Gravitational radiation reaction at the fourth-and-a-half post-Newtonian order

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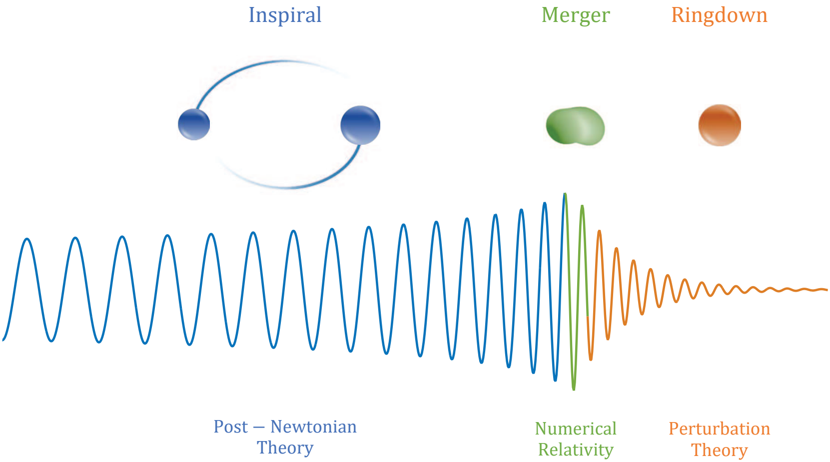
Introduction

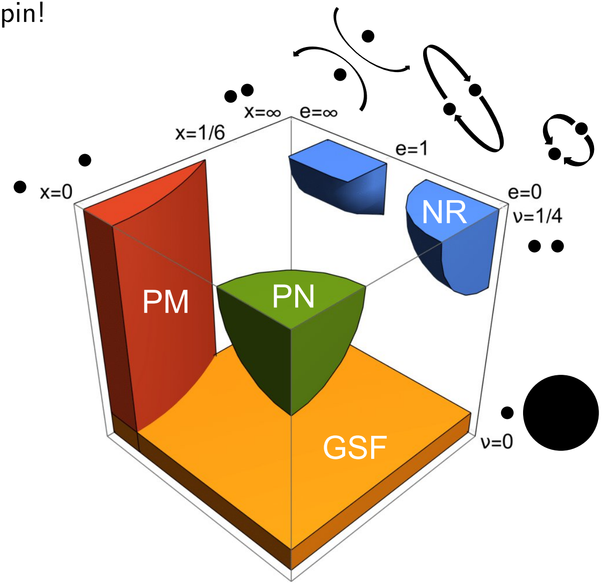
The three stages of a binary



[Antelis & Moreno (2017), arXiv:1610.03567]

Different techniques for different regions of parameter space

Ignoring spin!



Post-Newtonian results: what are they used for?

Post-Newtonian dynamics and waveforms are used:

- alone (in time or frequency domain)
- resummed (e.g. Padé resummations)
- inform EOB models (SEOB and TEOB)
- enter phenomenological waveform models (IMRPhenom)
- hybridized with NR
- hybridized with GSF

Advantages:

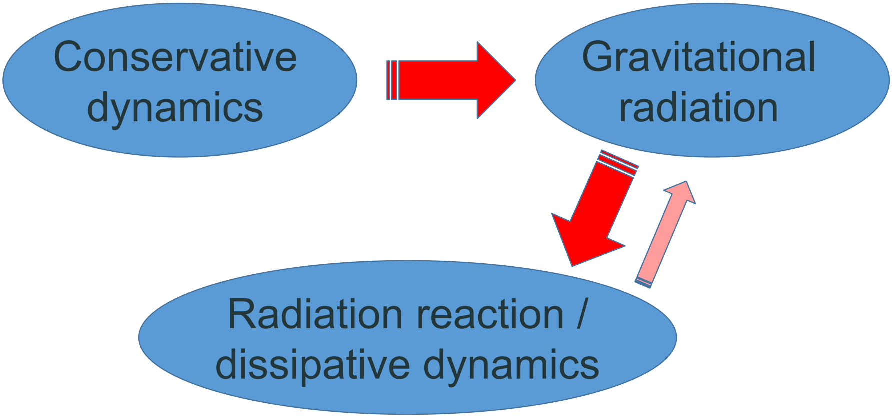
- first-principle method
- fully analytical
- fast to evaluate
- helps understand physics

Disadvantages:

- only valid in inspiral phase
- slow and oscillating convergence
- degrades for high eccentricity
- degrades for high mass-ratios

