“Measuring Lorentz invariance violations with gravitational waves and the SME formalism”

Meeting of the GdR’s working group « Formes d’ondes »

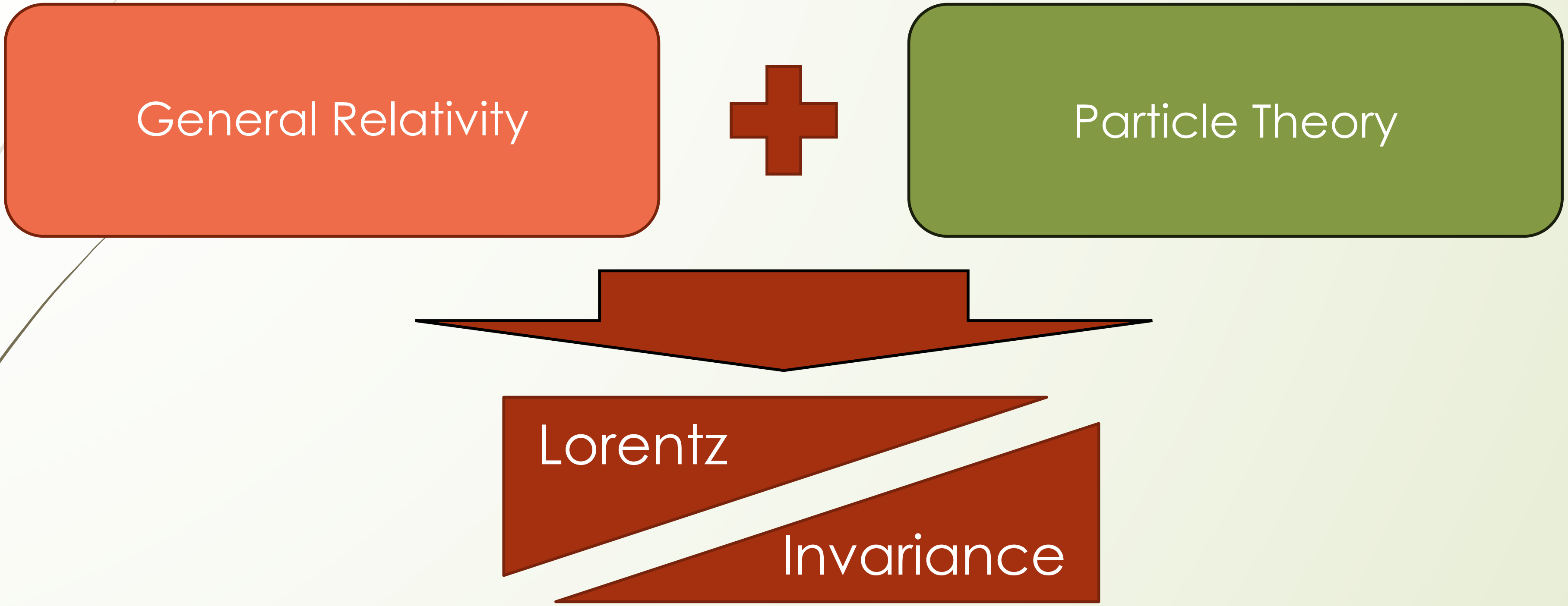
Samy AOULAD LAFKIH

Supervisors : Marie-Christine Angonin & Christophe Le Poncin-Lafitte

Work done with the help of N. A. Nilsson

Observatoire de Paris
SyRTE – Équipe Théorie et Métrologie

Scientific motivation



- **Lorentz Invariance** : a fundamental principle of Einstein theory of relativity
- The **Standard Model Extension (SME)** formalism parametrizes all possible violations to the **Lorentz invariance**
- Many different ways to estimate these parameters… like **gravitational wave generation** !

Tests of Lorentz invariance

Electromagnetism sector :

QED :

JPCS 1412, 032001 (2020)
Symmetry, 11 1220 (2019)
Nature, 575 310 (2017)
…

photon :

PRL 125, 221301 (2020)
PRD 78, 096008 (2008)
PRL 117, 241301 (2016)
…

Gravity sector :

AJL 848 L13 (2017)
PRD 92, 064049 (2013)
CQR 29, 175007 (2012)
PRD 94, 125030 (2016)
…

Particle sector :

electron :

PRD 92, 056002 (2020)
PRD 102, 056009 (2020)
…

proton :

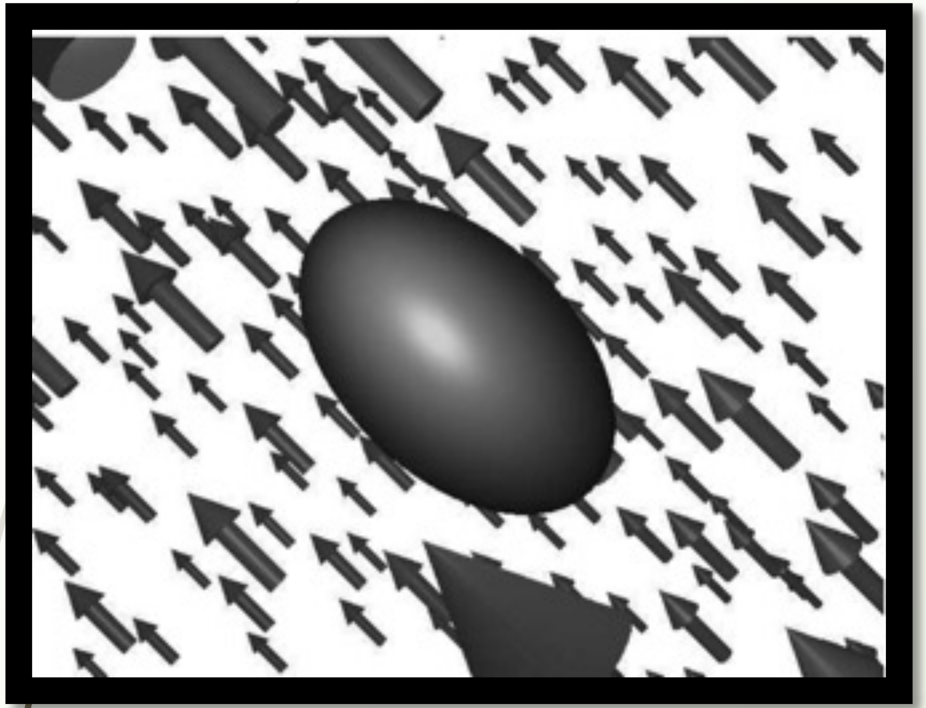
PRD 108, 076014 (2023)
Nature, 575 310 (2019)
…

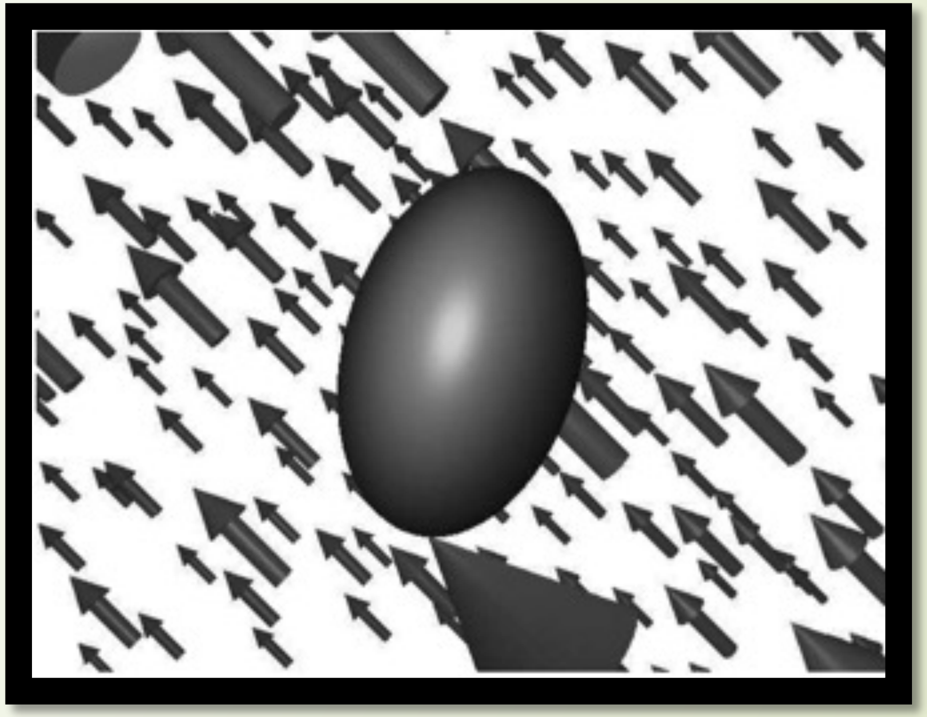
neutron :

PRD 98, 036003 (2018)
PRA 86, 012109 (2012)
…

…..

