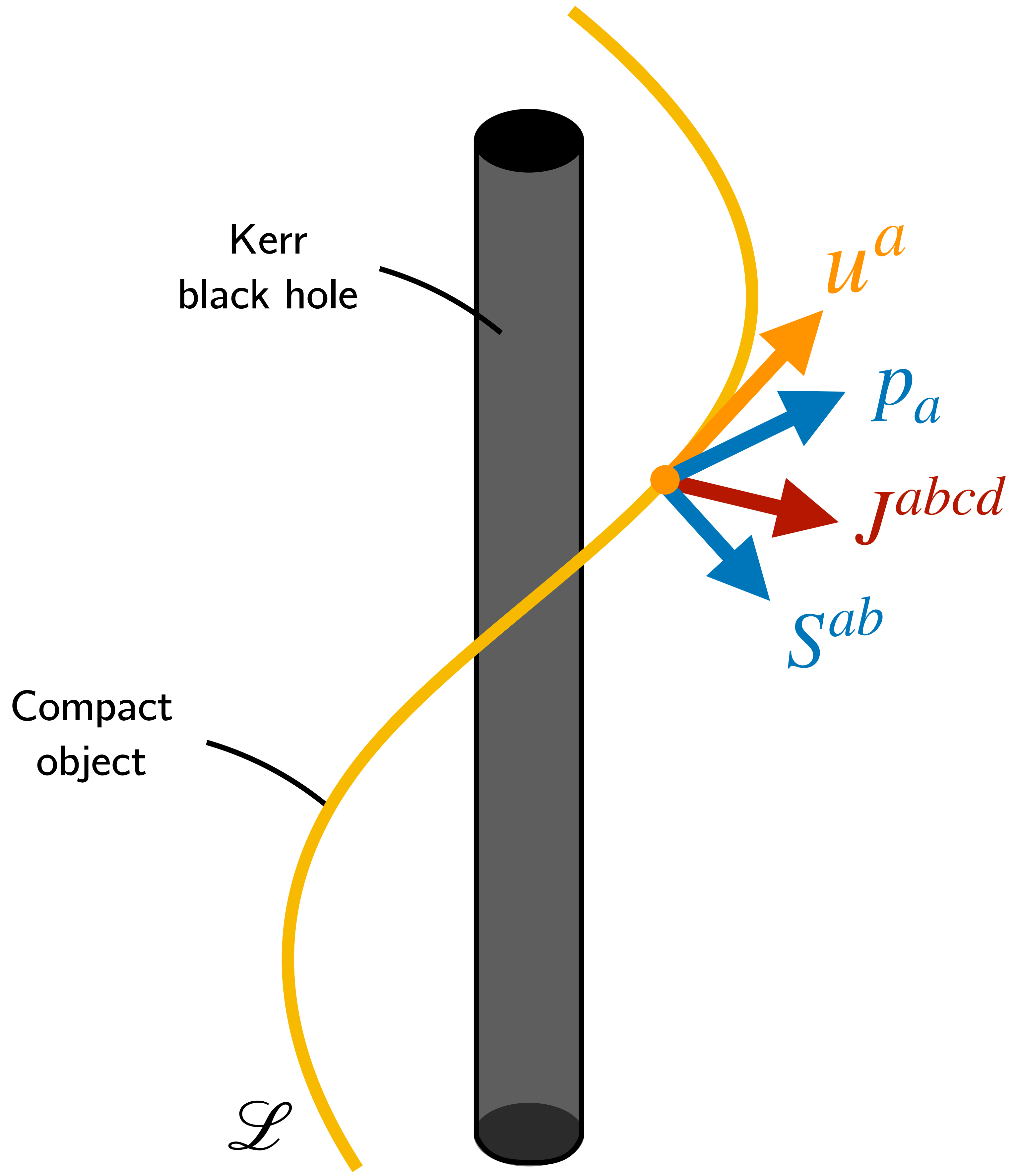
Dixon-Harte equations

Evolution = Kinematics + Dynamics

linear mom. driven by ...	spin-curvature coupling	multipolar "force"

$$\nabla_u p_a = R_{abcd} S^{bc} u^d + F_a$$

Evolution equations



Dixon-Harte equations

Evolution = Kinematics + Dynamics

linear mom. driven by ... | spin-curvature coupling | multipolar "force"

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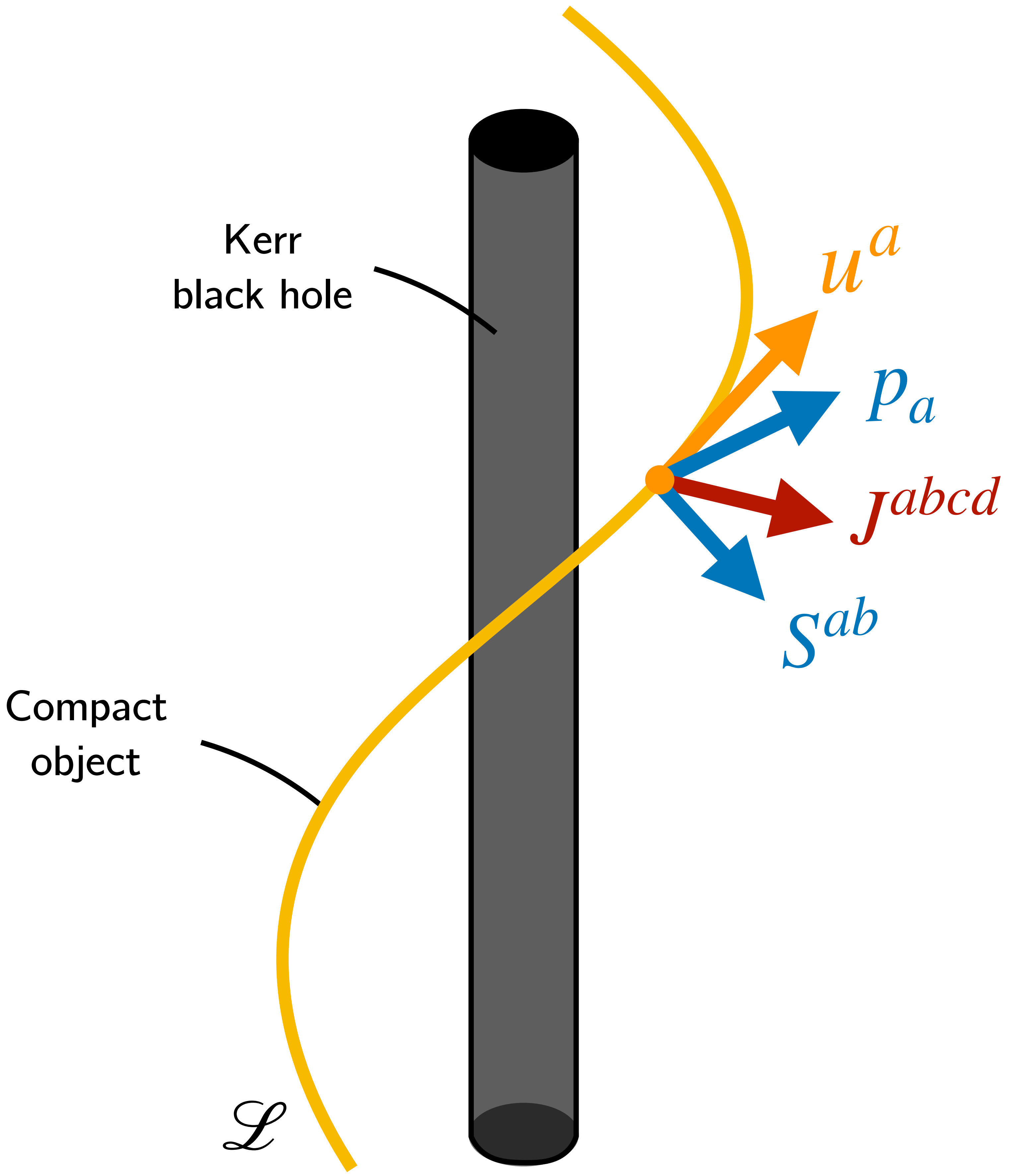

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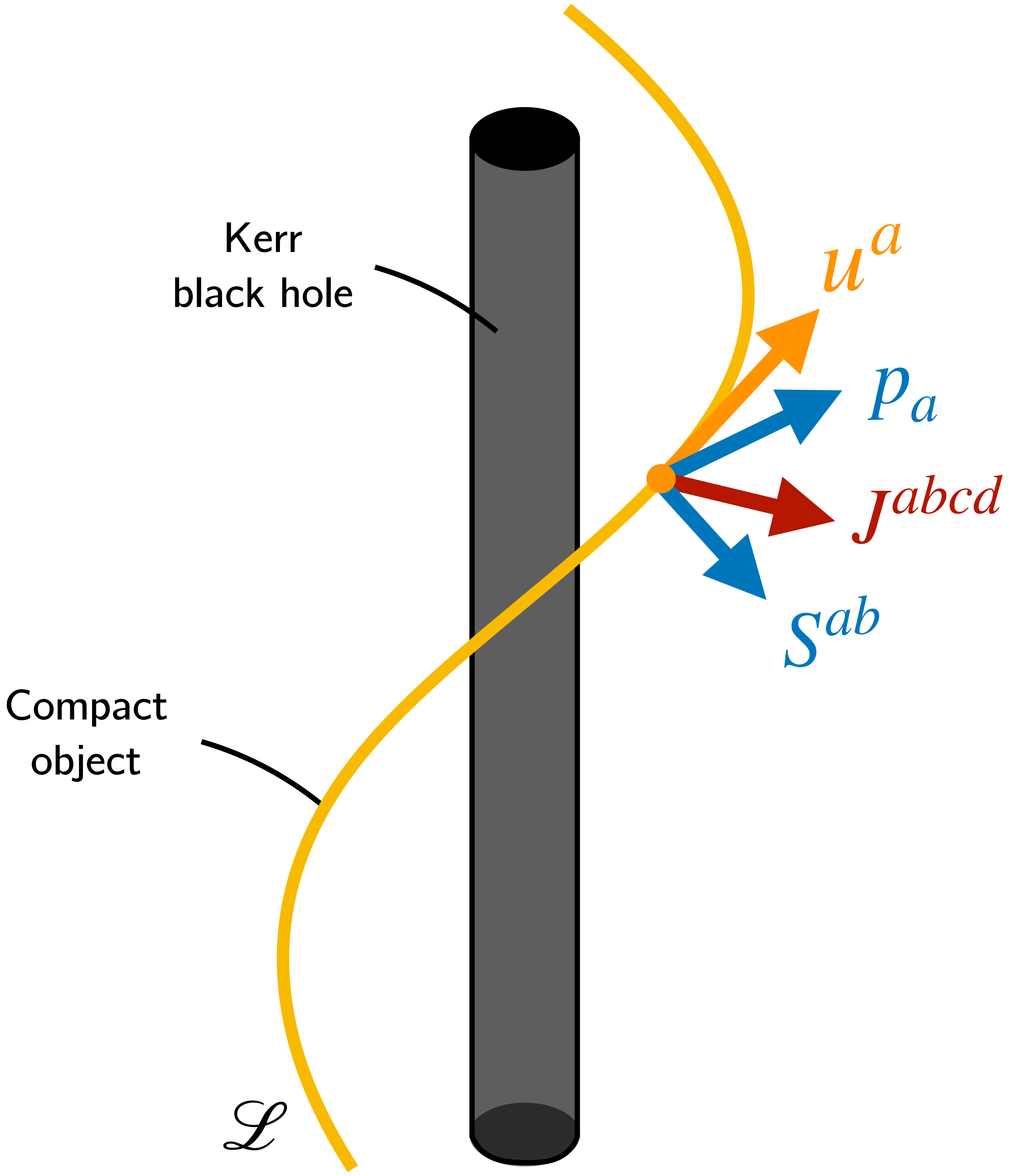

Dixon-Harte equations

Evolution = Kinematics + ~~Dynamics~~

linear mom. driven by … | spin-curvature coupling | ~~multipolar "force"~~

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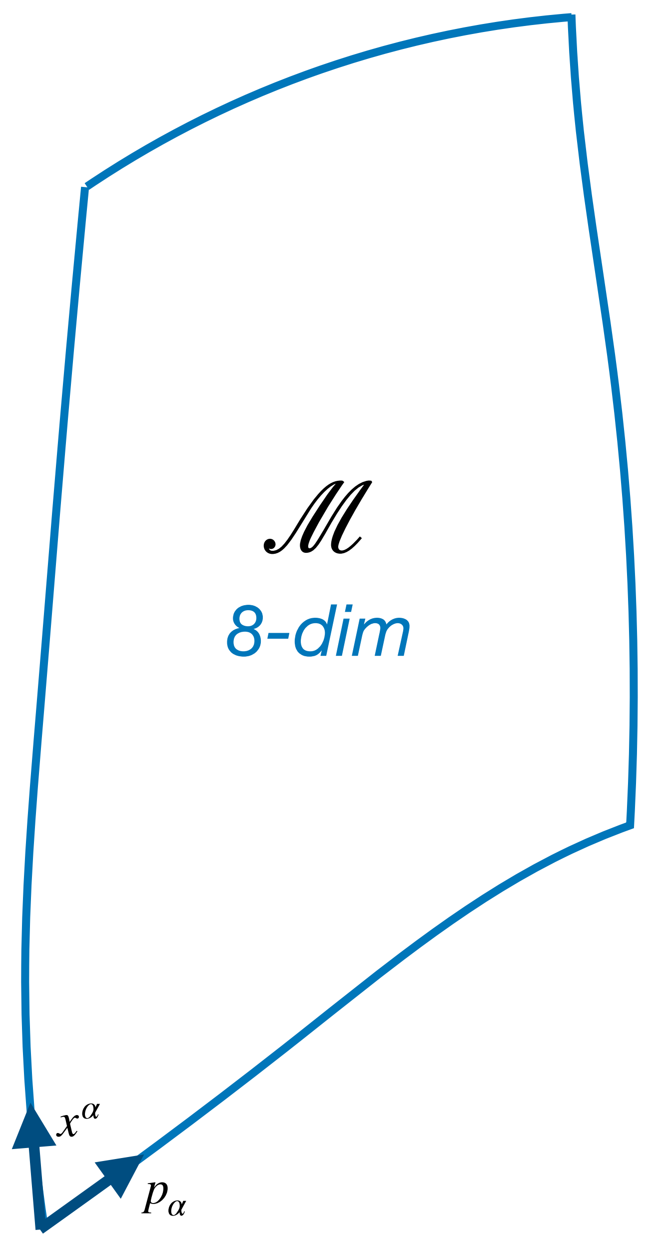
What happens to **Kerr integrability**
for the motion of **spinning objects** ?

1. ~~account for the object's spin~~
2. describe as a Hamiltonian system
3. find enough integrals of motion

Hamiltonian formulation of MPTD equations (monopolar)

Ham. system

- Phase space $\mathscr{M}$
- Poisson brackets $\{,\}$
- Hamiltonian H



Law of motion

$$\frac{dF}{d\lambda} = \{F, H\}$$

+ Leibniz rule

Phase space:

$$\mathscr{M} = \mathbb{R}^4 \times \mathbb{R}^4$$

$$(x^\alpha, p_\alpha)$$

Poisson brackets:

$$\{x^\alpha, p_\beta\} = \delta^\alpha_\beta,$$

Hamiltonian:

$$H := \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta$$

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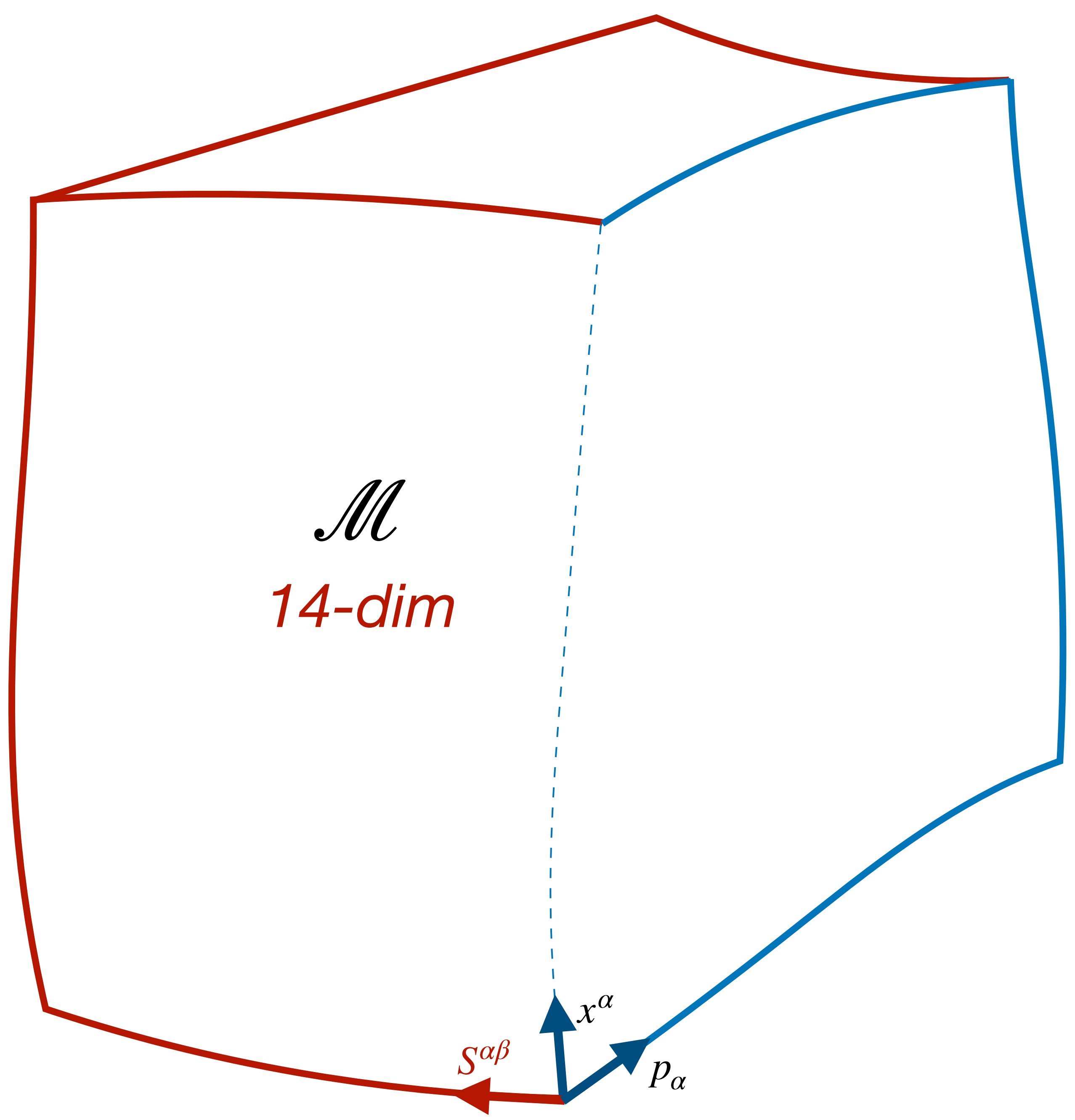

Law of motion

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Phase space:

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$$(x^\alpha, p_\alpha, S^{\alpha\beta})$$

Poisson brackets:

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$$\{p_\alpha, S^{\beta\gamma}\} = \dots$$

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Non-canonical brackets for the spin

$$\{x^\alpha, p_\beta\} = \delta^\alpha_\beta\,,$$

$$\{p_\alpha, p_\beta\} = R_{\alpha\gamma\delta\beta} S^{\gamma\delta}\,,$$

$$\{p_\alpha, S^{\beta\gamma}\} = 2\Gamma^{[\gamma}_{\delta\alpha} S^{\beta]\delta}\,,$$

$$\{S^{\alpha\beta}, S^{\gamma\delta}\} = 2(g^{\alpha[\delta} S^{\gamma]\beta} + g^{\beta[\gamma} S^{\delta]\alpha})\,,$$

$$\frac{dF}{d\lambda} = \{F, H\}$$

+ Leibniz rule

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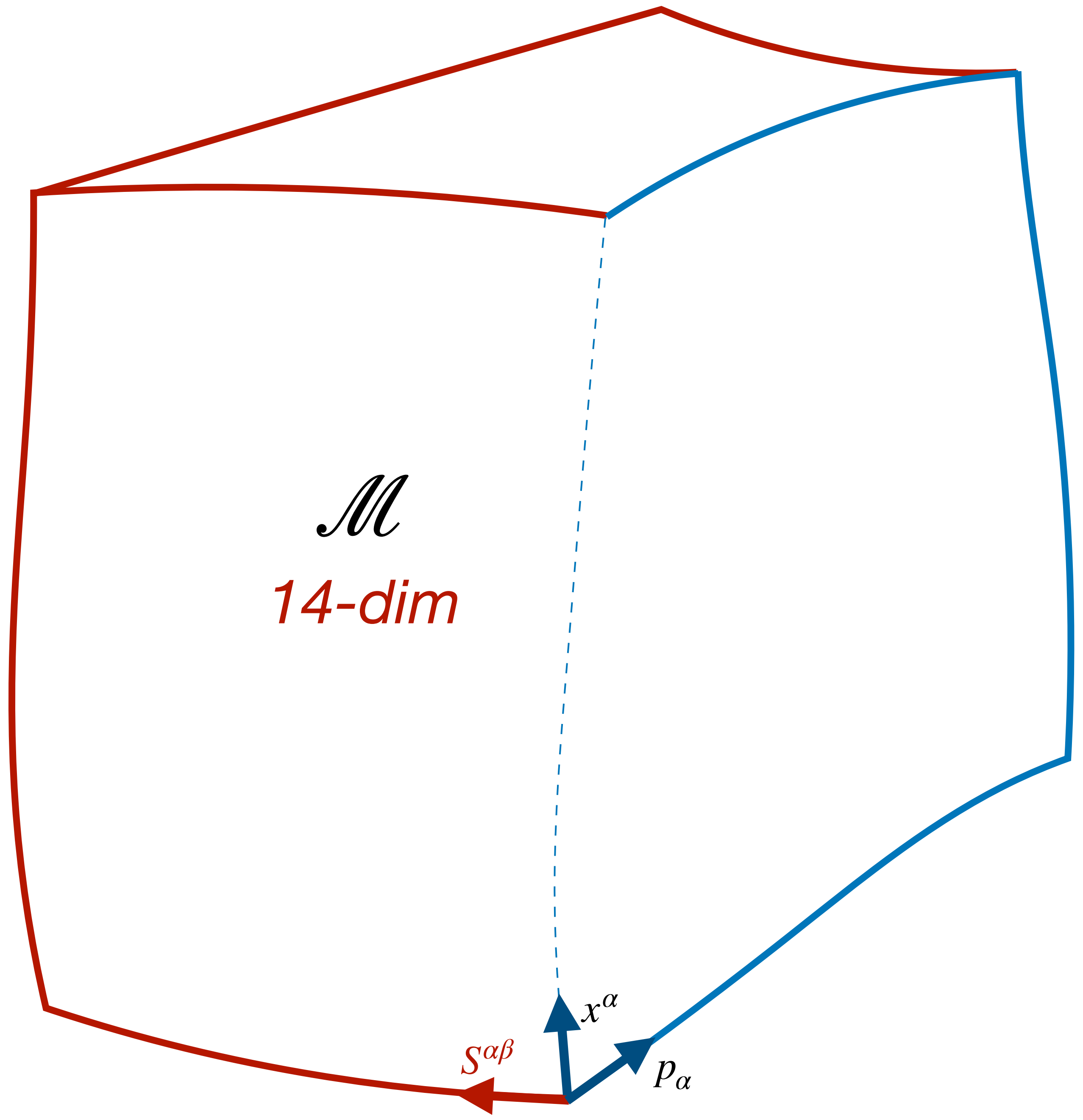

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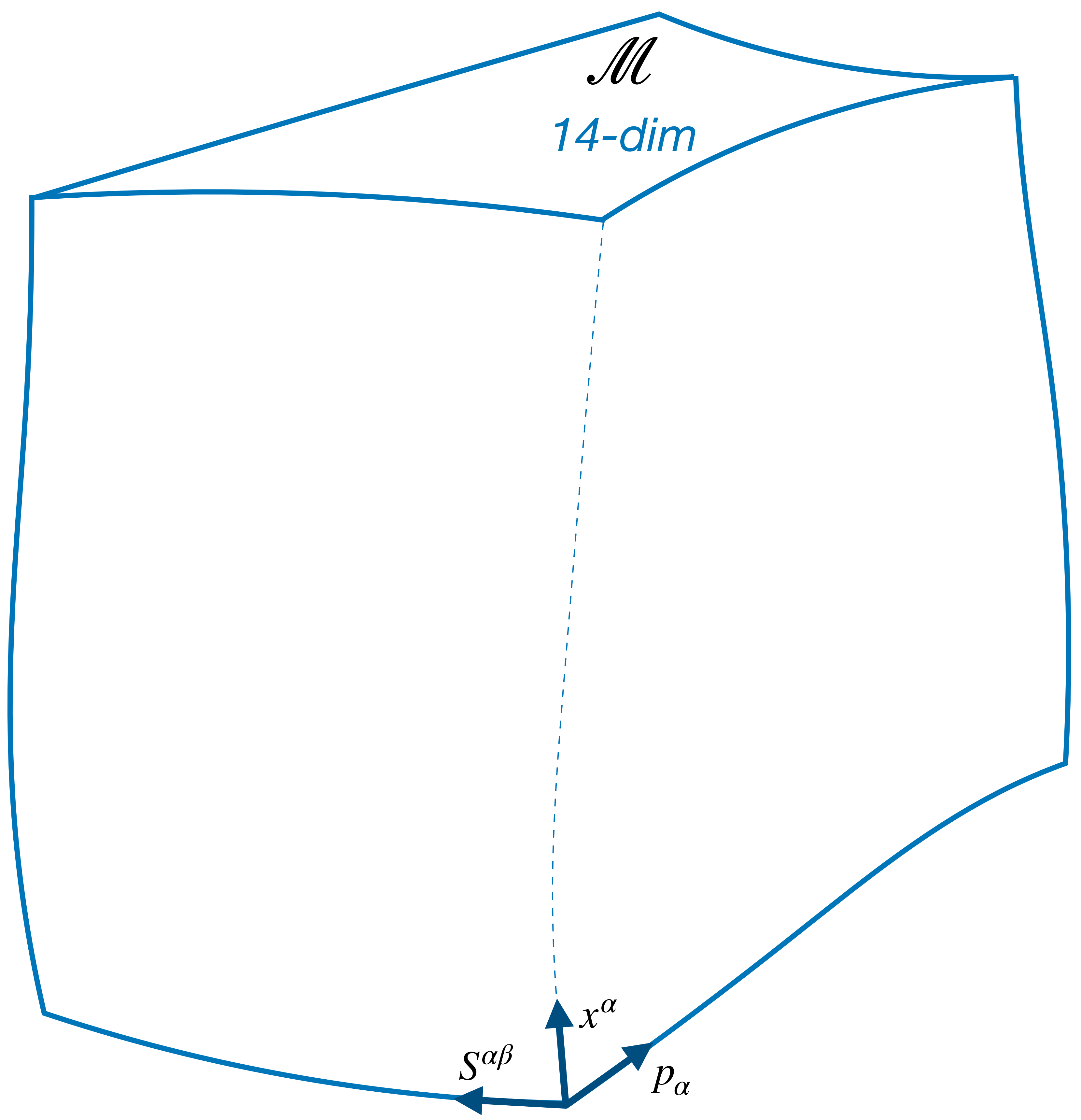
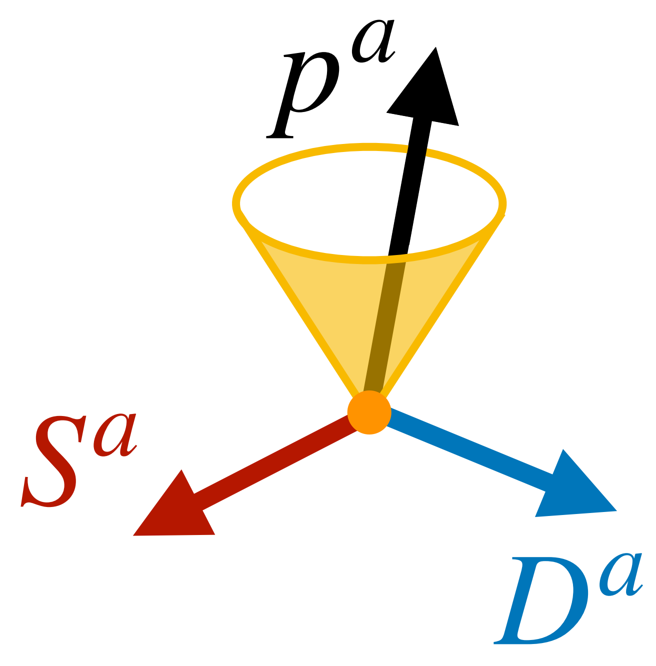
Geometry	Poisson brackets are degenerate → **GR local Lorentz invariance**	symplectic foliation of Poisson systems → **Casimir invariants**
Analysis	ODE system is not well-posed → **definition of spin in GR**	pullbacks of symplectic forms → **Poisson-Dirac brackets**

Problem I: definition of spin in GR



$$S^{ab} = \varepsilon^{abcd}\bar{p}_c S_d + 2\bar{p}^{[a}D^{b]}$$

(Faraday F^{ab} = Magnetic B^a + Electric E^a)



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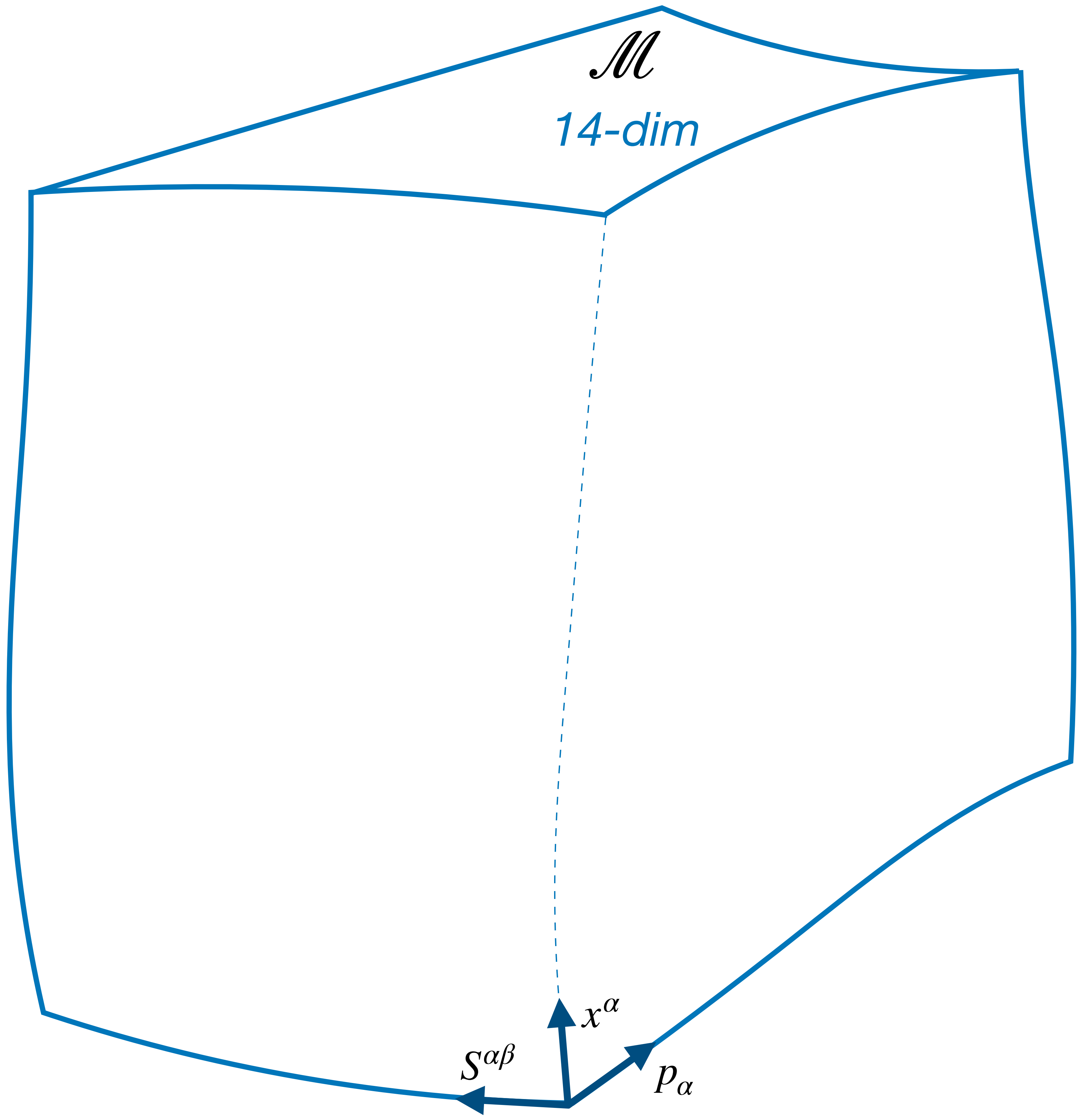

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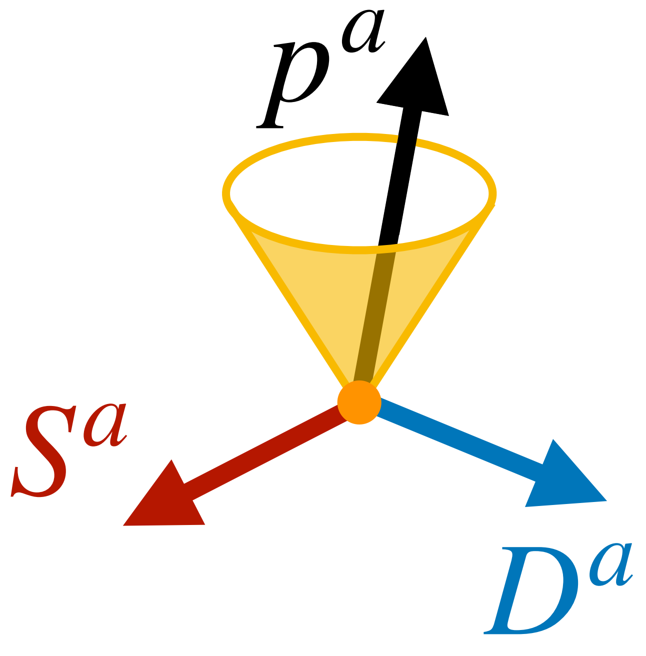