Review of recent post-Newtonian results

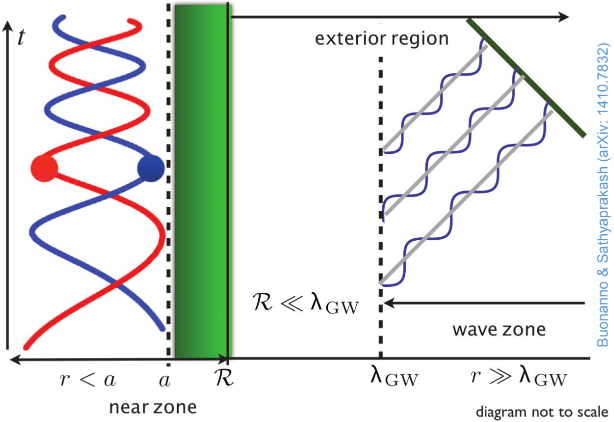
The three sectors of a PN computation



$$\boldsymbol{a} = \boldsymbol{a}_{\mathrm{N}} + \boldsymbol{a}_{1\mathrm{PN}} + \boldsymbol{a}_{2\mathrm{PN}} + \boldsymbol{a}_{2.5\mathrm{PN}} + \dots$$

$$F = F_{\mathrm{N}} + F_{1\mathrm{PN}} + F_{2\mathrm{PN}} + F_{2.5\mathrm{PN}} + \dots$$

Relating near-zone and exterior vacuum zone



- In NZ, obtain PN expansion of metric [up to homogeneous solution]
- In FZ, obtain PM expansion of metric [up to homogeneous solution]
- Both homogeneous solutions obtained by imposing asymptotic matching in buffer zone

Conservative dynamics (nonspinning)

For now, we **ignore** gravitational radiation and its backreaction

$\Longrightarrow$ focus only on time-even contributions

Dynamics described either by the acceleration or the Hamiltonian

How? / Order	Fokker	ADM	EFT
3PN	[ItFu '03][It '04] [BcDaEs '04]	[DaJaSc '01]	[FoSt '11]
4PN	[BeBcBoFaMs '17ab]	[DaJaSc '14] [DaJaSc '15]	[FoPoRoSt '19] [BüMiMqSc '20]
5PN (disputed)		[BiDaGe '20] (partial)	[BüMiMqSc '22]
6PN (partial)		[BiDaGe '20]	[BüMrMqSc '22]

Be = Bernard, Bi = Bini, Bc = Blanchet, Bo = Bohé, Bü = Blümlein, Da = Damour, Es = Esposito-Farèse, Fa = Faye, Fu = Futamase, Fo = Foffa, Iy = Itoh, Ja = Jaranowski, Mc = Marchand, Ms = Marsat, Mr = Maier, Mq = Marquard, Po = Porto, Ro = Rothstein, Sch = Schäfer, St= Sturani

Radiation: energy and angular momentum fluxes (nonspinning)

Order \ Orbit	Circular	Elliptic
2PN	[BcDaIy '95] [WiWs '96] [LbMiYa '19]	[GoIy '97]
3PN	[AmYaPo '24]	[ArBcIyQu '08ab] [ArBcIySh '09]
3.5PN	[BcFaIyJo '05]	
4PN	[BcFaHeLaTr '23]	
4.5PN	[McBcFa '16]	

Am = Amalberti, Bc = Blanchet, Da = Damour, Fa = Faye, Go = Gopakumar, He = Henry, Iy = Iyer, Jo = Joguet, La = Larrouturou, Lb = Leibovich, Ma = Maia, Mc = Marchand, Ms = Marsat, Po = Porto, Tr = Trestini, Wi = Will, Ws = Wiseman, Ya = Yang,

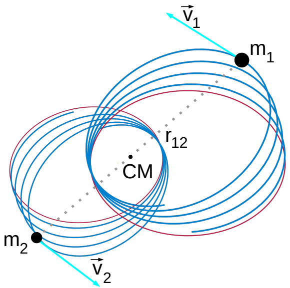
Radiation reaction: dissipative EOM (nonspinning)

Method / Order	Balance-laws	Matching	
Coordinates	Parametrized	Harmonic	Burke-Thorne
3.5PN	[IyWi '95]	[PtWi '02] [NiBc '05]	[Bc '93][Bc '97] [IyWi '95]
4PN (tails)		[BcDa '88]	
4.5PN	[GoIyIy '98]	[LbPrYa '23]	[BcFaTr '24]

Bc = Blanchet, Fa = Faye, Go = Gopakumar, Iy = Iyer, Lb = Leibovich, Ni = Nissanke, Pd = Pardo, Pt = Pati, Tr = Trestini, Wi = Will, Ya = Yang,

Equations of motion at 4.5PN

What are we computing exactly?



$$\frac{\mathrm{d}v_{12}^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \ldots\right]n_{12}^i + \ldots\right\}}^{\text{1PN}} + \overbrace{\frac{1}{c^4}\left[\ldots\right]}^{\text{2PN}}$$

$$+ \underbrace{\frac{1}{c^5}\left[\ldots\right]}_{\substack{\text{2.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^6}\left[\ldots\right]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}\left[\ldots\right]}_{\substack{\text{2.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^8}\left[\ldots\right]}_{\text{4PN}} + \underbrace{\frac{1}{c^9}\left[\ldots\right]}_{\substack{\text{2.5PN}\\\text{radiation reaction}}} + \mathcal{O}\left(\frac{1}{c^{10}}\right)$$