Violation of Lorentz invariance



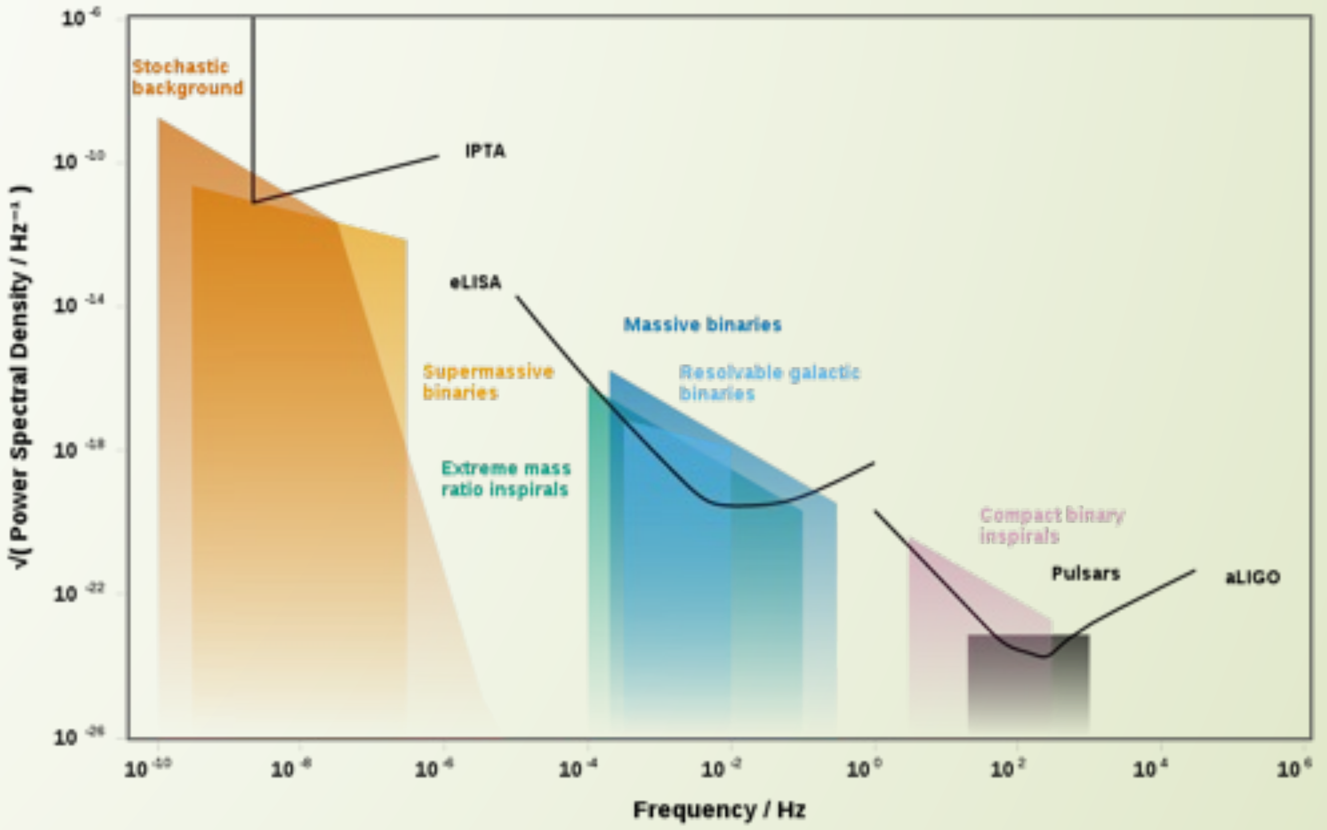


- Adding a background field that is **invariant under Lorentz transformations**

Images from Q. G. Bailey PhD manuscript

GW : Test of Lorentz invariance

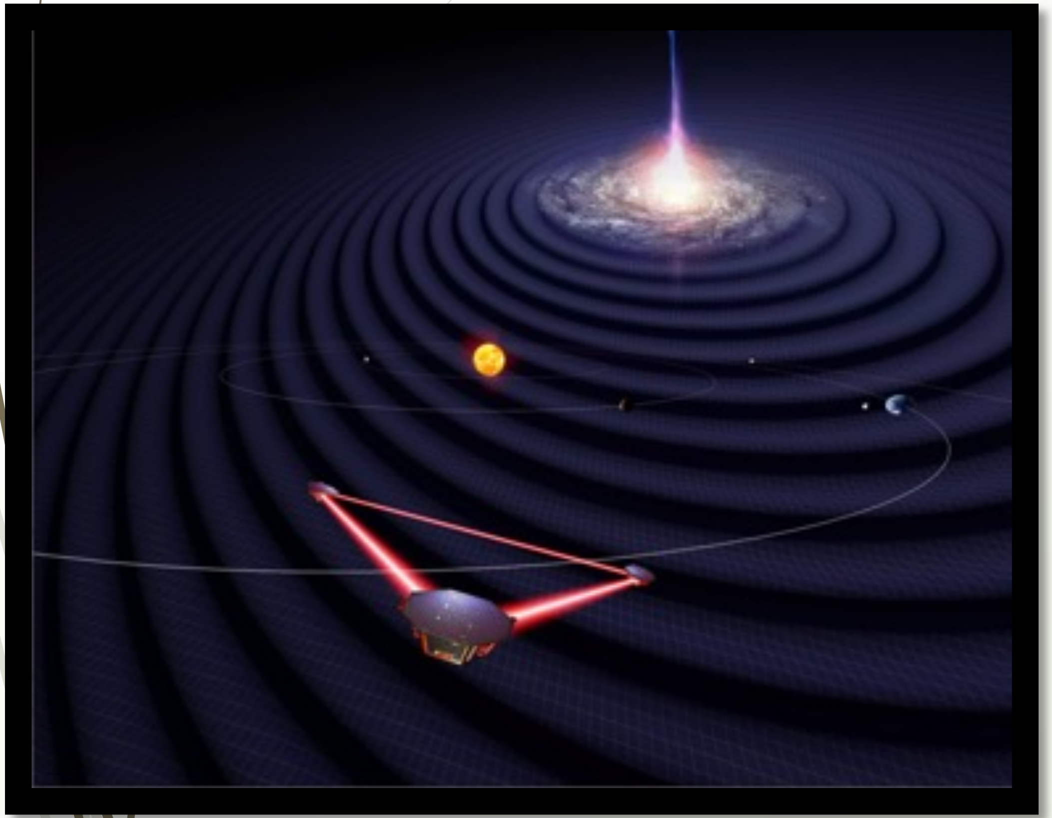
- **1st direct observation of GW** : 2015 with LIGO and VIRGO
- 1st direct observation of GW linked to an **electromagnetic counterpart** : 2017 : GW170817
- Allowed for a **test of Lorentz invariance** on propagation !



