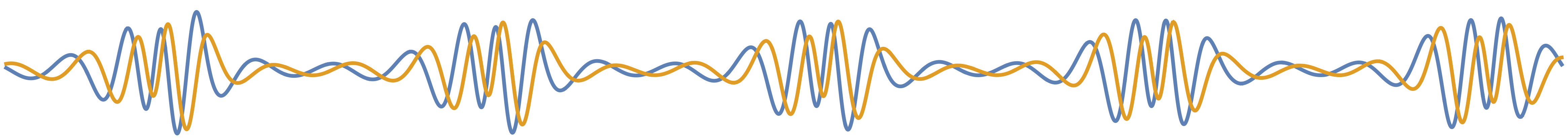
Comparison of 4.5PN and 2SF gravitational energy fluxes from quasicircular compact binaries



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24th September 2024

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Science Foundation Ireland sfi For what's next

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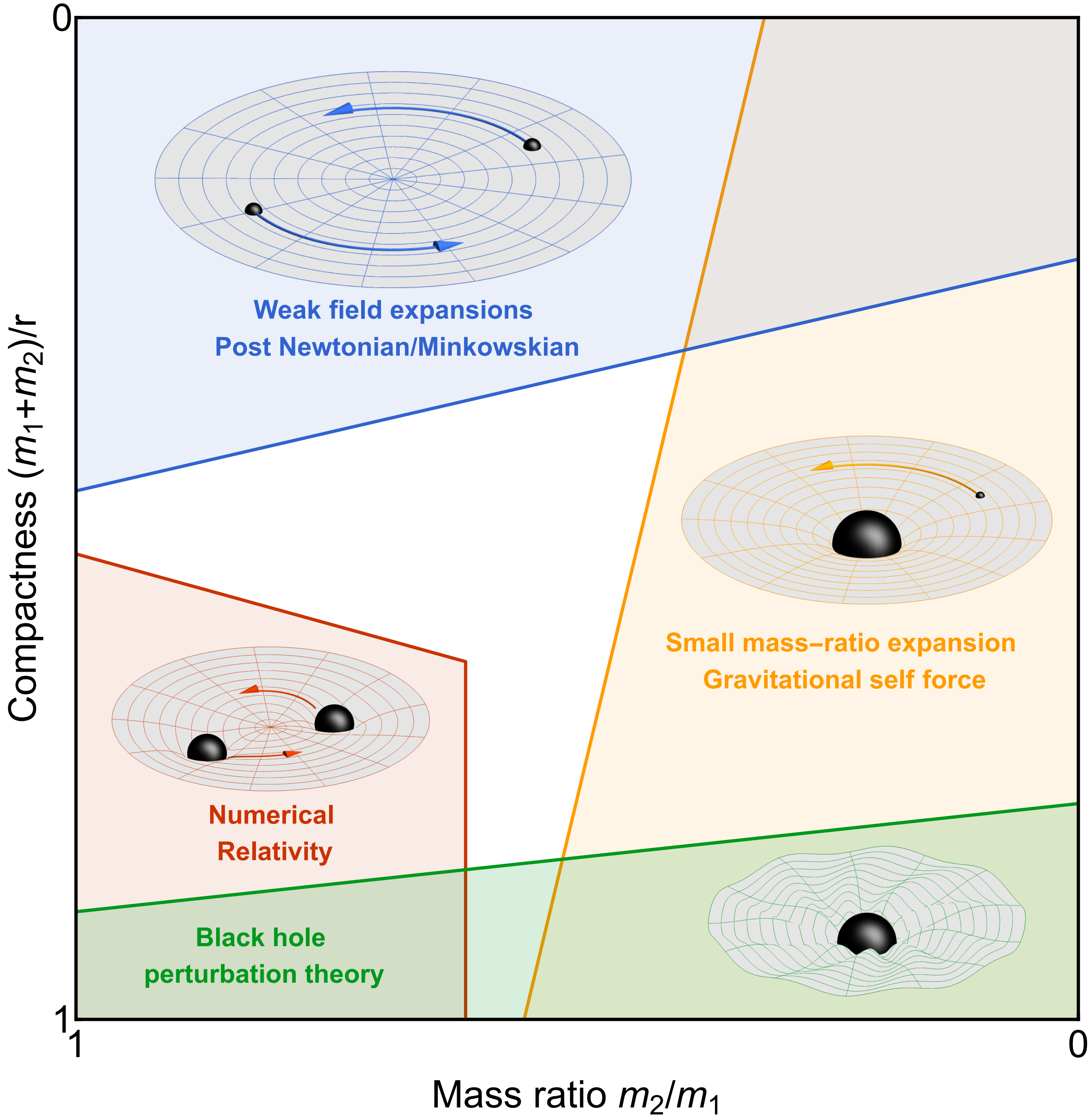
Overview and motivation