S^{ab} spin tensor, p^a time-like vector $\iff$ mass dipole $D^b = \bar{p}_a S^{ab}$, spin vector $S^a = \frac{1}{2}\varepsilon^{abcd}\bar{p}_b S_{cd}$

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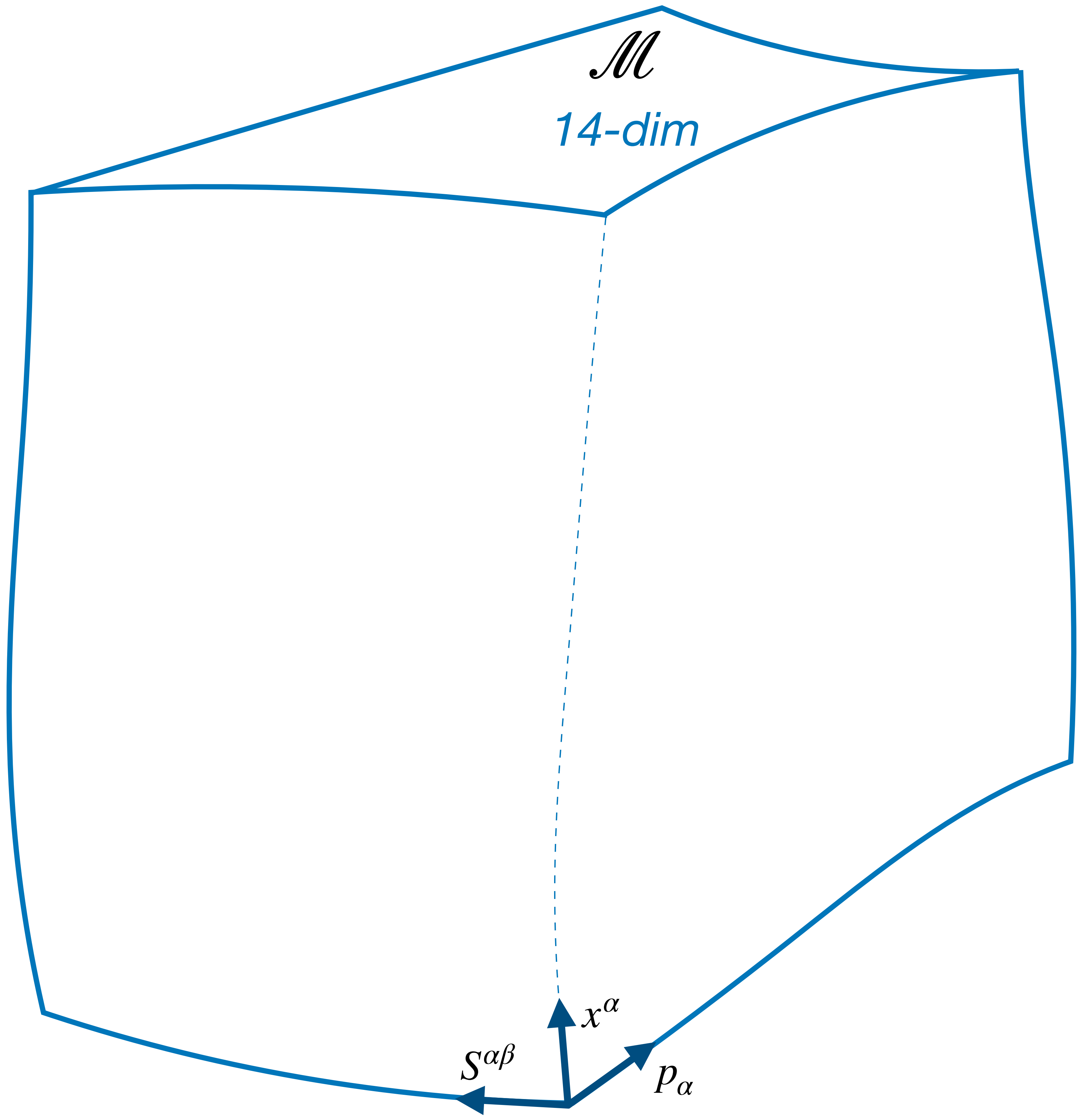

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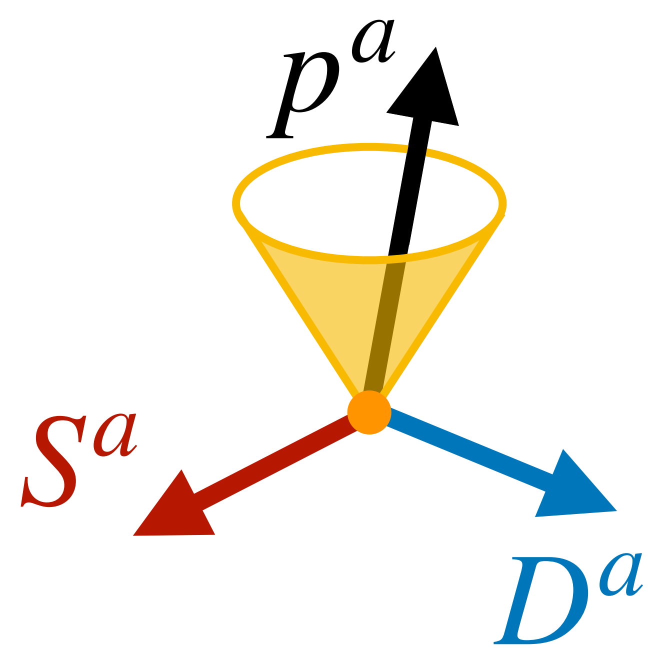

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$$D^b := p_a S^{ab} = 0$$

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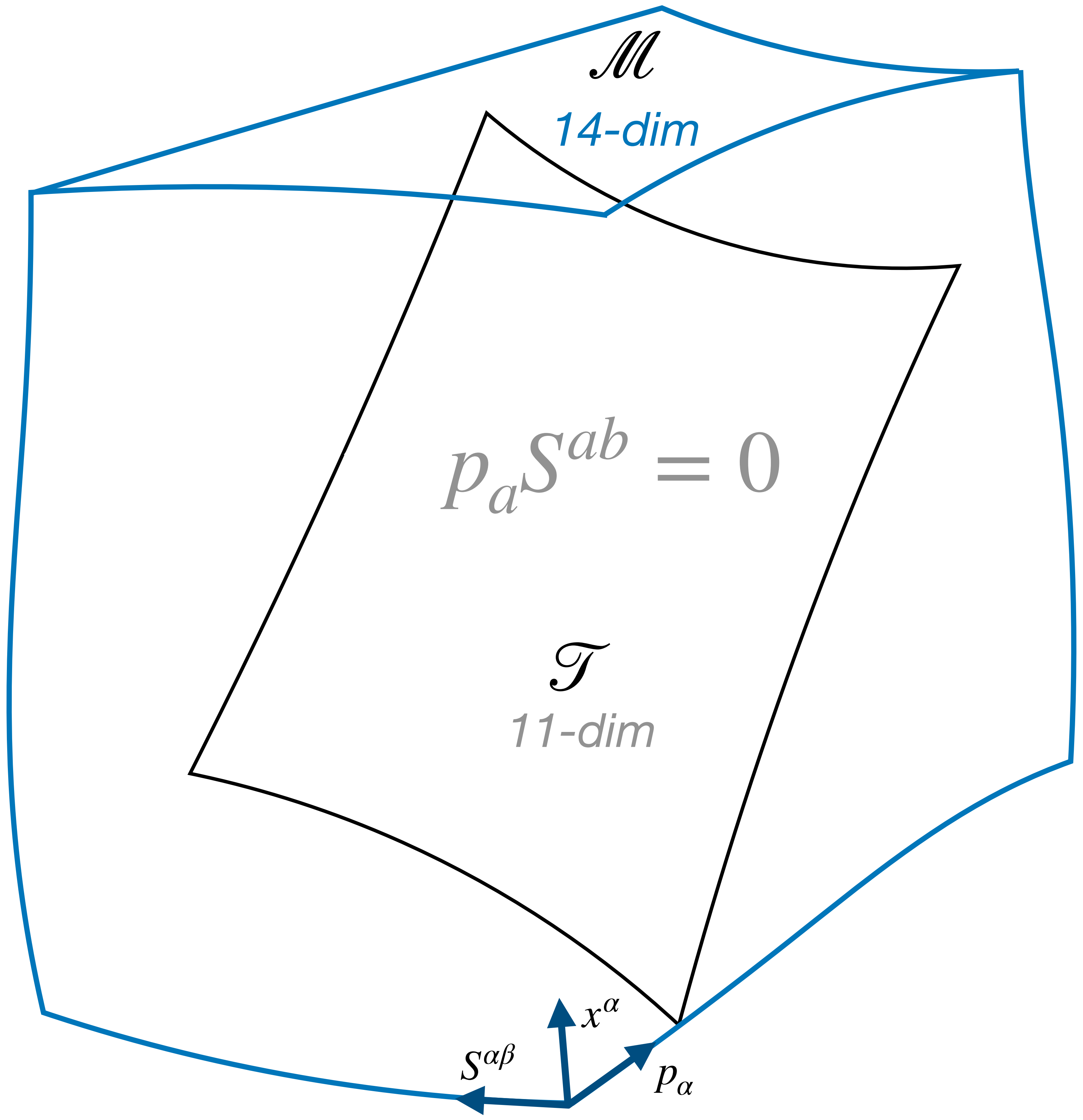

$$\color{red}{S^{ab}} = \varepsilon^{abcd}\bar{p}_c \color{red}{S_d} + 2\bar{p}^{[a}\color{blue}{D^{b]}}$$

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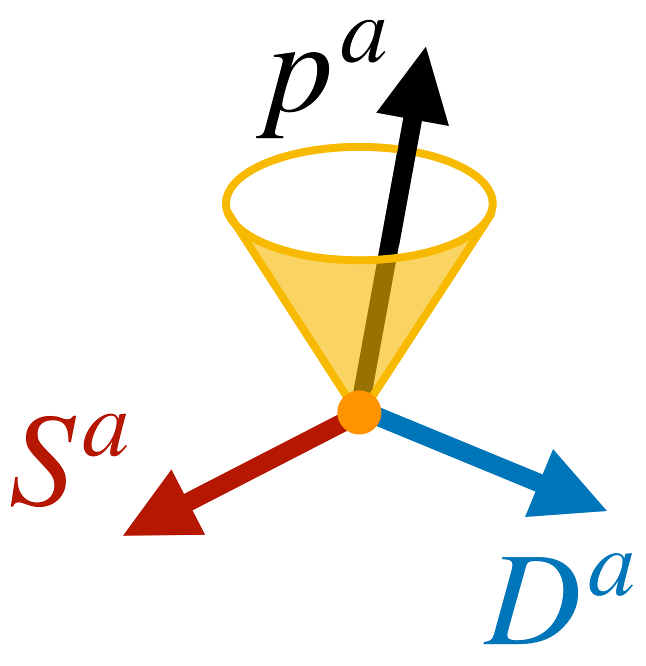

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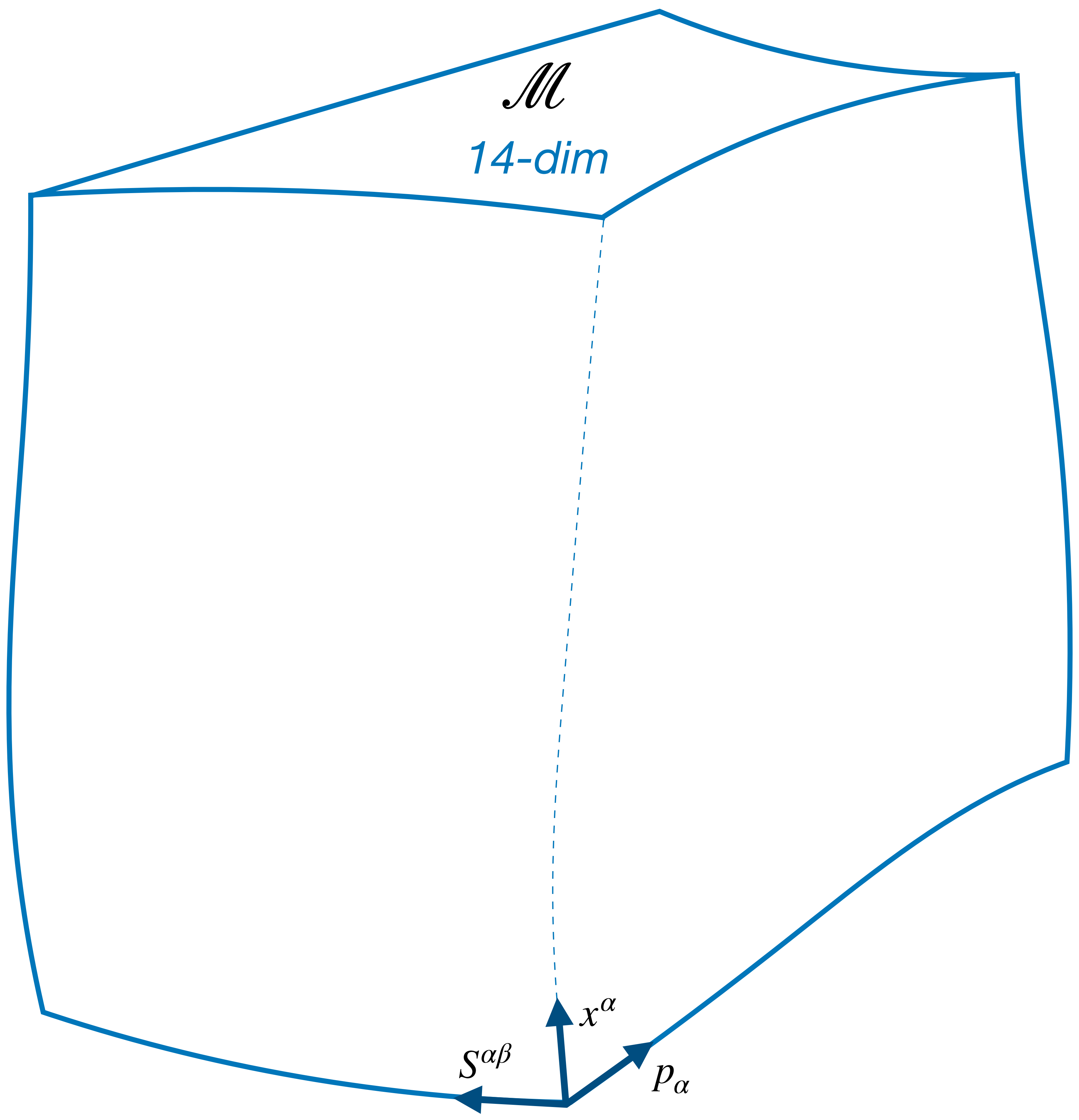
$\Longrightarrow$

Algebraic equation: restrict to **sub-manifold** 𝒯 in phase space ℳ

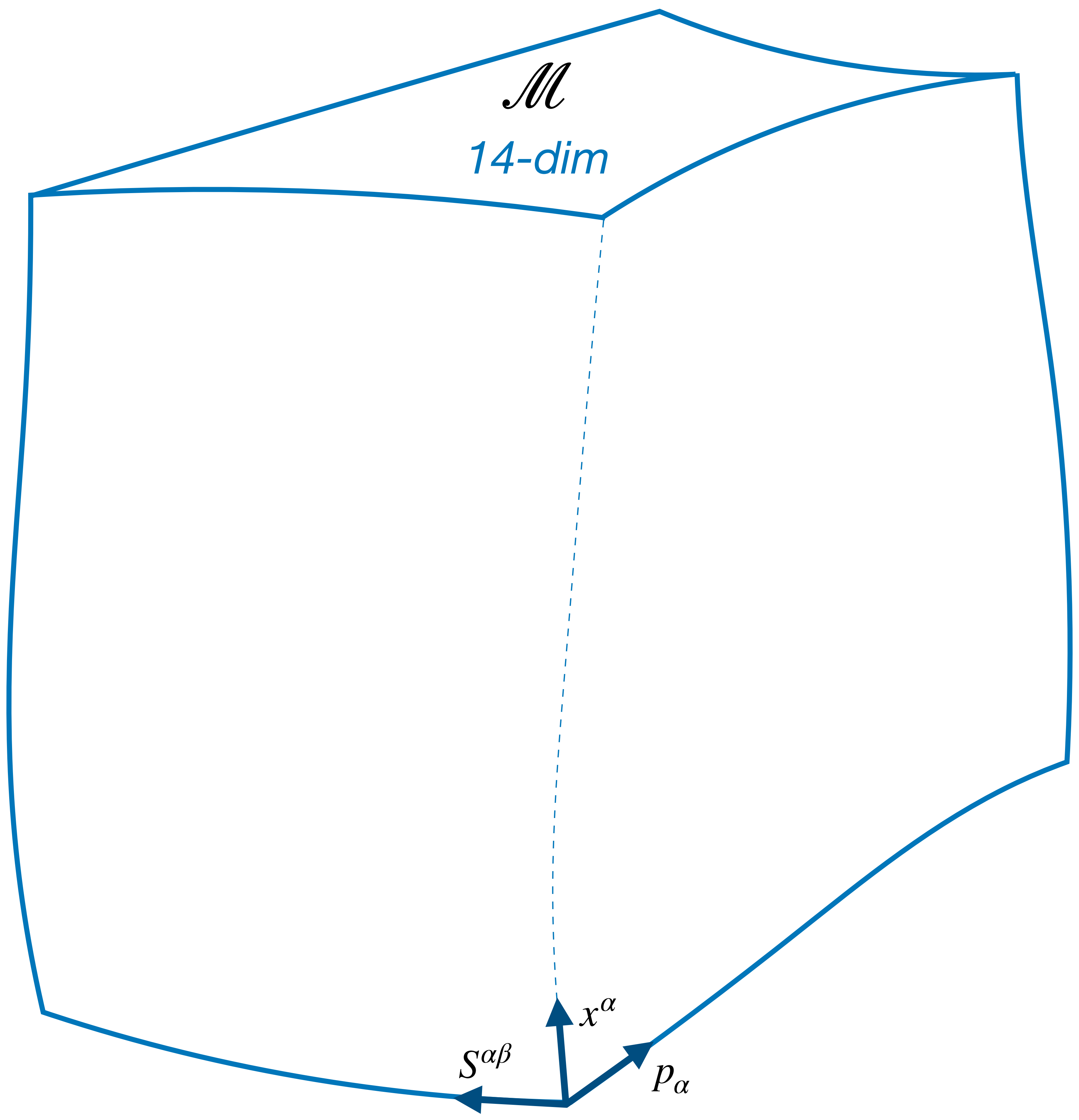
Problem II: local Lorentz invariance of GR



Poisson bracket for the angular momentum:

$$\{S^{\alpha\beta}, S^{\gamma\delta}\} = g^{\alpha\gamma}S^{\beta\delta} - g^{\alpha\delta}S^{\beta\gamma} + g^{\beta\delta}S^{\alpha\gamma} - g^{\beta\gamma}S^{\alpha\delta}$$

Problem II: local Lorentz invariance of GR



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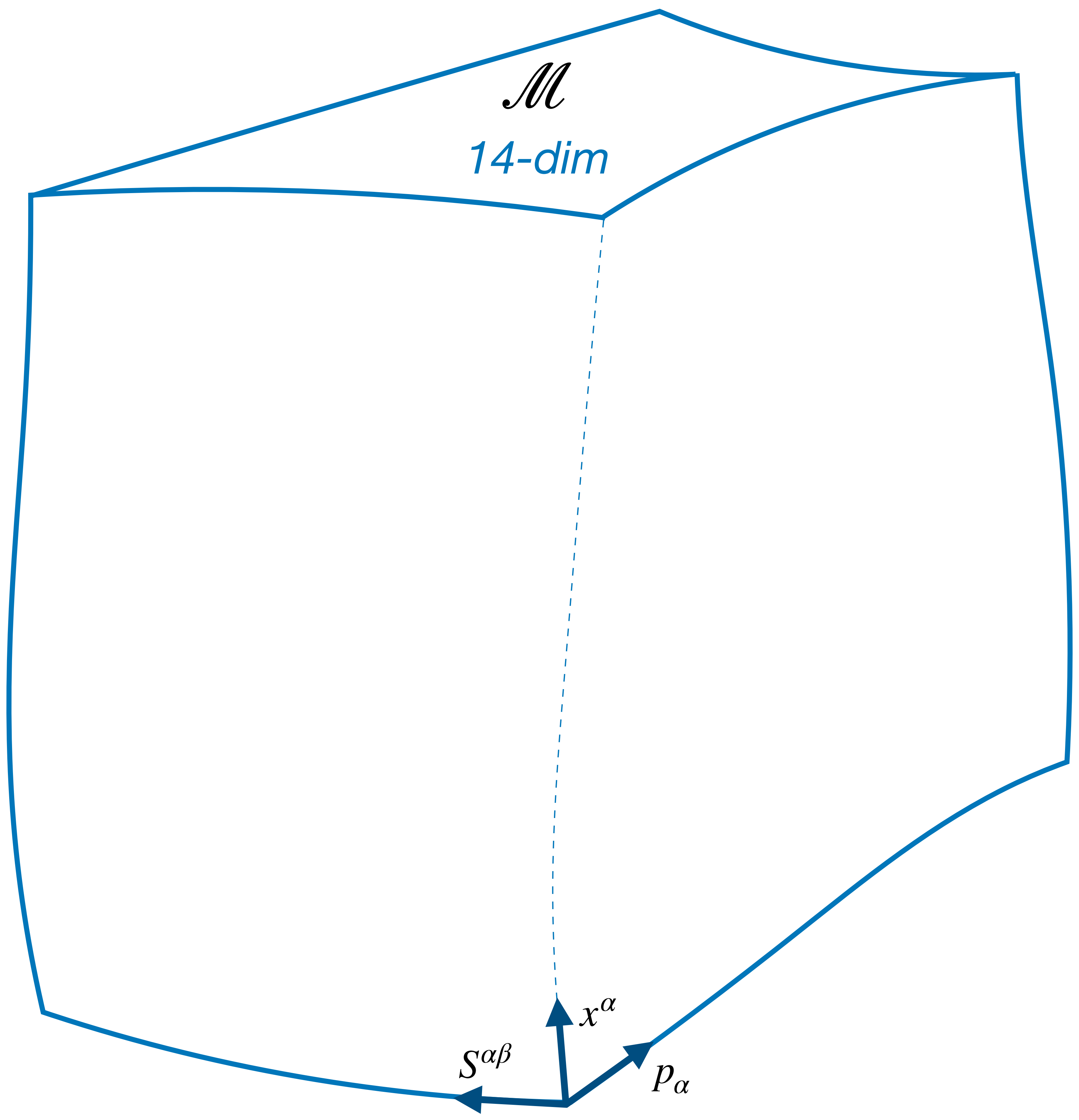

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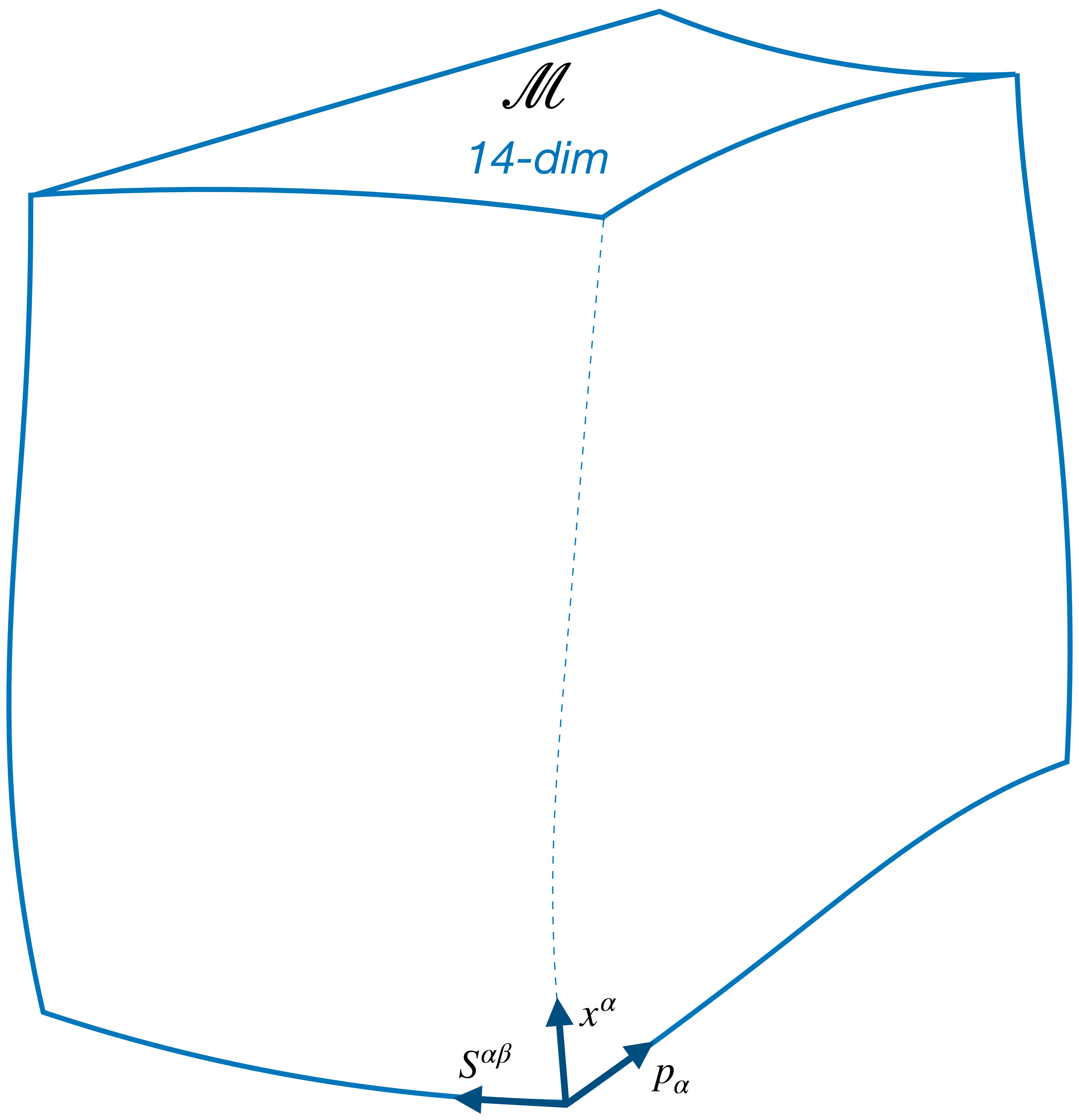

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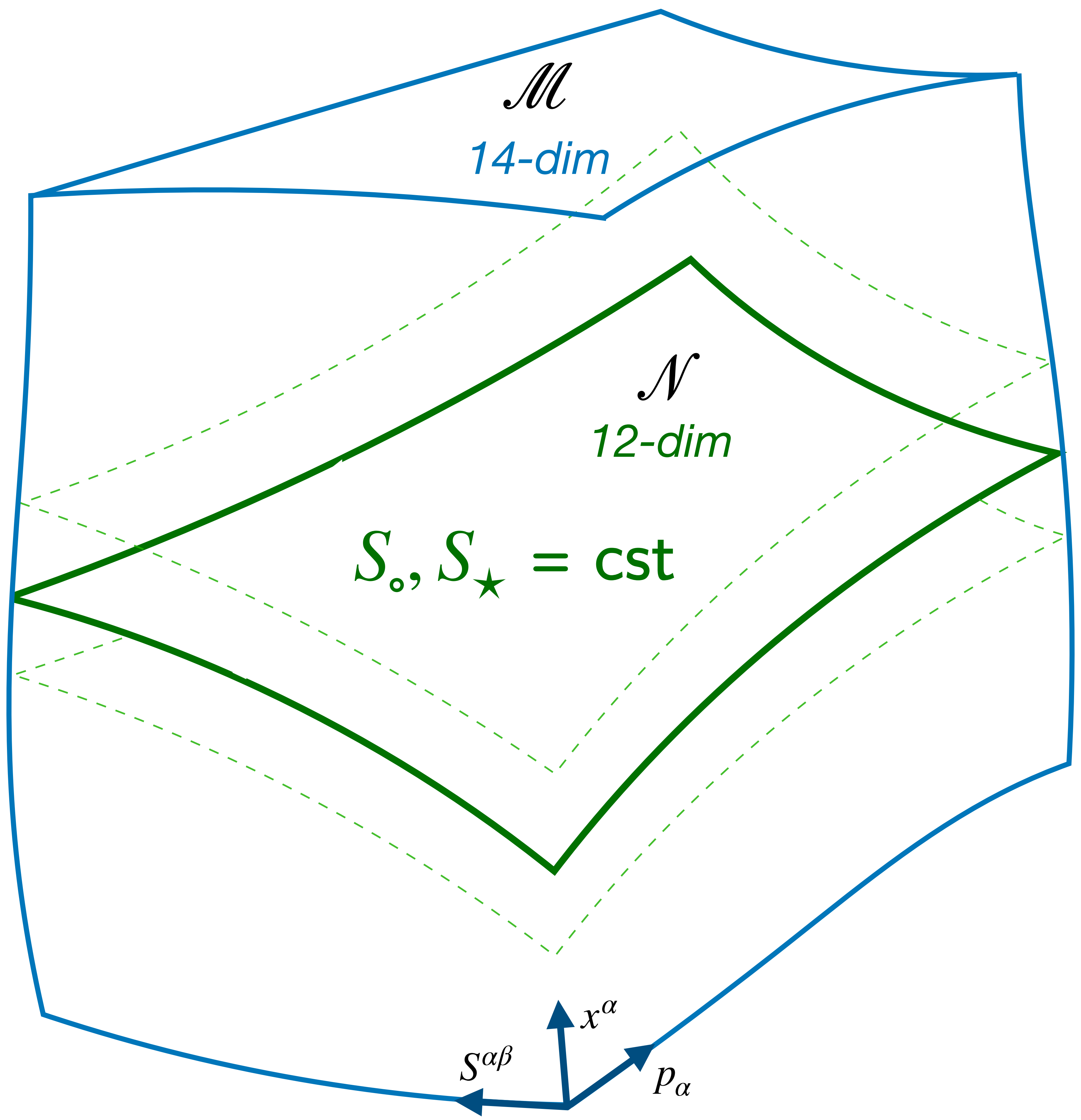
Such $F:\mathscr{M}\to\mathbb{R}$ si called a Casimir invariant of $\{,\}$ $\implies$ There are 2 Casimir invariants $S_\circ$ and $S_\star$:

$$S_\circ^2:=g_{\alpha\beta}g_{\gamma\delta}S^{\alpha\gamma}S^{\beta\delta} \qquad S_\star^2:=\varepsilon_{\alpha\beta\gamma\delta}S^{\alpha\beta}S^{\gamma\delta}$$

Problem II: local Lorentz invariance of GR



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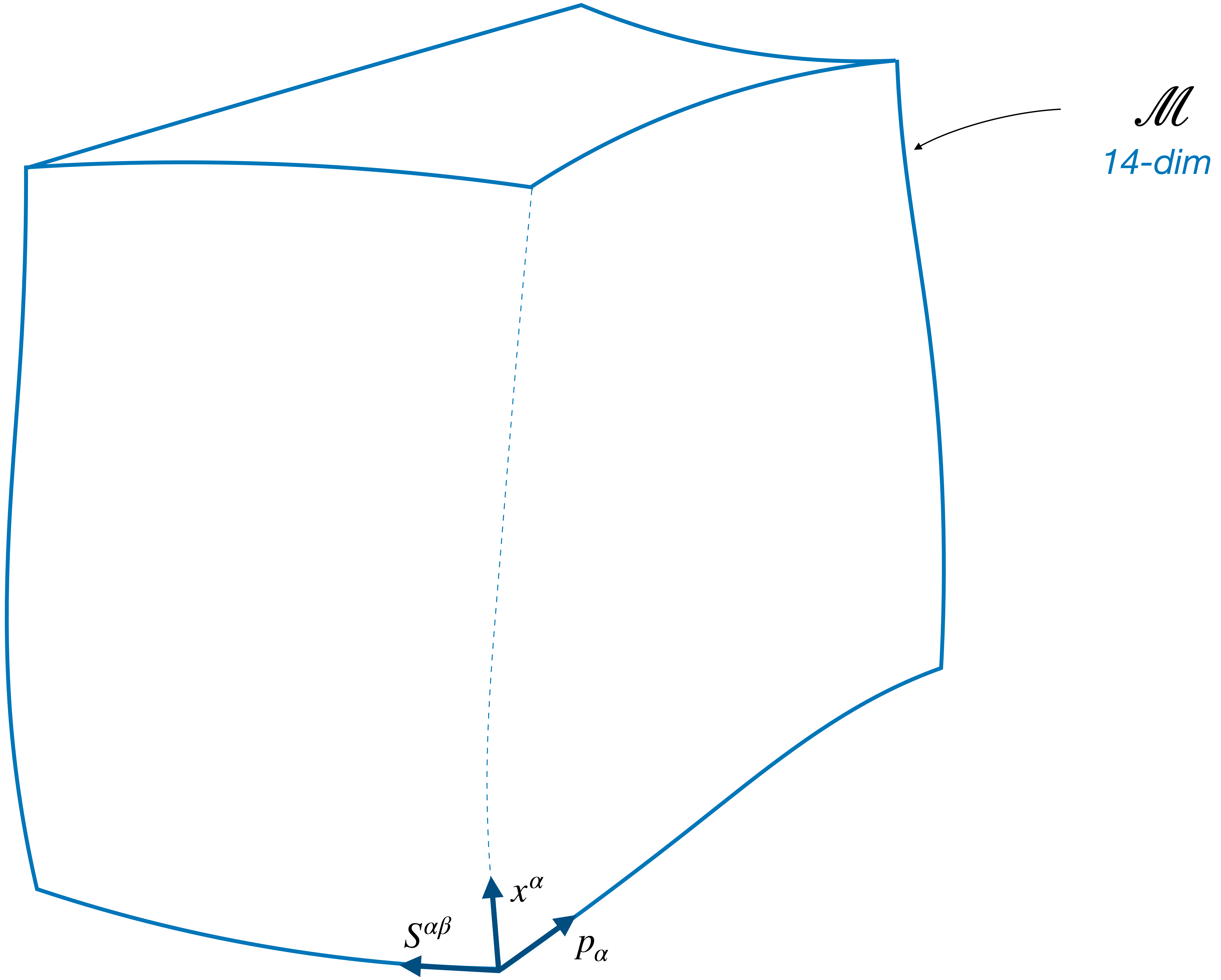
On $\mathscr{M}$, level sets of $(S_\circ, S_\star)$ are called symplectic leaves $\mathscr{N}$: they are non-degenerate