« Gravitational waves sensitivity curves », 2014

- LIGO-VIRGO on Earth probe GW from merging systems
- Frequency is quite high and the observation brief

The LISA mission



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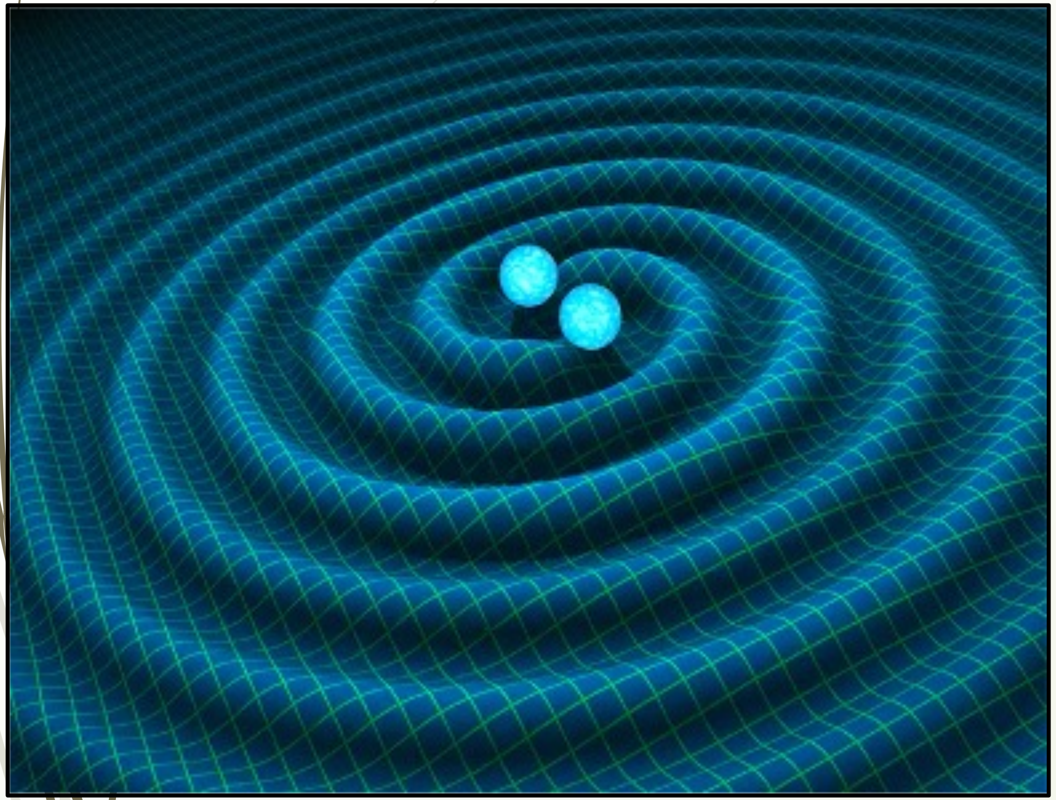
- Planned for **2037**
- Life expectancy : **4 years**
- Should be able to detect gravitational waves from galactic binaries (as well as others) with a frequency in the range of **0,1 mHz to 0,1 Hz**

- Constellation of 3 satellites exchanging lasers
- Heliocentric with 1 year period
- Arms of 2.5 million kilometers !

The LISA mission



- Exploitation of thousands of signals coming galactic binary systems with a strong SNR
- Compact objects binaries far from coalescence → **quasi monochromatic** signals
- Use the gravitational signal to perform a **parameter estimation** on Lorentz violating coefficients

Observation of a great many number of **quasi-monochromatic** periodic signals with LISA : possibility to create **robust statistical tools** !

Getting the wave equation

- Study possible Lorentz invariance violations in the generation of gravitational waves
- Modelisation of gravitational waves in the SME formalism with the coefficients *u*, *s* and *t*
- To do so : use the results of [*Q. G. Bailey and V. A. Kostelecký, PRD 74 045001 (2006)*]

$$L = L_{EH} + L_{LV}$$

$$= \sqrt{g}[(1-u)R$$

$$+ s^{\mu\nu} R_{\mu\nu}$$

$$+ t^{\lambda\kappa\mu\nu} R_{\lambda\kappa\mu\nu}]$$

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$$L = L_{EH} + L_{LV}$$

$$\Downarrow$$

$$G_{\mu\nu} = \kappa T_{\mu\nu} + \Phi^{LV}_{\mu\nu}(g, u, s)$$

Getting the wave equation

- Displace the modified Lagrangian in order to get the modified Einstein equation

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Getting the wave equation

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- Weak field approximation : linearisation in *h*

-

-

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Getting the wave equation

- Displace the modified Lagrangian in order to get the modified Einstein equation
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- We consider that the SME perturbations to the metric are sourced by the gravitational waves of GR at 1st order

$$G_{\mu\nu} = \kappa T_{\mu\nu} + \Phi^{LV}_{\mu\nu}(h)$$

$\Big\downarrow$ **Linearization in *h*, *u*, *s*, *t* + separation of h^{GR} and h^{LV}**

$$G^{GR}_{\mu\nu} + G^{LV}_{\mu\nu} = \kappa T_{\mu\nu} + \Phi^{LV}_{\mu\nu}(h^{GR})$$

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Linearization in *h, u, s, t* + separation of h^{GR} and h^{LV}

$$\cancel{G^{GR}_{\mu\nu}} + G^{LV}_{\mu\nu} = \cancel{\kappa T_{\mu\nu}} + \Phi^{LV}_{\mu\nu}(h^{GR})$$

Q. G. Bailey and V. A. Kostelecký, PRD 74 045001 (2006)

Wave equation

$$\begin{aligned} \Box \bar{h}^{LV}_{\mu\nu} = & \overbrace{\Box \left[\bar{u}\bar{h}^{GR}_{\mu\nu} + \eta_{\mu\nu}\bar{s}^{\alpha\beta}\bar{h}^{GR}_{\alpha\beta} - 2\bar{s}^{\alpha}{}_{(\mu}\bar{h}^{GR}_{\nu)\alpha} + \frac{1}{2}\bar{s}_{\mu\nu}\bar{h}^{GR} \right]}^{\text{« Local » source term}} \\ & \underbrace{- 2\bar{s}^{\alpha\beta}\left(\partial_\mu \partial_{[\nu} h^{GR}_{\beta]\alpha} + \partial_\alpha \partial_{[\beta} h^{GR}_{\nu]\mu}\right)}_{\text{« Global » source term}} \end{aligned}$$

Wave equation : model for h^{GR}

$$\bar{h}_{00}^{GR} = 4\frac{M}{r}$$

$$\bar{h}_{i0}^{GR} = 0$$

$$\bar{h}_{ij}^{GR} = -2\frac{\ddot{I}_{ij}}{r}$$

$$\bar{h}^{GR} = -2\frac{2M + \ddot{I}}{r}$$

M. P. Hobson, G. P. Efstathiou, A. N. Lasenby, Cambridge University Press (2006)

- Simple model of linearized gravitational waves far from the source : **quadrupole formula**

M is the total mass of the system

$\ddot{I}_{ij}$ is the second time derivative of the mass quadrupole

Wave equation : full form

$$
\begin{aligned}
\Box \bar{h}^{LV}_{ij} = &-12\bar{s}^{00}\hat{n}_{ij}\frac{M}{r^3} + 2\bar{s}^{ab}\left[\hat{n}_{ij}\left(3\frac{\ddot{I}_{ab}}{r^3} + 3\frac{\dddot{I}_{ab}}{r^2} + \frac{\ddddot{I}_{ab}}{r}\right) + \frac{\delta_{ij}}{3}\frac{\ddddot{I}_{ab}}{r}\right] \\
&+ 2\bar{s}^{00}\frac{\ddddot{I}_{ij}}{r} - 4\bar{s}^{0a}n_a\left(\frac{\dddot{I}_{ij}}{r^2} + \frac{\ddddot{I}_{ij}}{r}\right) + 2\bar{s}^{ab}\left[\hat{n}_{ab}\left(3\frac{\ddot{I}_{ij}}{r^3} + 3\frac{\dddot{I}_{ij}}{r^2} + \frac{\ddddot{I}_{ij}}{r}\right) + \frac{\delta_{ab}}{3}\frac{\ddddot{I}_{ij}}{r}\right] \\
&+ 4\bar{s}^{0a}n_{(j}\left(\frac{\dddot{I}_{i)a}}{r^2} + \frac{\ddddot{I}_{i)a}}{r}\right) - 4\bar{s}^{ab}\left[\hat{n}_{a(j}\left(3\frac{\ddot{I}_{i)b}}{r^3} + 3\frac{\dddot{I}_{i)b}}{r^2} + \frac{\ddddot{I}_{i)b}}{r}\right) + \frac{\delta_{a(j}}{3}\frac{\ddddot{I}_{i)b}}{r}\right] \\
&- 2\bar{s}_{(j}{}^{0}n_{i)}\left(\frac{\dddot{I}}{r^2} + \frac{\ddddot{I}}{r}\right) + 2\bar{s}_{(j}{}^{a}\left[\hat{n}_{i)a}\left(3\frac{2M + \ddot{I}}{r^3} + 3\frac{\dddot{I}}{r^2} + \frac{\ddddot{I}}{r}\right) + \frac{\delta_{i)a}}{3}\frac{\ddddot{I}}{r}\right] \\
&+ \delta_{ij}\left[-\bar{s}^{00}\frac{\ddddot{I}}{r} + 2\bar{s}^{0a}n_a\left(\frac{\dddot{I}}{r^2} + \frac{\ddddot{I}}{r}\right) - \bar{s}^{ab}\left(\hat{n}_{ab}\left(3\frac{2M + \ddot{I}}{r^3} + 3\frac{\dddot{I}}{r^2} + \frac{\ddddot{I}}{r}\right) + \frac{\delta_{ab}}{3}\frac{\ddddot{I}}{r}\right)\right]
\end{aligned}
$$