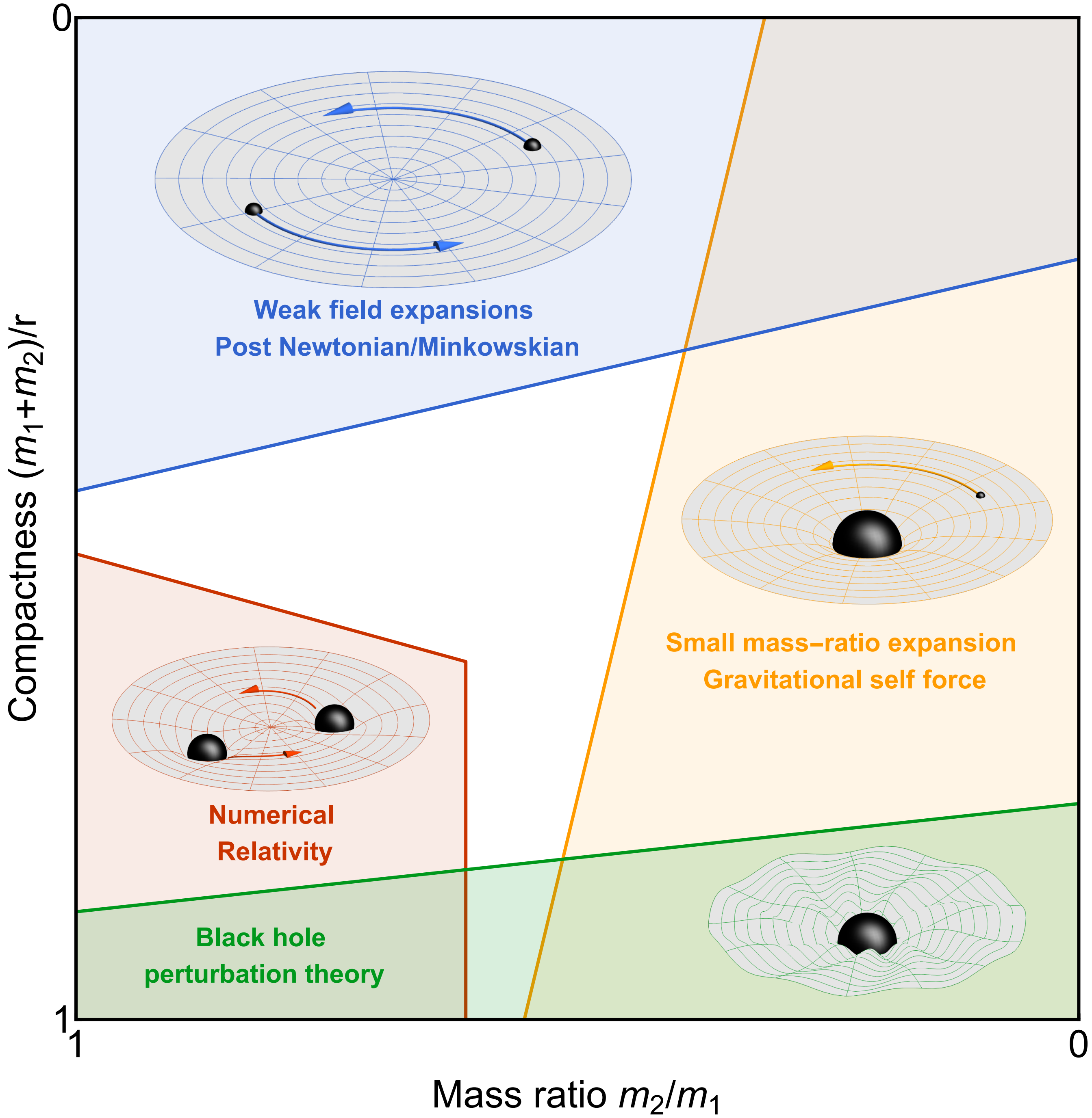
- Many approaches need to model GWs emission from compact binaries