Fixing the problems: a summary



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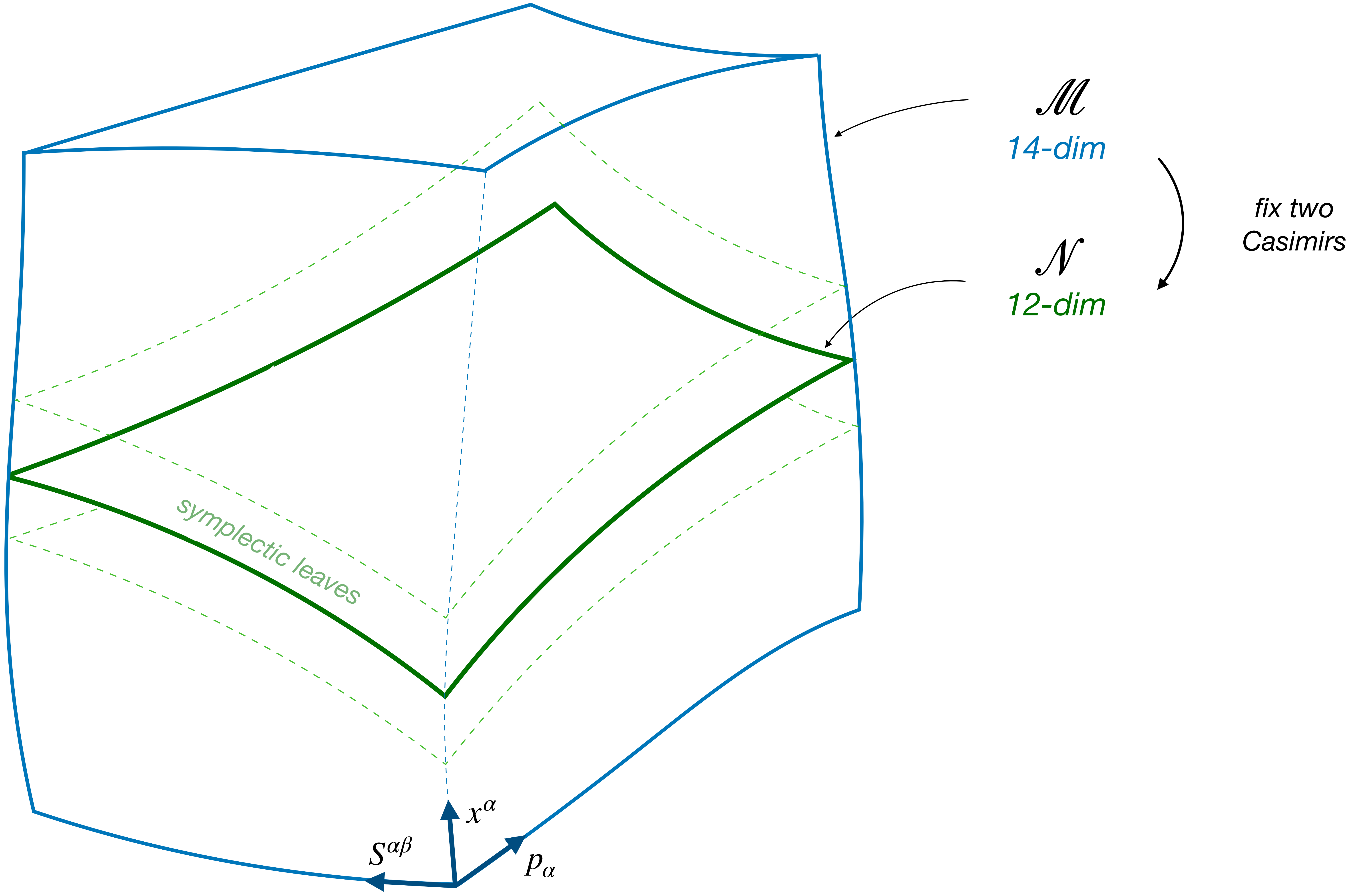

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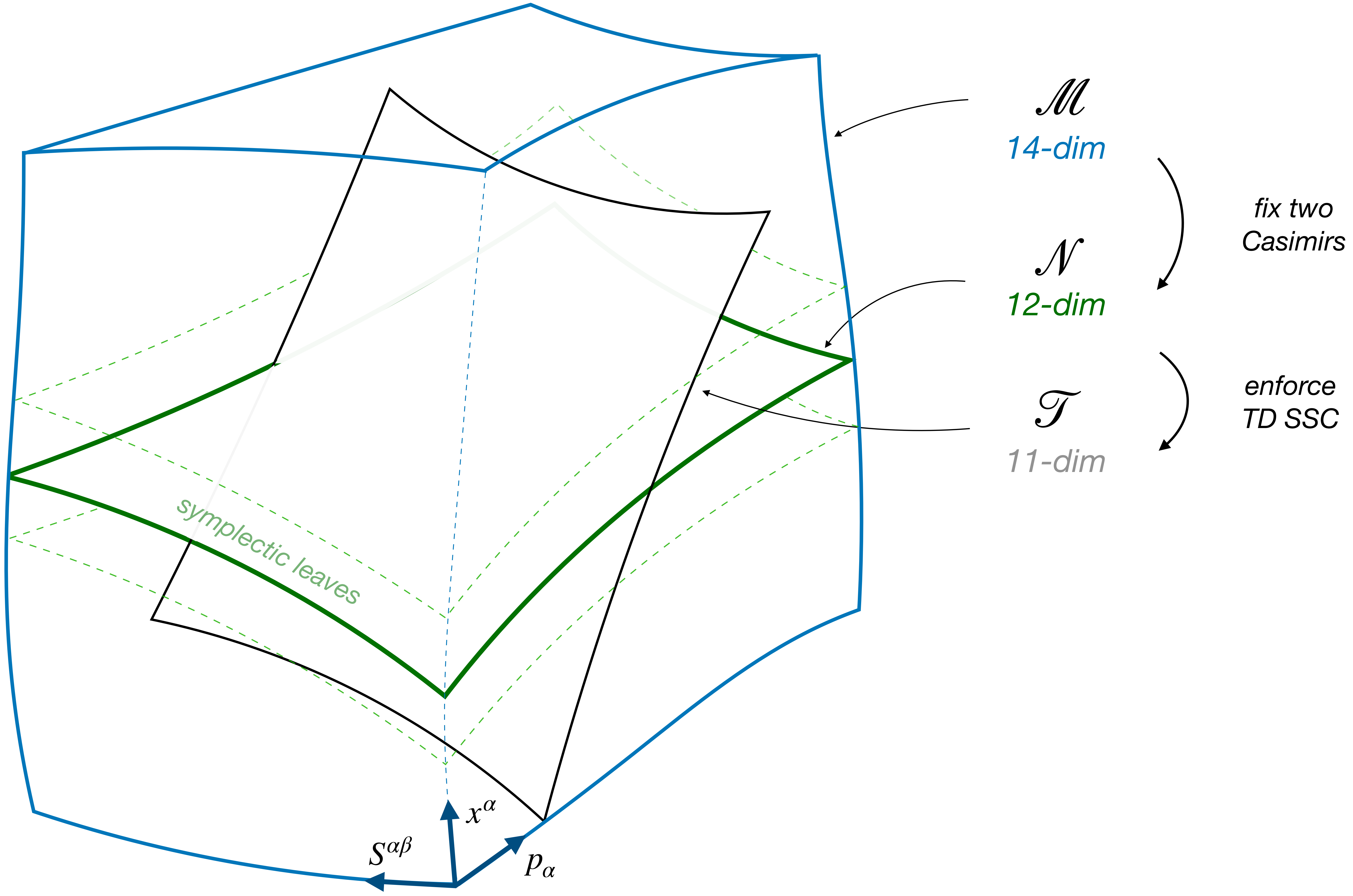

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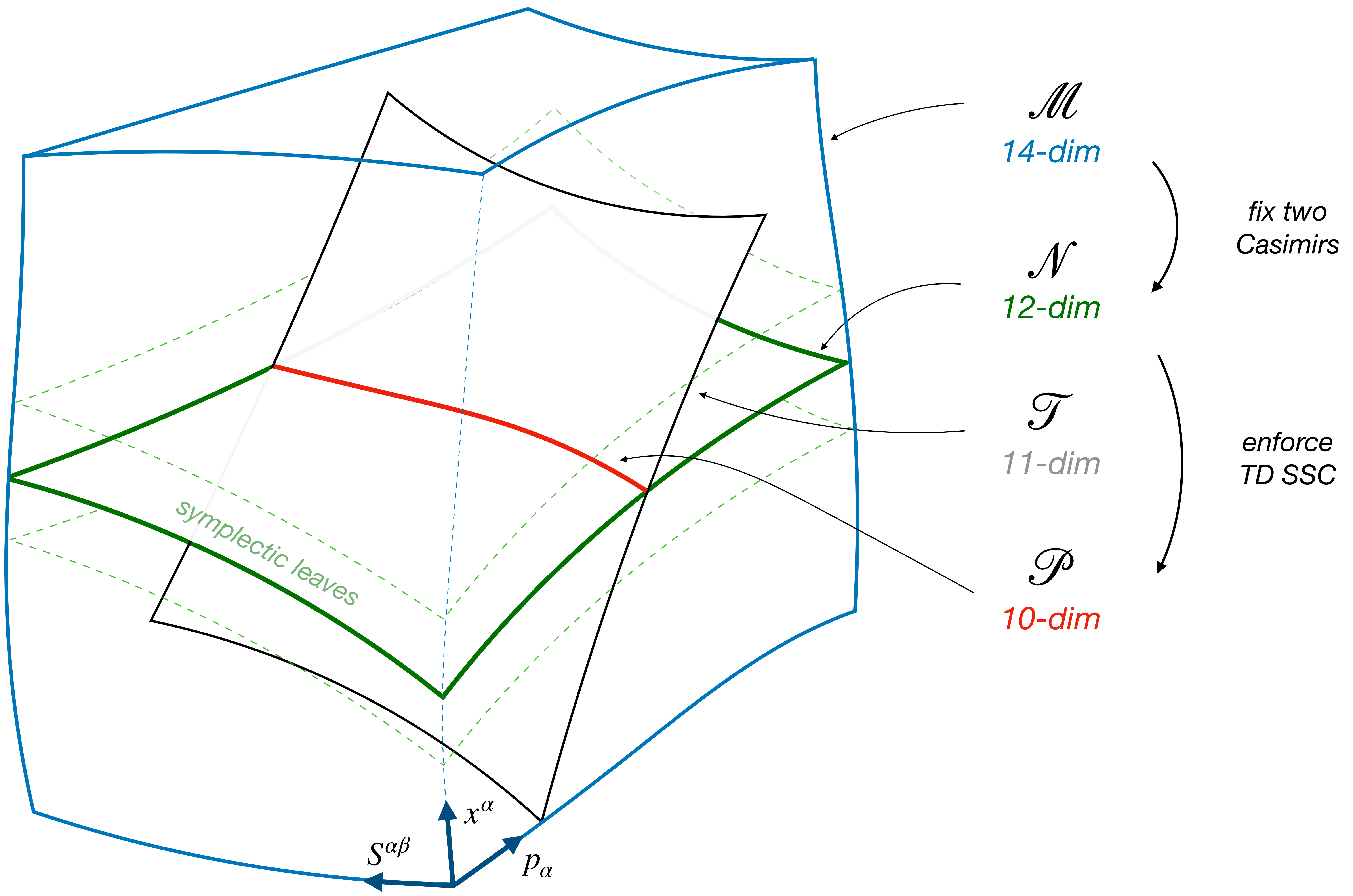

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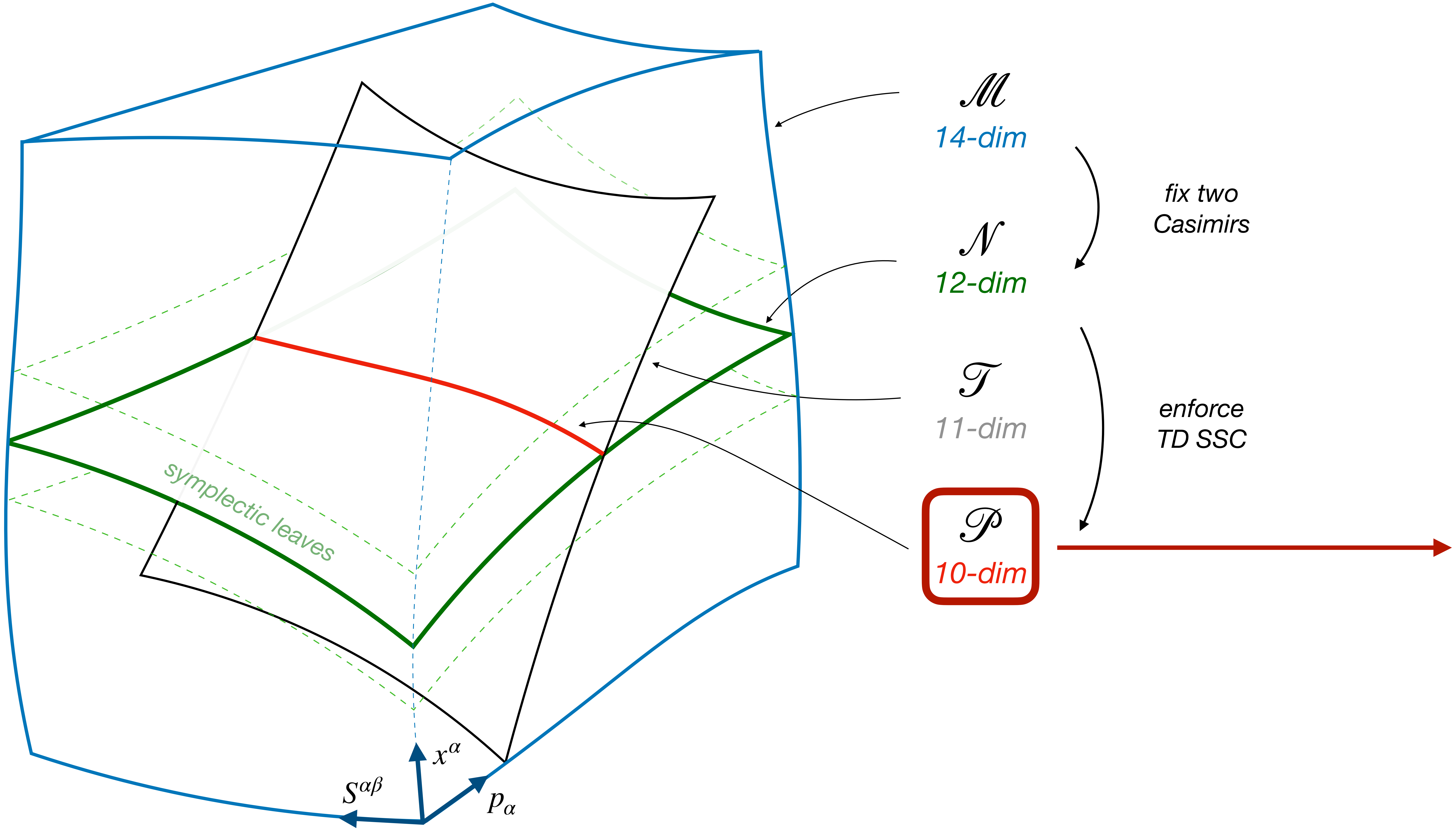

Where one can
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integrability !

$\Updownarrow$

**Where physical
trajectories lie !**

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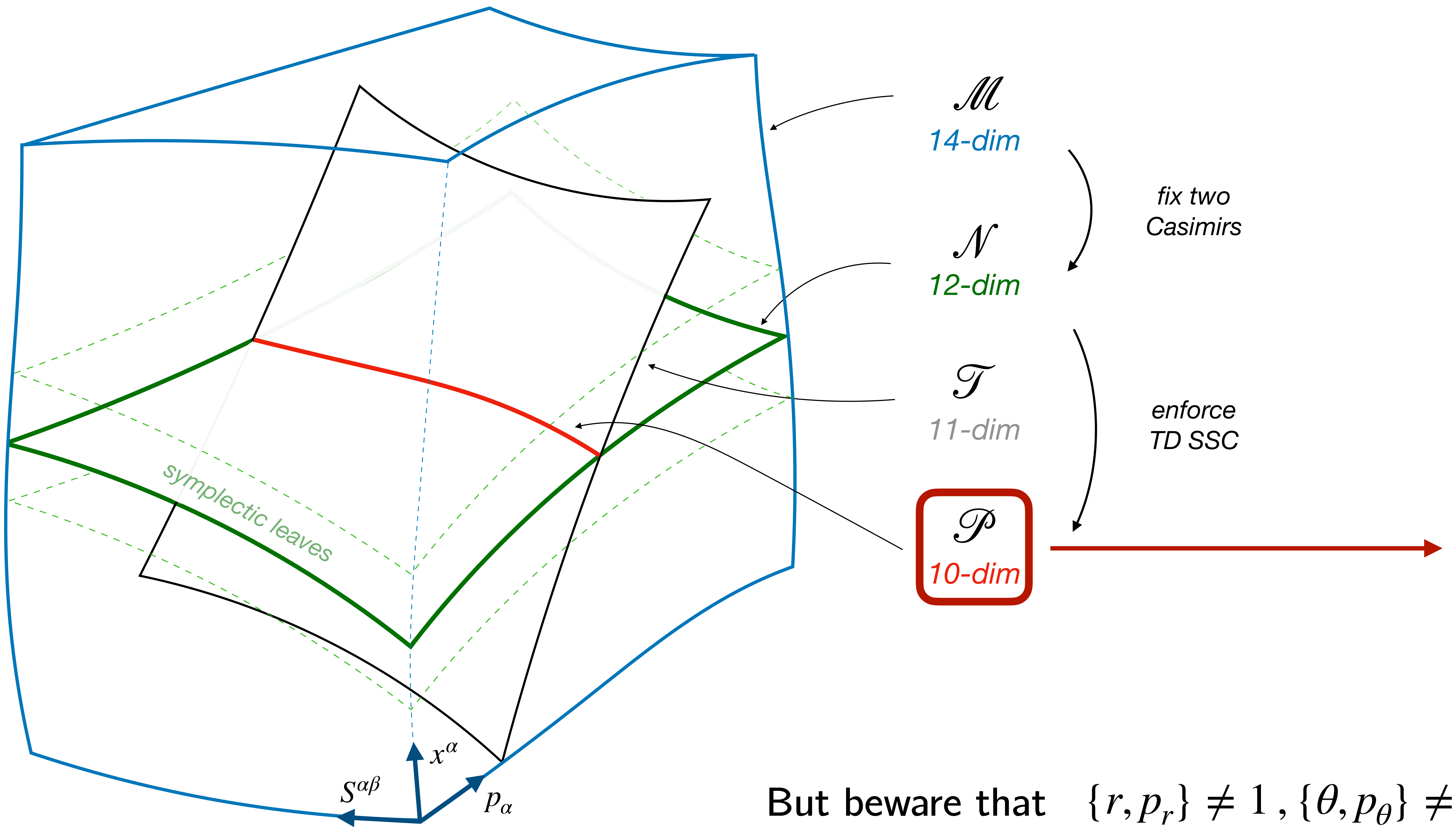

Where one can finally ask about integrability !

⇕

Where physical trajectories lie !

But beware that $\{r, p_r\} \neq 1\,, \{\theta, p_\theta\} \neq 1\,, \{\theta, r\} \neq 0\,, \dots$ on $\mathscr{P}$

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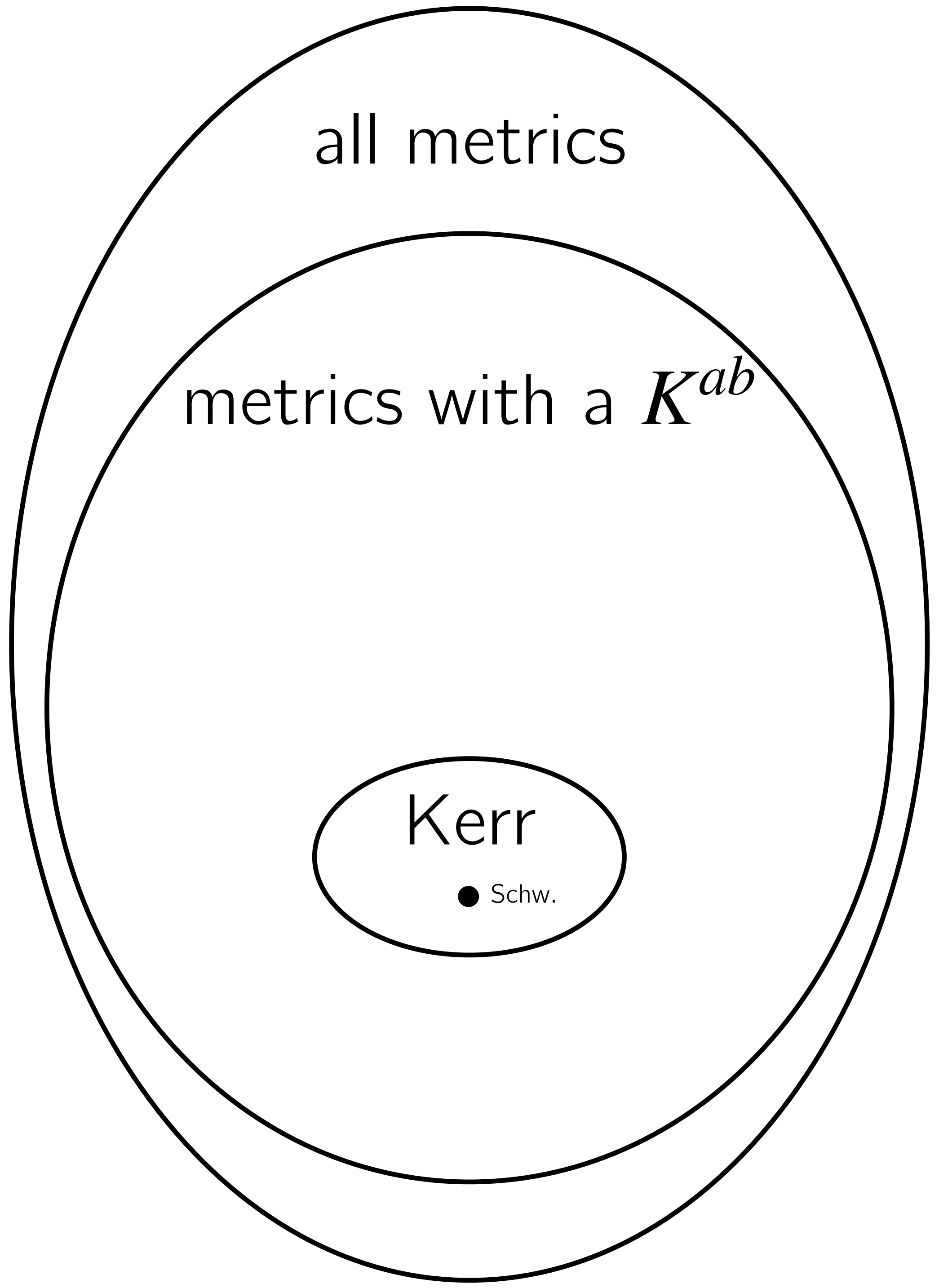
Geodesic integrals of motion

Killing field	Definition	Integral in g_{ab} (**geodesics**) (***any*** *compact object*)
Killing vector	$\nabla_{(a}k_{b)} = 0$	$k^\alpha p_\alpha$
Killing-Stäckel tensor	$\nabla_{(a}K_{bc)} = 0$ + sym	$K^{\alpha\beta}p_\alpha p_\beta$



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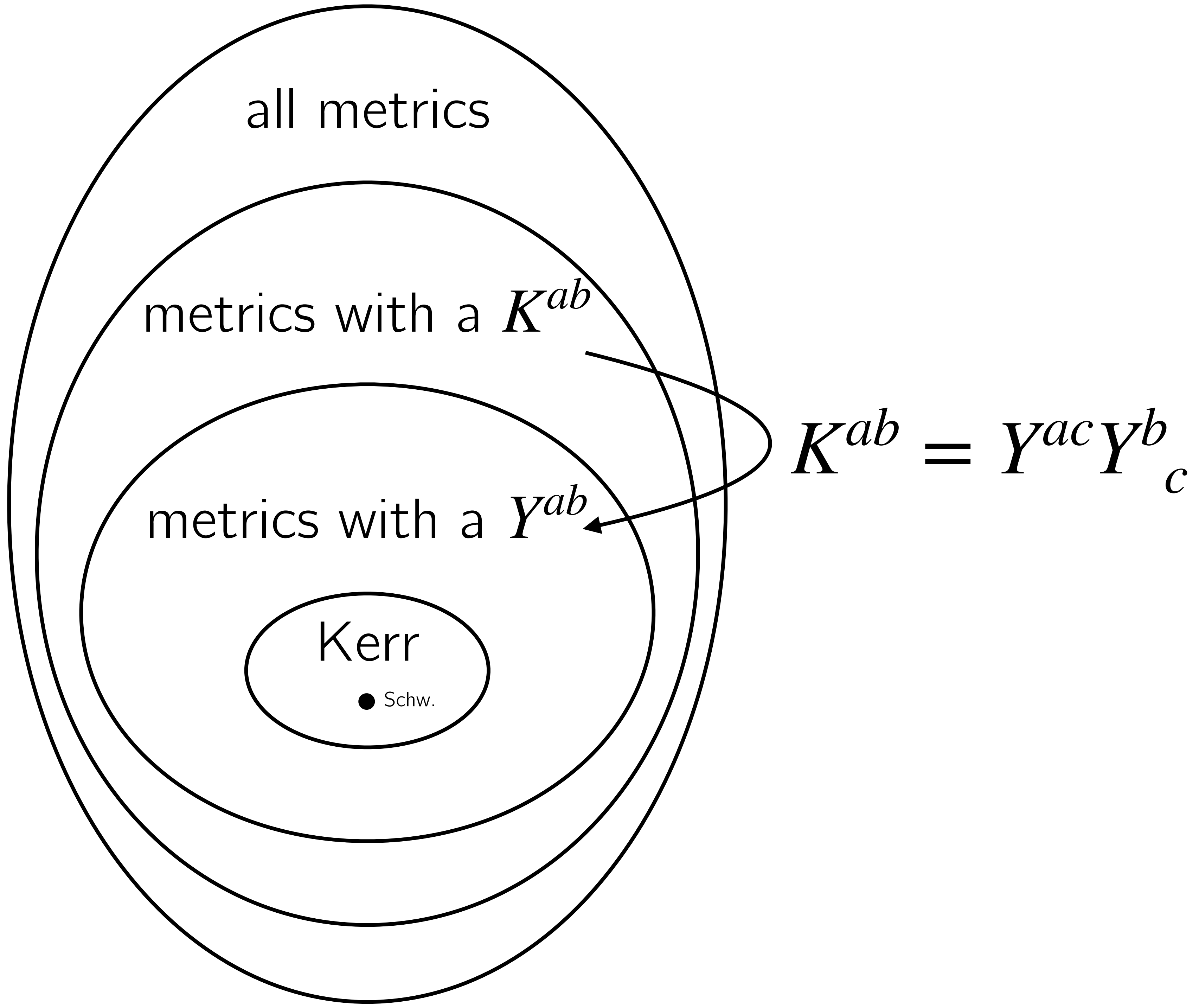

Beyond-geodesic integrability around **black holes**

Killing field	Definition	Integral in g_{ab} (**geodesics**) (**any** *compact object*)	Integral **in Kerr** (***linear**-in-spin order*) (**any** *compact object*)
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Killing-Yano tensor	$\nabla_{(a}Y_{b)c} = 0$ + anti-sym	$Y^{\alpha\beta} p_\alpha Y_{\beta\gamma} p^\gamma$	*Rüdiger (1981)* $\mathfrak{Q}^{(1)} = K^{\alpha\beta} p_\alpha p_\beta + L^\alpha{}_{\beta\gamma} p_\alpha S^{\beta\gamma}$ $\mathfrak{K} = \epsilon_{\alpha\beta\gamma\delta} Y^{\alpha\beta} S^{\gamma\delta}$
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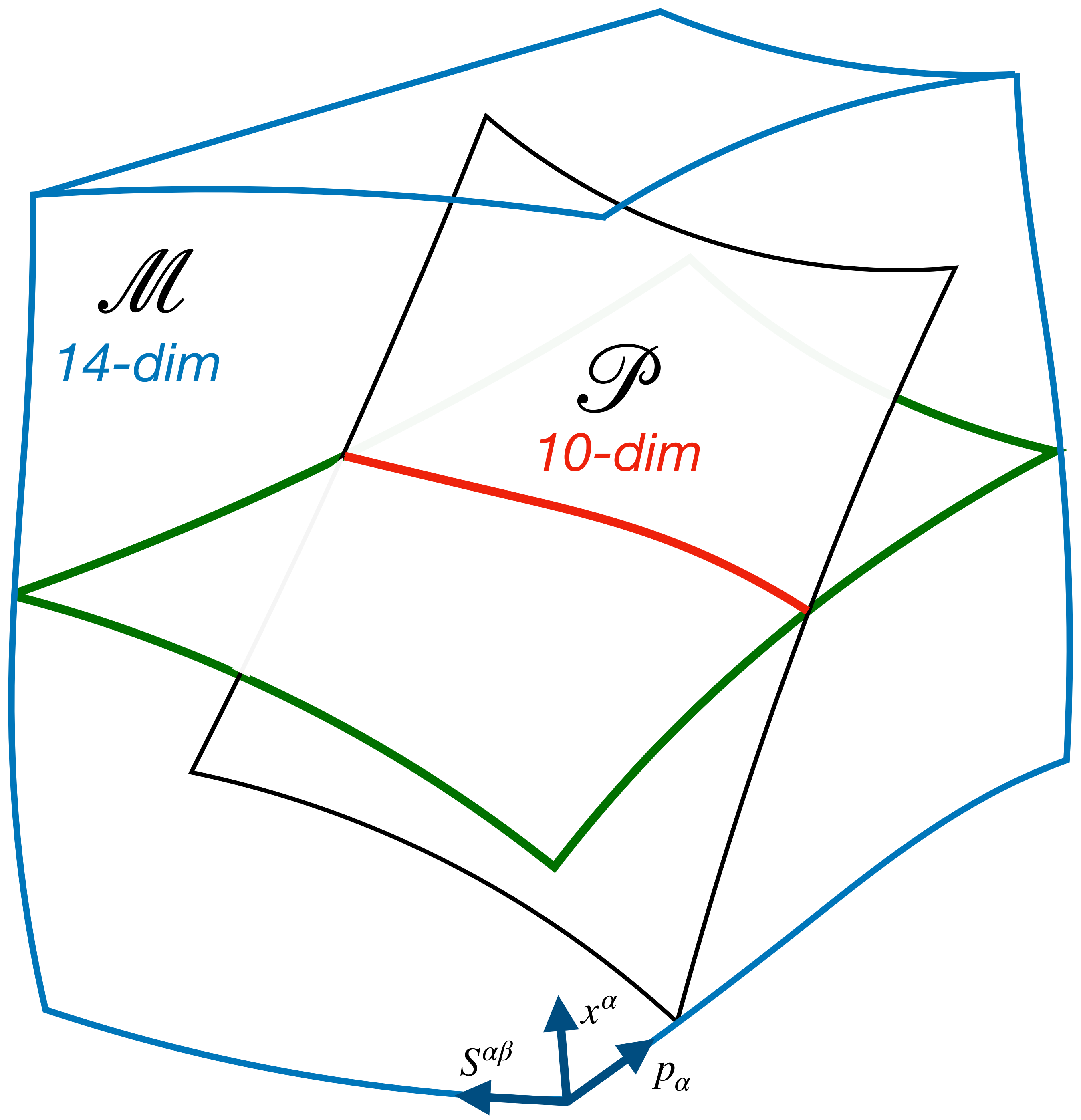
Not much left to do now...