Metric

The metric reads

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left[\hat{X} + V_i V_i + \frac{1}{6}V^3\right] + \mathcal{O}(8,13)\,,$$
$$g_{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i + \mathcal{O}(7,12)\,,$$
$$g_{ij} = \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2\right] + \frac{4}{c^4}\hat{W}_{ij} + \mathcal{O}(6,11)$$

where potentials divided into *symmetric* and *radiation reaction* parts

$$V = V_{\mathrm{sym}} + V_{\mathrm{RR}}\,, \qquad V_i = V^i_{\mathrm{sym}} + V^i_{\mathrm{RR}}\,, \qquad \hat{W}_{ij} = \hat{W}^{ij}_{\mathrm{sym}} + V^{ij}_{\mathrm{RR}}$$

Usually, we use retarded operator $\widetilde{\Box}^{-1}_{\mathrm{ret}}$ (conservative + dissipative effects). Here, RR piece is obtained by matching to the exterior metric, and other piece defined using the symmetric operator:

$$\widetilde{\Box}^{-1}_{\mathrm{sym}}\left[\overline{\tau}^{\alpha\beta}\right] \equiv \underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\mathsf{sym}}\left[\widetilde{r}^B\overline{\tau}^{\alpha\beta}\right] = \sum_{k=0}^{+\infty}\left(\frac{\partial}{c\partial t}\right)^{2k}\underset{B=0}{\mathrm{FP}}\,\Delta^{-k-1}\left[\widetilde{r}^B\overline{\tau}^{\alpha\beta}\right]$$

Time-antisymmetric piece of metric

The matching procedure tells us that the RR piece of the inner metric = time-antisymmetric piece of exterior metric (regular in the source!)

$$h^{00}_{\mathrm{RR}\,1} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\big\{\mathrm{M}_L\big\}\,,$$

$$h^{0i}_{\mathrm{RR}\,1} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left[\partial_{L-1}\big\{\mathrm{M}^{(1)}_{iL-1}\big\} + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\big\{\mathrm{S}_{bL-1}\big\}\right]\,,$$

$$h^{ij}_{\mathrm{RR}\,1} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left[\partial_{L-2}\big\{\mathrm{M}^{(2)}_{ijL-2}\big\} + \frac{2\ell}{\ell+1}\partial_{aL-2}\big\{\varepsilon_{ab(i}\mathrm{S}^{(1)}_{j)bL-2}\big\}\right]\,,$$

where

$$\big\{f\big\}(t,r) \equiv \frac{f(t-r/c)-f(t+r/c)}{2r} = -\frac{f'(t)}{c} + \mathcal{O}\left(\frac{r^2}{c^3}\right)$$

Generalized Burke-Thorne gauge

The previous metric has the structure:

$$h^{00}_{\mathrm{RR}\,1} = \mathcal{O}(c^{-2}), \qquad h^{0i}_{\mathrm{RR}\,1} = \mathcal{O}(c^{-3}), \qquad h^{ij}_{\mathrm{RR}\,1} = \mathcal{O}(c^{-4})$$

Generalized Burke-Thorne gauge transform: $h'^{\mu\nu}_{\mathrm{RR}\,1} = h^{\mu\nu}_{\mathrm{RR}\,1} + (\partial\xi_1)^{\mu\nu}$
[Blanchet, PRD 47, 4392 (1993)]

Obtain the structure:

$$h'^{00}_{\mathrm{RR}\,1} = \mathcal{O}(c^{-7}), \qquad h'^{0i}_{\mathrm{RR}\,1} = \mathcal{O}(c^{-6}), \qquad h'^{ij}_{\mathrm{RR}\,1} = \mathcal{O}(c^{-5})$$

Radiation reaction corrections enter at much higher PN order in this gauge! Thus, define the RR potentials:

$$h'^{00}_{\mathrm{RR}\,1} = -\frac{4}{Gc^2}V_{\mathrm{RR}}, \quad h'^{0i}_{\mathrm{RR}\,1} = -\frac{4}{Gc^3}V^i_{\mathrm{RR}}, \quad h'^{ij}_{\mathrm{RR}\,1} = -\frac{4}{Gc^4}V^{ij}_{\mathrm{RR}}$$