[Image credit: LISA Consortium Waveform Working Group]

Overview and motivation

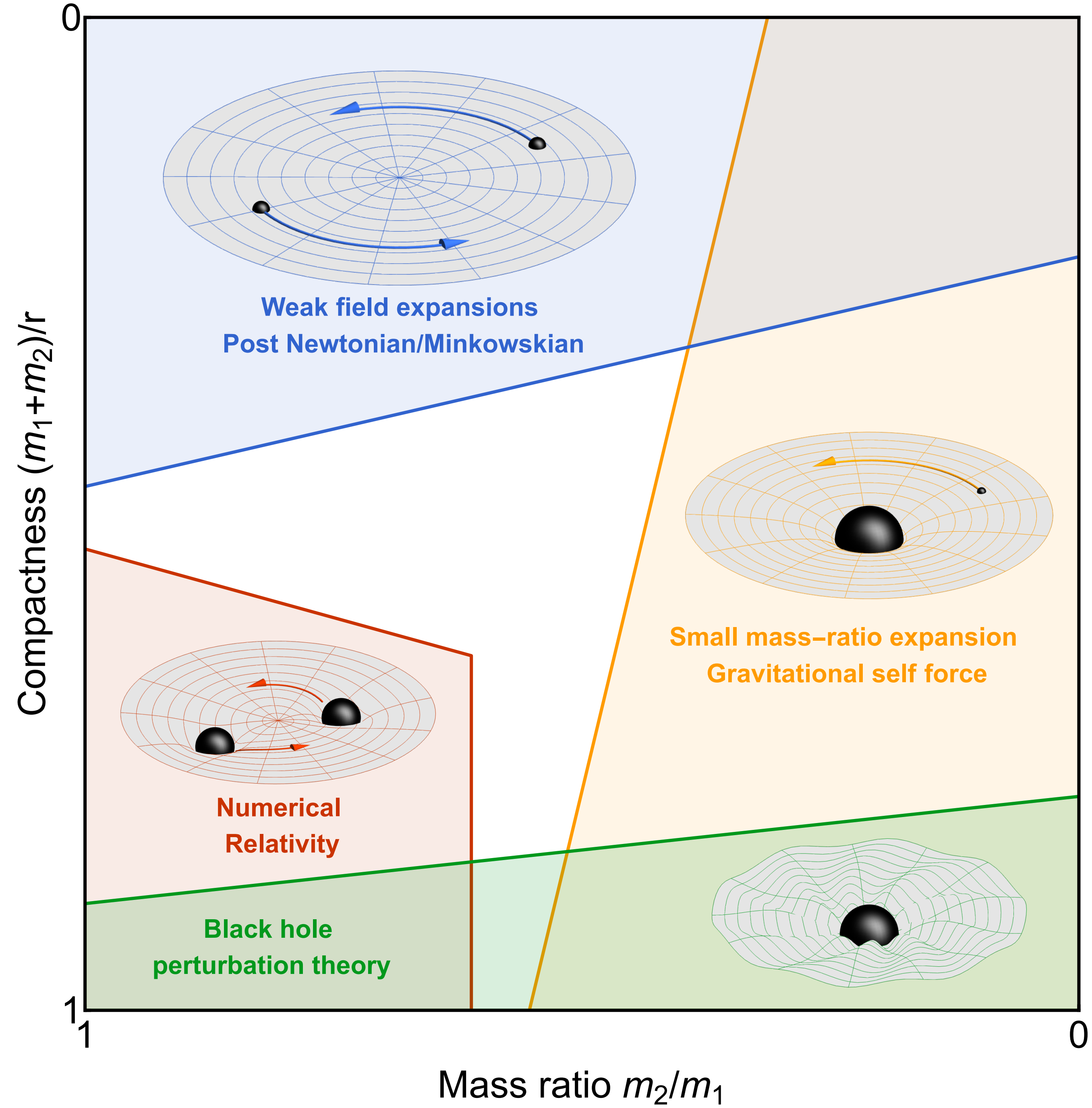
- Many approaches need to model GWs emission from compact binaries
- Each approach is complex and it is very important make comparisons between them