1. Work in the **correct**, physical phase space $\mathscr{P}$



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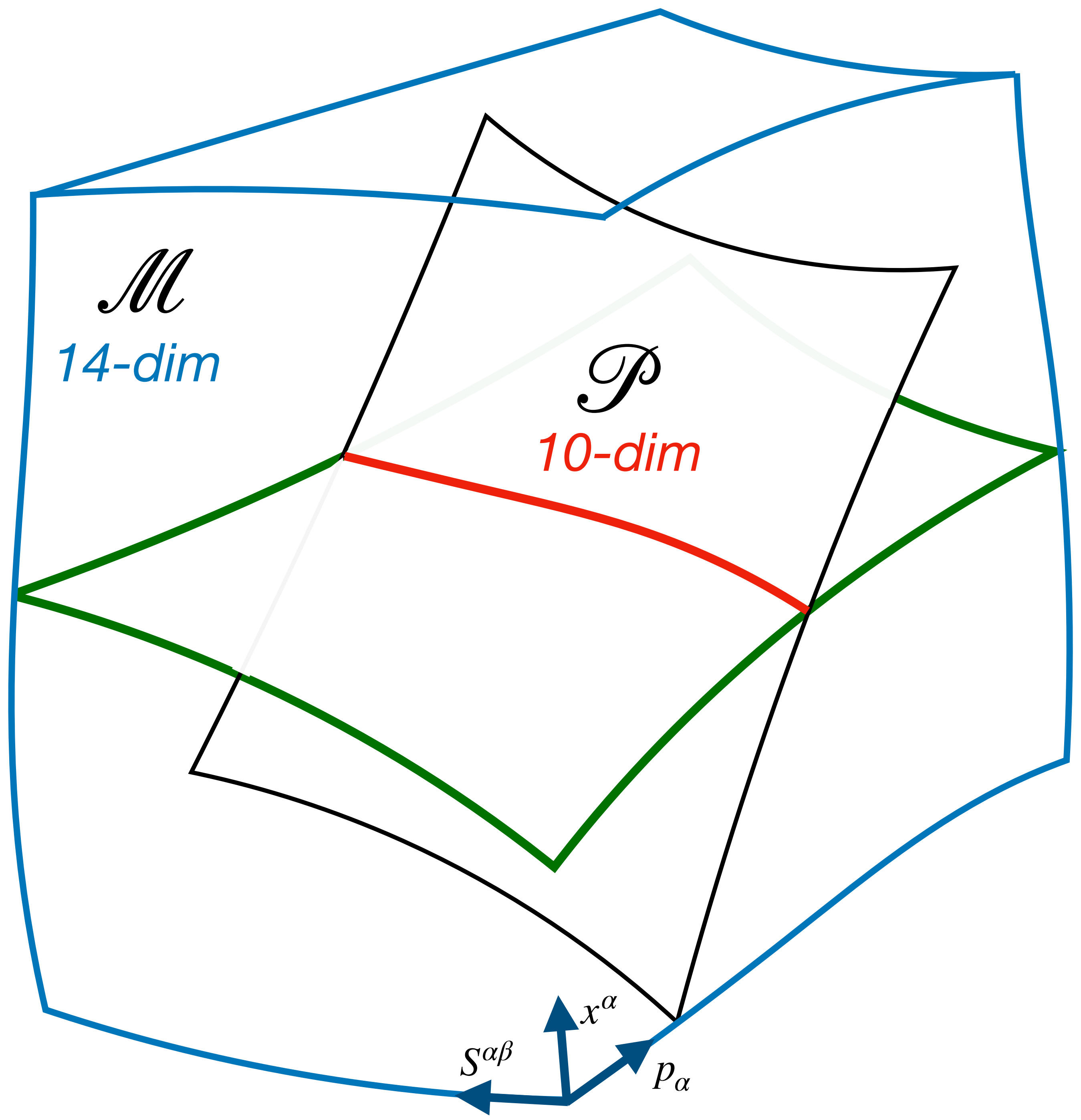

2. Take the 5 (Kerr) invariants in the literature

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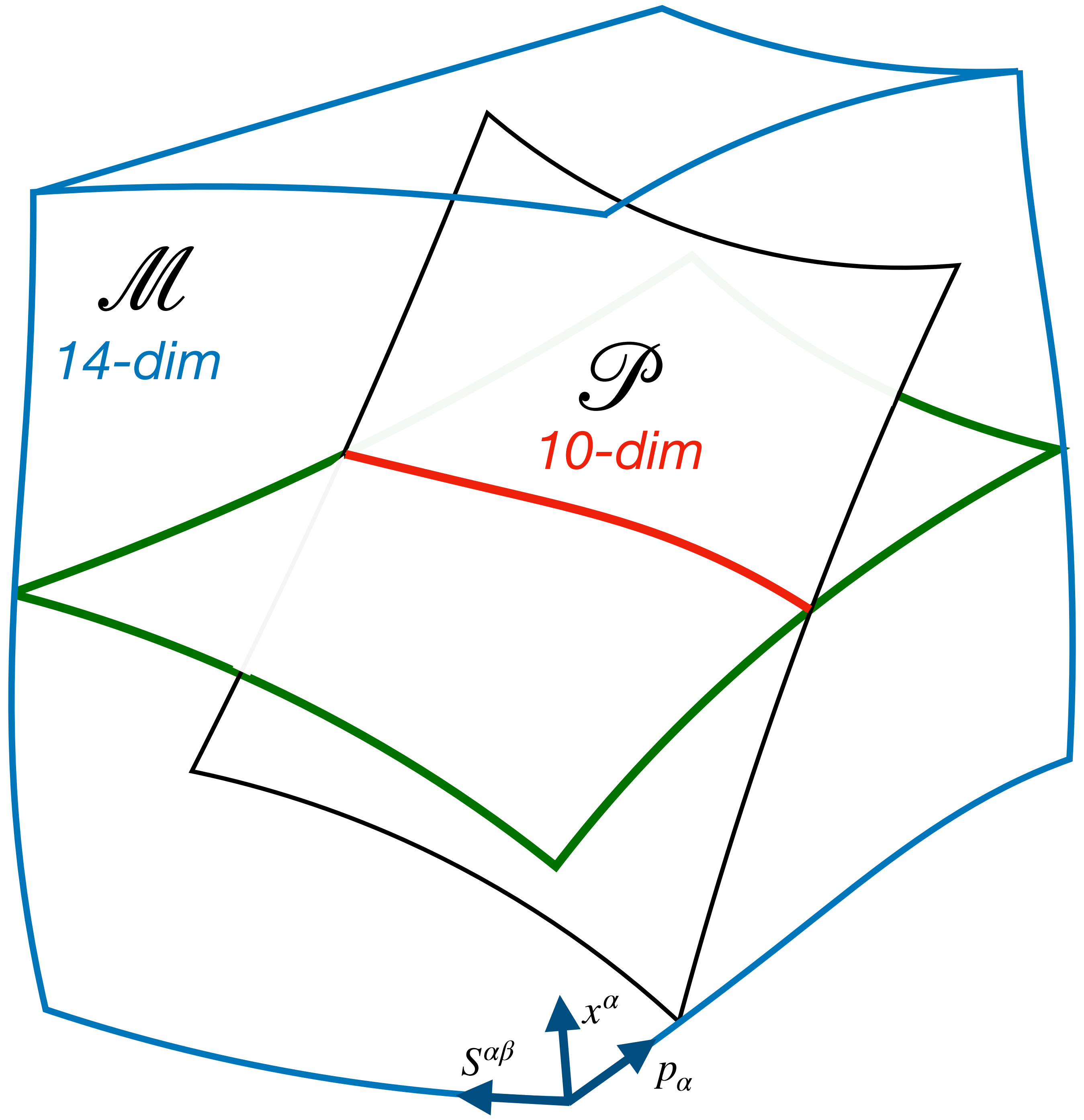

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3. Compute the relevant Poisson brackets

$$\{F, G\}^{\mathscr{P}} := \{F, G\}^{\mathscr{M}} + \text{corr}$$

Poisson brackets of first integrals on N

```
randomValues = {πt → RandomReal[{1, 5}], πr → RandomReal[{3, 4}], πφ → RandomReal[{4, 8}], πθ → RandomReal[{2, 7}], a → RandomReal[{0, 1}], M -
   θ → RandomReal[{0, π}]};

PBNQH = (PBN[Q, H] // QuadSpin) /. randomValues;
PBNKH = (PBN[K, H] // QuadSpin) /. randomValues;
PBNKQ = (PBN[K, Q] // QuadSpin) /. randomValues;
```

Define Poisson brackets on 14-dim M

```
Print["{Q,H}lin= ", PBNQH // LinSpin // Simplify // Chop]
Print["{K,H}lin= ", PBNKH // LinSpin // Simplify // Chop]
Print["{K,Q}lin= ", PBNKQ // LinSpin // Simplify // Chop]
```

Compute them at linear order in spin

$\{Q,H\}_{lin} = 0$

$\{K,H\}_{lin} = 3.49963\,D1 + 0.359485\,D2 + 0.831548\,D3 - 0.0187377\,S1 + 0.182414\,S2$

$\{K,Q\}_{lin} = -3.45461\,D1 - 0.354861\,D2 - 44.921\,D3 - 4.01725\,S1 + 39.1084\,S2$

Some don't vanish ⇒ may explain some literature "claims" on non-integrability

Poisson brackets of first integrals on N + applied SSC (not P-bracket yet, but sufficient thanks to properties of H and Q)

```
randomValues = {πt → RandomReal[{0, 1}], πr → RandomReal[{0, 1}], πφ → RandomReal[{0, 1}], πθ → RandomReal[{0, 1}], a → RandomReal
    M → RandomReal[{0, 1}], r → RandomReal[{2, 10}], θ → RandomReal[{0, π / 2}]};

PBNKHwithSSC = (PBN[K, H] // ToP // QuadSpin) /. randomValues;
PBNQHwithSSC = (PBN[Q, H] // ToP // QuadSpin) /. randomValues;
PBNKQwithSSC = (PBN[K, Q] // ToP // QuadSpin) /. randomValues;
```

Define Poisson brackets on 10-dim P

```
Print["{Q,H}lin+SSC= ", PBNKHwithSSC // LinSpin // Simplify // Chop]
Print["{K,H}lin+SSC= ", PBNQHwithSSC // LinSpin // Simplify // Chop]
Print["{K,Q}lin+SSC= ", PBNKQwithSSC // LinSpin // Simplify // Chop]
```

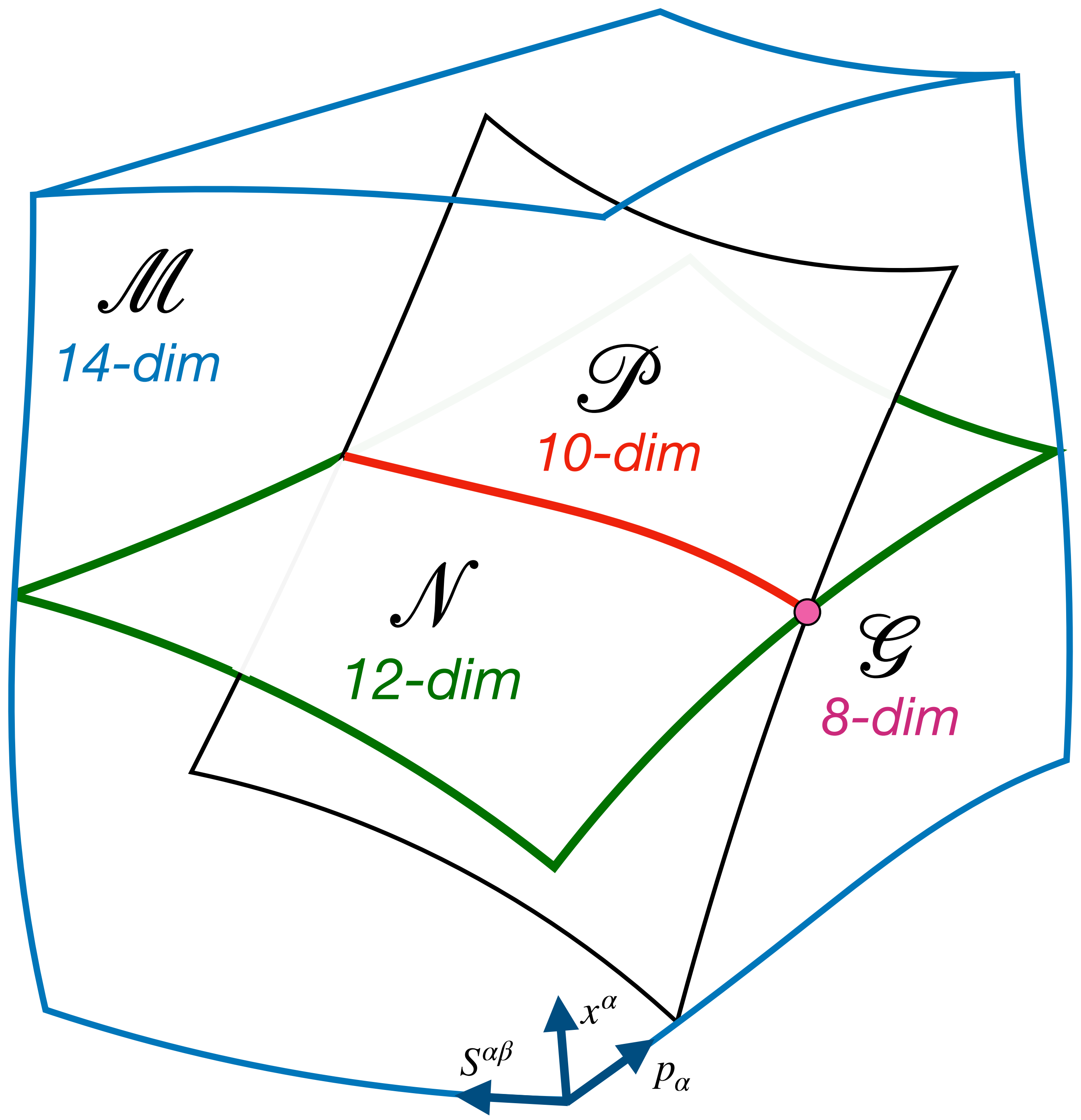
Compute them at linear order in spin

```
{Q,H}lin+SSC= 0
{K,H}lin+SSC= 0
{K,Q}lin+SSC= 0
```

All (non-trivial ones) vanish ⇒ linear in spin integrability in Kerr

Not much left to do now...

1. Work in the correct, physical phase space $\mathscr{P}$



2. Take the (Kerr) invariants in the literature

$$H, E, L_z, \mathfrak{K}, \mathfrak{Q}^{(1)}$$

3. Compute the relevant Poisson brackets

$$\{F, G\}^{\mathscr{P}} := \{F, G\}^{\mathscr{M}} + \text{corr}$$

changes everything !

4. Conclude integrability (in Kerr, lin-in-spin)

At **linear order in spin**, the motion of **any test body** in a **Kerr background** can be described by a **5-dimensional Hamiltonian system** that is **integrable**.

Plan

I. Geodesics

1. geodesic motion
2. hamiltonian formulation
3. integrable systems

II. Adding spin

1. linear-in-spin motion
2. hamiltonian formulation
3. integrability in Kerr

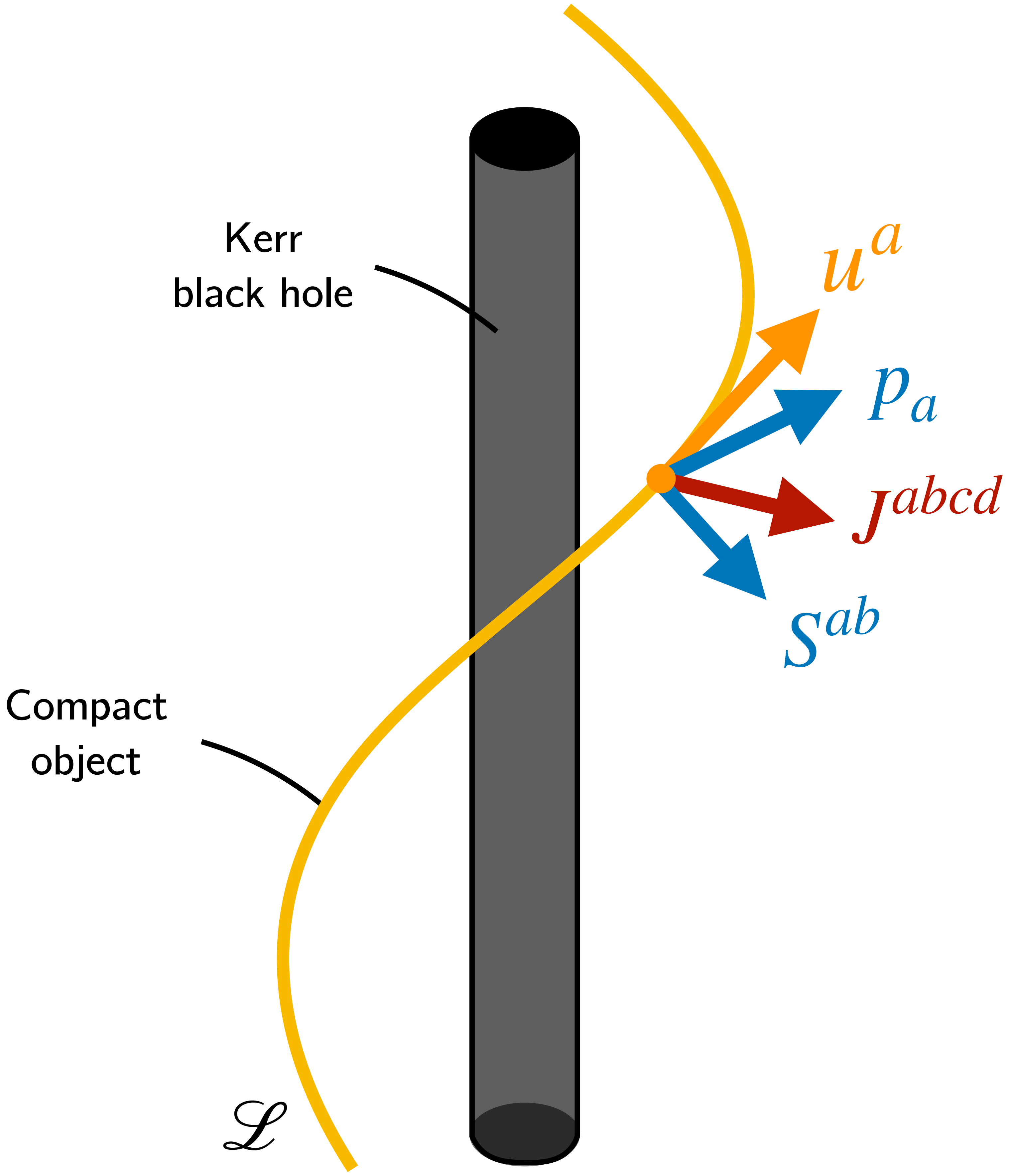
III. Quadrupoles

1. quadratic-in-spin motion
2. hamiltonian formulation
3. "integrability" in Kerr

What happens to **Kerr integrability**
for the motion of **deforming objects** ?

1. account for the object's deformation
2. describe as a Hamiltonian system
3. find enough integrals of motion

Summary at quadratic-in-spin order

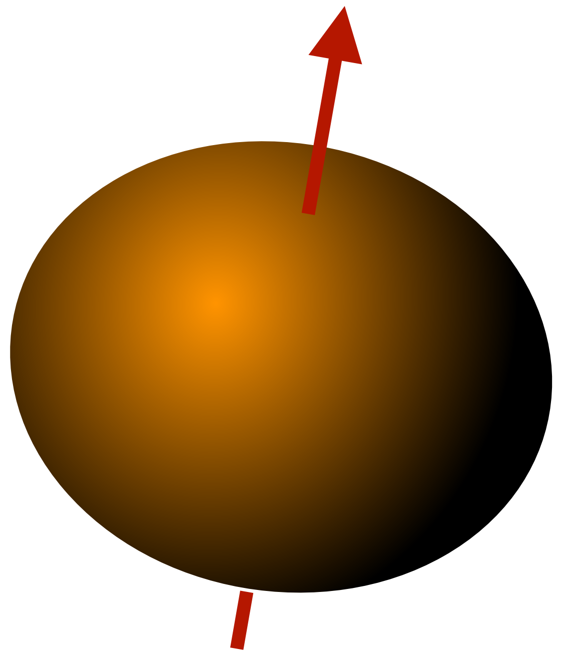


Dixon-Harte equations at quad. order

$$\nabla_u p_a = R_{abcd} S^{bc} u^d + F_a[J]$$ — quadrupole "force"

$$\nabla_u S^{ab} = 2p^{[a} u^{b]} + N^{ab}[J]$$ — quadrupolar "torque"

Spin-induced quadrupole



$$J^{abcd} := \kappa \cdot \frac{3}{\mu^3} p^{[a} S^{b]e} S_e{}^{[c} p^{d]}$$

κ → *deformability coefficient*

- $\kappa = 1$ for black holes
- $\kappa > 1$ for neutron stars
- $\kappa \gg 1$ for white dwarfs

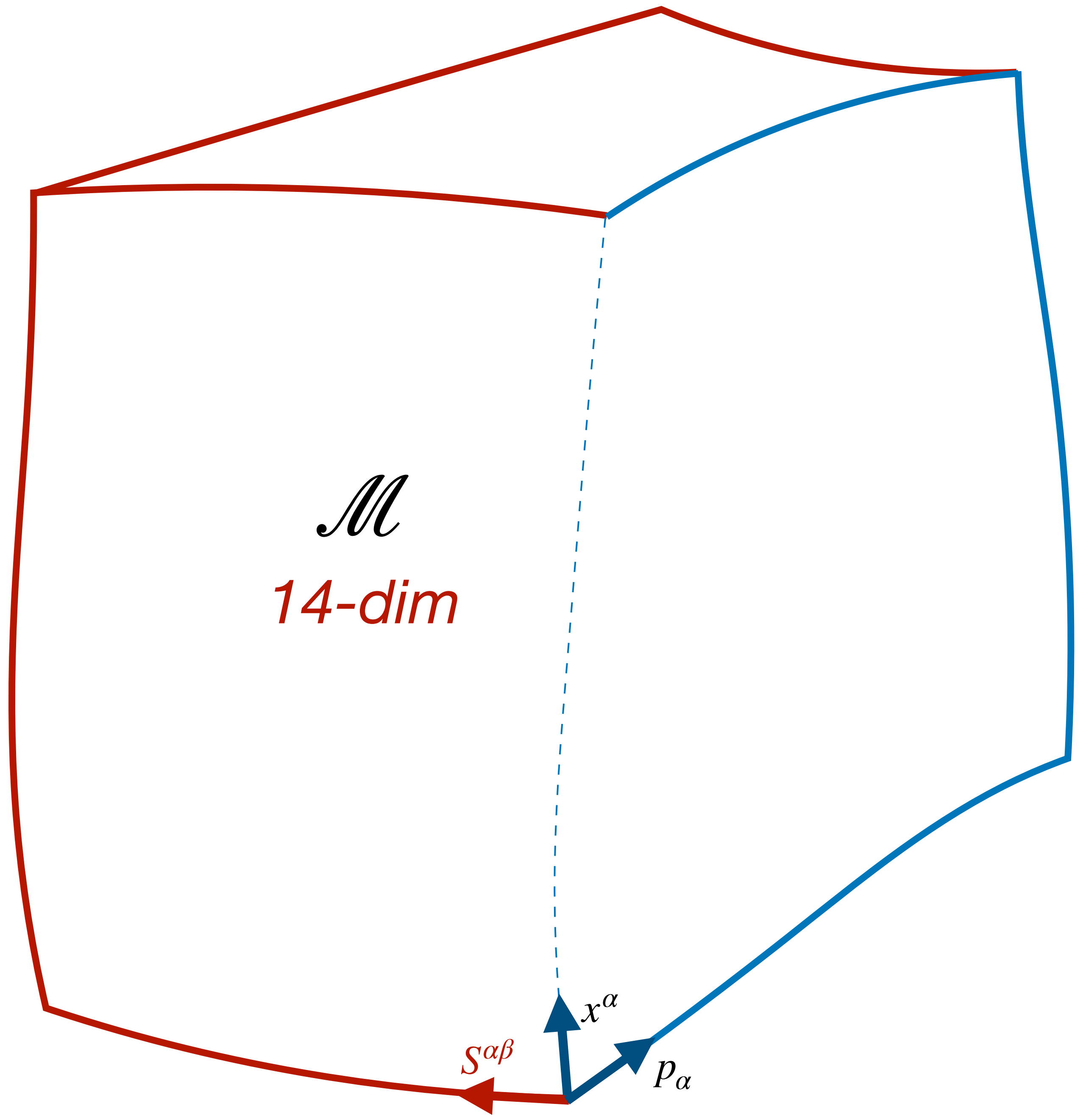
What happens to **Kerr integrability**
for the motion of **deforming objects** ?

1. ~~account for the object's deformation~~
2. describe as a Hamiltonian system
3. find enough integrals of motion

Hamiltonian formulation of MPTD equations (dipolar)

Ham. system

- Phase space $\mathscr{M}$
- Poisson brackets $\{,\}$
- Hamiltonian H



Law of motion

$$\frac{dF}{d\lambda} = \{F, H\}$$

+ Leibniz rule

Phase space:

$$\mathscr{M} = \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^6$$

$$(x^\alpha, p_\alpha, S^{\alpha\beta})$$

Poisson brackets:

$$\{x^\alpha, p_\beta\} = \delta^\alpha_\beta,$$

$$\{p_\alpha, p_\beta\} \neq 0,$$

$$\{p_\alpha, S^{\beta\gamma}\} = \dots$$

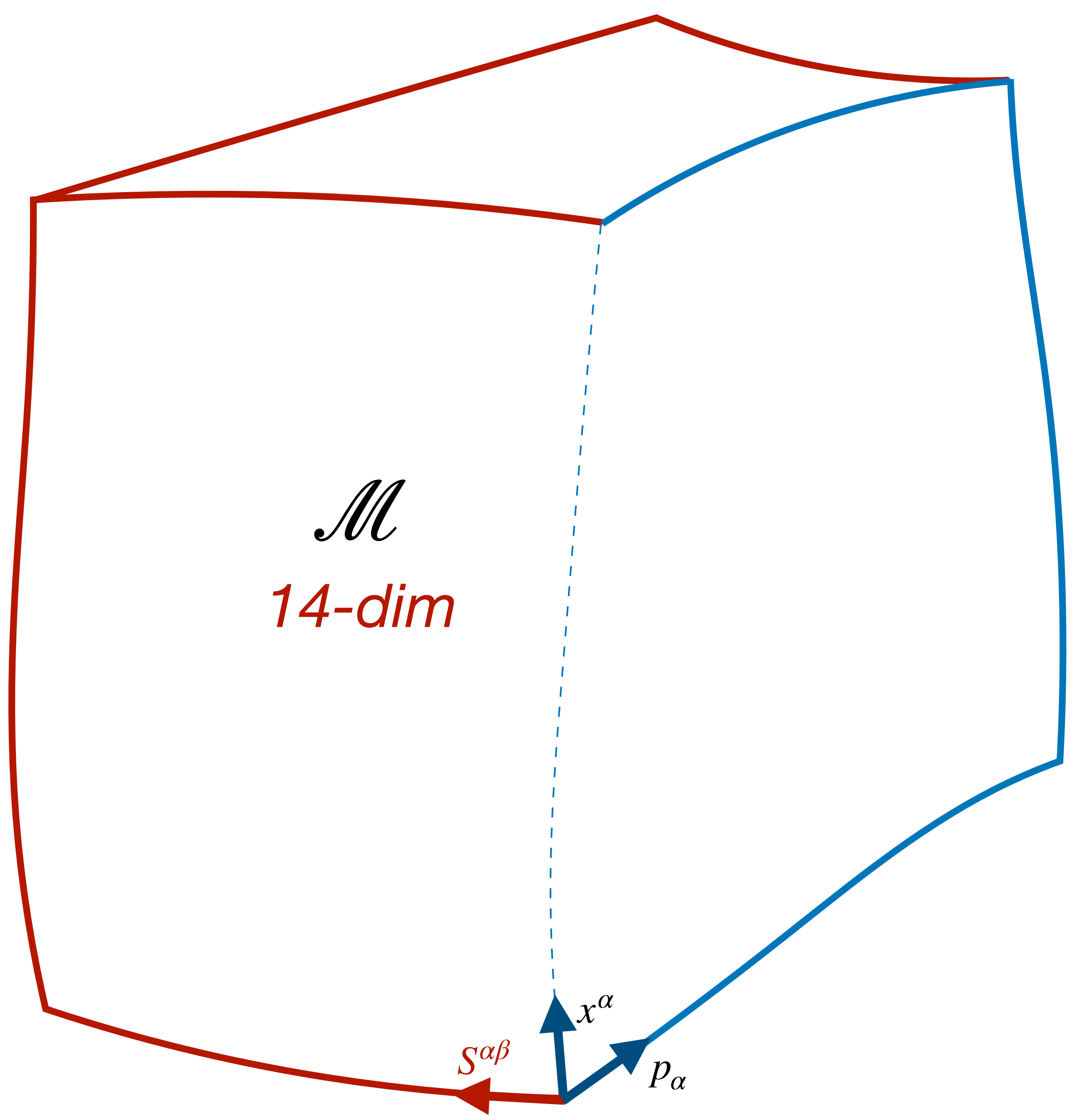
Hamiltonian:

$$H := \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta$$

Hamiltonian formulation of MPTD equations (quadrupole)

Ham. system

- Phase space $\mathscr{M}$
- Poisson brackets $\{,\}$
- Hamiltonian H



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$$\{p_\alpha, S^{\beta\gamma}\} = \dots$$

Hamiltonian:

$$H := \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta$$

$$+ D_{\alpha\beta\gamma\delta} S^{\alpha\beta} S^{\gamma\delta} + \kappa Q_{\alpha\beta\gamma\delta} S^{\alpha\beta} S^{\gamma\delta}$$

What happens to **Kerr integrability**
for the motion of **deforming objects** ?

1. ~~account for the object's deformation~~
2. ~~describe as a Hamiltonian system~~
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Beyond-geodesic integrability around **black holes**

Killing field	Definition	Integral in g_{ab} (***geodesics***) (***any*** *compact object*)	Integral **in Kerr** (***linear****-in-spin* order) (***any*** *compact object*)
Killing vector	$\nabla_{(a}k_{b)} = 0$	$k^\alpha p_\alpha$	*Dixon (1964)* $k^\alpha p_\alpha + \frac{1}{2} S^{ab} \nabla_a k_b$
Killing-Yano tensor	$\nabla_{(a}Y_{b)c} = 0$ + anti-sym	$Y^{\alpha\beta} p_\alpha Y_{\beta\gamma} p^\gamma$	*Rüdiger (1981)* $\mathfrak{Q}^{(1)} = K^{\alpha\beta} p_\alpha p_\beta + L^\alpha{}_{\beta\gamma} p_\alpha S^{\beta\gamma}$ $\mathfrak{K} = \epsilon_{\alpha\beta\gamma\delta} Y^{\alpha\beta} S^{\gamma\delta}$
Killing-Stäckel tensor	$\nabla_{(a}K_{bc)} = 0$ + sym	$K^{\alpha\beta} p_\alpha p_\beta$	-

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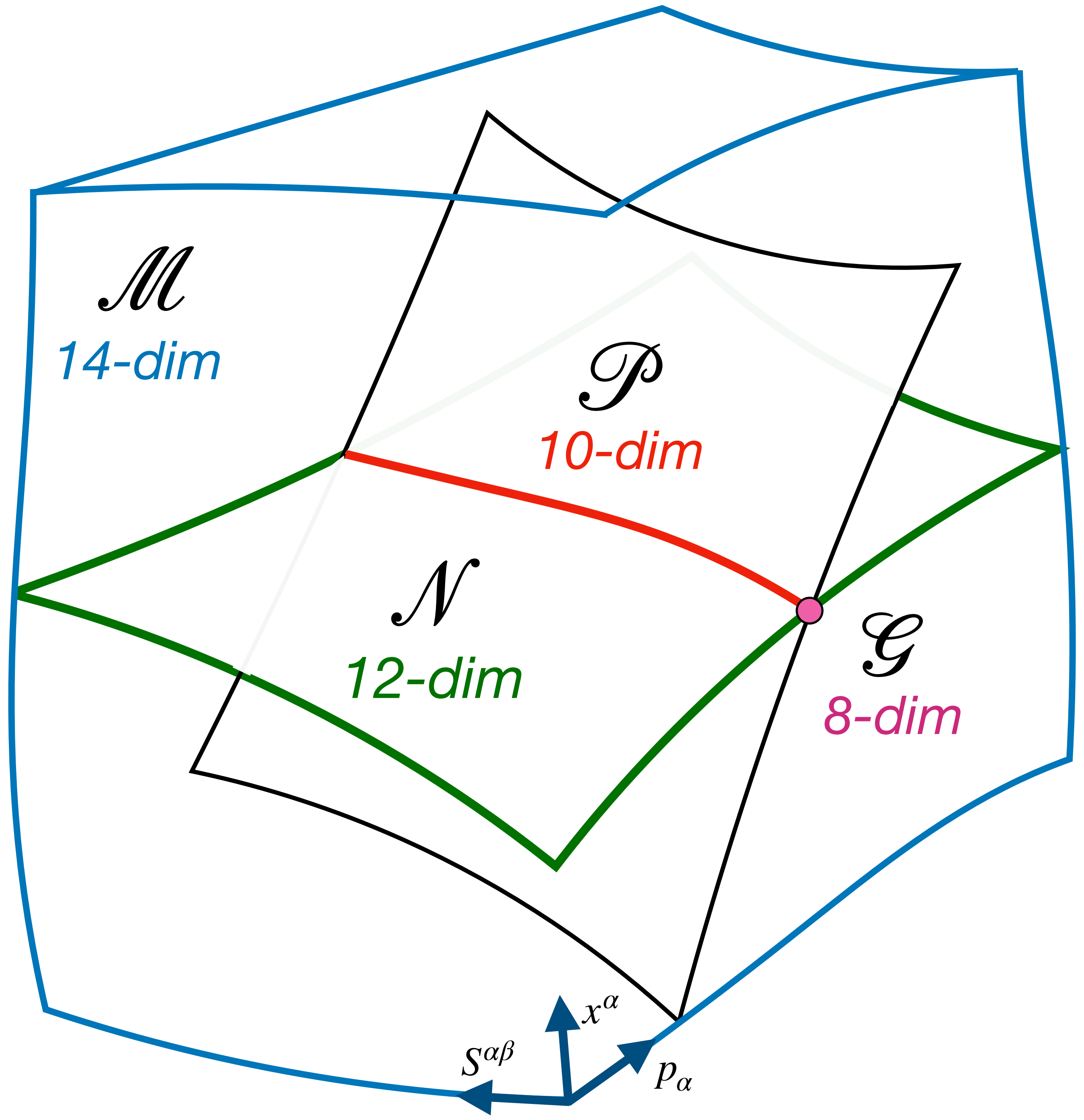
Killing-Stäckel tensor	$\nabla_{(a}K_{bc)} = 0$ + sym	$K^{\alpha\beta}p_\alpha p_\beta$	-	-

Same recipe (Mathematica just takes longer...)