Radiation reaction potentials in terms of $(\mathrm{M}_L, \mathrm{S}_L)$

$$
\begin{aligned}
V_{\mathrm{RR}} = &-\frac{G}{5c^5} x^{ab} \mathrm{M}^{(5)}_{ab} + \frac{G}{c^7}\left[\frac{1}{189} x^{abc}\mathrm{M}^{(7)}_{abc} - \frac{1}{70} r^2 x^{ab} \mathrm{M}^{(7)}_{ab}\right] \\
&+ \frac{G}{c^9}\left[-\frac{1}{9072} x^{abcd}\mathrm{M}^{(9)}_{abcd} + \frac{1}{3402} r^2 x^{abc}\mathrm{M}^{(9)}_{abc} \right.\\
&\qquad \left. - \frac{1}{2520} r^4 x^{ab}\mathrm{M}^{(9)}_{ab}\right] + \mathcal{O}\left(\frac{1}{c^{11}}\right) \\
V^i_{\mathrm{RR}} = &\frac{G}{c^5}\left[\frac{1}{21}\hat{x}^{iab}\mathrm{M}^{(6)}_{ab} - \frac{4}{45}\varepsilon_{iab} x^{ac}\mathrm{S}^{(5)}_{bc}\right] \\
&+ \frac{G}{c^7}\left[-\frac{1}{972}\hat{x}^{iabc}\mathrm{M}^{(8)}_{abc} + \frac{1}{378} r^2 \hat{x}^{iab}\mathrm{M}^{(8)}_{ab} \right.\\
&\qquad \left. + \frac{1}{336}\varepsilon_{iab}\hat{x}^{acd}\mathrm{S}^{(7)}_{bcd} - \frac{2}{315}\varepsilon_{iab} r^2 \hat{x}^{ac}\mathrm{S}^{(7)}_{bc}\right] + \mathcal{O}\left(\frac{1}{c^9}\right) \\
V^{ij}_{\mathrm{RR}} = &\frac{G}{c^5}\left[-\frac{1}{108}\hat{x}^{ijab}\mathrm{M}^{(7)}_{ab} + \frac{2}{63}\varepsilon_{ab(i}\hat{x}^{j)ac}\mathrm{S}^{(6)}_{bc}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)
\end{aligned}
$$

We know the $(\mathrm{M}_L, \mathrm{S}_L)$ at this order, so this is very easy to compute

Symmetric potentials

The symmetric potentials are defined as

$$\begin{aligned}
V_{\text{sym}} &= \widetilde{\Box}^{-1}_{\text{sym}}\Big[-4\pi G\sigma\Big]\,,\\
V^i_{\text{sym}} &= \widetilde{\Box}^{-1}_{\text{sym}}\Big[-4\pi G\sigma_i\Big]\,,\\
\hat{W}^{ij}_{\text{sym}} &= \widetilde{\Box}^{-1}_{\text{sym}}\Big[-4\pi G\big(\sigma_{ij}-\delta_{ij}\sigma_{kk}\big)-\partial_i V\partial_j V\Big]\,,\\
\hat{R}^i_{\text{sym}} &= \widetilde{\Box}^{-1}_{\text{sym}}\Big[-4\pi G\big(V\sigma_i - V_i\sigma\big)-2\partial_k V\partial_i V_k - \frac{3}{2}\partial_t V\partial_i V\Big]\,,\\
\hat{X}_{\text{sym}} &= \widetilde{\Box}^{-1}_{\text{sym}}\Big[-4\pi G V\sigma_{kk}+\hat{W}_{ij}\partial_{ij}V+2V_i\partial_t\partial_i V+V\partial_t^2 V\\
&\qquad\qquad + \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j\partial_j V_i\Big]\,.
\end{aligned}$$

They involve time derivatives (so the RR acceleration appears), and they are sourced by $V = V_{\text{sym}} + V_{\text{RR}}$, etc., **so they also contribute to the RR !**

This contribution can be expressed in terms of $(\mathrm{M}_L, \mathrm{S}_L)$.

Acceleration in terms of potentials

The equations of motion are obtained by writing $\nabla_\beta T^{\alpha\beta} = 0$ where

$$T^{\mu\nu} = \frac{m_1 v_1^\mu v_1^\nu \delta(\mathbf{x}-\mathbf{y}_1)}{\sqrt{-(g)_1}\sqrt{-(g_{\alpha\beta})_1 v_1^\mu v_1^\nu / c^2}} + \frac{m_2 v_2^\mu v_2^\nu \delta(\mathbf{x}-\mathbf{y}_2)}{\sqrt{-(g)_2}\sqrt{-(g_{\alpha\beta})_2 v_2^\mu v_2^\nu / c^2}}$$

Metric evaluated on each particle with Hadamard regularization.

In terms of potentials, reads

$$\begin{aligned} a_1^i = \Bigg(& \partial_i V \;+ \frac{1}{c^2}\Big[(v_1^2 - 4V)\partial_i V + 4\partial_t V_i - 8 v^j \partial_{[i} V_{j]} - 3 v_1^i \partial_t V - 4 v_1^i v_1^j \partial_j V\Big] \\ & + \frac{1}{c^4}\Big[4 v_1^i V_j \partial_j V + 4 v_1^i v_1^j v_1^k \partial_j V_k + 8 v_1^j V_i \partial_j V + 8 \partial_t \hat{R}^i + v_1^i v_1^2 \partial_t V + 4 V_i \partial_t V \\ & - 8 V \partial_t V_i - 4 v_1^j \partial_t \hat{W}_{ij} + 8 v_1^j \partial_j \hat{R}_i - 8 V v_1^j \partial_j V_i - 4 \hat{W}_{ij} \partial_j V - 4 v_1^j v_1^k \partial_k \hat{W}_{ij} \\ & - 8 v_1^j \partial_i \hat{R}_j + 8 V^2 \partial_i V + 8 V v_1^j \partial_i V_j + 8 V_j \partial_i V_j + 2 v_1^j v_1^k \partial_i \hat{W}_{jk} + 4 \partial_t \partial_i \hat{X}\Big]\Bigg)_1 + \mathcal{O}(6, 11) \end{aligned}$$

RR contributions come from both piece of $V = V_{\text{sym}} + V_{\text{RR}}$, etc.