Wave equation

Classical $\Box^{-1}$ operator for linearised gravitational wave :

$$\Box F = S$$

$$\longrightarrow$$

$$F(ct,\vec{x}) = \int_{\mathbb{R}^3} d\overrightarrow{x'}\,\frac{S(ct-|\vec{x}-\overrightarrow{x'}|,\overrightarrow{x'})}{|\vec{x}-\overrightarrow{x'}|}$$

M. P. Hobson, G. P. Efstathiou, A. N. Lasenby, Cambridge University Press (2006)

Convenient when *S* is proportional to a dirac distribution

Here : **complicated** to extract a « straight-forward » analytical formula

Wave equation

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Convenient when *S* is proportional to a dirac distribution

Here : **complicated** to extract a « straight-forward » analytical formula

Use the formulas established for the Post-Minkowskian (PM) method

Inverse d'Alembertian

$$\Box_R^{-1}\left(\hat{n}_L r^{B-k} F(t-r)\right) = \frac{1}{D(B-k)} \int_{-\infty}^{t'-r'} \mathrm{d}s F(s) \hat{\partial}'_L \left[\frac{(t'-r'-s)^{B-k+l+2} - (t'+r'-s)^{B-k+l+2}}{r'} \right]$$

$$D(B-k) = 2^{B-k+3}(B-k+2)(B-k+1)...(B-k+2-l)$$

L. Blanchet and T. Damour, Phil.Trans.R.Soc.Lond. A 320 379-430, 1986

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$\hat{n}_L$ → Symmetric Trace-free tensor composed with coordinates of unitary direction vectors $\vec{\boldsymbol{n}}$

r^B → Complex power in order to **regularize** our potentials around $r = 0$
$B \to 0$ at the end of the procedure

Particular solution

$$
\begin{aligned}
\Box^{-1}\left[\Box \bar{h}^{LV}_{ij}\right] =& 2\bar{s}^{00}\hat{n}_{ij}\frac{M}{r} - \bar{s}^{ab}\left[\hat{n}_{ij}\left(\frac{\ddot{I}_{ab}}{r} + \dddot{I}_{ab}\right) + \frac{\delta_{ij}}{3}\dddot{I}_{ab}\right] \\
& - \bar{s}^{00}\dddot{I}_{ij} + 2\bar{s}^{0a}n_a\dddot{I}_{ij} - \bar{s}^{ab}\left[\hat{n}_{ab}\left(\frac{\ddot{I}_{ij}}{r} + \dddot{I}_{ij}\right) - \frac{\delta_{ab}}{3}\dddot{I}_{ij}\right] \\
& - 2\bar{s}^{0a}n_{(j}\dddot{I}_{i)a} + 2\bar{s}^{ab}\left[\hat{n}_{a(j}\left(\frac{\ddot{I}_{i)b}}{r} + \dddot{I}_{i)b}\right) + \frac{\delta_{a(j}}{3}\dddot{I}_{i)b}\right] \\
& + \bar{s}_{(j}{}^{0}n_{i)}\dddot{I} - \bar{s}_{(j}{}^{a}\left[\hat{n}_{i)a}\left(\frac{2M + \ddot{I}}{r} + \dddot{I}\right) + \frac{\delta_{i)a}}{3}\ddddot{I}\right] \\
& - \frac{\delta_{ij}}{2}\left[-\bar{s}^{00}\dddot{I} + 2\bar{s}^{0a}n_a\dddot{I} - \bar{s}^{ab}\left(\hat{n}_{ab}\left(\frac{2M + \ddot{I}}{r} + \dddot{I}\right) + \frac{\delta_{ab}}{3}\dddot{I}\right)\right]
\end{aligned}
$$

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\end{aligned}
$$

Apparition of STF directional-multipoles, even though $\bar{h}^{GR}_{\mu\nu}$ was **spherically symmetric**

Particular solution

$$
\begin{aligned}
\Box^{-1}\left[\Box \bar{h}^{LV}_{ij}\right] =& 2\bar{s}^{00}\hat{n}_{ij}\frac{M}{r} - \bar{s}^{ab}\left[\hat{n}_{ij}\left(\frac{\ddot{I}_{ab}}{r} + \dddot{I}_{ab}\right) + \frac{\delta_{ij}}{3}\dddot{I}_{ab}\right] \\
& - \bar{s}^{00}\dddot{I}_{ij} + 2\bar{s}^{0a}n_a\dddot{I}_{ij} - \bar{s}^{ab}\left[\hat{n}_{ab}\left(\frac{\ddot{I}_{ij}}{r} + \dddot{I}_{ij}\right) - \frac{\delta_{ab}}{3}\dddot{I}_{ij}\right] \\
& - 2\bar{s}^{0a}n_{(j}\dddot{I}_{i)a} + 2\bar{s}^{ab}\left[\hat{n}_{a(j}\left(\frac{\ddot{I}_{i)b}}{r} + \dddot{I}_{i)b}\right) + \frac{\delta_{a(j}}{3}\dddot{I}_{i)b}\right] \\
& + \bar{s}_{(j}{}^{0}n_{i)}\dddot{I} - \bar{s}_{(j}{}^{a}\left[\hat{n}_{i)a}\left(\frac{2M + \ddot{I}}{r} + \dddot{I}\right) + \frac{\delta_{i)a}}{3}\ddddot{I}\right] \\
& - \frac{\delta_{ij}}{2}\left[-\bar{s}^{00}\dddot{I} + 2\bar{s}^{0a}n_a\dddot{I} - \bar{s}^{ab}\left(\hat{n}_{ab}\left(\frac{2M + \ddot{I}}{r} + \dddot{I}\right) + \frac{\delta_{ab}}{3}\dddot{I}\right)\right]
\end{aligned}
$$

Static terms proportional to the system mass

Particular solution

$$
\begin{aligned}
\Box^{-1}\left[\Box\bar{h}^{LV}_{ij}\right] = & 2\bar{s}^{00}\hat{n}_{ij}\frac{M}{r} - \bar{s}^{ab}\left[\hat{n}_{ij}\left(\frac{\ddot{I}_{ab}}{r} + \dddot{I}_{ab}\right) + \frac{\delta_{ij}}{3}\dddot{I}_{ab}\right] \\
& - \bar{s}^{00}\dddot{I}_{ij} + 2\bar{s}^{0a}n_a\dddot{I}_{ij} - \bar{s}^{ab}\left[\hat{n}_{ab}\left(\frac{\ddot{I}_{ij}}{r} + \dddot{I}_{ij}\right) - \frac{\delta_{ab}}{3}\dddot{I}_{ij}\right] \\
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\end{aligned}
$$

Fastest decreasing terms are in **same power of 1/r** as the GR solution that was injected in the source terms

Particular solution

$$
\begin{aligned}
\Box^{-1}\left[\Box \bar{h}^{LV}_{ij}\right] = & 2\bar{s}^{00}\hat{n}_{ij}\frac{M}{r} - \bar{s}^{ab}\left[\hat{n}_{ij}\left(\frac{\ddot{I}_{ab}}{r} + \dddot{I}_{ab}\right) + \frac{\delta_{ij}}{3}\dddot{I}_{ab}\right] \\
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\end{aligned}
$$