[Image credit: LISA Consortium Waveform Working Group]

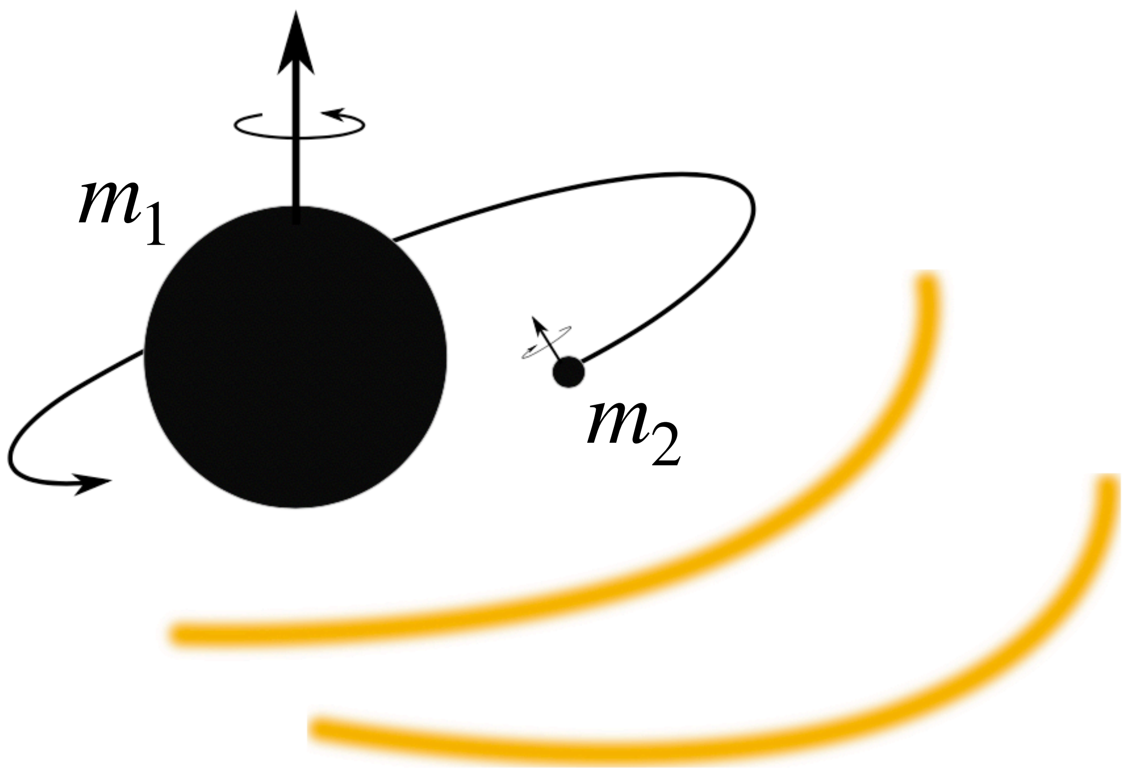
Overview and motivation

- Many approaches need to model GWs emission from compact binaries
- Each approach is complex and it is very important make comparisons between them
- Post-Newtonian (PN) and self-force (SF) are perturbative expansions
- Recently both have extended the order of their expansions: 4.5PN and 2SF



[Image credit: LISA Consortium Waveform Working Group]

Gravitational self-force approach



[Image credit: Adam Pound]