Acceleration in terms of multipolar moments

We obtain the acceleration in terms of the multipolar moments at 4.5PN
[Blanchet, Faye & DT (2024), 2407.18295]

$$a^i_{2.5\text{PN}\,1} = -\frac{2G}{5c^5} y_1^a \mathrm{M}^{(5)}_{ia}$$

$$\begin{aligned}
a^i_{3.5\text{PN}\,1} = \frac{G}{c^7}\Big\{ & -\frac{11}{105} y_1^b \mathrm{M}^{(7)}_{ib} y_1^2 + \frac{17}{105} y_1^{iab} \mathrm{M}^{(7)}_{ab} - \frac{8}{15} y_1^b \mathrm{M}^{(6)}_{ib} (v_1 y_1) \\
& + \mathrm{M}^{(6)}_{ab} \left(\frac{8}{15} y_1^{bi} v_1^a + \frac{3}{5} v_1^i y_1^{ab} \right) - \frac{2}{5} y_1^b \mathrm{M}^{(5)}_{ib} v_1^2 \\
& + \frac{G \mathrm{M}^{(5)}_{ia}}{r_{12}} \left(\frac{7}{5} m_2 n_{12}^a r_{12} + \frac{1}{5} m_2 y_1^a \right) \\
& + \mathrm{M}^{(5)}_{ab} \left[\frac{8}{5} v_1^{bi} y_1^a + \frac{G}{r_{12}} \left(\frac{1}{5} n_{12}^{bi} m_2 y_1^a - \frac{m_2 n_{12}^i}{r_{12}} y_1^{ab} \right) \right] \\
& + \frac{1}{63} \mathrm{M}^{(7)}_{iab} y_1^{ab} - \frac{16}{45} \varepsilon_{ibj} \mathrm{S}^{(6)}_{aj} y_1^{ab} - \frac{16}{45} \varepsilon_{ibj} v_1^a y_1^b \mathrm{S}^{(5)}_{aj} \\
& - \frac{32}{45} \varepsilon_{iaj} v_1^a y_1^b \mathrm{S}^{(5)}_{bj} + \frac{16}{45} \varepsilon_{abj} v_1^a y_1^b \mathrm{S}^{(5)}_{ij} \Big\}
\end{aligned}$$

$$a^i_{4.5\text{PN}\,1} = \text{(very long !)}$$

We can also replace the multipolar moments and get the acceleration in terms of $(\mathbf{y}_1, \mathbf{y}_2, \mathbf{v}_1, \mathbf{v}_2)$

Flux balance laws

We know with 4PN accuracy the four Poincaré invariants $E_{\rm cons}$, $J^i_{\rm cons}$, $P^i_{\rm cons}$, and $G^i_{\rm cons}$ which are conserved by the conservative dynamics.

We also know their associated fluxes at infinity, $\mathcal{F}_E$, $\mathcal{F}^i_J$, $\mathcal{F}^i_P$, $\mathcal{F}^i_G$

We have explicitly proven all four balance laws with 2PN accuracy:
[Blanchet, Faye & DT (2024), 2407.18295]

$$\frac{\mathrm{d}}{\mathrm{d}t}[H_{\rm cons} + H_{\rm RR}] = -\mathcal{F}_H$$

where

- H stands generically for E, J^i, P^i, and G^i
- $H_{\rm RR}$ is a Schott term which we control
- the time derivative is taken with our newly computed 4.5PN acceleration

Defining the center-of-mass frame

Integrating the flux balance equations yields

$$P^i(t) = P_0^i - \int_{t_0}^{t} \mathrm{d}t' \mathcal{F}_{\boldsymbol{P}}(t')$$

$$G^i(t) = G_0^i + P_0^i(t - t_0) - \int_{t_0}^{t} \mathrm{d}t' \mathcal{F}_{\boldsymbol{G}}(t') - \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' \mathcal{F}_{\boldsymbol{P}}(t'')$$

where $t_0 =$ initial time, before emission of GWs

Apply Lorentz boost ⇒ rest frame of initial system: $P_0^i = 0$ and $G_0^i = 0$

Send $t_0 \to -\infty$. The conditions to be in the CM frame are:

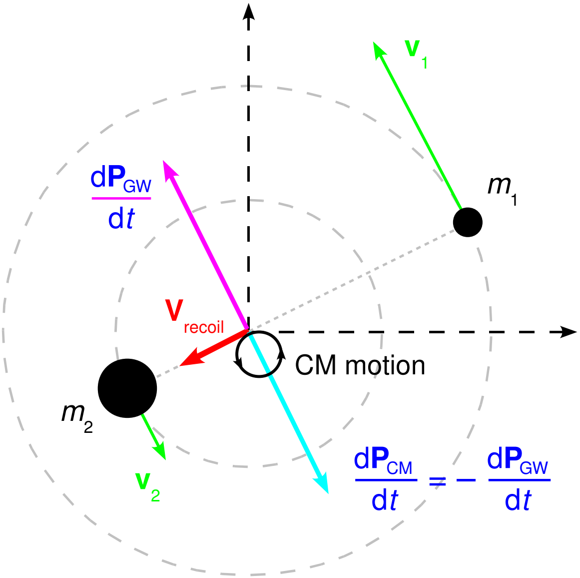
$$G^i(t) + \Gamma^i(t) = 0 \quad \Longrightarrow \quad P^i(t) + \Pi^i(t) = 0$$

where

$$\Pi^i(t) = \int_{-\infty}^{t} \mathrm{d}t' \, \mathcal{F}_{\boldsymbol{P}}(t')$$

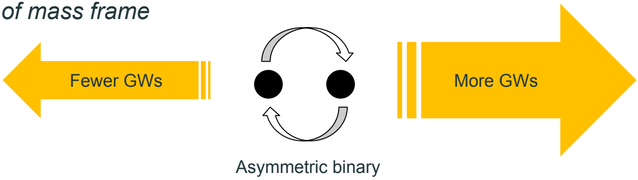
$$\Gamma^i(t) = \int_{-\infty}^{t} \mathrm{d}t' \, \mathcal{F}_{\boldsymbol{G}}(t') + \int_{-\infty}^{t} \mathrm{d}t' \, \Pi^i(t')$$

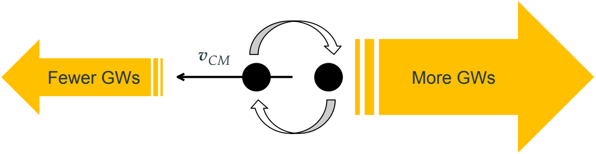
Gravitational recoin: circular orbits



Gravitational recoil: secular effect for eccentric orbits

In the center of mass frame





Gravitational recoil

GWs that escaped to infinity contribute to the center-of-mass frame

Passage to the CM frame

Solving iteratively for the y_1^i in $G^i + \Gamma^i = 0$, we find

$$y_1^i = \underbrace{x^i\Big(X_2 + \nu\Delta\,\mathcal{P}\Big) + \nu\Delta\,\mathcal{Q}\,v^i}_{\substack{\text{matter contribution} \\ \text{obtained by solving for } y_1^i \text{ in } G^i = 0}} + \underbrace{\mathcal{R}^i}_{\text{radiation contribution}}$$

where [Blanchet, Faye & DT (2024), 2407.18295]

$$\mathcal{R}^i = \underbrace{-\frac{\Gamma^i}{m}}_{\text{3.5PN}} + \underbrace{\frac{\nu}{mc^2}\left[\left(\frac{v^2}{2} - \frac{Gm}{r}\right)\Gamma^i + v^j\Big(\Pi^j + \mathcal{F}_G^j\Big)x^i\right]}_{\text{4.5PN}} + \mathcal{O}(11)$$

Why previous results are unaffected

This new nonlocal term enters at 3.5PN! Does it affect previous results ?

- The EOM are **not** affected at 3.5PN:

$$a_{12}^i = \frac{Gm}{r_{12}} n_{12}^i + \text{(higher order terms)}$$

$$\underset{\text{go to CM frame}}{\Longrightarrow} \qquad a^i = \frac{Gm}{r} n^i + 0 + \text{(higher order terms)}$$

  But this affects 4.5PN EOM of [Leibovich, Pardo & Yang (2023), 2302.11016]

- The 3.5PN flux is **not** affected because of the structure of the quadrupole moment:

$$\mathrm{I}_{ij} = m_1 y_1^{\langle i} y_1^{j\rangle} + m_2 y_2^{\langle i} y_2^{j\rangle} + \text{(higher order terms)}$$

$$\underset{\text{go to CM frame}}{\Longrightarrow} \qquad (\mathrm{I}_{ij})_{\mathrm{CM}} = m\nu r^2 n^{\langle i} n^{j\rangle} + 0 + \text{(higher order terms)}$$

  This is not true for other moments!