Elephant in the room : the **non-decreasing** terms
Do not seem to be a gauge artifact
No-go theorem for the SME coefficients in front of them

Gauge condition

Our metric correction $\bar{h}^{LV}_{\mu\nu}$ must respect **3 conditions** :

$$\Box \bar{h}^{LV}_{\mu\nu} = \Lambda_{\mu\nu} \quad \text{and} \quad \partial^{\mu} \bar{h}^{LV}_{\mu\nu} = 0 \quad \text{and} \quad \bar{h}^{LV}_{[\mu\nu]} = 0$$

This condition is guaranteed by our **particular solution**, it will always be respected as long as we only add **homogeneous solutions** to it

Gauge condition

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Our particular solution naturally respects this, **only symmetric homogeneous solution** are permissible

Gauge condition

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The particular solution **does not repect our gauge condition naturally**, we must impose it through our homogeneous solution

Gauge condition

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$$\downarrow$$

$$\Box v_{\mu\nu} = 0 \quad \text{and} \quad \partial^\mu v_{\mu\nu} = -\partial^\mu \Box^{-1} \left[\Lambda_{\mu\nu}\right] \quad \text{and} \quad v_{[\mu\nu]} = 0$$

Gauge condition

$$\begin{aligned}\Box\bar{h}^{LV}_{\mu\nu} =& -\bar{s}^{\alpha\beta}\left(\partial_\mu\partial_\nu\bar{h}^{GR}_{\alpha\beta} - \partial_\mu\partial_\beta\bar{h}^{GR}_{\alpha\nu} + \partial_\alpha\partial_\beta\bar{h}^{GR}_{\mu\nu} - \partial_\alpha\partial_\nu\bar{h}^{GR}_{\mu\beta}\right)\\ & -\frac{1}{2}\left(\bar{s}_\nu{}^\beta\partial_\mu\partial_\beta\bar{h}^{GR} - \bar{s}^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{GR} + \bar{s}^\alpha{}_\mu\partial_\alpha\partial_\nu\bar{h}^{GR}\right)\end{aligned}$$

Gauge condition

$$\Box \bar{h}^{LV}_{\mu\nu} = -\,\bar{s}^{\alpha\beta}\left(\partial_\mu\partial_\nu \bar{h}^{GR}_{\alpha\beta} - \partial_\mu\partial_\beta \bar{h}^{GR}_{\alpha\nu} \ldots\right.$$

$$\left(\partial_\mu\partial_\nu h^{GR}_{\alpha\beta} - \partial_\mu\partial_\beta h^{GR}_{\alpha\nu} \ldots\right.$$

$$-\frac{1}{2}\left(\bar{s}_\nu{}^\beta \partial_\mu\partial_\beta \bar{h}^{GR} - \bar{s}^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha\partial_\beta \bar{h}^{GR} + \bar{s}^\alpha{}_\mu \partial_\alpha\partial_\nu \bar{h}^{GR}\right)$$

We need to use the harmonic gauge condition on $\bar{h}^{GR}_{\mu\nu}$ to calculate the associated $v_{\mu\nu}$, but our model only contains the main terms

Gauge condition

$$\Box \bar{h}^{LV}_{\mu\nu} = -\,\bar{s}^{\alpha\beta}\left(\partial_\mu\partial_\nu \bar{h}^{GR}_{\alpha\beta} - \partial_\mu\partial_\beta \bar{h}^{GR}_{\alpha\nu} \ldots\right.$$

$$\left.-\frac{1}{2}\left(\bar{s}_\nu{}^\beta \partial_\mu\partial_\beta \bar{h}^{GR} - \bar{s}^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha\partial_\beta \bar{h}^{GR} + \bar{s}^{\alpha}{}_{\mu}\partial_\alpha\partial_\nu \bar{h}^{GR}\right)\right.$$

$$\left(\partial_\mu\partial_\nu h^{GR}_{\alpha\beta} - \partial_\mu\partial_\beta h^{GR}_{\alpha\nu}\right.$$

We need to use the harmonic gauge condition on $\bar{h}^{GR}_{\mu\nu}$ to calculate the associated $v_{\mu\nu}$, but our model only contains the main terms

By imposing a divergence-free model, we manage to calculate almost all $v_{\mu\nu}$

Full solution

$$
\begin{aligned}
\bar{h}^{LV}_{ij} &= 2\bar{s}^{00}\hat{n}_{ij}\frac{M}{r} - \bar{s}^{ab}\left[\hat{n}_{ij}\left(\frac{\ddot{I}_{ab}}{r} + \dddot{I}_{ab}\right) + \frac{\delta_{ij}}{3}\dddot{I}_{ab}\right] \\
&\quad - \bar{s}^{00}\dddot{I}_{ij} + 2\bar{s}^{0a}n_a\dddot{I}_{ij} - \bar{s}^{ab}\left[\hat{n}_{ab}\left(\frac{\ddot{I}_{ij}}{r} + \dddot{I}_{ij}\right) + \frac{\delta_{ab}}{3}\dddot{I}_{ij}\right] \\
&\quad - 2\bar{s}^{0a}n_{(j}\dddot{I}_{i)a} + 2\bar{s}^{ab}\left[\hat{n}_{a(j}\left(\frac{\ddot{I}_{i)b}}{r} + \dddot{I}_{i)b}\right) + \frac{\delta_{a(j}}{3}\dddot{I}_{i)b}\right] + \frac{2}{3}\bar{s}^{a}{}_{(j}\frac{\ddot{I}_{i)a}}{r} - \delta_{ij}\left(-4\bar{s}^{00}\frac{M}{r} + \frac{1}{3}\bar{s}^{ab}\frac{\ddot{I}_{ab}}{r}\right) \\
&\quad - \frac{1}{2}\left(\bar{s}_{(j}{}^{0}n_{i)}\dddot{I} - \xi\bar{s}_{(j}{}^{a}\left[\hat{n}_{i)a}\left(\frac{2M+\ddot{I}}{r} + \dddot{I}\right) + \frac{\delta_{i)a}}{3}\ddddot{I}\right]\right. \\
&\quad - \frac{\delta_{ij}}{2}\left[-\bar{s}^{00}\dddot{I} + 2\bar{s}^{0a}n_a\dddot{I} - \bar{s}^{ab}\left(\hat{n}_{ab}\left(\frac{2M+\ddot{I}}{r} + \dddot{I}\right) + \frac{\delta_{ab}}{3}\dddot{I}\right)\right] - \frac{1}{3}\left(\bar{s}_{ij} + \bar{s}^{00}\delta_{ij}\right)\frac{2M+\ddot{I}}{r}
\end{aligned}
$$