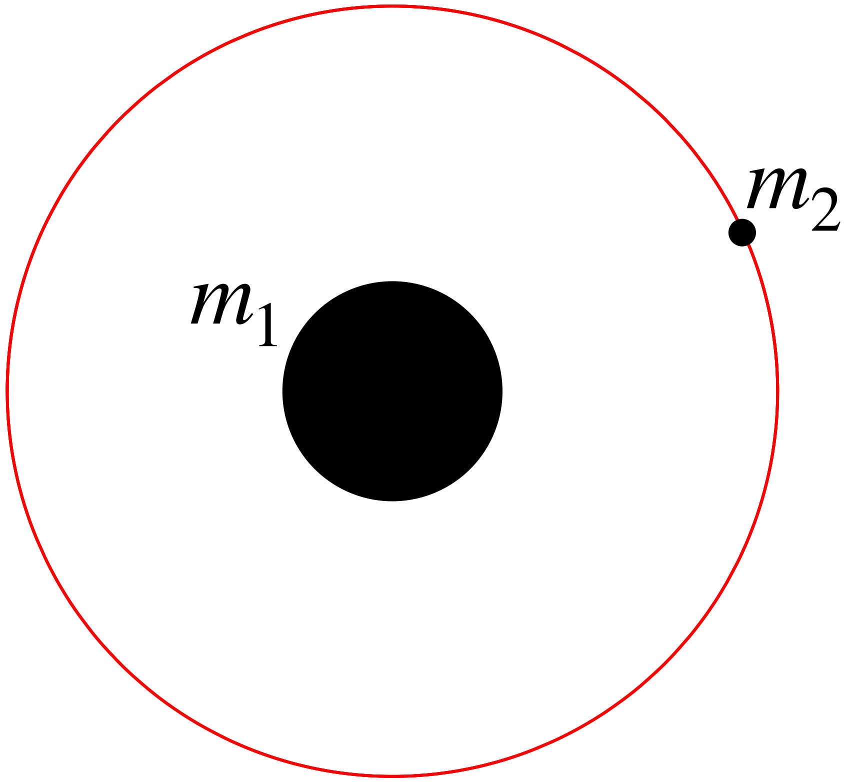
- $\epsilon = 1/q = m_2/m_1 \ll 1$
- Small body perturbs spacetime:

$$g_{\alpha\beta} = g^{\text{Kerr}}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots$$

- Perturbation affects m_2's motion:

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon f^\mu_{(1)} + \epsilon^2 f^\mu_{(2)} + \ldots$$

Zeroth order: circular geodesics in Schwarzschild



- constants $J_A = (m_1, \Omega)$
- Azimuthal phase φ_p with frequency Ω

- Simple ODEs:

$$\frac{d\varphi_p}{dt} = \Omega \qquad \frac{dm_1}{dt} = 0$$

$$\frac{d\Omega}{dt} = 0$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

- evolution due to the self-force:

$$\frac{d\varphi_p}{dt} = \Omega \qquad \frac{dm_1}{dt} = \epsilon \mathscr{F}^{(1)}_{\mathscr{H}}$$

$$\frac{d\Omega}{dt} = \epsilon \left[F^{(0)}(\Omega) + \epsilon F^{(1)}(\Omega) + \mathcal{O}(\epsilon^2)\right]$$

- Truncation in forcing terms is sufficient to ensure error in $\varphi_p \propto \epsilon$ over a radiation reaction timescale

- waveform:

$$h_+ - ih_\times = \sum_{lm} \left[\epsilon h^{(1)}_{lm}(\Omega) + \epsilon^2 h^{(2)}_{lm}(\Omega)\right] {}_{-2}Y_{lm}(\theta, \varphi_p)$$

Native rapid waveform generation

Offline step

- solve field equations for waveform amplitudes $h_{lm}^{(n)}$ and forcing functions $F^{(n-1)}$ on a grid of Ω values