1. Work in the correct, physical phase space $\mathscr{P}$



2. Take the (Kerr) invariants in the literature

$$H, E, L_z, \mathfrak{K}, \mathfrak{Q}^{(2)}$$

3. Compute the relevant Poisson brackets

$$\{F, G\}^{\mathscr{P}} := \{F, G\}^{\mathscr{M}} \boxed{+\,\text{corr}}$$

changes everything !

4. Conclude integrability (in Kerr, lin-in-spin)

At **quadratic order in spin**, the motion of **a test black hole** in a **Kerr background** can be described by a **5-dimensional Hamiltonian system** that is **integrable**.

To sumarize

In any background spacetime:

- **Spin** comes with **degeneracies**: SO(1,3) invariance + center-of-mass condition
-
-

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- There exists **adapted tools** in Ham mechanics to **lift these degeneracies**
-

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In any background spacetime:

- **Spin** comes with **degeneracies**: SO(1,3) invariance + center-of-mass condition
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In a Kerr background:

- Carter's **integrability persists** at **linear-in-spin order** for **any test object** (5 integrals)
-
-

To sumarize

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- Carter's **integrability persists** at **linear-in-spin order** for **any test object** (5 integrals)
- Carter's **integrability persists** at **quadratic-in-spin** order for **a test BH** (5 integrals)
-

To sumarize

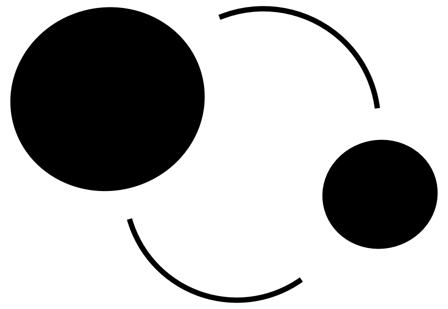
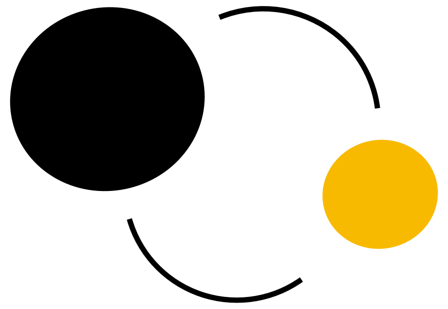
In any background spacetime:

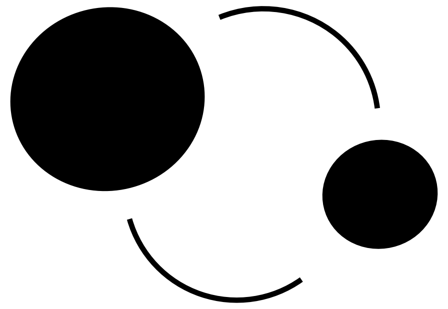
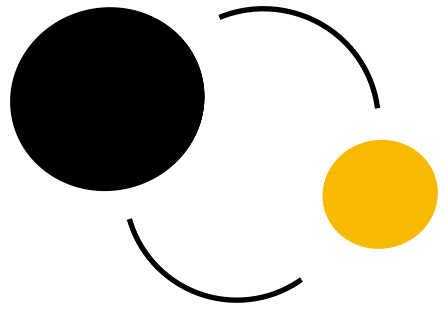
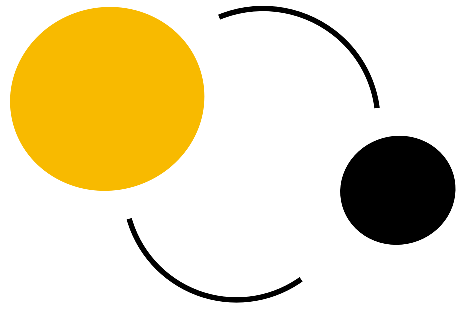
- **Spin** comes with **degeneracies**: SO(1,3) invariance + center-of-mass condition
- There exists **adapted tools** in Ham mechanics to **lift these degeneracies**
- Resulting **Hamiltonian** formulation is covariant, non-degenerate and 10D

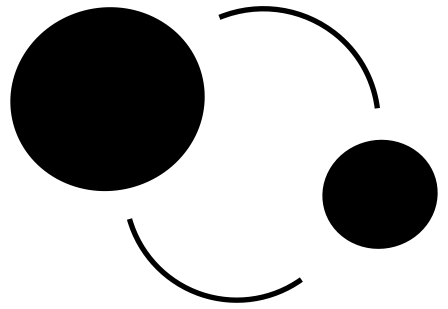
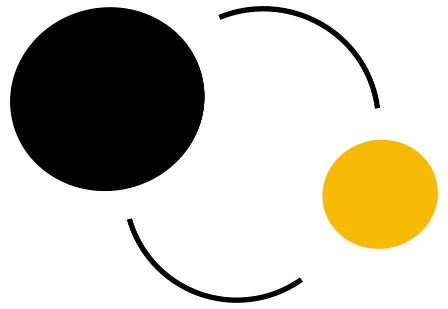
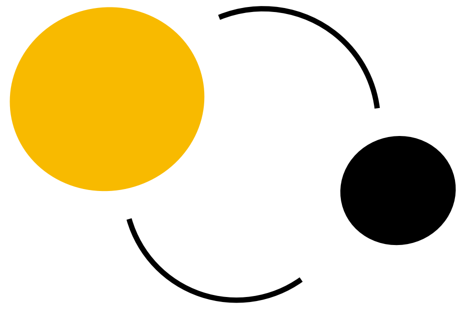
In a Kerr background:

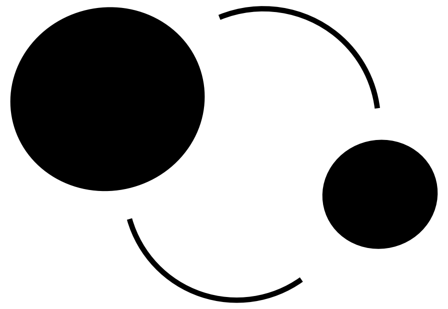
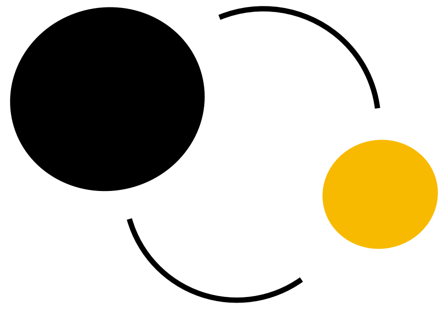
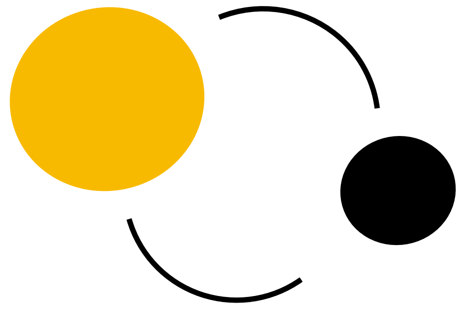
- Carter's **integrability persists** at **linear-in-spin order** for **any test object** (5 integrals)
- Carter's **integrability persists** at **quadratic-in-spin** order for **a test BH** (5 integrals)
- Still **thanks to the "hidden" symmetry** in Kerr (Killing-Yano tensor)

Take-away slide(s)

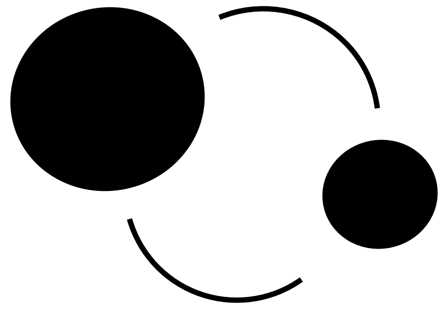
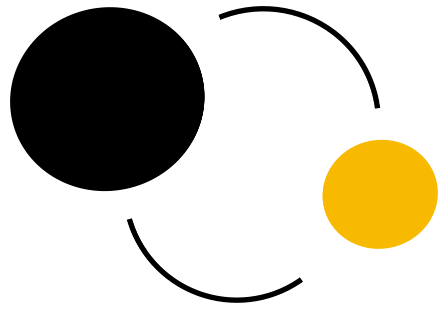
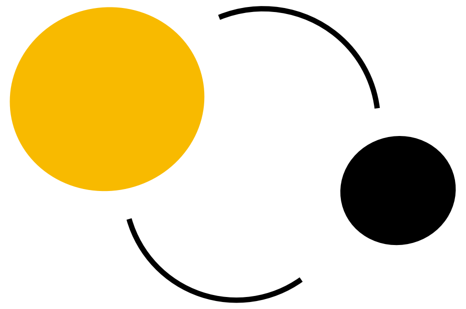
	BH-*BH* 		BH-*NS* 			
Monopolar order (geodesic)	*Constants of motion:* 4/4	*Integrable system?* YES	*Constants of motion:* 4/4	*Integrable system?* YES		

	BH-*BH* 		BH-*NS* 		NS-*BH* 	
Monopolar order (geodesic)	*Constants of motion:* 4/4	*Integrable system?* YES	*Constants of motion:* 4/4	*Integrable system?* YES	*Constants of motion:* 3/4	*Integrable system?* NO

	 BH-*BH*		 BH-*NS*		 NS-*BH*	
Monopolar order (geodesic)	*Constants of motion:* 4/4	*Integrable system?* YES	*Constants of motion:* 4/4	*Integrable system?* YES	*Constants of motion:* 3/4	*Integrable system?* NO
Dipolar order (linear-in-spin)	*Constants of motion:* 5/5	*Integrable system?* YES	*Constants of motion:* 5/5	*Integrable system?* YES	*Constants of motion:* 3/5	*Integrable system?* NO

	BH-*BH* 		BH-*NS* 		NS-*BH* 	
Monopolar order (geodesic)	*Constants of motion:* 4/4	*Integrable system?* YES	*Constants of motion:* 4/4	*Integrable system?* YES	*Constants of motion:* 3/4	*Integrable system?* NO
Dipolar order (linear-in-spin)	*Constants of motion:* 5/5	*Integrable system?* YES	*Constants of motion:* 5/5	*Integrable system?* YES	*Constants of motion:* 3/5	*Integrable system?* NO
Quadrupolar order (quadratic-in-spin)	*Constants of motion:* 5/5	*Integrable system?* YES	*Constants of motion:* 4/5	*Integrable system?* NO	*Constants of motion:* 3/5	*Integrable system?* NO

Simple ↓ Realistic

	BH-*BH* 		BH-*NS* 		NS-*BH* 	
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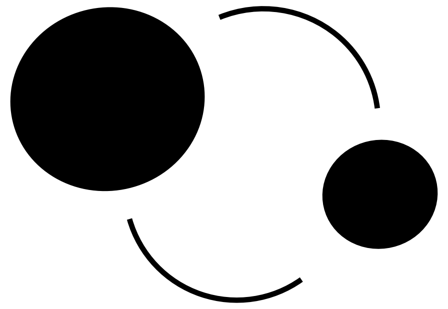
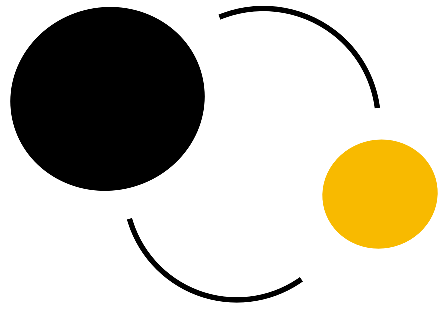
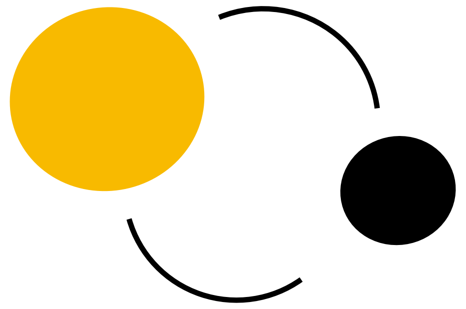
Dipolar order (linear-in-spin)	*Constants of motion:* 5/5	*Integrable system?* YES	*Constants of motion:* 5/5	*Integrable system?* YES	*Constants of motion:* 3/5	*Integrable system?* NO
Quadrupolar order (quadratic-in-spin)	*Constants of motion:* 5/5	*Integrable system?* YES	*Constants of motion:* 4/5	*Integrable system?* NO	*Constants of motion:* 3/5	*Integrable system?* NO

More symmetries → Less symmetries

Simple ↓ Realistic

	BH-*BH* 		BH-*NS* 		NS-*BH* 	
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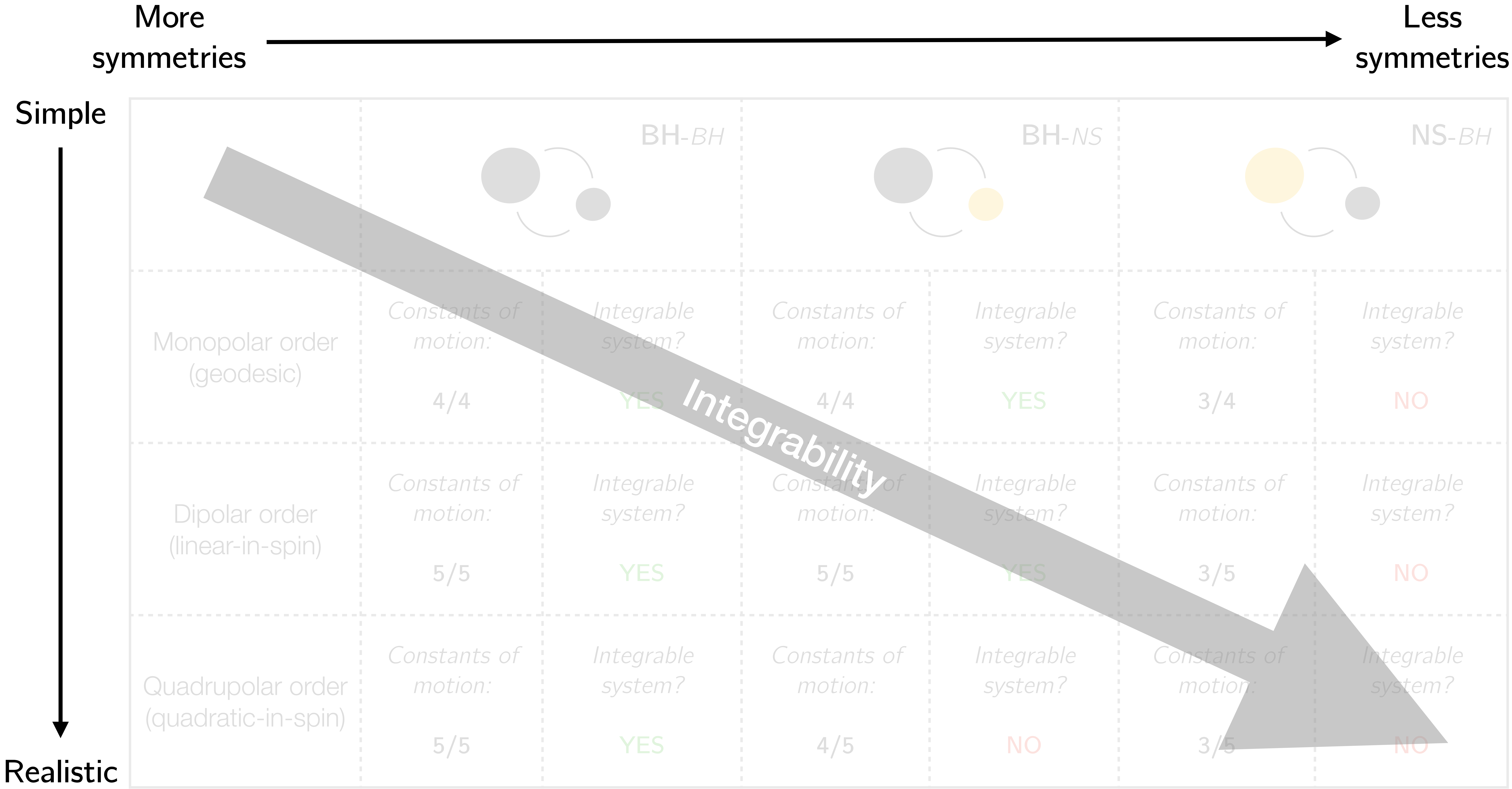
More symmetries
Less symmetries
Simple
Realistic
BH-BH
BH-NS
NS-BH
Monopolar order (geodesic)
Dipolar order (linear-in-spin)
Quadrupolar order (quadratic-in-spin)
Constants of motion:
Integrable system?
4/4
YES
4/4
YES
3/4
NO
5/5
YES
5/5
YES
3/5
NO
5/5
YES
4/5
NO
3/5
NO
Integrability

Thank you !

If you are interested in bridging Ham mechanics and GR motion, please reach out: *paul.ramond@obspm.fr*

More details in a series of works (with collaborators) :

Paper 0: arxiv.org/abs/2402.02670 (integrability results in Kerr)
Paper I: arxiv.org/abs/2210.03866 (details and math. foundations)
Paper II: arxiv.org/abs/2402.05049 (applications in Schwarzschild linear-in-spin)
Other extensions coming soon(-ish)...

Extra content

Beyond-geodesic integrability around **black holes**

Killing field	Definition	Integral in g_{ab} (***geodesics***) (***any*** *compact object*)	Integral **in Kerr** (***linear****-in-spin order*) (***any*** *compact object*)	Integral **in Kerr** (***quadratic****-in-spin order*) (***Kerr-like*** *object*)
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Killing-Yano tensor	$\nabla_{(a}Y_{b)c} = 0$ + anti-sym	$Y^{\alpha\beta} p_\alpha Y_{\beta\gamma} p^\gamma$	*Rüdiger (1981)* $\mathfrak{Q}^{(1)} = K^{\alpha\beta} p_\alpha p_\beta + L^\alpha{}_{\beta\gamma} p_\alpha S^{\beta\gamma}$ $\mathfrak{K} = \varepsilon_{\alpha\beta\gamma\delta} Y^{\alpha\beta} S^{\gamma\delta}$	
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Killing field	Definition	Integral in g_{ab} (***geodesics***) (***any** compact object*)	Integral **in Kerr** (***linear**-in-spin order*) (***any** compact object*)	Integral **in Kerr** (***quadratic**-in-spin order*) (***Kerr-like** object*)
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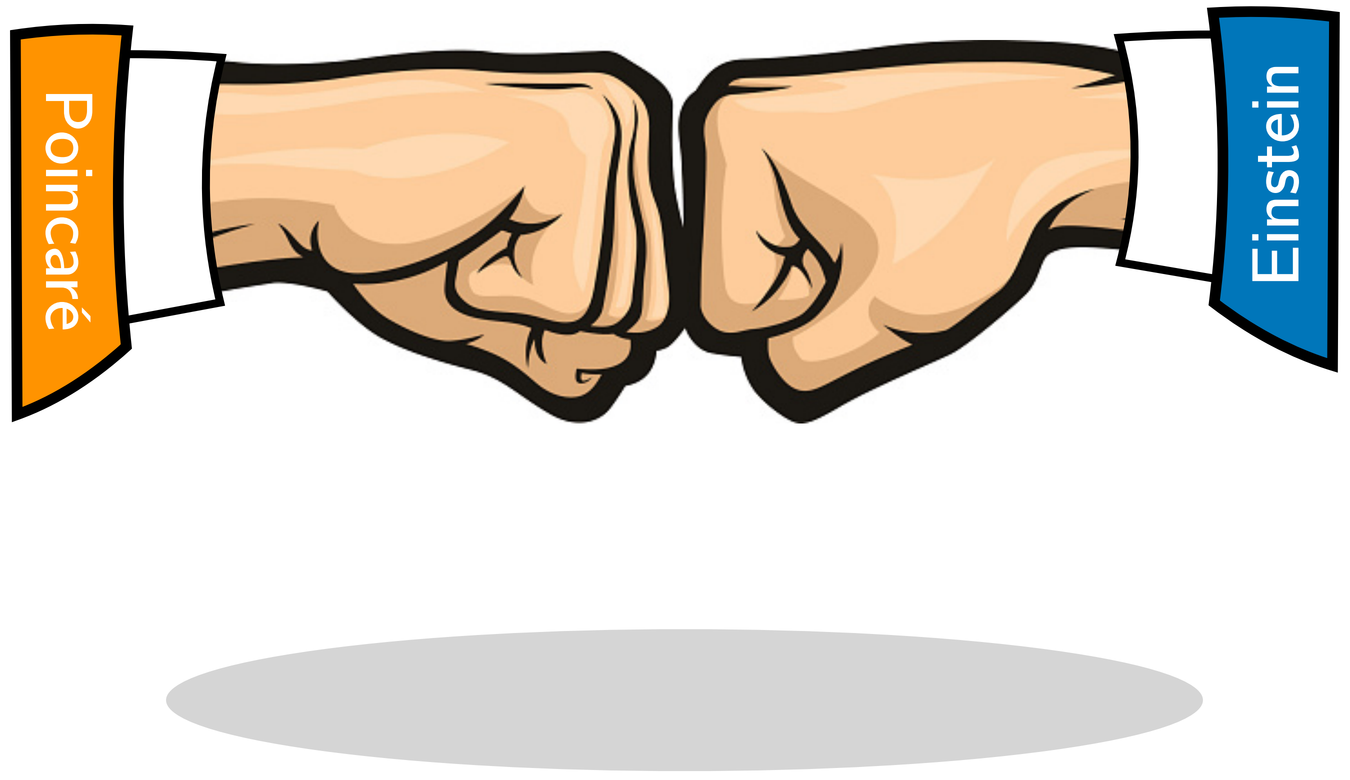
Killing-Yano tensor	$\nabla_{(a}Y_{b)c} = 0$ + anti-sym	$Y^{\alpha\beta} p_\alpha Y_{\beta\gamma} p^\gamma$	*Rüdiger (1981)* $\mathfrak{Q}^{(1)} = K^{\alpha\beta} p_\alpha p_\beta + L^{\alpha}{}_{\beta\gamma} p_\alpha S^{\beta\gamma}$ $\mathfrak{K} = \varepsilon_{\alpha\beta\gamma\delta} Y^{\alpha\beta} S^{\gamma\delta}$	*Druart-Compère-Vines (2023)* $\mathfrak{Q}^{(2)} = \mathfrak{Q}^{(1)} + M_{\alpha\beta\gamma\delta} S^{\alpha\beta} S^{\gamma\delta}$ $\mathfrak{K}$
Killing-Stäckel tensor	$\nabla_{(a}K_{bc)} = 0$ + sym	$K^{\alpha\beta} p_\alpha p_\beta$	-	-

Thank you !



If you are interested in bridging Ham mechanics and GR motion, please reach out: *paul.ramond@obspm.fr*

Thank you !

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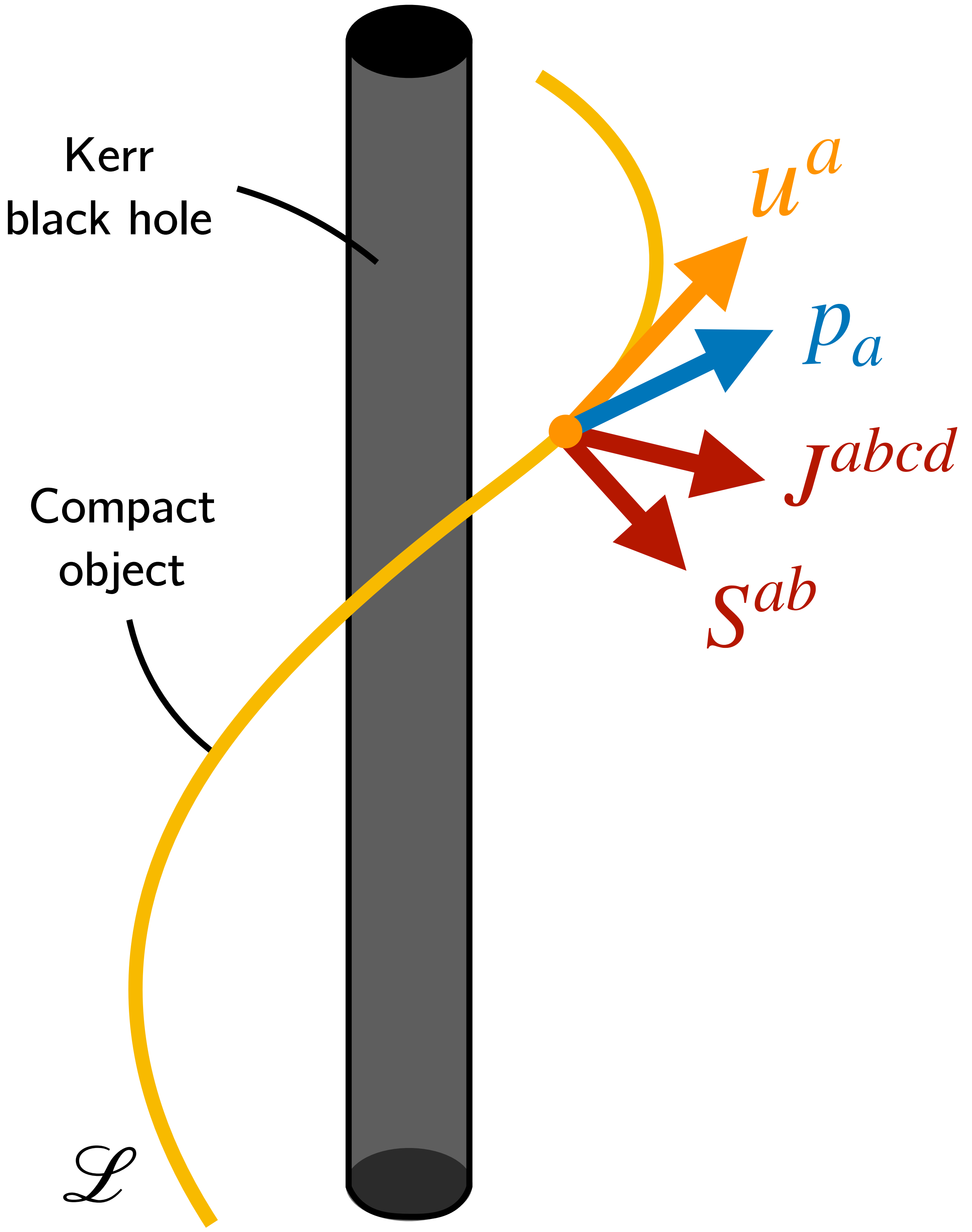
More details in a 6-part series of works (with collaborators) :

Paper 0: arxiv.org/abs/2402.02670 (integrability results in Kerr)
Paper I: arxiv.org/abs/2210.03866 (math. foundations, linear-in-spin)
Paper II: arxiv.org/abs/2402.05049 (applications in Schwarzschild linear-in-spin)
Paper III, IV and V coming soon(-ish)...

What about quadratic-in-spin order ?

At quadratic-in-spin order,
spin-induced quadrupole J^{abcd}

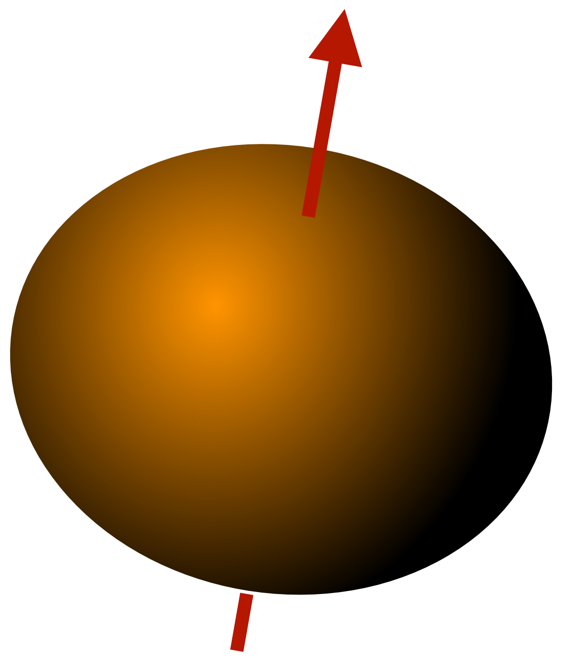


MPTD equations at quad. order

$$\nabla_u p_a = R_{abcd} S^{bc} u^d + F_a[J]$$ — quadrupole "force"

$$\nabla_u S^{ab} = 2 p^{[a} u^{b]} + N^{ab}[J]$$ — quadrupolar "torque"

Spin-induced quadrupole



$$J^{abcd} := \kappa \cdot \frac{3}{\mu^3} p^{[a} S^{b]e} S_e{}^{[c} p^{d]}$$

κ → *deformability coefficient*

- $\kappa = 1$ for black holes
- $\kappa > 1$ for neutron stars
- $\kappa \gg 1$ for white dwarfs

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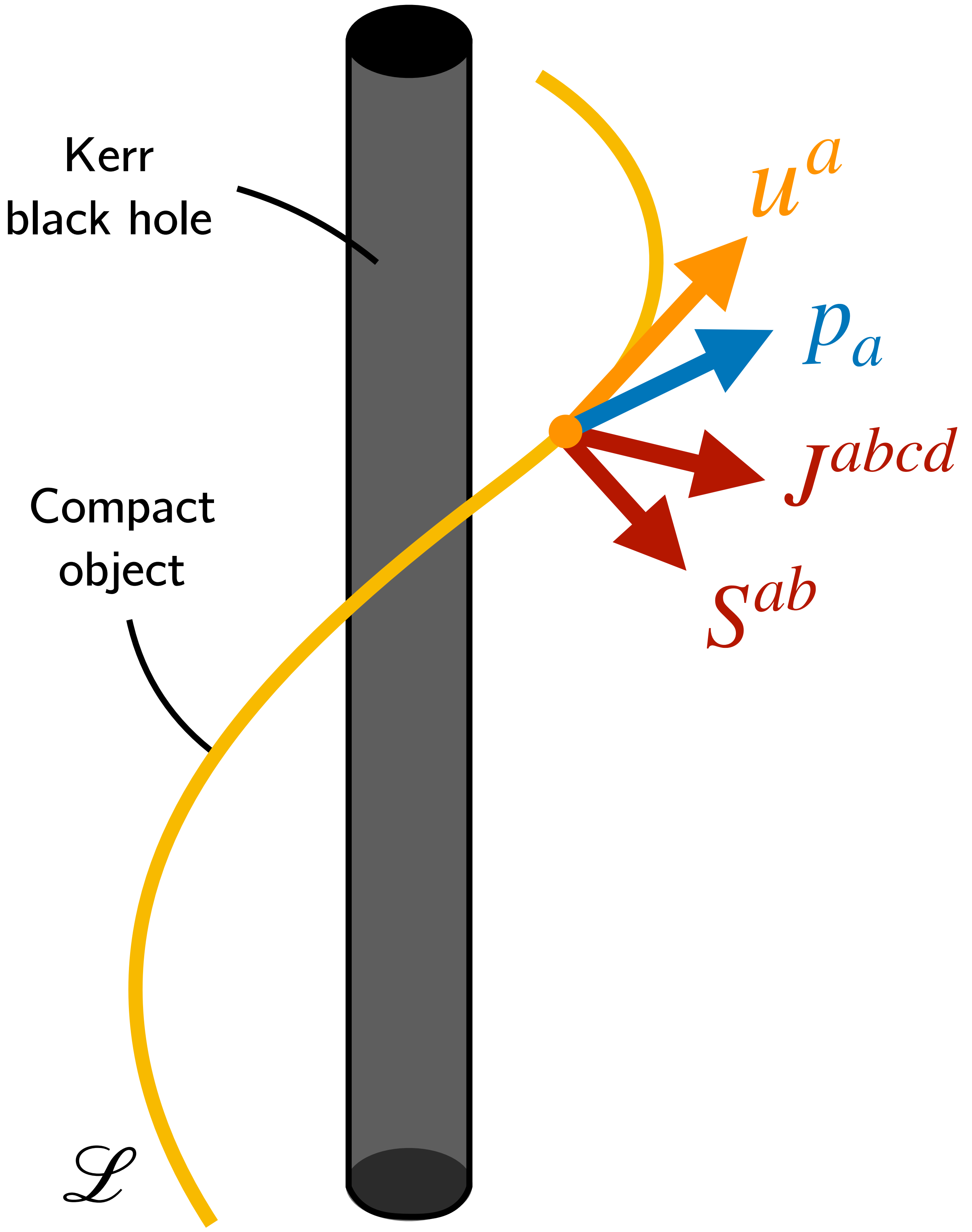
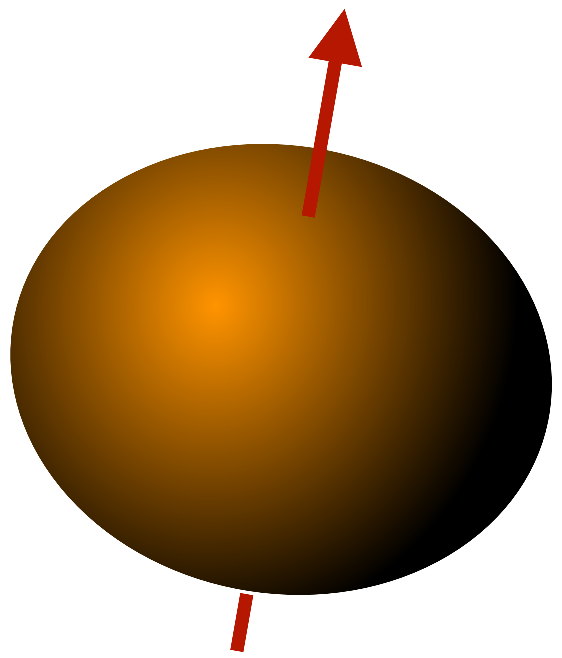