Observable : the Riemann tensor

$$
\begin{aligned}
2R_{0i0j} =& \frac{1}{7}\bar{s}^{00}\left[\hat{n}_{ij}\left(\frac{9}{2}{}^{(5)}I + \frac{61}{6}\frac{\ddddot{I}}{r} + 9\frac{\dddot{I}}{r^2} + 9\frac{\ddot{I}}{r^3} + 52\frac{M}{r^3}\right) - \frac{\delta_{ij}}{45}\left(27{}^{(5)}I + 14\frac{\ddddot{I}}{r}\right)\right] \\
& + 18\bar{s}^{00}\hat{n}_{ij}\frac{M}{r^3} + \frac{4}{3}\bar{s}^{00}\,{}^{(5)}I_{ij} \\
& - \bar{s}^{0a}\left[\hat{n}_{bij}\left(5\frac{\dddot{I}_{ab}}{r^2} + \frac{\ddddot{I}_{ab}}{r} + {}^{(5)}I_{ab}\right) + \frac{1}{5}\delta_{ij}n_b\left(2\frac{\dddot{I}_{ab}}{r^2} - 2\frac{\ddddot{I}_{ab}}{r} + {}^{(5)}I_{ab}\right)\right] \\
& + \bar{s}^{0a}\left[\frac{1}{2}\hat{n}_{aij}\left(9\frac{\dddot{I}}{r^2} - \frac{\ddddot{I}}{r} + {}^{(5)}I\right) - \frac{\delta_{ij}n_a}{5}\left(-5\frac{\dddot{I}}{r^2} + \frac{\ddddot{I}}{r} + 2{}^{(5)}I\right)\right] \\
& + \frac{1}{5}\bar{s}_{(j}{}^{0}n_{i)}\left({}^{(5)}I - 12\frac{\ddddot{I}}{r}\right) - 2\bar{s}^{0a}n_a\,{}^{(5)}I_{ij} + \frac{2}{5}\bar{s}^{a0}n_{(i}\left[3\frac{\dddot{I}_{j)a}}{r^2} + 7\frac{\ddddot{I}_{j)a}}{r} - {}^{(5)}I_{j)a}\right] \\
& + \bar{s}^{ab}\hat{n}_{abij}\left({}^{(5)}I + 2\frac{\ddddot{I}}{r} - 5\frac{\dddot{I}}{r^2} - 5\frac{\ddot{I}+2M}{r^3}\right) + \frac{4}{7}\bar{s}^{a}{}_{(i}\hat{n}_{j)b}\left({}^{(5)}I + 2\frac{\ddddot{I}}{r} + 9\frac{\dddot{I}}{r^2} + 9\frac{\ddot{I}+2M}{r^3}\right) \\
& - \frac{1}{42}\bar{s}^{ab}\hat{n}_{ab}\delta_{ij}\left(-13{}^{(5)}I + 30\frac{\ddddot{I}}{r} + 60\frac{\dddot{I}}{r^2} + 60\frac{\ddot{I}+2M}{r^3}\right) + \bar{s}^{ab}\hat{n}_{ab}\left(\frac{\ddddot{I}_{ij}}{r} + {}^{(5)}I_{ij}\right) \\
& + \frac{1}{15}\bar{s}_{ij}\left(2{}^{(5)}I + 19\frac{\ddddot{I}}{r}\right) - 2\bar{s}^{a}{}_{(i}\frac{\ddddot{I}_{j)a}}{r} \\
& - \left(-\frac{72}{7}\bar{s}^{a}{}_{(i}n_{j)} - \frac{2}{7}\bar{s}^{ab}\delta_{ij}\hat{n}_{ab} + \frac{12}{7}\bar{s}^{00}\hat{n}_{ij} - \frac{4}{5}\bar{s}_{ij} + \frac{4}{15}\bar{s}^{00}\right)\frac{M}{r^3}
\end{aligned}
$$

Polarisations

$$R_{0i0j} = (\delta_{ij} - n_i n_j)\, A + n_i n_j B + 2 n_{(i} C_{j)} + D_{ij}^{TT}$$

K. Schumacher, N. Yunes, K.Yagi, PRD 108 104038, 2023

Polarisations

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K. Schumacher, N. Yunes, K.Yagi, PRD 108 104038, 2023

$$n^i C_i = 0$$

$$n^i D_{ij}^{TT} = 0 \quad \text{and} \quad \delta^{ij} D_{ij}^{TT} = 0$$

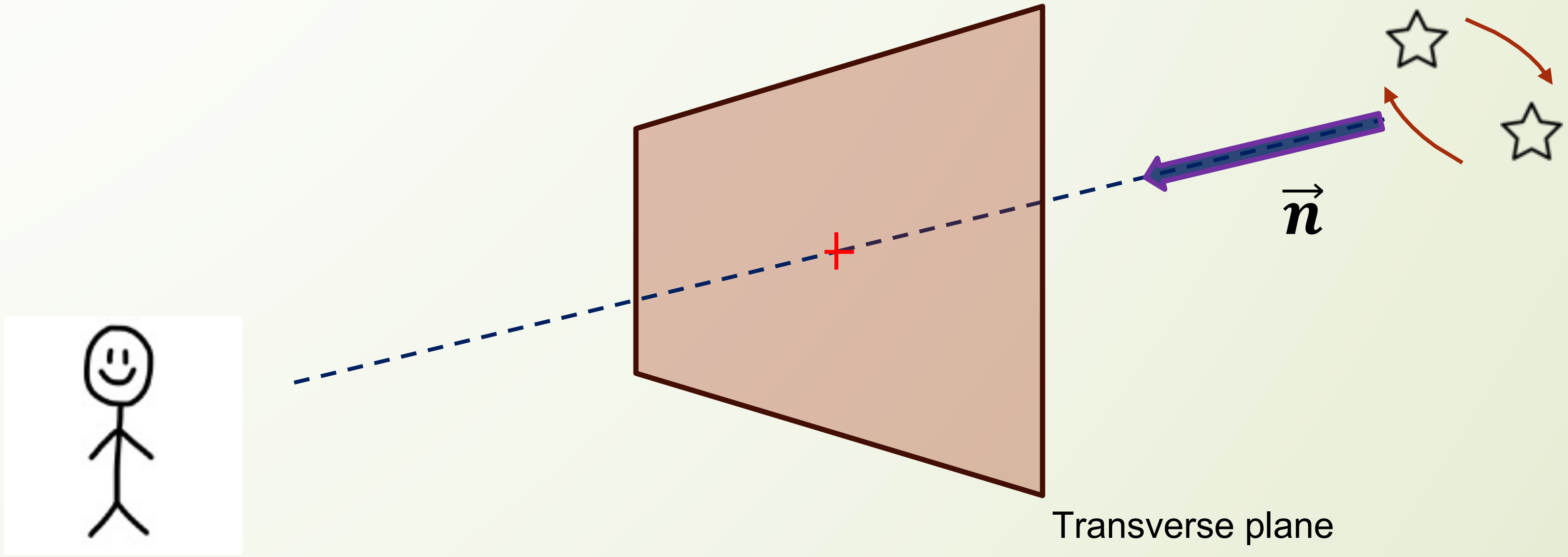
Polarisations

$$\boxed{R_{0i0j} = (\delta_{ij} - n_i n_j)\, A + n_i n_j B + 2n_{(i}C_{j)} + D_{ij}^{TT}}$$

K. Schumacher, N. Yunes, K.Yagi, PRD 108 104038, 2023

$$\boxed{n^i C_i = 0}$$

$$\boxed{n^i D_{ij}^{TT} = 0 \quad \text{and} \quad \delta^{ij} D_{ij}^{TT} = 0}$$



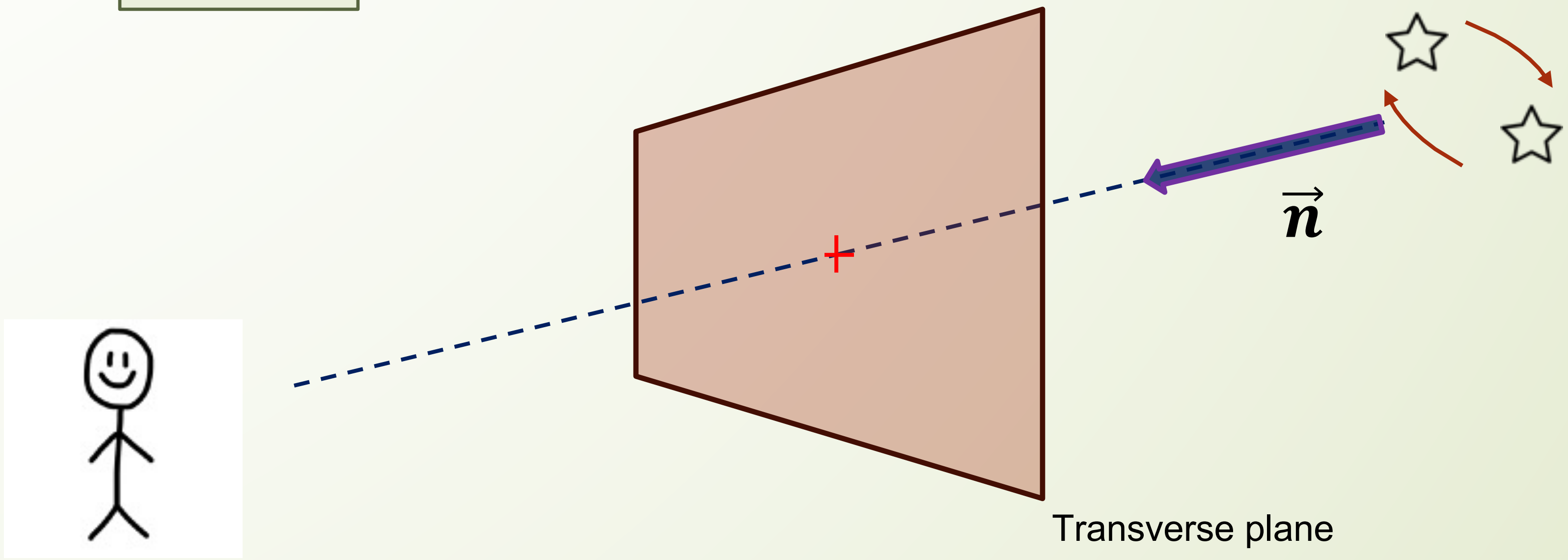
Polarisations

$$R_{0i0j} = (\delta_{ij} - n_i n_j)\, A + n_i n_j B + 2 n_{(i} C_{j)} + D_{ij}^{TT}$$