The equations of motion in the CM frame

In the CM frame, we find [Blanchet, Faye & DT (2024), 2407.18295]

$$a^i_{\text{RR}} = a^i_{\text{2.5PN}} + a^i_{\text{3.5PN}} + a^i_{\text{4.5PN}}\Big|_{\text{mat}} + a^i_{\text{4.5PN}}\Big|_{\text{rad}}\,.$$

where

$$a^i_{\text{2.5PN}} = \frac{8G^2m^2\nu}{c^5r^3}\left[v^i\Big(\frac{2Gm}{5r} + 3\dot{r}^2 - \frac{6}{5}v^2\Big) + n^i\dot{r}\Big(\frac{2Gm}{15r} - 5\dot{r}^2 + \frac{18}{5}v^2\Big)\right]$$

$$\begin{aligned}a^i_{\text{3.5PN}} = \frac{8G^2m^2\nu}{c^7r^3}\Bigg[&v^i\Bigg(\Big(-\frac{776}{105} - \frac{11}{3}\nu\Big)\frac{G^2m^2}{r^2} + \Big(\frac{5}{2} - \frac{35}{2}\nu\Big)\dot{r}^4 + \Big(-\frac{39}{10} + \frac{111}{10}\nu\Big)\dot{r}^2v^2\\
&+ \frac{Gm}{r}\left[\Big(-\frac{2591}{60} - \frac{97}{5}\nu\Big)\dot{r}^2 + \Big(\frac{4861}{420} + \frac{58}{15}\nu\Big)v^2\right] + \frac{27}{70}v^4\Bigg)\\
&+ n^i\dot{r}\Bigg(\Big(\frac{32}{7} + \frac{11}{3}\nu\Big)\frac{G^2m^2}{r^2} + \Big(-\frac{7}{2} + \frac{7}{2}\nu\Big)\dot{r}^4 + \Big(\frac{5}{2} + \frac{25}{2}\nu\Big)\dot{r}^2v^2\\
&+ \frac{Gm}{r}\left[\Big(\frac{1353}{20} + \frac{133}{5}\nu\Big)\dot{r}^2 + \Big(-\frac{5379}{140} - \frac{136}{15}\nu\Big)v^2\right] + \Big(\frac{87}{70} - \frac{48}{5}\nu\Big)v^4\Bigg)\Bigg]\end{aligned}$$

$$a^i_{\text{4.5PN}}\Big|_{\text{mat}} = (\ldots\ldots)$$

and

$$a^i_{\text{4.5PN}}\Big|_{\text{rad}} = \frac{G\Delta}{r^2c^2}\left(2n^iv^j + n^jv^i\right)\left[\Pi^j + \mathcal{F}^j_{\boldsymbol{G}}\right].$$

Balance equations in the CM frame

Express 2PN conserved quantities, E and J^i, in the CM frame:

$$E = E\Big|_{\text{mat}} + E\Big|_{\text{rad}} \qquad\qquad J^i = J^i\Big|_{\text{mat}} + J^i\Big|_{\text{rad}}$$

where

$$E\Big|_{\text{rad}} = \frac{\nu\Delta}{c^2} v^2 v^i \Big[\Pi^i + \mathcal{F}^i_{\boldsymbol{G}}\Big] \qquad\qquad E\Big|_{\text{mat}} = (.....)$$

$$J^i\Big|_{\text{rad}} = \frac{\nu\Delta}{c^2} \varepsilon_{ijk} x^j v^k v^q \Big[\Pi^l + \mathcal{F}^l_{\boldsymbol{G}}\Big] \qquad\qquad J^i\Big|_{\text{mat}} = (....)$$

We check explicitly that:

$$\frac{\mathrm{d}E_{\text{CM}}}{\mathrm{d}t}\Big|_{a^i_{\text{CM}}} = -\left(\mathcal{F}_E\right)_{\text{CM}} \qquad\qquad \frac{\mathrm{d}J^i_{\text{CM}}}{\mathrm{d}t}\Big|_{a^i_{\text{CM}}} = -\left(\mathcal{F}_{\boldsymbol{J}}\right)_{\text{CM}}$$

Important consistency check ! If we had we ignored the radiation contribution (i.e. set $\mathcal{R}^i = 0$ like [Leibovich, Pardo & Yang (2023), 2302.11016]), then we would not recover this balance equation in the CM frame!

Link with Gopakumar, Iyer and Iyer (GII) 1998

In [Gopakumar, Iyer & Iyer (1998), gr-qc/9703075], the EOM of motion are obtained by assuming *a general parametrized ansatz* for the structure of a^i_{CM}, E_{CM} and J^i_{CM}, and putting constraints of the parameters by asking that *the balance equations for E and J be satisfied in the CM frame*.

Different accelerations associated to different parameters are proven to be related by a gauge transformation.

But they make a crucial assumption: a^i_{CM}, E_{CM} and J^i_{CM} are taken to be *local-in-time* ! And we have proven the contrary.

The parametrization cannot be correct because it does not feature a nonlocal term at 4.5PN!