MPTD equations at quad. order

$$\nabla_u p_a = R_{abcd} S^{bc} u^d - \frac{1}{6} J^{bcde} \nabla_a R_{bcde}$$

$$\nabla_u S^{ab} = 2 p^{[a} u^{b]} - \frac{4}{3} J^{[a}{}_{cde} R^{b]cde}$$

Spin-induced quadrupole



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- Rüdiger 1981 → integrals of motion at dipolar order
- **Compère & Druart 2023 → integrals of motion at quadrupolar order**

GR & geodesics

Schw. geodesics
& integrability

Kerr geodesics
& integrability

Beyond geodesics
+ first integrals

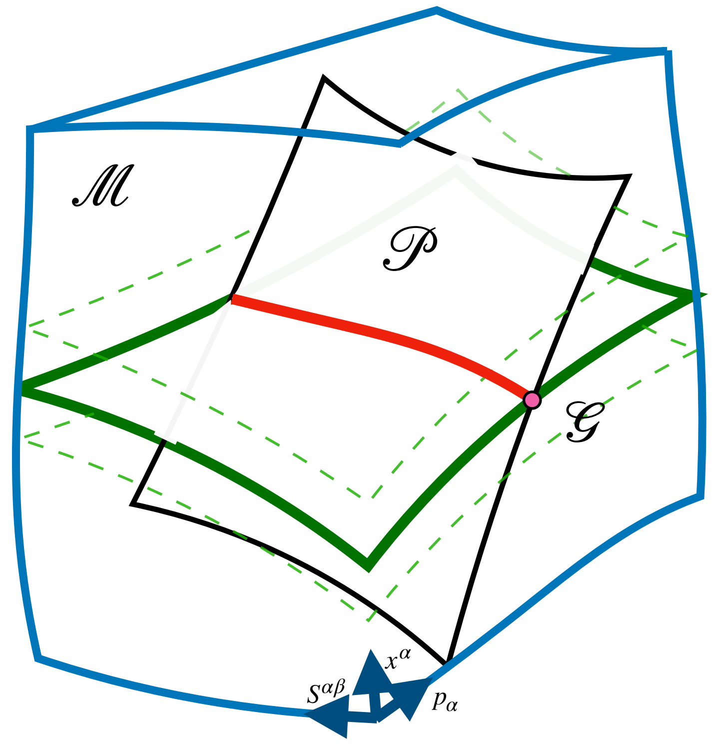
**Extended-body
integrability ?**

Beyond-geodesic integrability around **black holes**

Killing field in $(\mathscr{E}, g_{ab})$	Definition	Integral in g_{ab} (***geodesics***) (***any** compact object*)	Integral **in Kerr** (***linear**-in-spin order*) (***any** compact object*)	Integral **in Kerr** (***quadratic**-in-spin order*) (***Kerr-like** object*)
Killing vector k^a	$\nabla_{(a}k_{b)} = 0$	$k^\alpha p_\alpha$	*Dixon/Harte (1964/2012)* $\mathfrak{C} := k^\alpha p_\alpha + \frac{1}{2} S^{ab} \nabla_a k_b$	*Dixon/Harte (1964/2012)* $\mathfrak{C}$
Killing-Yano tensor Y^{ab} $\Uparrow$	$\nabla_{(a}Y_{b)c} = 0$ + anti-sym	$Y^{\alpha\beta}p_\alpha Y_{\beta\gamma}p^\gamma$	*Rüdiger (1981)* $\mathfrak{Q}^{(1)} = K^{\alpha\beta}p_\alpha p_\beta + L^\alpha{}_{\beta\gamma}p_\alpha S^{\beta\gamma}$ $\mathfrak{K} = \varepsilon_{\alpha\beta\gamma\delta}Y^{\alpha\beta}S^{\gamma\delta}$	*Druart-Compère-Vines (2023)* $\mathfrak{Q}^{(2)} = \mathfrak{Q}^{(1)} + M_{\alpha\beta\gamma\delta}S^{\alpha\beta}S^{\gamma\delta}$ $\mathfrak{K}$
Killing-Stäckel tensor K^{ab}	$\nabla_{(a}K_{bc)} = 0$ + sym	$K^{\alpha\beta}p_\alpha p_\beta$	-	-

Quadratic-in-spin integrability

1. Work in the correct, physical phase space



2. Construct a quadratic-in-spin, covariant Ham. from scratch

3. Take the (Kerr) invariants in the literature

$$\tilde{\mu}, E, L_z, \mathfrak{K}, \mathfrak{Q}^{(2)}$$

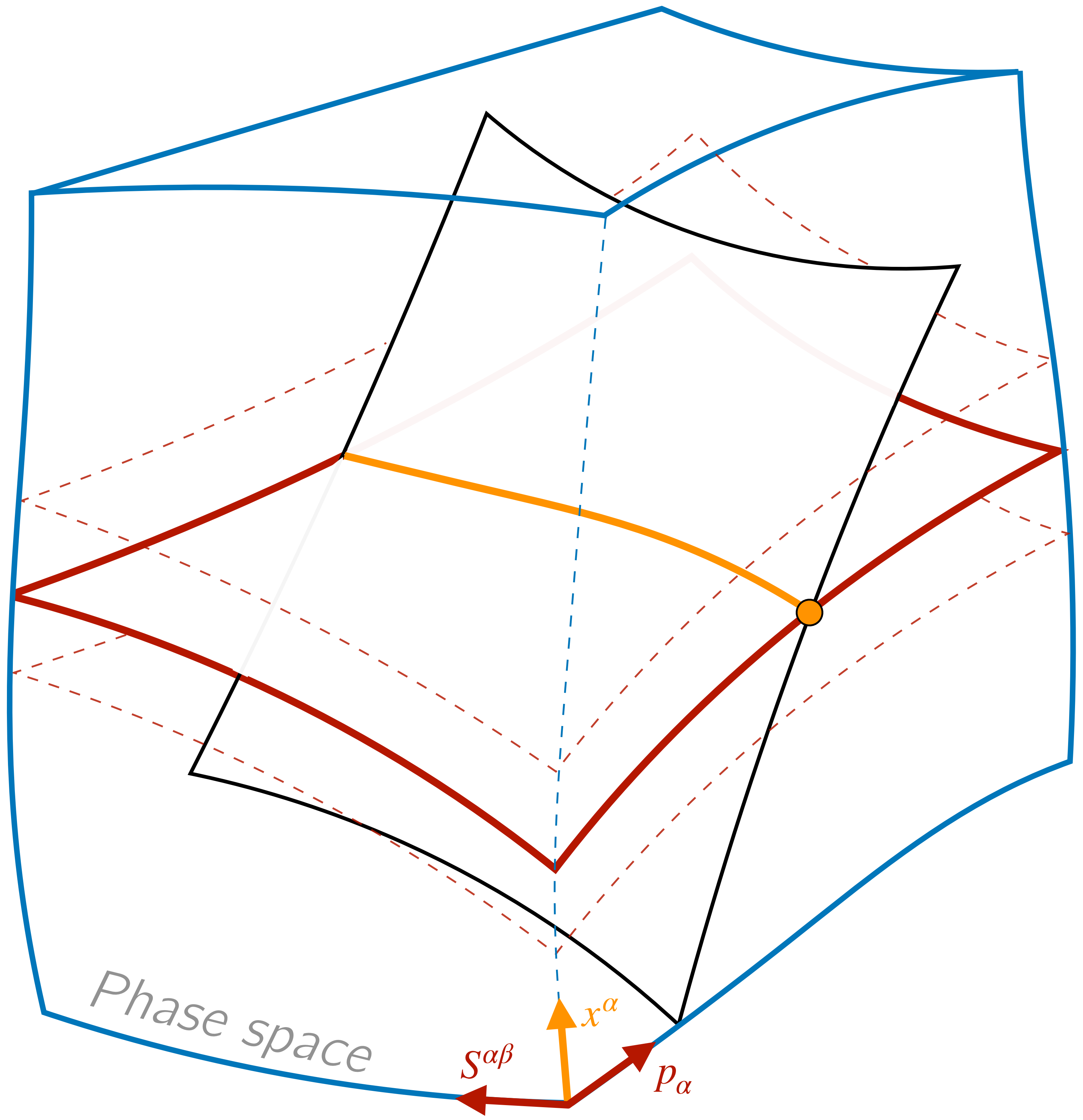
4. Compute the relevant Poisson brackets

$$\{F, G\}^{\mathscr{P}} = \{F, G\}^{\mathscr{M}} + \text{corr}$$

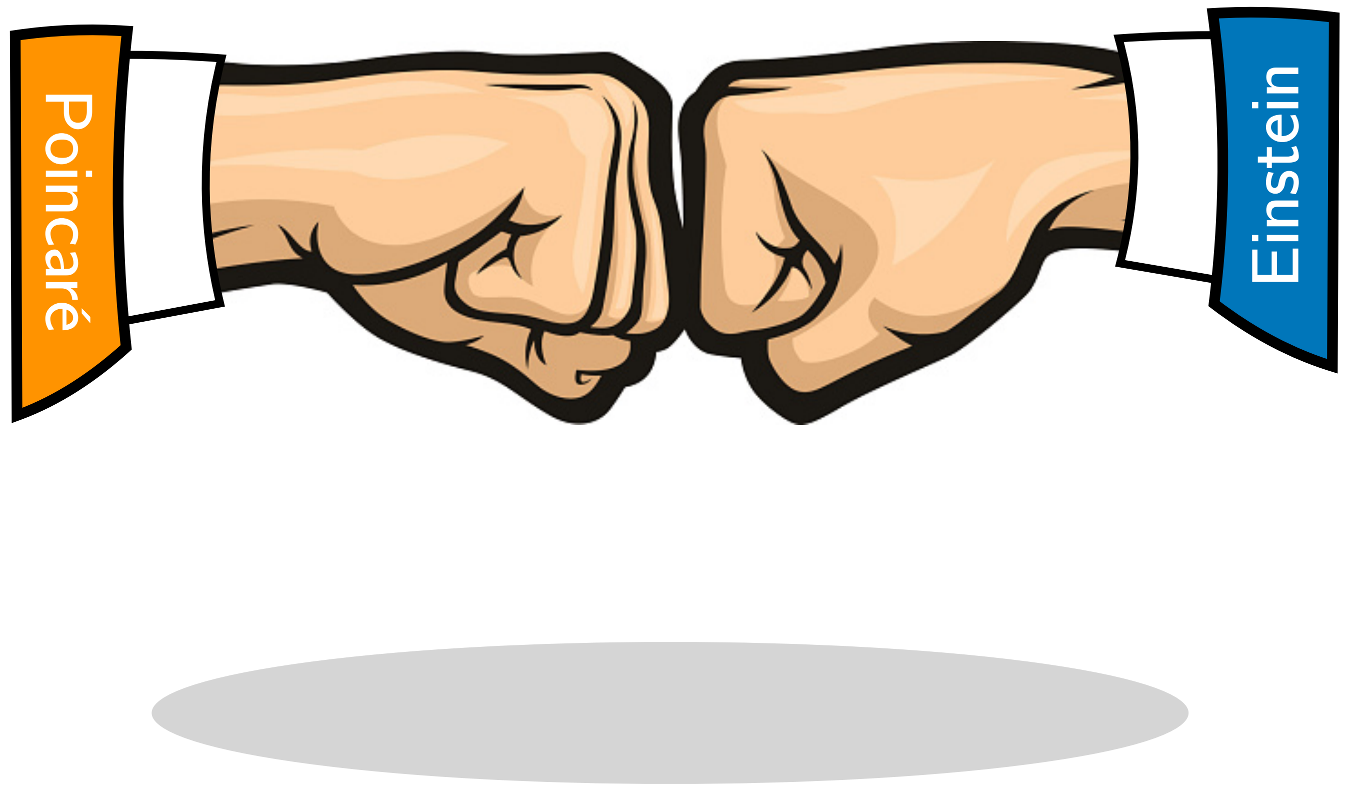
changes everything ! (again)

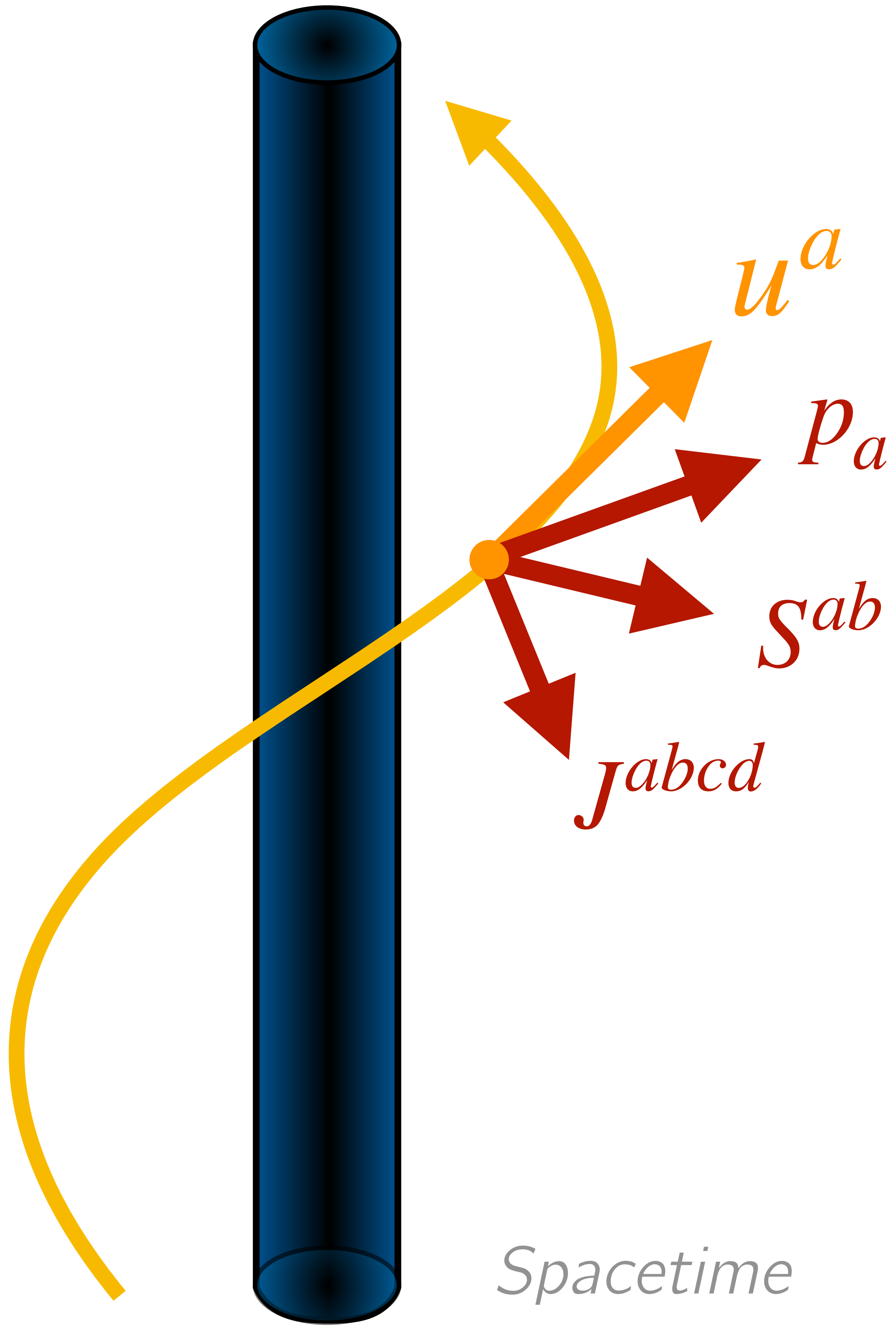
5. Conclude integrability

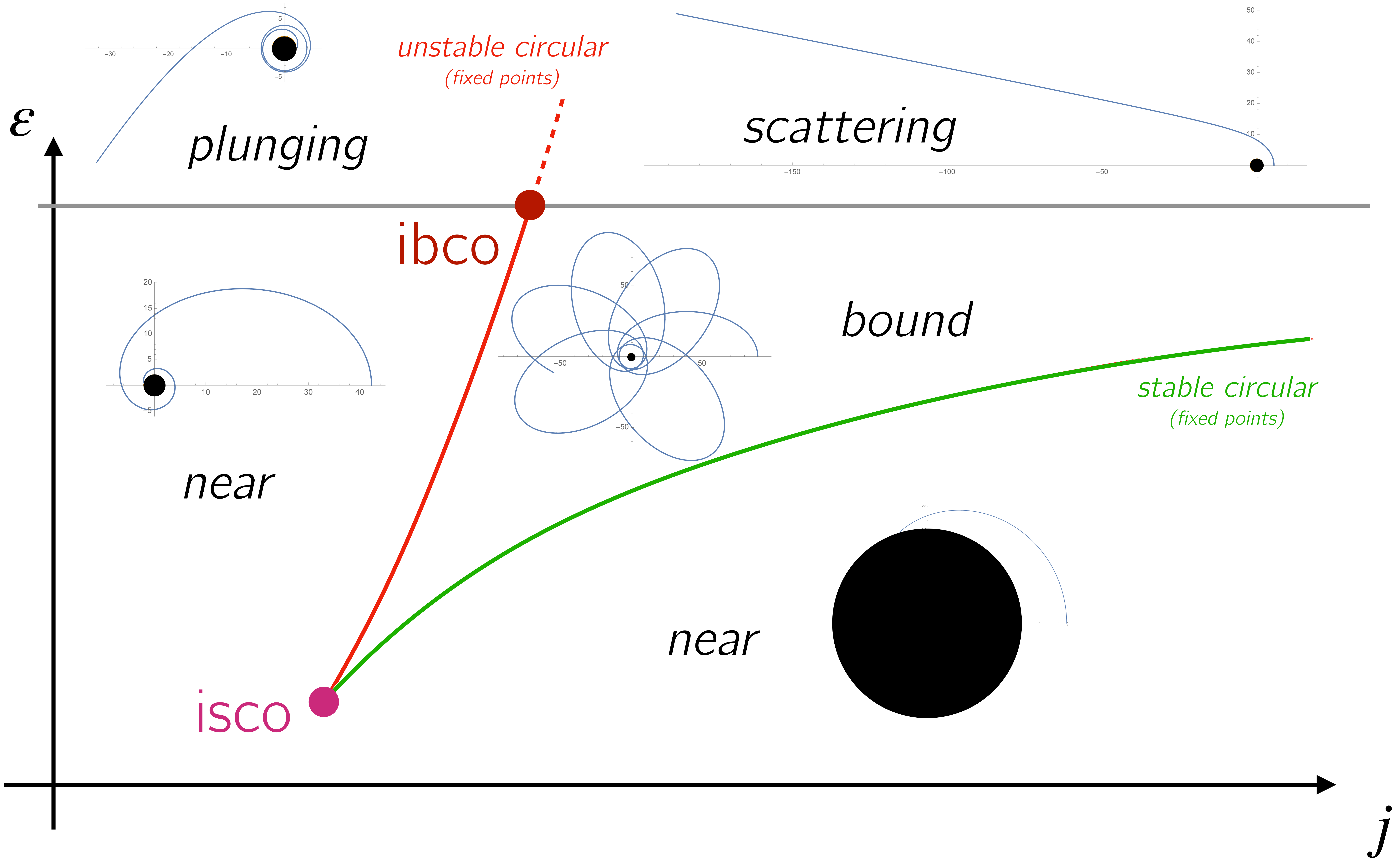
At **quadratic order in spin**, the motion of **a Kerr-like** body in a **Kerr background** can be described by a 5-dimensional Hamiltonian system that is **integrable**.


Phase space
$S^{\alpha\beta}$
x^α
p_α

Thank you !

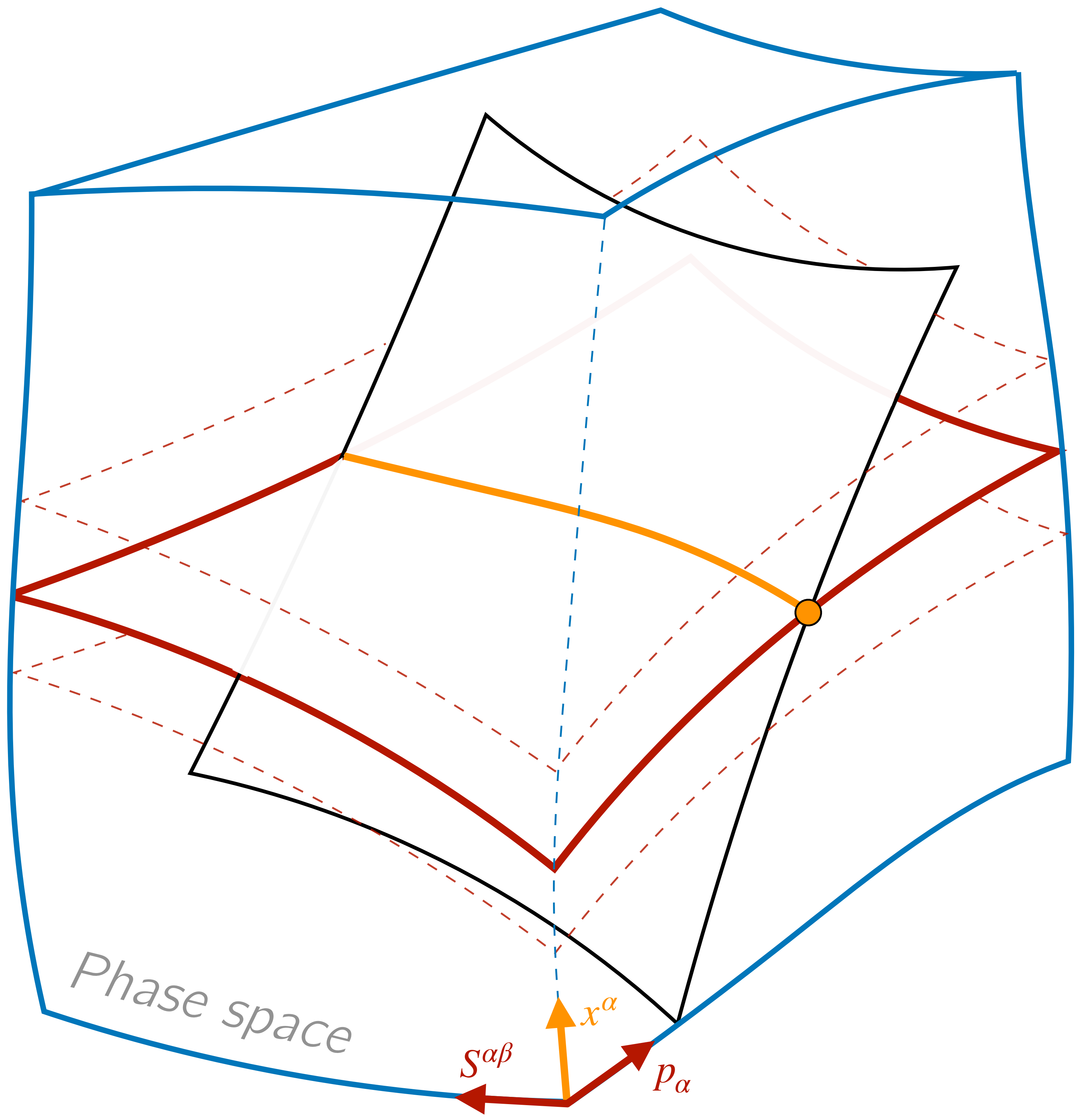

Poincaré
Einstein

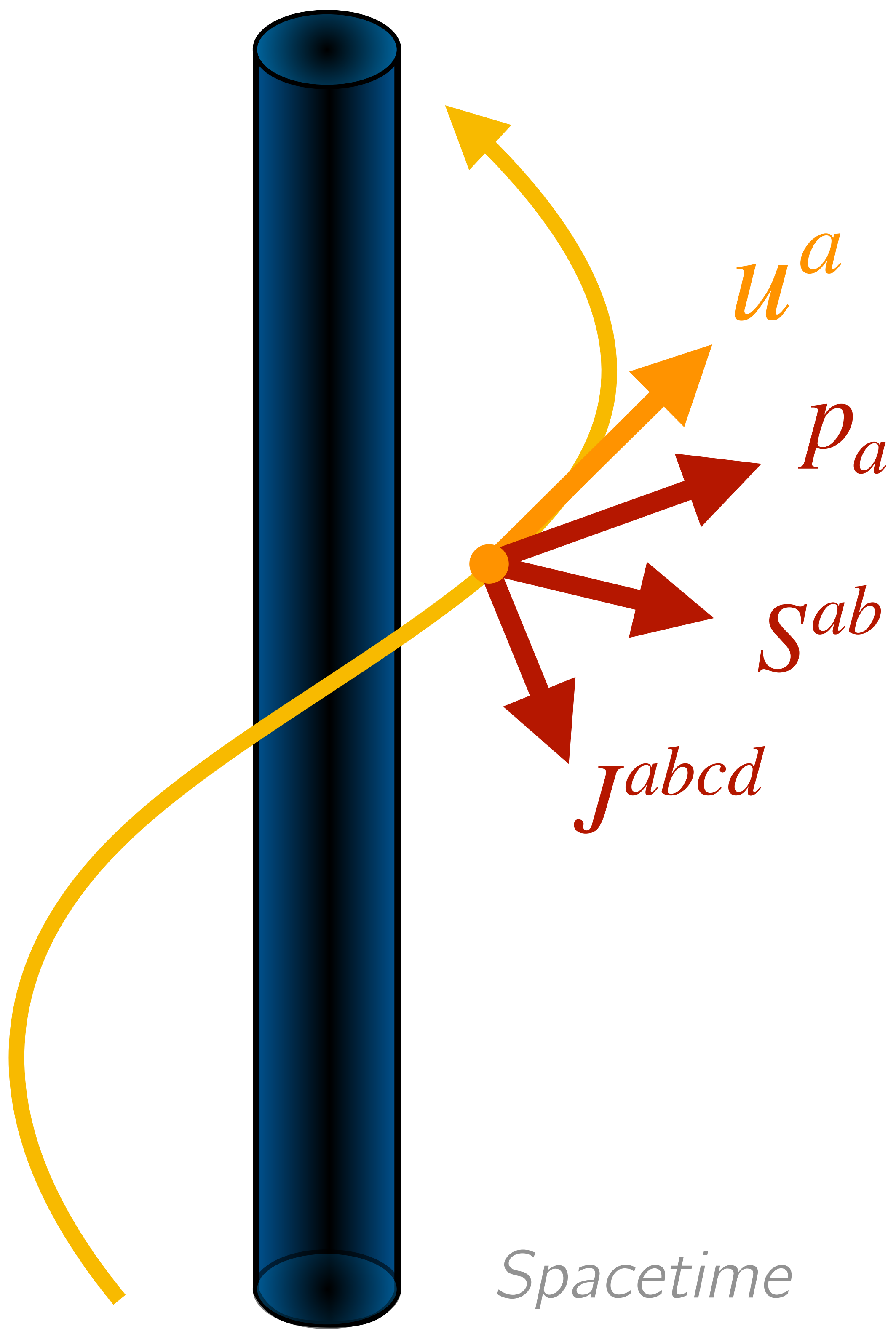

u^a
p_a
S^{ab}
J^{abcd}
Spacetime



unstable circular
(fixed points)
ε
plunging
scattering
ibco
bound
stable circular
(fixed points)
near
near
isco
j

Integrable dynamics of extended test bodies around rotating black holes





P. Ramond - IMCCE

LUTH - Mar 7th, 2024

time

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GR & geodesics

time

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GR & geodesics

Schw. geodesics
& integrability

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GR & geodesics

Schw. geodesics
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Kerr geodesics
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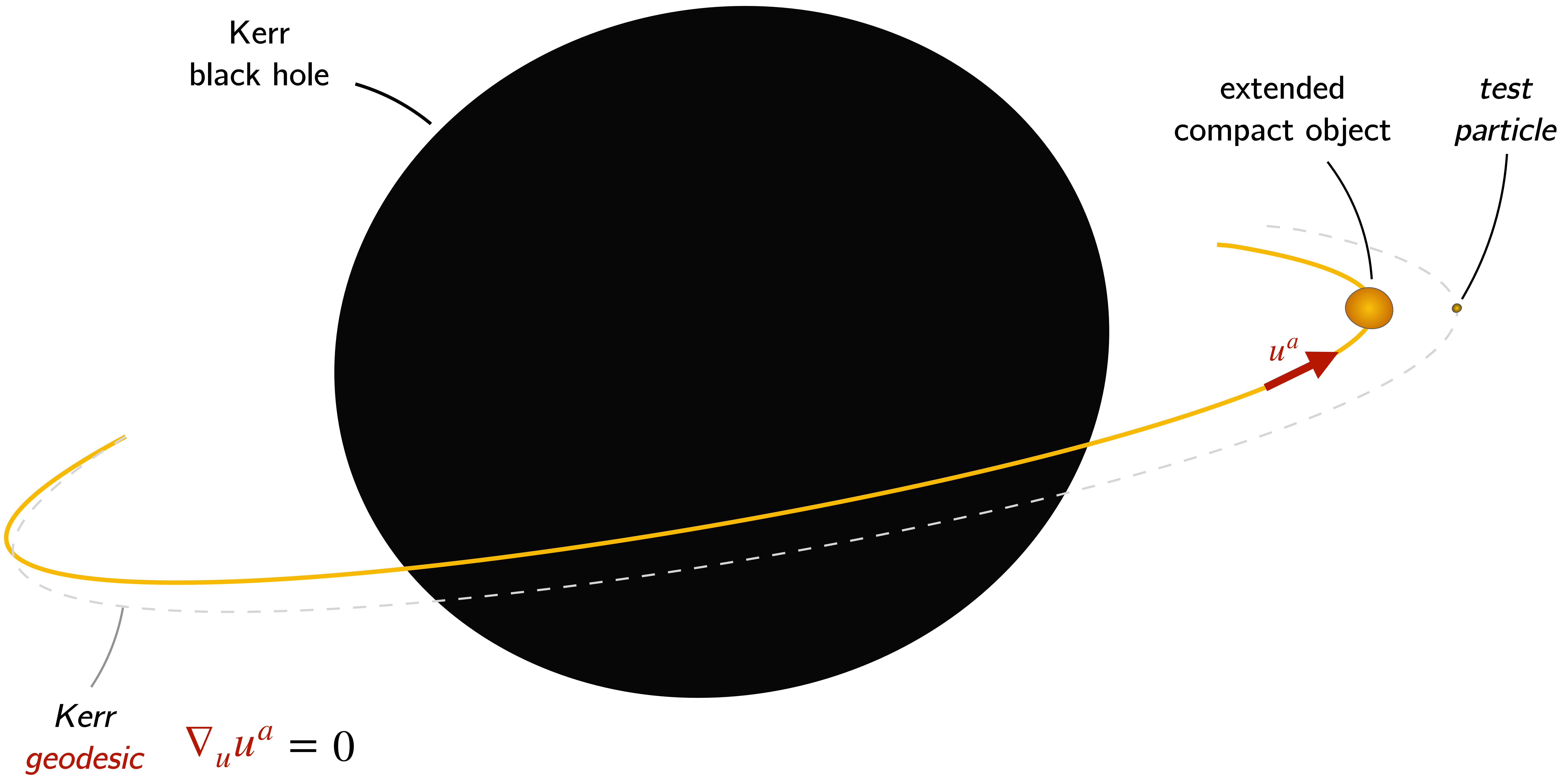
How do things **really** fall around black holes?