$$n^i C_i = 0$$

$$n^i D_{ij}^{TT} = 0 \quad \text{and} \quad \delta^{ij} D_{ij}^{TT} = 0$$

GR : ≠ 0



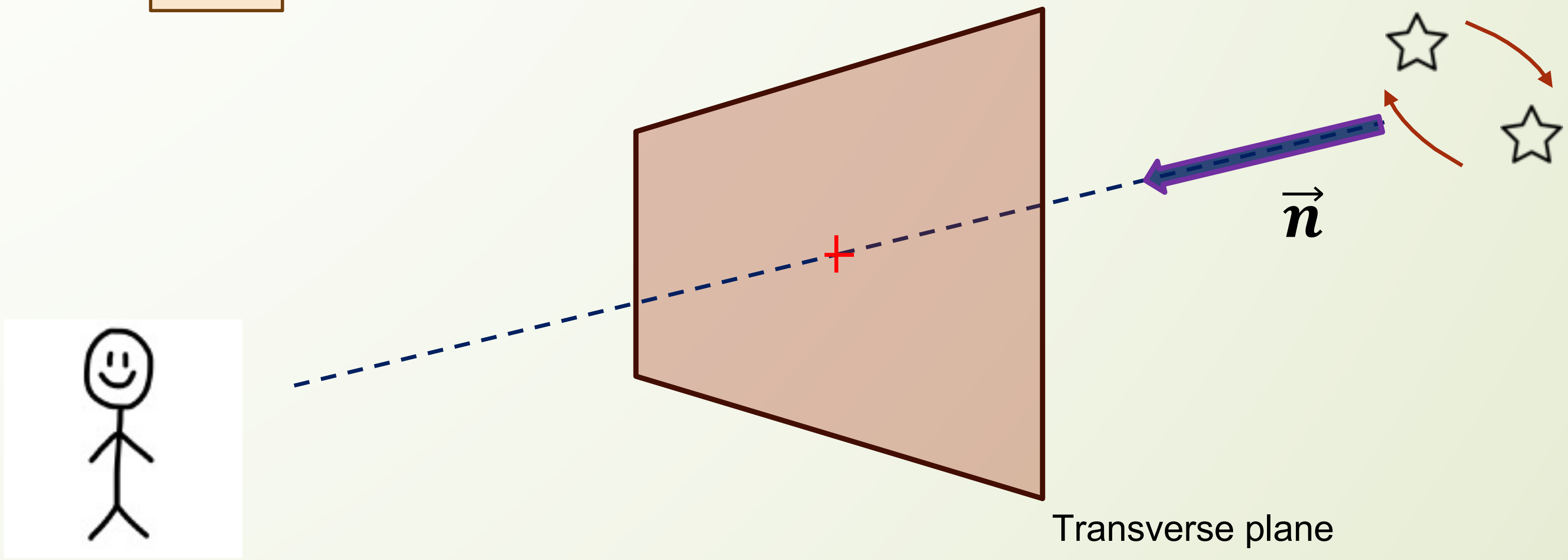
Polarisations

$$R_{0i0j} = (\delta_{ij} - n_i n_j)\, A + n_i n_j B + 2 n_{(i} C_{j)} + D^{TT}_{ij}$$

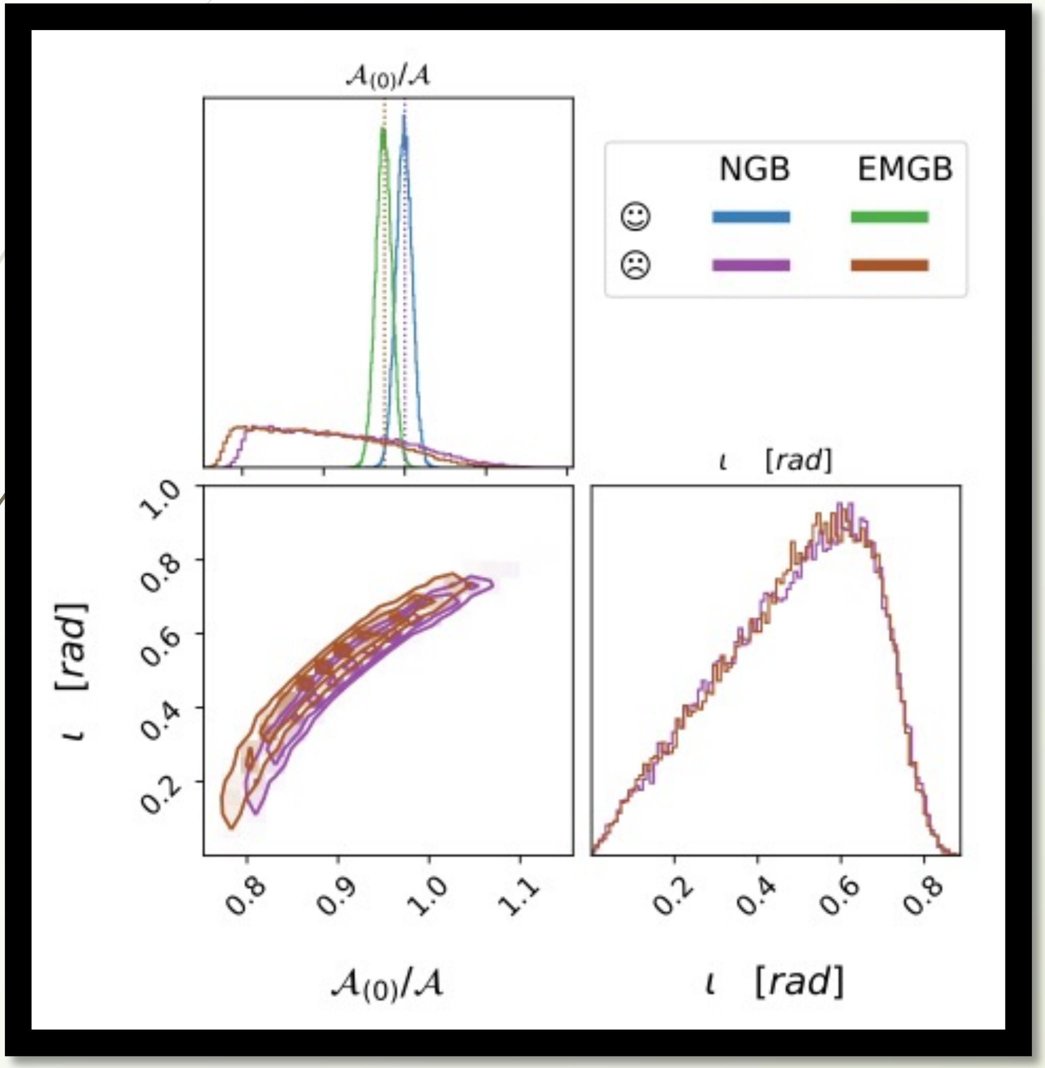
$\neq 0$

$$n^i C_i = 0$$

$$n^i D^{TT}_{ij} = 0 \quad \text{and} \quad \delta^{ij} D^{TT}_{ij} = 0$$



Futur : Bayesian Analysis



- Once the waveforms (and observables) have been verified
- Use a Bayesian analysis in order to perform a parameter estimation
- On the left : example of one such analysis for the electromagnetic properties of a compact object binary