Correction to the parametrization of GII

We correct the parametrization

$$\widetilde{a}^i_{\mathsf{RR}} = a^{i\,\mathsf{GII}}_{\mathsf{RR}} + \frac{G\Delta}{r^2c^2}\left(2n^iv^j + n^jv^i\right)\left[\Pi^j + \mathcal{F}^j_{\boldsymbol{G}}\right] - \frac{\Delta}{mc^2}v^iv^j\left[\mathcal{F}^j_{\boldsymbol{P}} + \dot{\mathcal{F}}^j_{\boldsymbol{G}}\right]$$
$$\widetilde{E}_{\mathsf{RR}} = E^{\mathsf{GII}}_{\mathsf{RR}} + \frac{\nu\Delta}{c^2}v^2v^i\left[\Pi^i + \mathcal{F}^i_{\boldsymbol{G}}\right],$$
$$\widetilde{J}_{\mathsf{RR}} = J^{i\,\mathsf{GII}}_{\mathsf{RR}} + \frac{\nu\Delta}{c^2}\varepsilon_{ijk}x^jv^kv^l\left[\Pi^l + \mathcal{F}^l_{\boldsymbol{G}}\right].$$

These new expressions are also compatible with the flux-balance equations and feature the correct nonlocal terms.

Parameters of (corrected) GII corresponding to our result

The solution we found corresponds to the set of parameters:

$$\alpha_3 = 5 \qquad \beta_2 = 4$$
$$\xi_1 = -\frac{99}{14} + 27\nu \qquad \xi_2 = 5 - 20\nu$$
$$\xi_3 = \frac{274}{7} + \frac{67}{21}\nu \qquad \xi_4 = \frac{5}{2} - \frac{5}{2}\nu$$
$$\xi_5 = -\frac{292}{7} - \frac{57}{7}\nu \qquad \rho_5 = \frac{51}{28} + \frac{71}{14}\nu$$
$$\psi_1 = -\frac{94}{63} + \frac{4325}{168}\nu - \frac{1663}{12}\nu^2 \qquad \psi_2 = -\frac{2347}{42} + \frac{13649}{56}\nu - \frac{925}{84}\nu^2$$
$$\psi_3 = \frac{2746}{21} - \frac{80723}{252}\nu + \frac{148}{3}\nu^2 \qquad \psi_4 = \frac{870}{7} - \frac{12725}{24}\nu + \frac{1730}{7}\nu^2$$
$$\psi_5 = -\frac{541}{14} - \frac{4885}{42}\nu + \frac{803}{21}\nu^2 \qquad \psi_6 = -\frac{50263}{189} + \frac{110122}{189}\nu + \frac{18832}{189}\nu^2$$
$$\psi_7 = -\frac{1145}{18} + \frac{9395}{36}\nu - \frac{8815}{72}\nu^2 \qquad \psi_8 = \frac{7856}{63} - \frac{58025}{252}\nu - \frac{947}{9}\nu^2$$
$$\psi_9 = \frac{9101}{126} + \frac{1831}{12}\nu - \frac{9103}{189}\nu^2 \qquad \chi_6 = -\frac{16309}{504} + \frac{11315}{84}\nu - \frac{827}{56}\nu^2$$
$$\chi_8 = \frac{5465}{126} - \frac{11075}{84}\nu + \frac{4175}{168}\nu^2 \qquad \chi_9 = -\frac{191}{756} - \frac{5167}{378}\nu + \frac{36499}{252}\nu^2$$

Any other set of parameters is equally correct, but in a different gauge!

Localizing the nonlocal term for circular orbits

Suppose that the motion has been eternally quasicircular, departing from exact circularity only adiabatically due to radiation reaction.

We can compute the nonlocal terms:

$$\Pi^i = -\frac{464}{105}\,\frac{G^4 m^5}{c^7 r^4}\,\Delta\,\nu^2\,n^i + \mathcal{O}(9) \tag{1}$$

$$\Gamma^i = \frac{48}{5}\,\frac{G^3 m^4}{c^7 r^2}\,\Delta\,\nu^2\,v^i + \mathcal{O}(9) \tag{2}$$

Injecting into the full (conservative + dissipative) accelerations, we find::

$$a^i_{\text{circ}} = -\omega^2 x^i - \frac{32 G^3 m^3 \nu}{5 c^5 r^4} v^i \Bigg[1 + \gamma \left(-\frac{743}{336} - \frac{11}{4}\nu \right) + 4\pi \gamma^{3/2} + \gamma^2 \left(-\frac{34639}{18144} - \frac{12521}{2016}\nu + \frac{5}{4}\nu^2 \right) + \mathcal{O}(\gamma^{5/2}) \Bigg]$$

where $\gamma = \frac{Gm}{c^2 r}$ and the conservative section ω is known at 4PN

Conclusion

- Radiation reaction at 2PN $\Leftrightarrow$ equation of motion obtained at 4.5PN
- Prove flux-balance laws at 2PN for E, $\boldsymbol{J}$, $\boldsymbol{P}$, $\boldsymbol{G}$
- Going to CM frame requires accounting for radiation contribution
- Nonlocal contribution in passage to CM frame at 3.5PN
- Nonlocal contribution in equations of motion at 4.5PN
- Recover flux balance laws at 2PN in CM frame (nonlocal !)
- Corrected Gopakumar-Iyer-Iyer parametrization
- Localization for circular orbits