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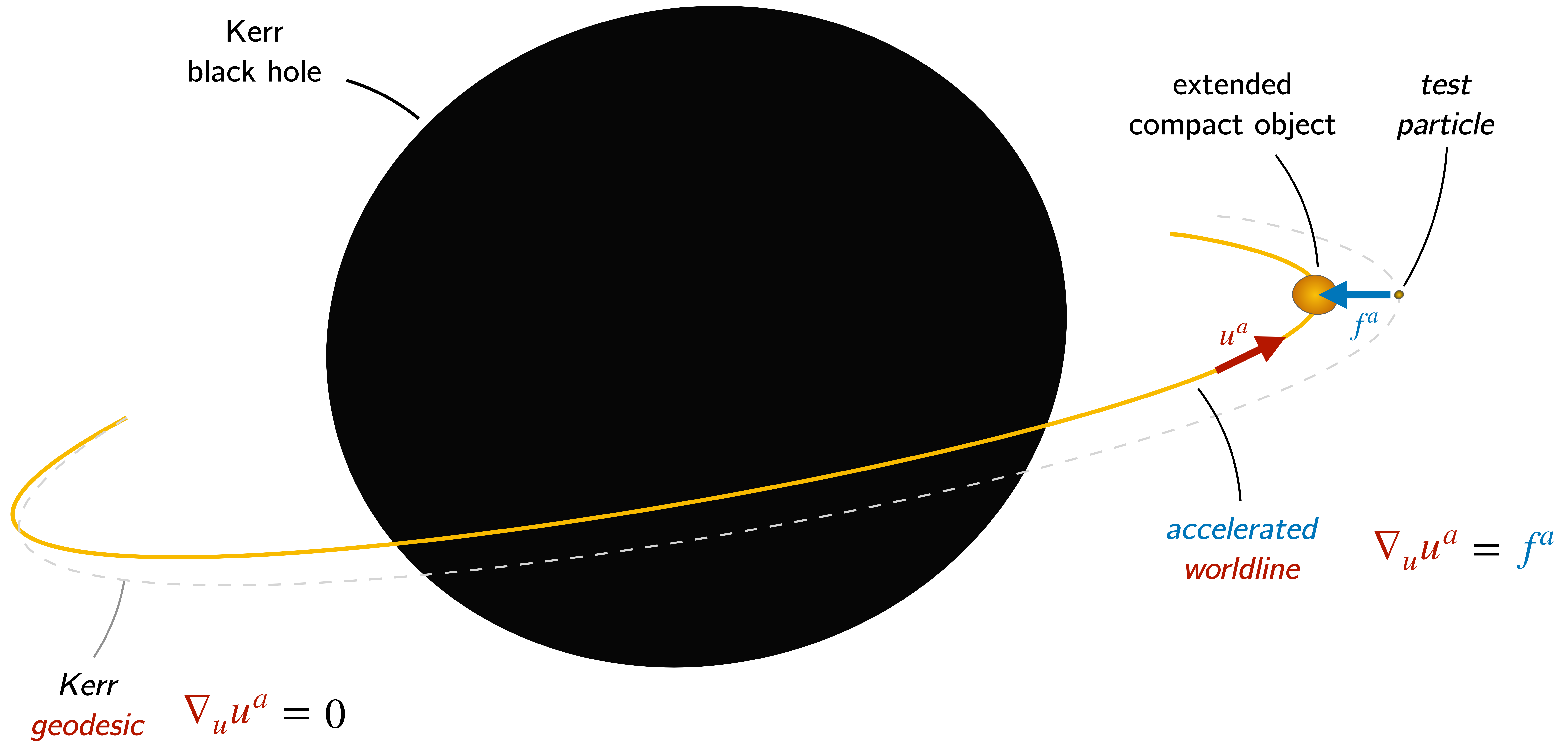

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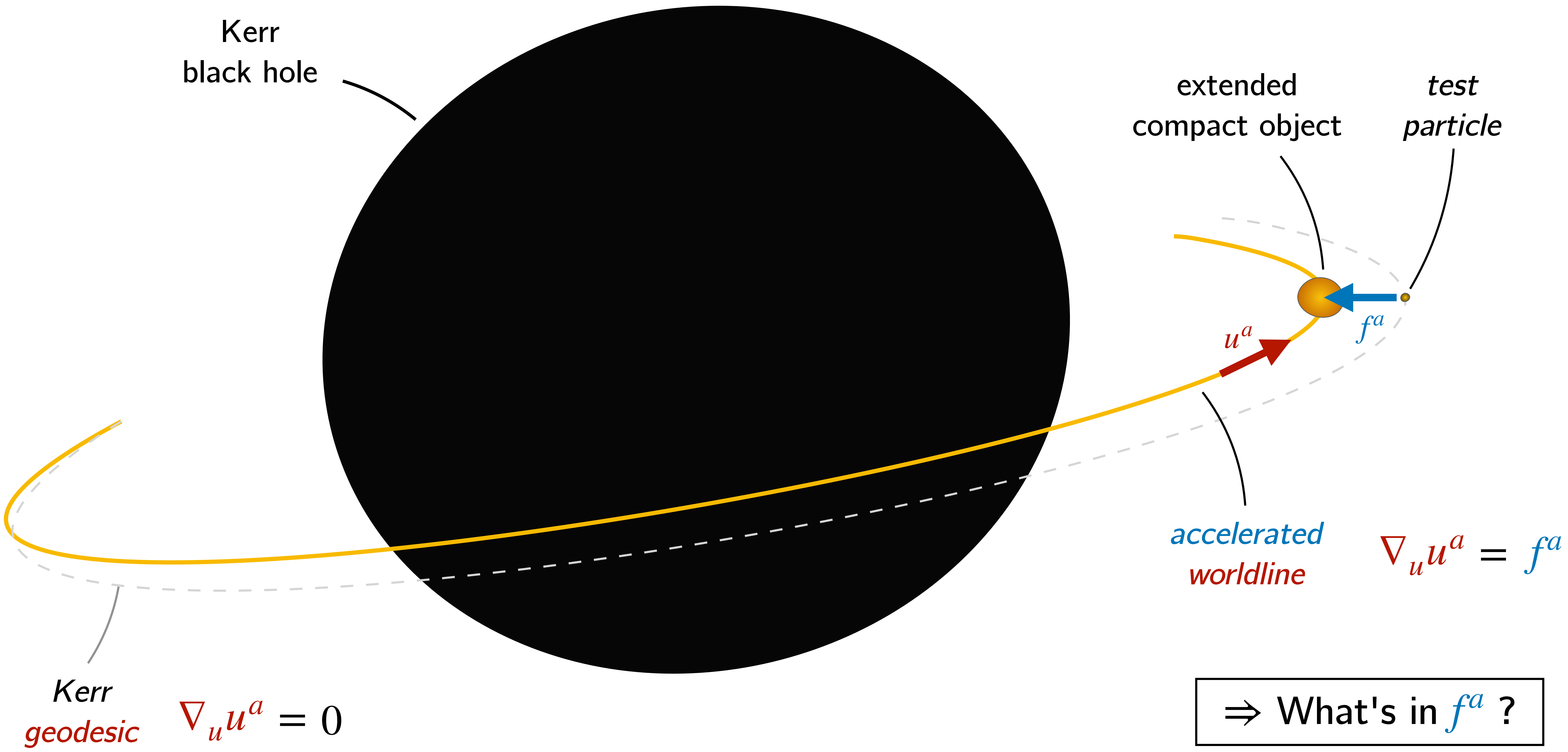


$\Rightarrow$ What's in f^a ?

non-GR, hairs, environment, etc...

Corrections to geodesic motion

only
"feels"

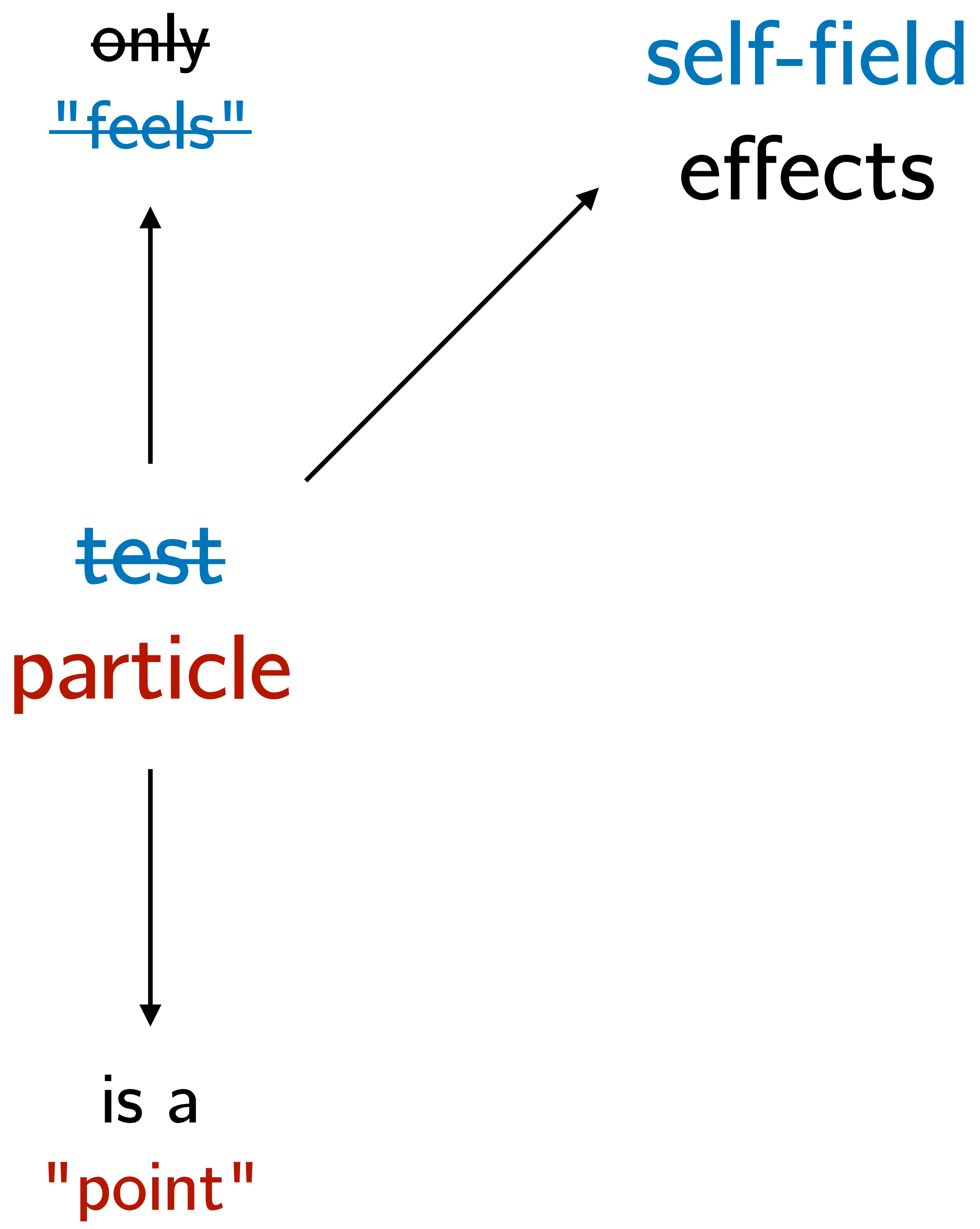
↑

test
particle

↓

is a
"point"

Corrections to geodesic motion



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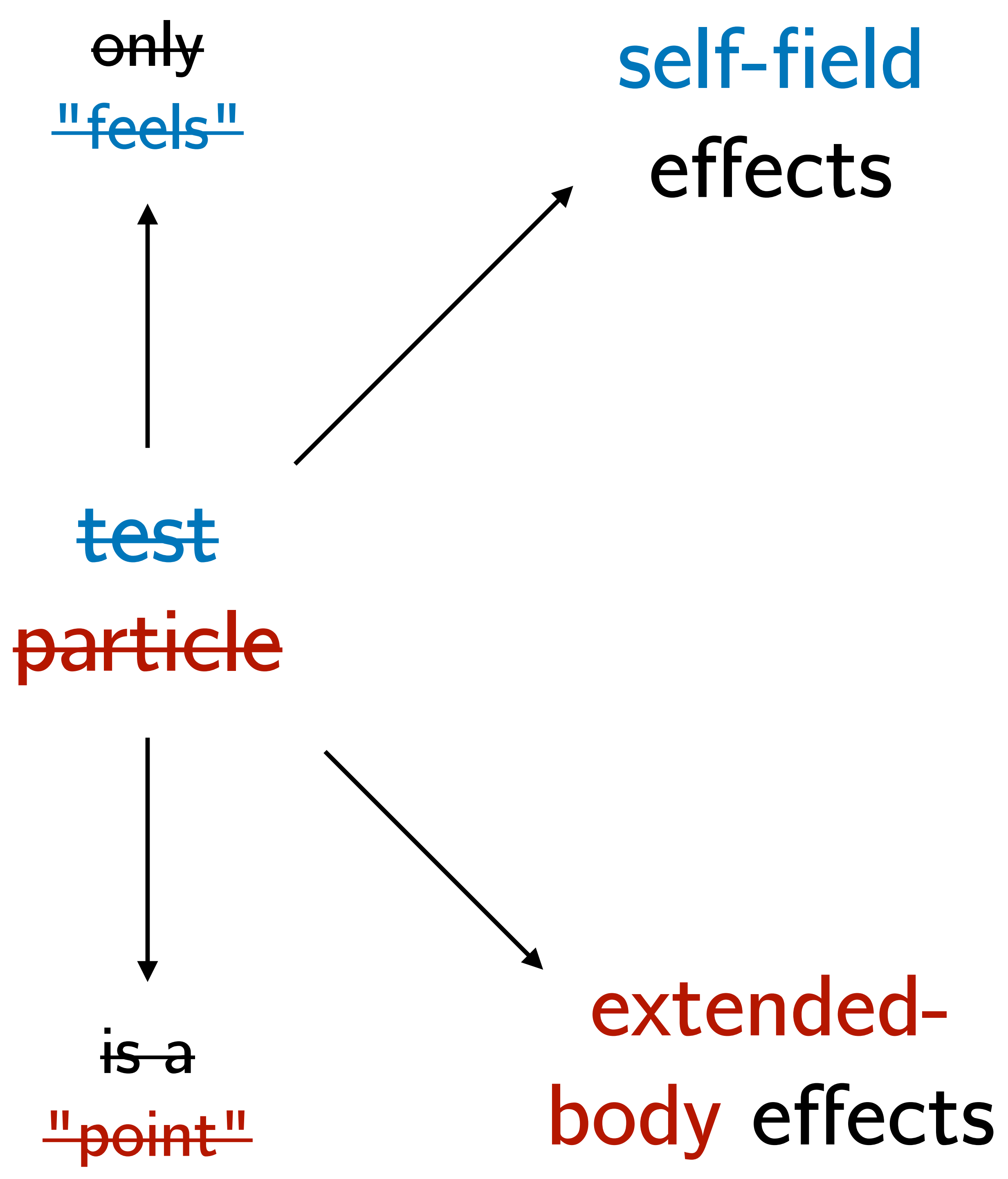

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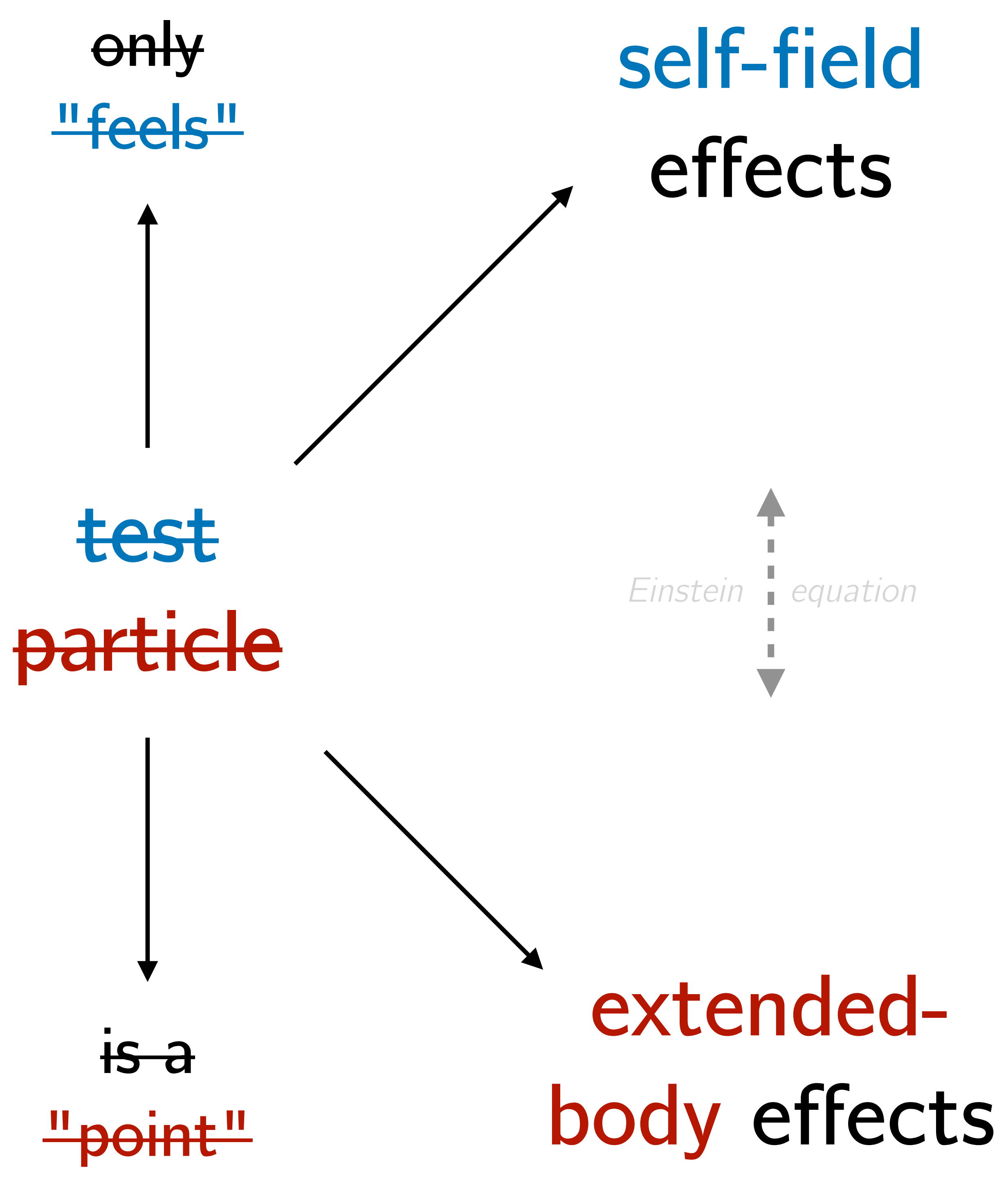

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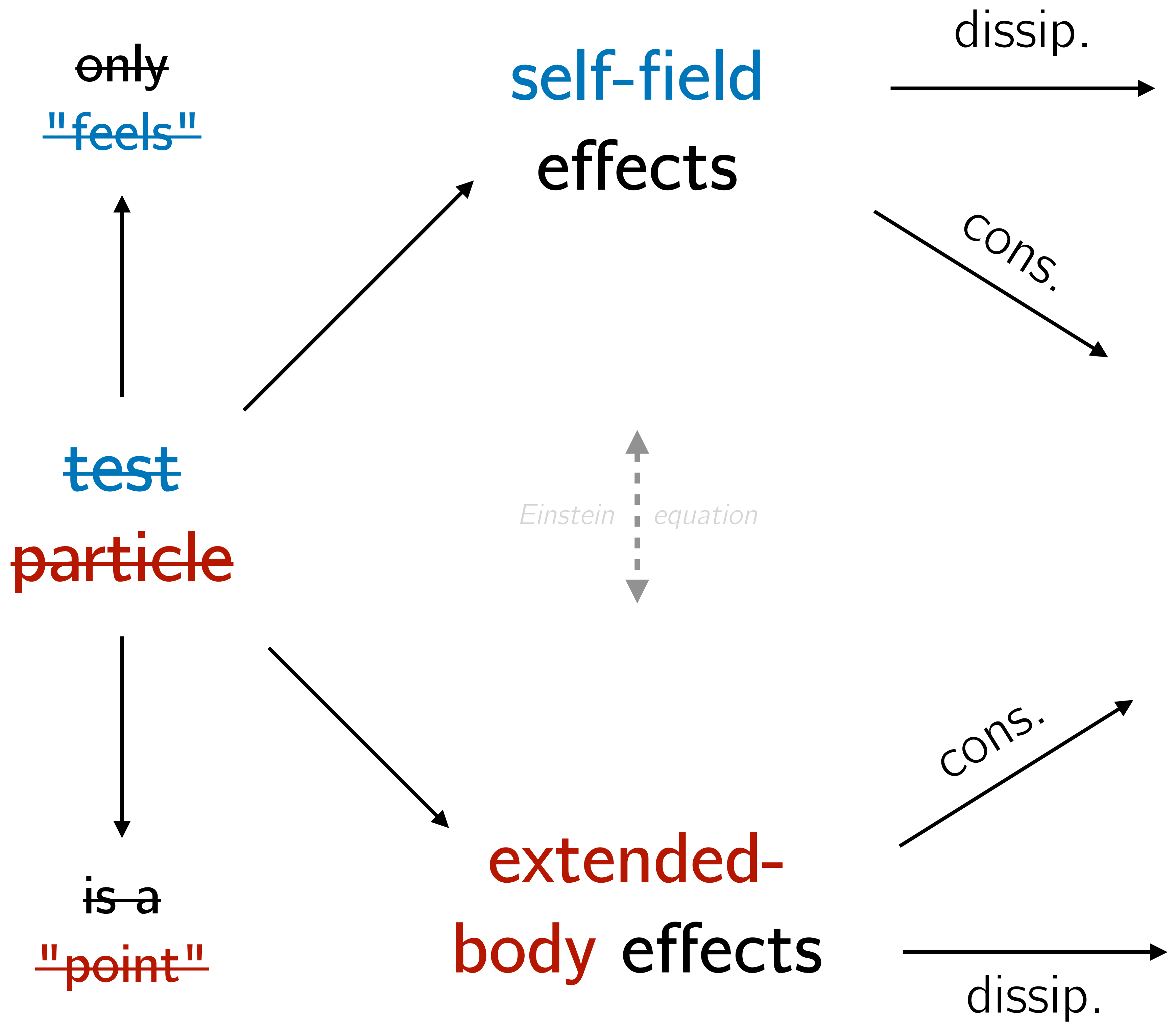

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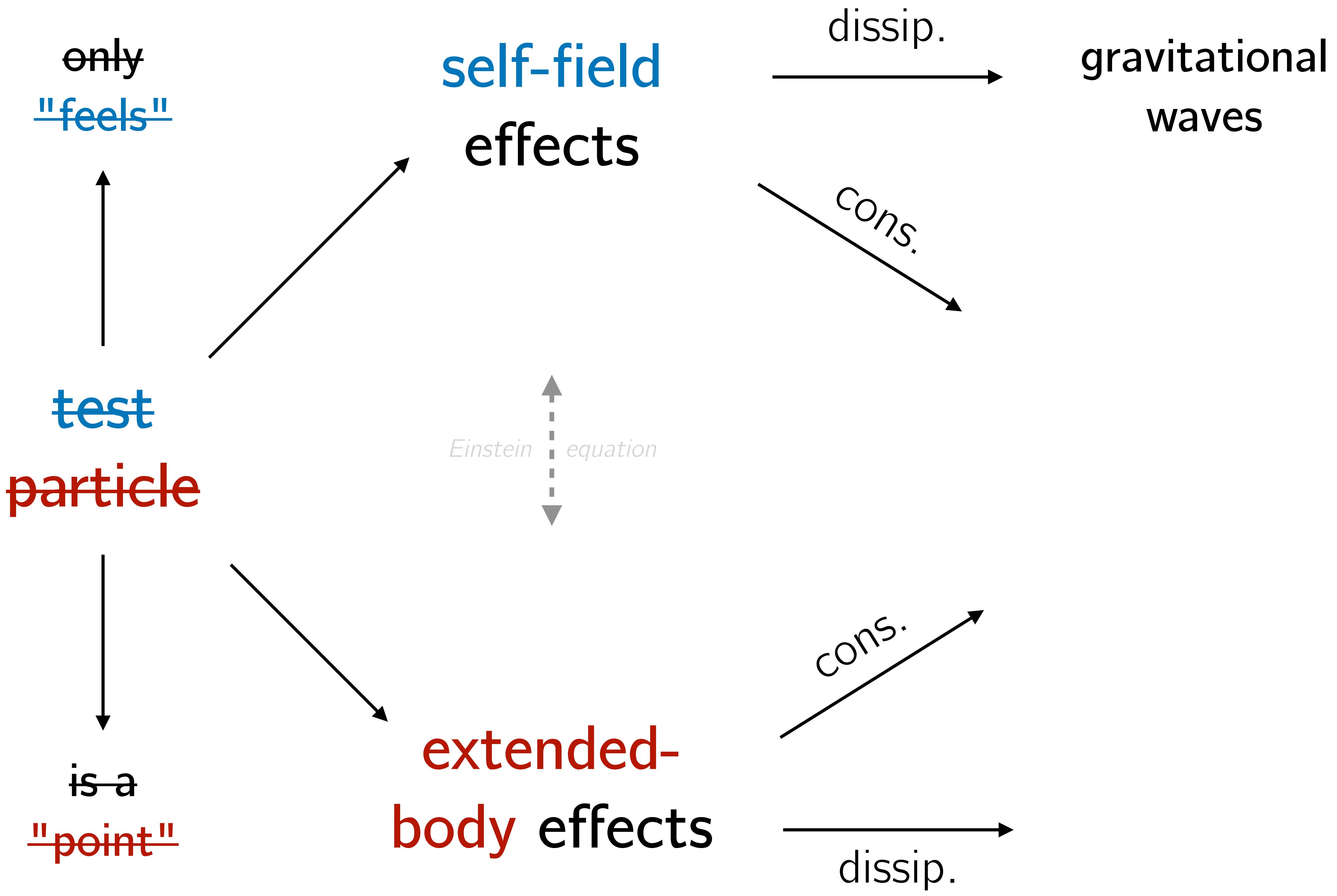

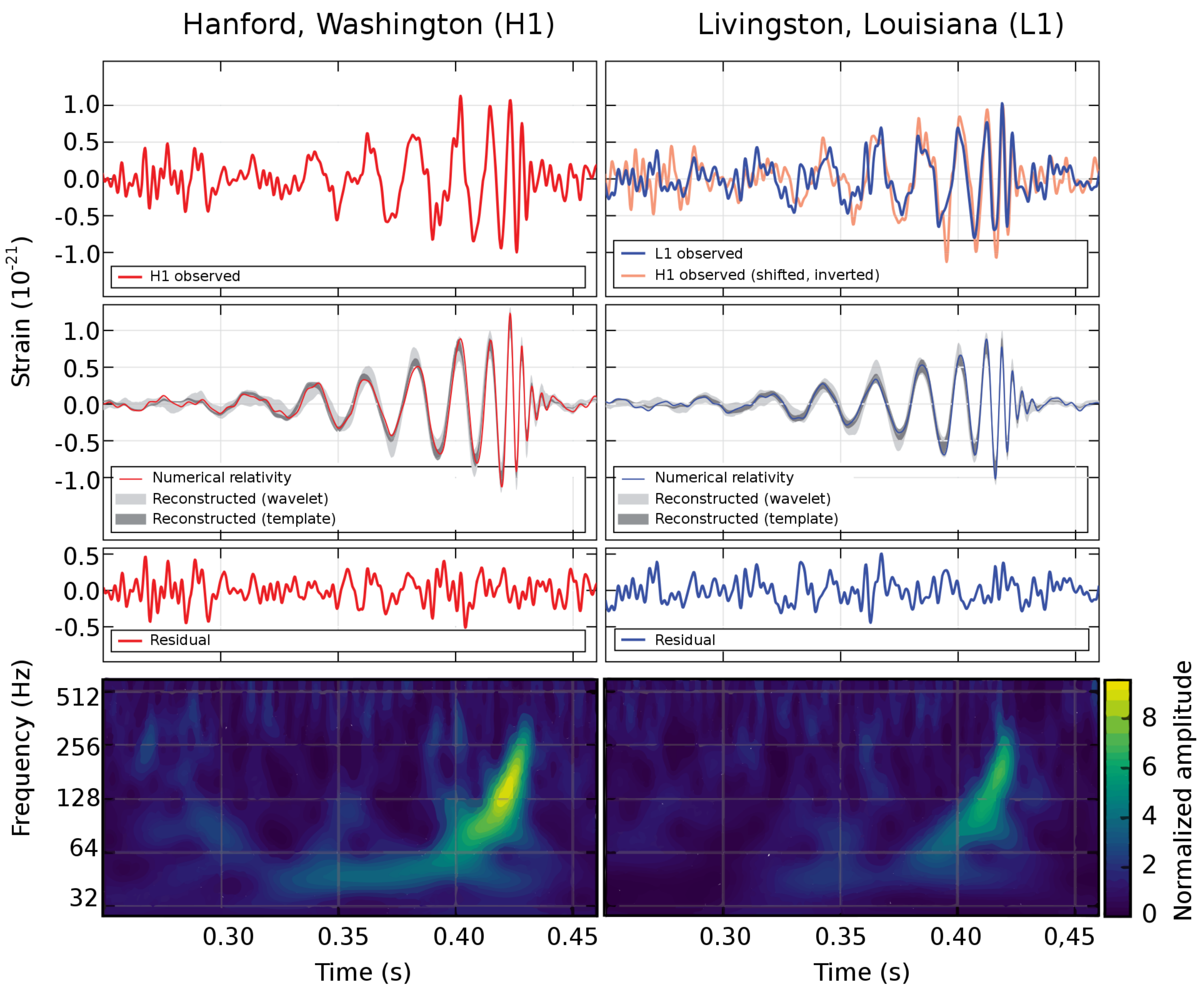

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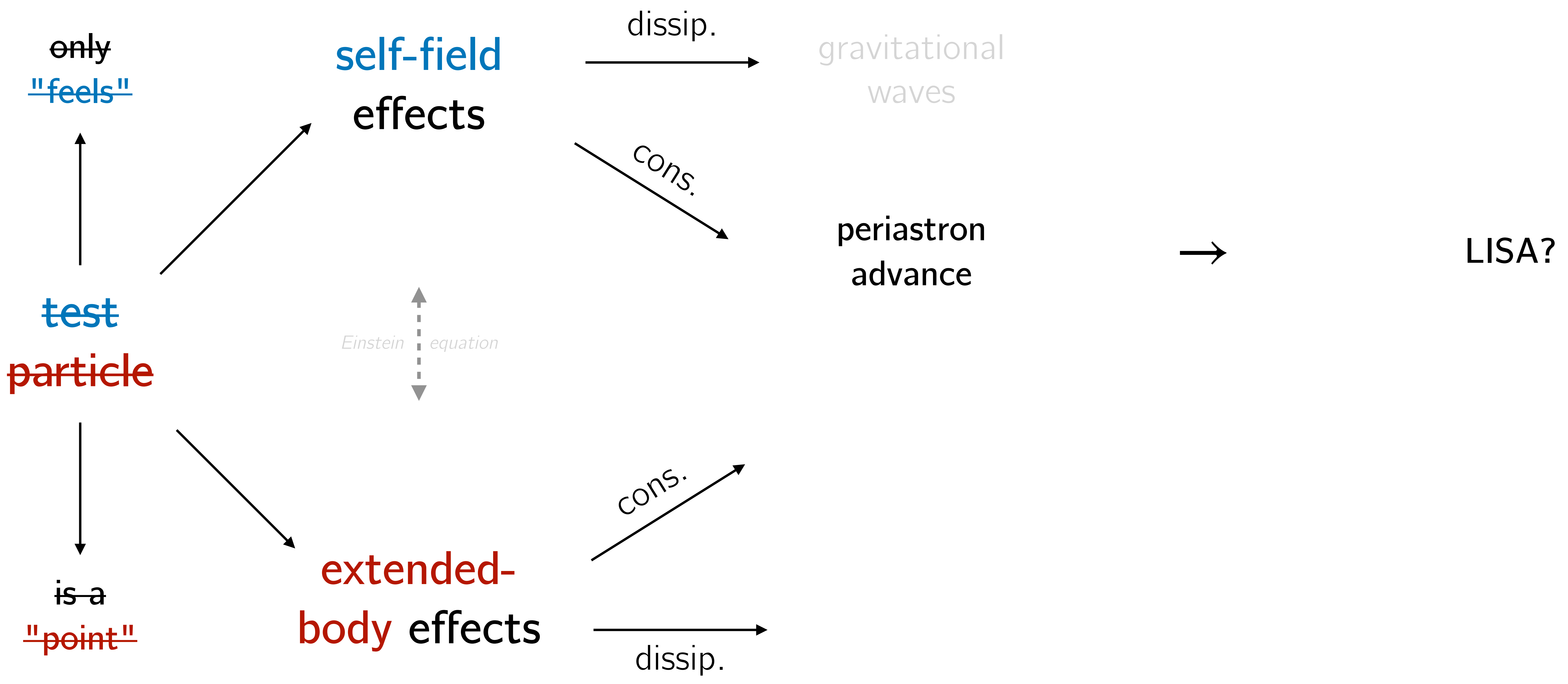