E. Savalle, A. Bourgoin, C. Le Poncin-Lafitte, S. Mathis, M-C. Angonin, C. Aykroyd, PRD 109 083003, 2024

Conclusion

- LISA shapes up to be **very promising** for Lorentz violations probing
- **PM methods** very useful despite a complicated outlook
- Interesting **non-decreasing** terms in the full solutions
- New polarisations
- Code a **parameter estimator** for LISA Data for the SME coefficients
- Use different SME formalism with Einstein-Lifshitz formulation

Thank you for your attention

Annexe

Geodesic deviation with Riemann tensor

$$\boxed{\frac{d^2\xi_j}{dt^2} = -R_{0j0k}\xi^k}$$

K. Schumacher, N. Yunes, K.Yagi, PRD 108 104038, 2023

Full differential equations and solutions

$$\Box^{-1}\left[\Box \bar{h}^{LV}_{00}\right] = -\bar{s}^{ab}\dddot{I}_{ab} + 2\bar{s}^{ab}\hat{n}_{ab}\frac{M}{r} + \frac{1}{2}\bar{s}^{00}\dddot{I} - \frac{1}{2}\bar{s}^{ab}\left[\hat{n}_{ab}\left(\frac{\ddot{I}+2M}{r} + \dddot{I}\right) + \frac{\delta_{ab}}{3}\dddot{I}\right]$$

$$\begin{aligned}\Box^{-1}\left[\Box \bar{h}^{LV}_{0j}\right] =& 2\bar{s}^{ab}n_{[j}\dddot{I}_{a]b} + \bar{s}^{a0}\left(-2\hat{n}_{aj}\frac{M}{r} + \dddot{I}_{aj}\right)\\ &- \frac{1}{2}\bar{s}_j{}^{0}\dddot{I} + \frac{1}{2}\bar{s}_j{}^{a}n_a\dddot{I} - \frac{1}{2}\bar{s}^{00}n_j\dddot{I} + \frac{1}{2}\bar{s}^{0a}\left[\hat{n}_{aj}\left(\frac{\ddot{I}+2M}{r} + \dddot{I}\right) + \frac{\delta_{aj}}{3}\dddot{I}\right]\end{aligned}$$

Full differential equations and solutions

$$\bar{h}^{LV}_{00} = -\bar{s}^{ab}\dddot{I}_{ab} + 2\bar{s}^{ab}\hat{n}_{ab}\frac{M}{r} + \bar{s}^{ab}\frac{\ddot{I}_{ab}}{r} + \frac{1}{2}\bar{s}^{00}\dddot{I} - \frac{1}{2}\bar{s}^{ab}\left[\hat{n}_{ab}\left(\frac{\ddot{I}+2M}{r} + \dddot{I}\right) + \frac{\delta_{ab}}{3}\dddot{I}\right] - \frac{2}{3}\bar{s}^{00}\frac{\ddot{I}}{r}$$

$$\begin{aligned}\bar{h}^{LV}_{0j} =& 2\bar{s}^{ab}n_{[j}\dddot{I}_{a]b} + \bar{s}^{a0}\left(-2\hat{n}_{aj}\frac{M}{r} + \dddot{I}_{aj}\right) - \frac{8}{3}\bar{s}_j{}^0\frac{M}{r} - \bar{s}^{0a}\frac{\ddot{I}_{aj}}{r} \\ & - \frac{1}{2}\bar{s}_j{}^0\dddot{I} + \frac{1}{2}\bar{s}_j{}^a n_a\dddot{I} - \frac{1}{2}\bar{s}^{00}n_j\dddot{I} + \frac{1}{2}\bar{s}^{0a}\left[\hat{n}_{aj}\left(\frac{\ddot{I}+2M}{r} + \dddot{I}\right) + \frac{\delta_{aj}}{3}\dddot{I}\right] + \frac{2}{3}\xi\bar{s}^0{}_j\frac{\ddot{I}+2M}{r}\end{aligned}$$

The 5 assumptions of this SME model

- In asymptotically inertial Cartesian coordinates :
- The dominant effects are linear in the vacuum values
- There are no relevant couplings of the SME with matter
- The independently conserved piece of the trace-reversed energy-momentum tensor vanishes
- Linear combinations of twice derivated $h_{\mu\nu}$, $\eta_{\mu\nu}$, and the SME vacuum values are used to construct the undetermined fluctuation terms

$$
\begin{aligned}
u &= \overline{u} + \tilde{u}, \\
s^{\mu\nu} &= \overline{s}^{\mu\nu} + \tilde{s}^{\mu\nu}, \\
t^{\kappa\lambda\mu\nu} &= \overline{t}^{\kappa\lambda\mu\nu} + \tilde{t}^{\kappa\lambda\mu\nu}.
\end{aligned}
$$

$$
\begin{aligned}
\partial_\alpha \overline{u} &= 0, \\
\partial_\alpha \overline{s}^{\mu\nu} &= 0, \\
\partial_\alpha \overline{t}^{\kappa\lambda\mu\nu} &= 0.
\end{aligned}
$$

Q. G. Bailey and V. A. Kostelecký, PRD 74 045001 (2006)

Divergence-free source term

$$\Box\bar{h}^{LV}_{\mu\nu} = \Lambda_{\mu\nu} \quad \text{and} \quad \partial^\nu\Lambda_{\mu\nu} = 0$$

$$\begin{aligned}
&\Rightarrow \quad \Box^{-1}\left(\Box\bar{h}^{LV}_{\mu\nu}\right) = \Box^{-1}\Lambda_{\mu\nu} \\
&\Rightarrow \quad \partial^\nu\Box^{-1}\left(\Box\bar{h}^{LV}_{\mu\nu}\right) = \partial^\nu\Box^{-1}\Lambda_{\mu\nu} \\
&\Rightarrow \quad \Box\left(\partial^\nu\Box^{-1}\left(\Box\bar{h}^{LV}_{\mu\nu}\right)\right) = \Box\left(\partial^\nu\Box^{-1}\Lambda_{\mu\nu}\right) = \partial^\nu\left(\Box\Box^{-1}\Lambda_{\mu\nu}\right) = \partial^\nu\Lambda_{\mu\nu} = 0 \\
&\Rightarrow \quad \Box\left(\partial^\nu\Box^{-1}\left(\Lambda_{\mu\nu}\right)\right) = 0
\end{aligned}$$

Full Einstein equation

$$G^{\mu\nu} - (T^{Rstu})^{\mu\nu} = \kappa (T_g)^{\mu\nu}, \tag{3.5}$$

where

$$\begin{aligned}(T^{Rstu})^{\mu\nu} \equiv\ & -\tfrac{1}{2} D^\mu D^\nu u - \tfrac{1}{2} D^\nu D^\mu u + g^{\mu\nu} D^2 u + u G^{\mu\nu} + \tfrac{1}{2} s^{\alpha\beta} R_{\alpha\beta} g^{\mu\nu} \\ & + \tfrac{1}{2} D_\alpha D^\mu s^{\alpha\nu} + \tfrac{1}{2} D_\alpha D^\nu s^{\alpha\mu} - \tfrac{1}{2} D^2 s^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} D_\alpha D_\beta s^{\alpha\beta} \\ & + \tfrac{1}{2} t^{\alpha\beta\gamma\mu} R_{\alpha\beta\gamma}{}^{\nu} + \tfrac{1}{2} t^{\alpha\beta\gamma\nu} R_{\alpha\beta\gamma}{}^{\mu} + \tfrac{1}{2} t^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} g^{\mu\nu} \\ & - D_\alpha D_\beta t^{\mu\alpha\nu\beta} - D_\alpha D_\beta t^{\nu\alpha\mu\beta}. \end{aligned} \tag{3.6}$$

Q. G. Bailey and V. A. Kostelecký, PRD 74 045001 (2006)

Precise definitions

$$\boxed{M = \int T_{00}\, d\vec{x}}$$

$$\boxed{I^{ij} = \int T_{00} x^i x^j \, d\vec{x}}$$

Examples of theories that break Lorentz invariance : Hořava-Lifshitz