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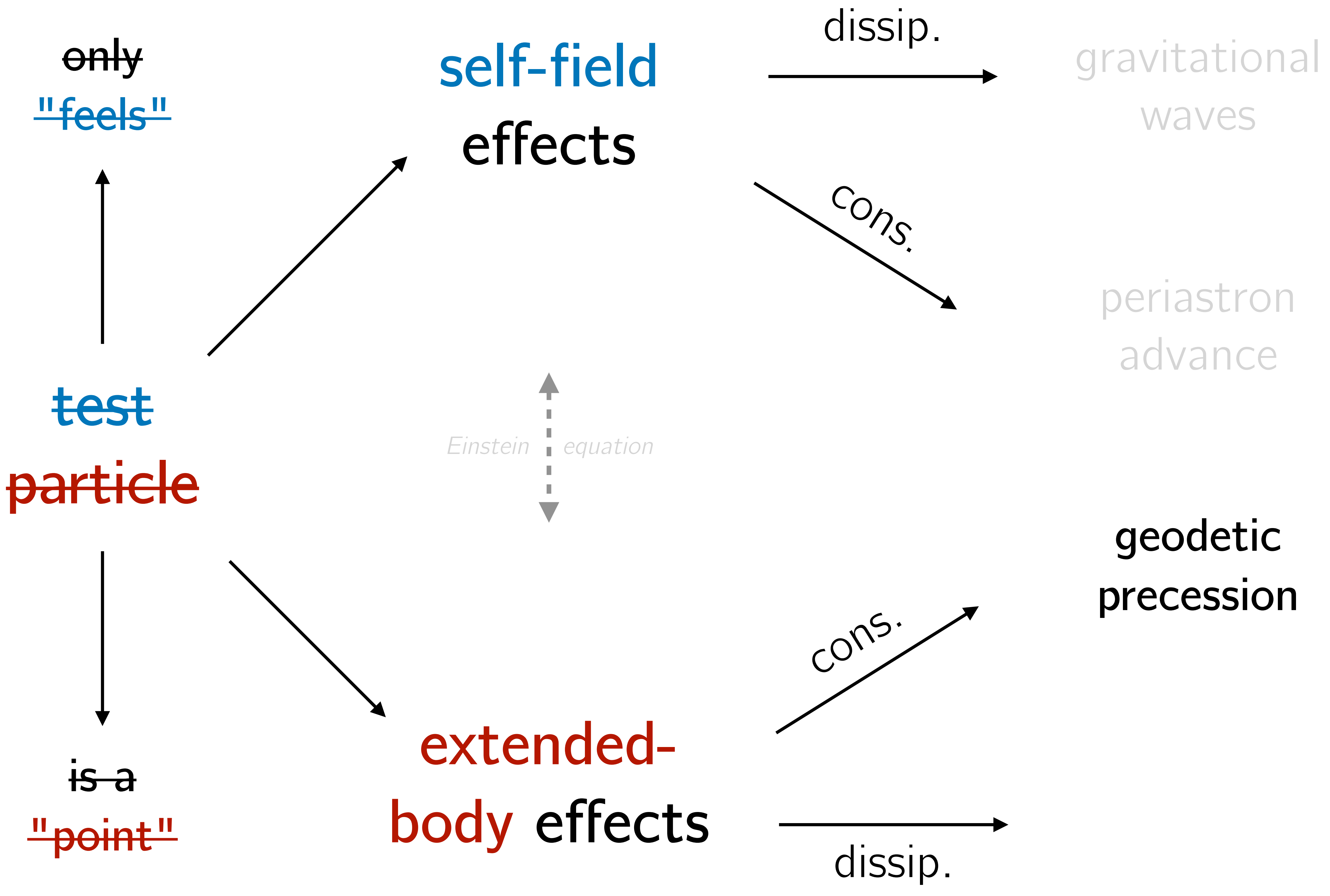
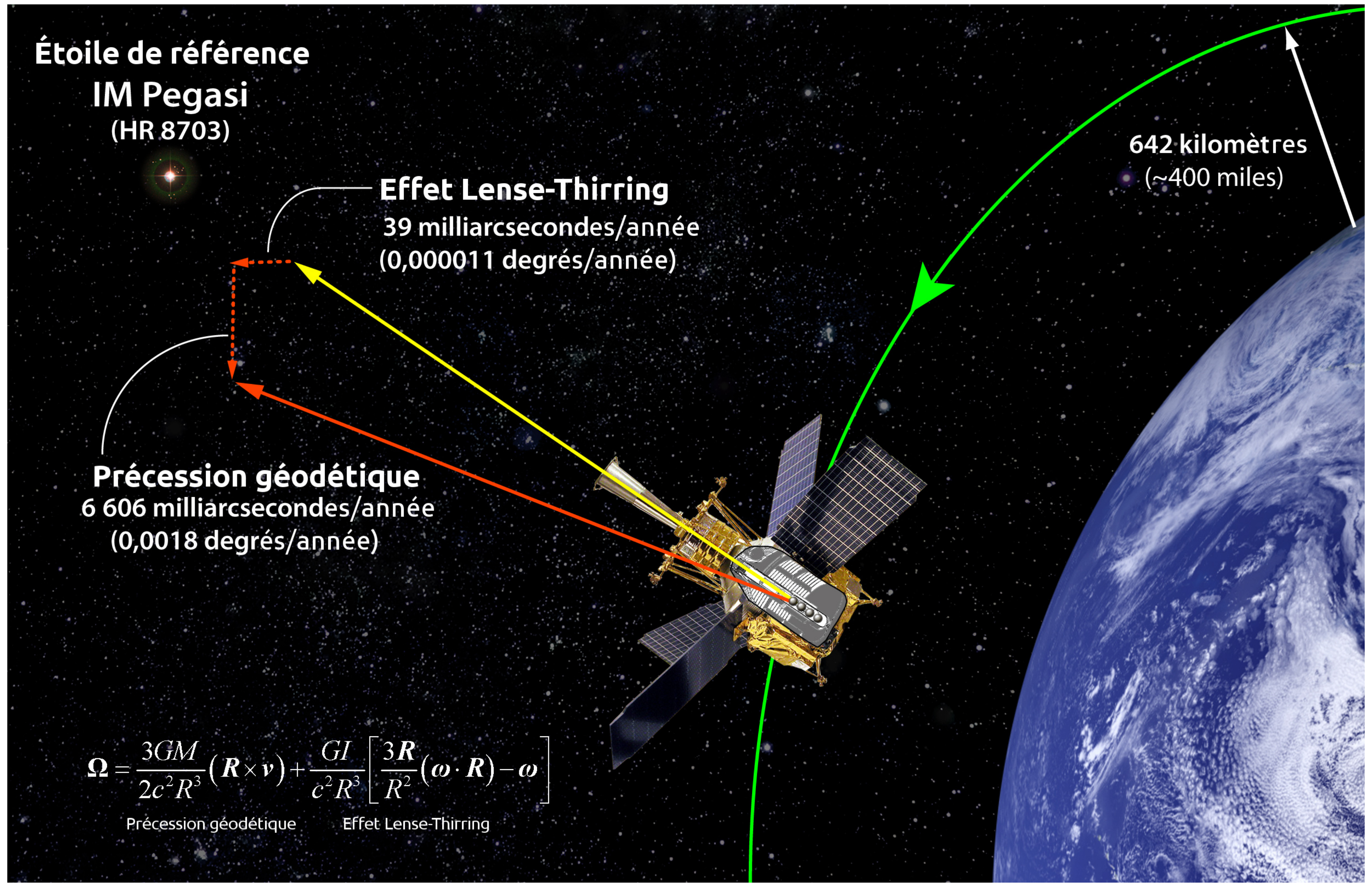
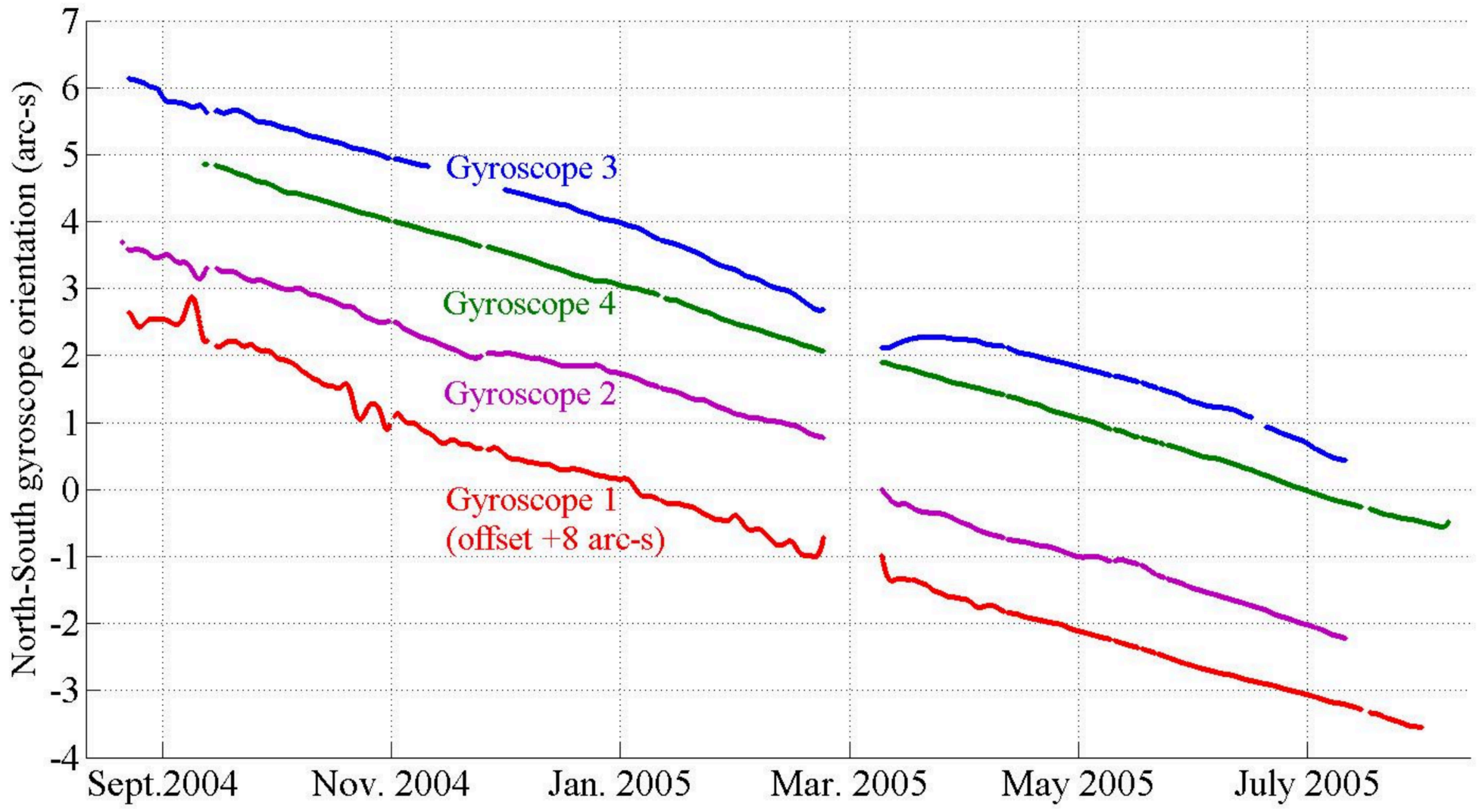
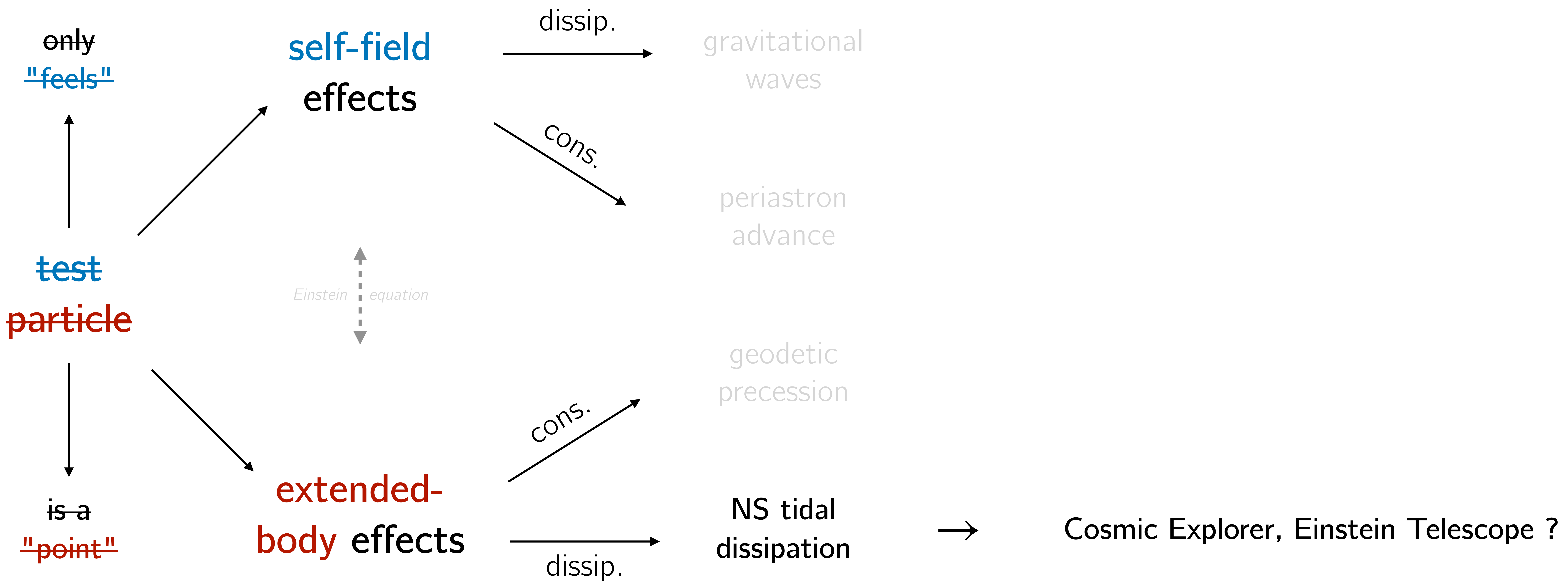






Everitt; et al. (2011). "Gravity Probe B: Final Results of a Space Experiment to Test General Relativity". *Physical Review Letters.* 106 (22): 221101.

Corrections to geodesic motion



self-field
effects

$\Uparrow$

This Talk

$\Downarrow$

extended-
body effects

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-
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GR & geodesics

Schw. geodesics
& integrability

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GR & geodesics

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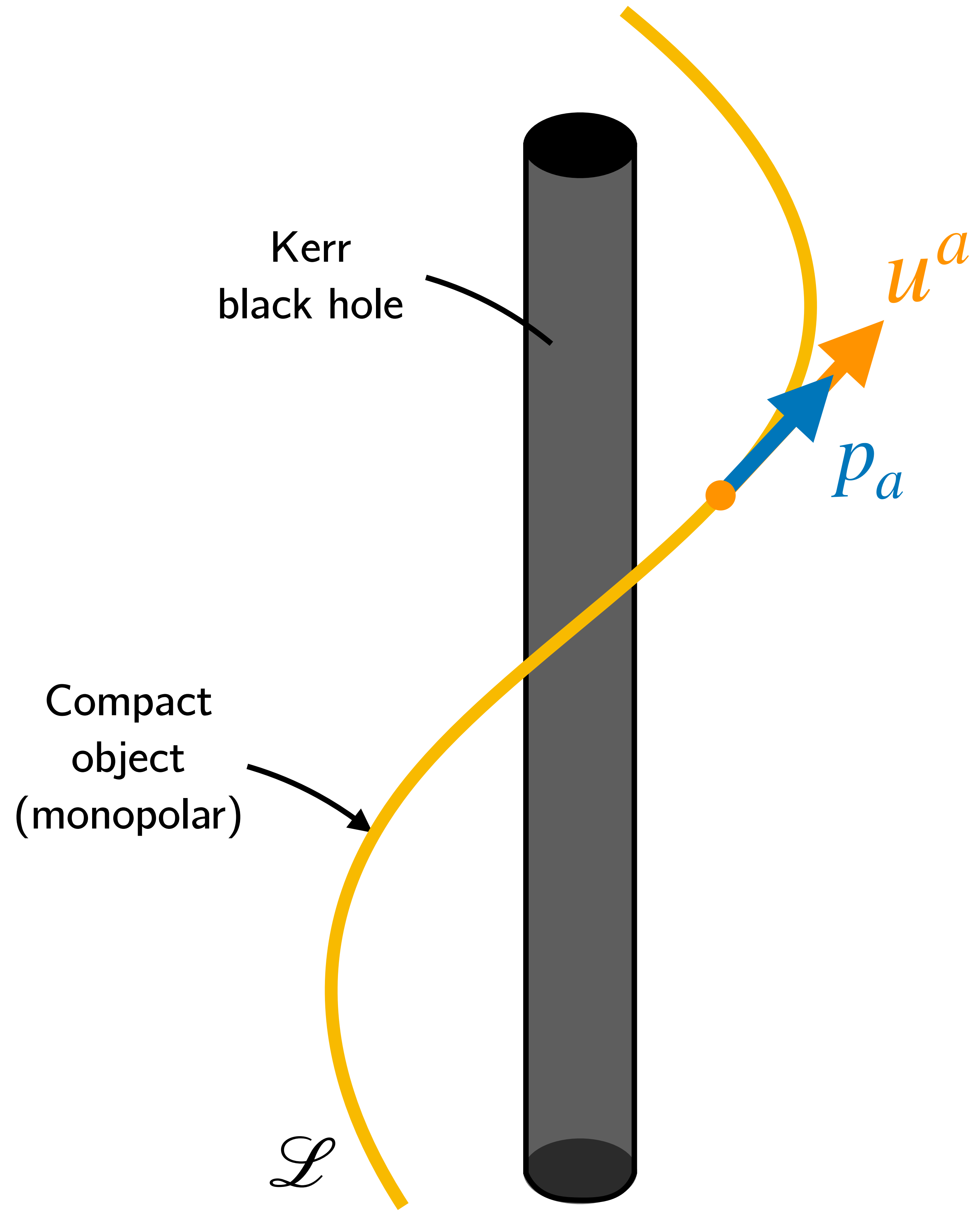
Beyond geodesics

Monopolar order

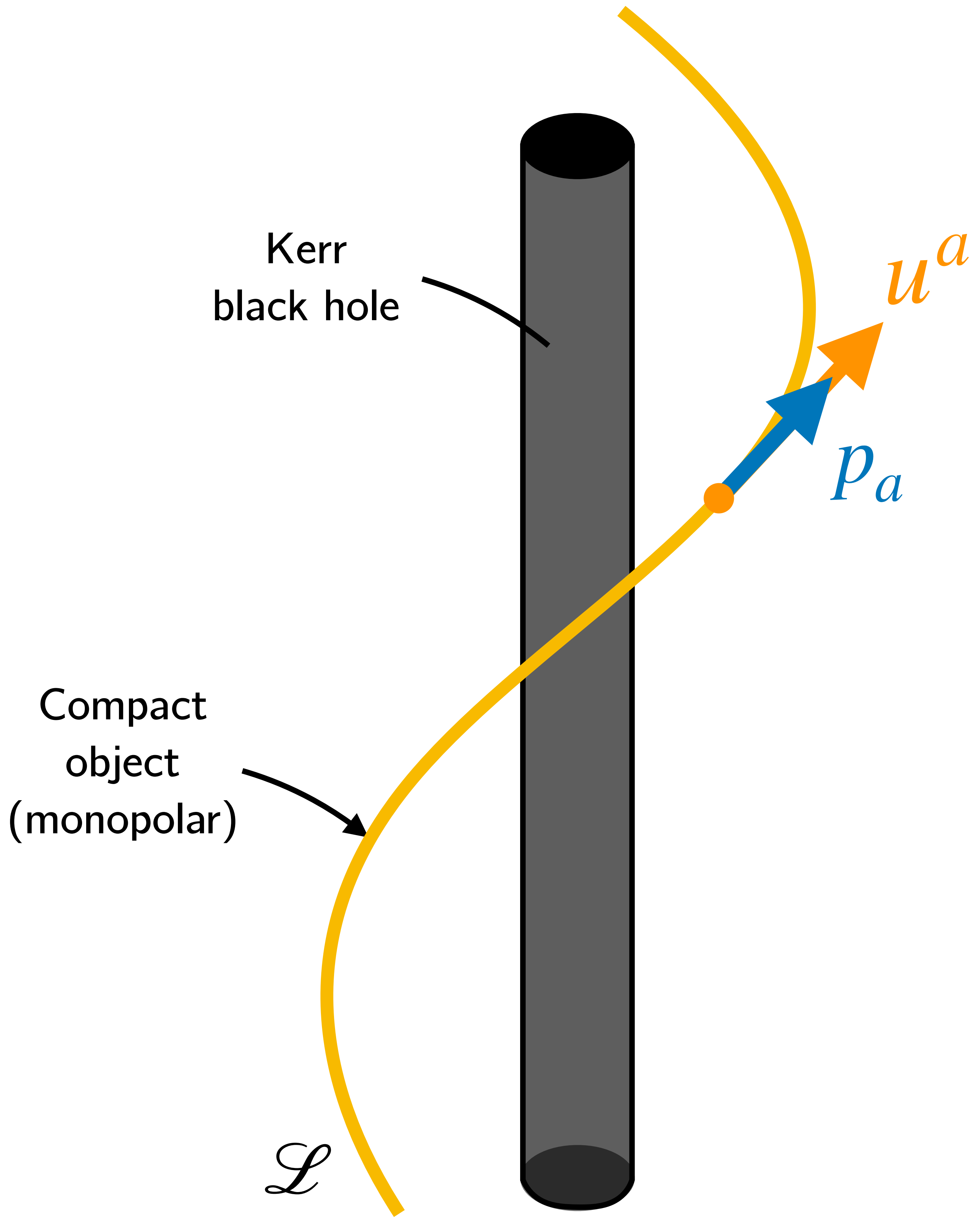


Monopolar $\Rightarrow S^{ab} = F_a = N^{ab} = 0$

$$\nabla_u p_a = 0 + 0$$

$$0 = 2p^{[a}u^{b]} + 0$$

Monopolar order



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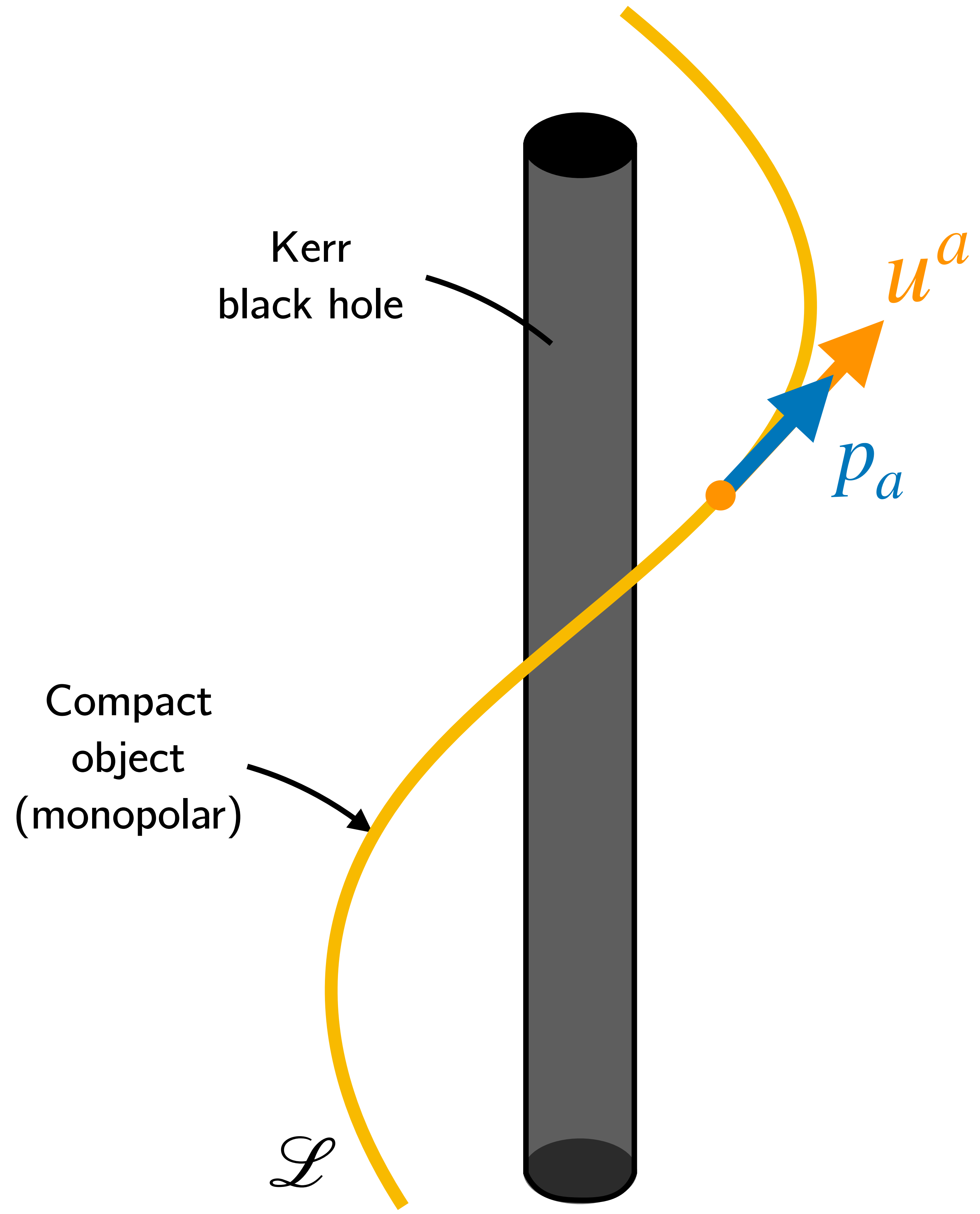
$\Rightarrow$ $p^a = \mu u^a$

μ = cst

$\mathscr{L}$ = geodesic

Monopolar order



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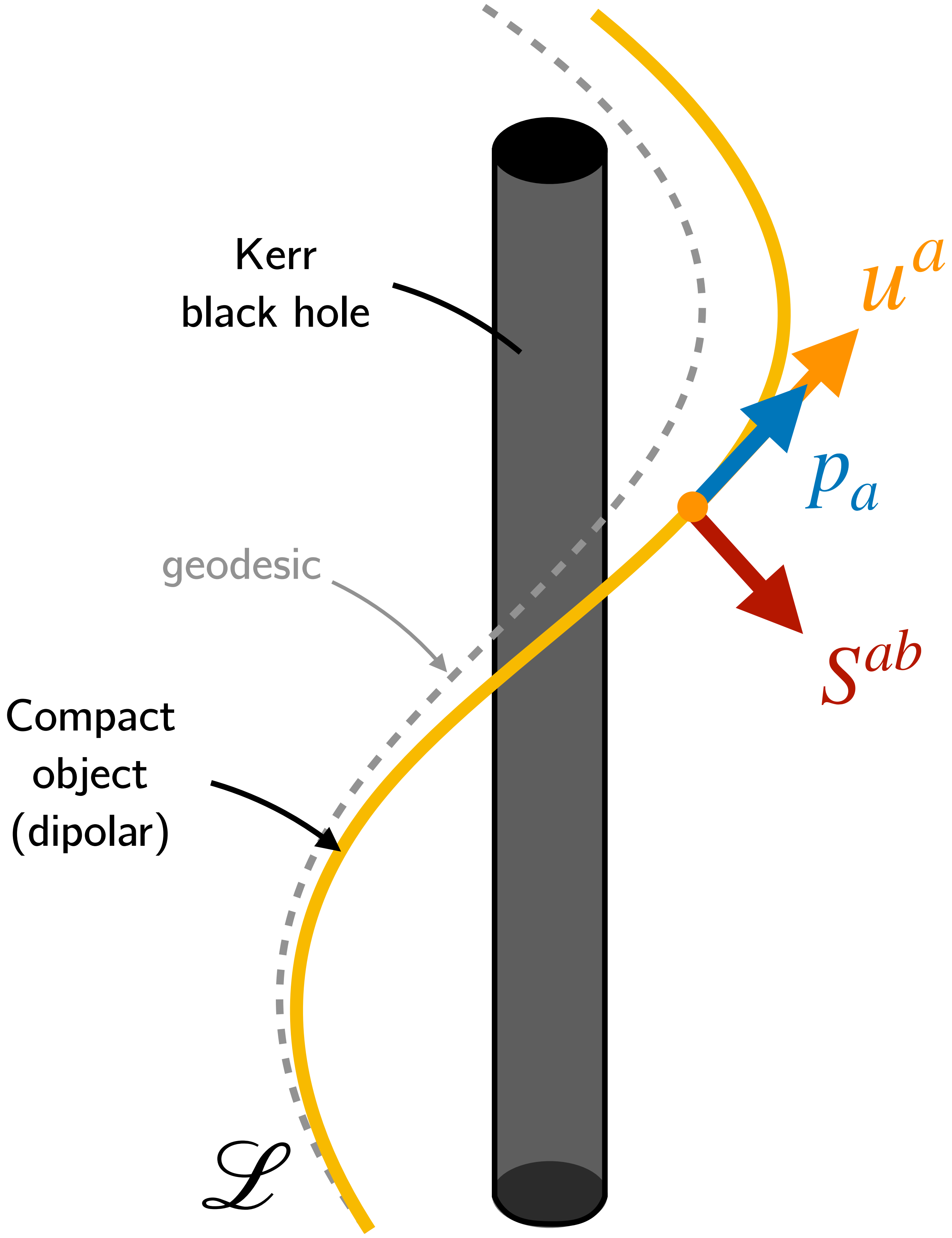
Universality: monopolar test objects follow spacetime geodesics

Dipolar order



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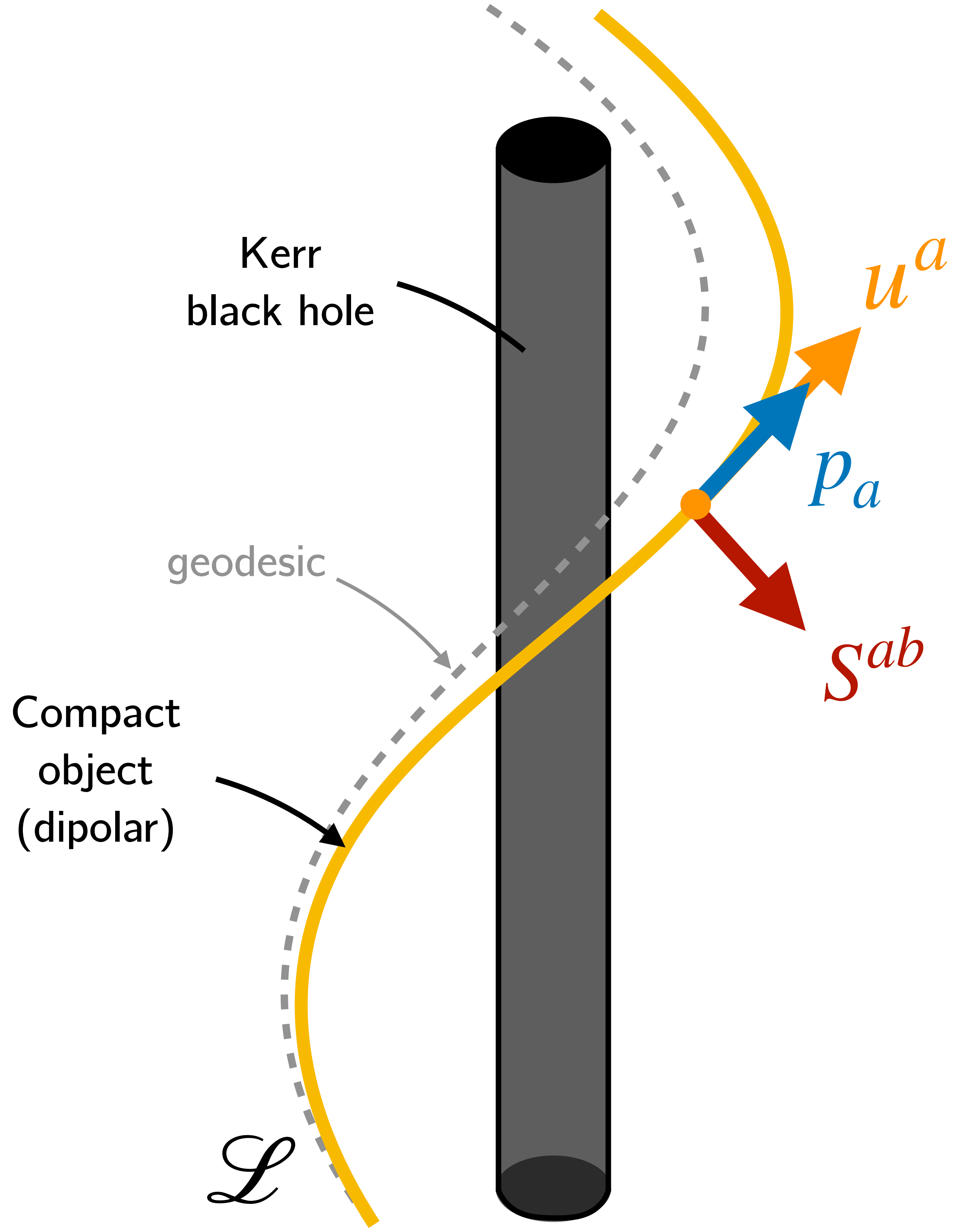
Dipolar order



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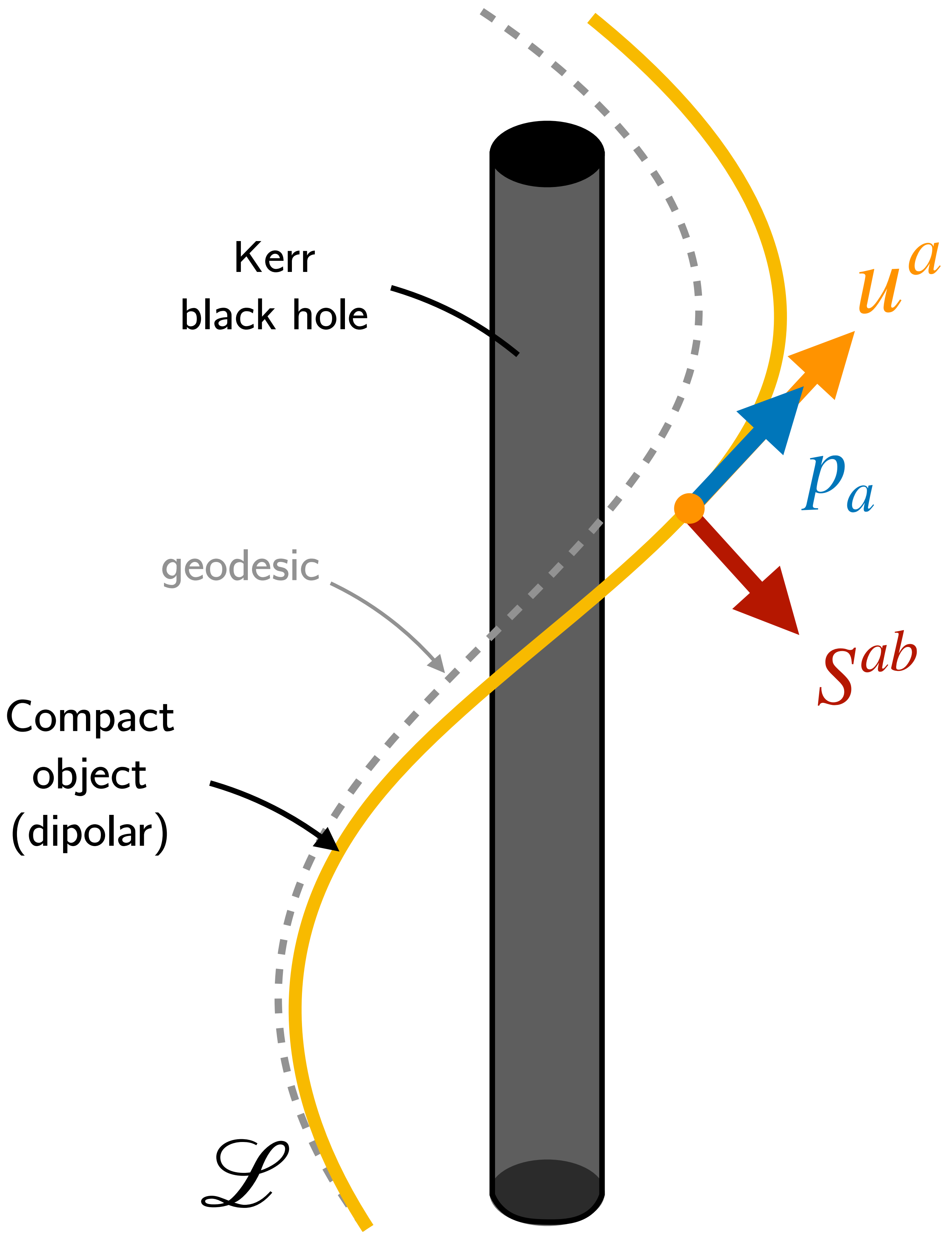
$\mathscr{L} \neq$ geodesic

Dipolar order



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$p^a = \mu u^a$

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Beyond geodesics
+ first integrals

Example: Schwarzschild geodesics

Phase space trajectories $\Longleftrightarrow$ **Spacetime orbits**