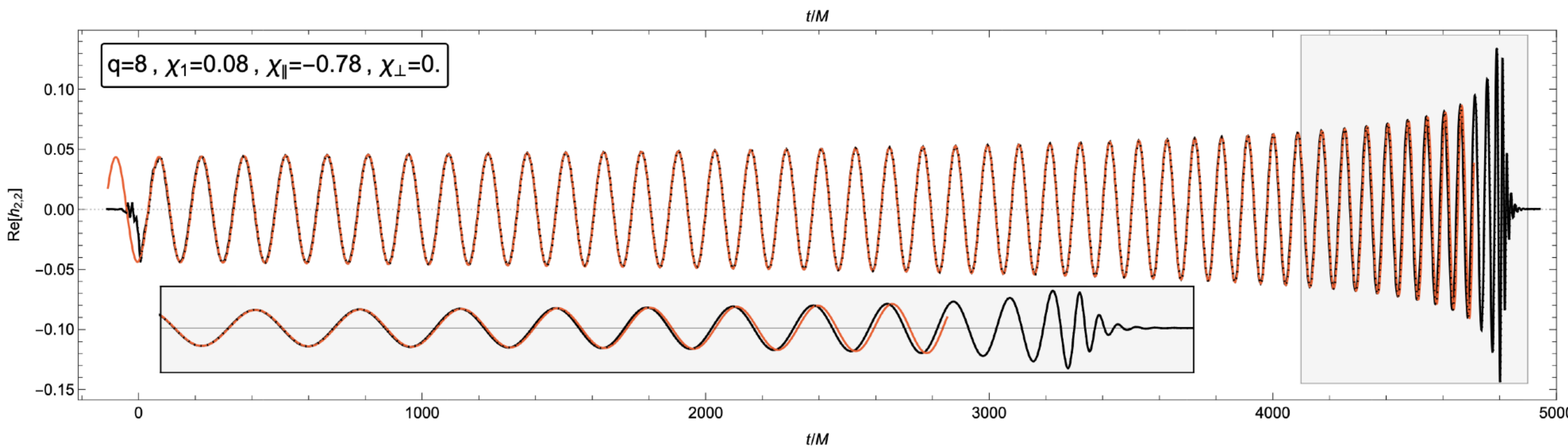
Online step

- solve ODEs for φ_p and Ω
- Add up the mode amplitudes $h_{\ell m}^{(n)}$ at each sample time
- `FastEMRIWaveforms` (**FEW**) software package can compute a 2-year long EMRI waveform in ~ 10 - 100ms

Complete 1PA inspiral waveforms

Pound, NW, Wardell,
Durkan, Miller, Mathews

Comparison with NR waveform from SXS collaboration



Complete quasi-circular 1PA inspiral model with generic (precessing) secondary spin, linear primary spin and evolving m_1 and χ_1

Work on merger and ringdown portion of the waveform ongoing

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

$$G_{\alpha\beta}[g^{\mathrm{Kerr}}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

$$G_{\alpha\beta}[g^{\text{Kerr}}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

- Expand the Einstein and stress-energy tensors as

$$T_{\alpha\beta} = \epsilon T^{(1)}_{\alpha\beta} + \epsilon^2 T^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3),$$

$$G_{\alpha\beta}[g] = \epsilon \delta G_{\alpha\beta}[h^{(1)}] + \epsilon^2 \left[\delta G_{\alpha\beta}[h^{(2)}] + \delta^2 G_{\alpha\beta}[h^{(1)}, h^{(1)}]\right] + \mathcal{O}(\epsilon^3),$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

$$G_{\alpha\beta}[g^{\rm Kerr}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

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- Write metric as product of slowly evolving amplitudes and a rapidly evolving phase:

$$h^{(n)}_{\alpha\beta} = \sum_m h^{(n,m)}_{\alpha\beta}(\Omega; x^i) e^{-im\varphi_p} \qquad x^i = \{r, \theta, \phi\}$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller,
Pound, Moxon, Grant

$$\epsilon \delta G_{\alpha\beta}[h^{(1)}] + \epsilon^2 \left[\delta G_{\alpha\beta}[h^{(2)}] + \delta^2 G_{\alpha\beta}[h^{(1)}, h^{(1)}] \right] = \epsilon T^{(1)}_{\alpha\beta} + \epsilon^2 T^{(2)}_{\alpha\beta}$$

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- Substitute multiscale expansion into the Einstein field equations. By treating t as a function of (Ω, φ_p) time derivatives can be computed via:

$$\partial_t = \dot{\varphi}_p \partial_{\varphi_p} + \dot{\Omega} \partial_\Omega = \Omega \partial_{\varphi_p} + \epsilon F^{(0)}(\Omega) \partial_\Omega + \mathcal{O}(\epsilon^2)$$

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- Expand the linearised and second-order Einstein tensors as

$$\delta G_{\alpha\beta} = \delta G^{[0]}_{\alpha\beta} + \epsilon \delta G^{[1]}_{\alpha\beta} + \mathcal{O}(\epsilon^2)$$

$$\delta^2 G_{\alpha\beta} = \delta^2 G^{[0]}_{\alpha\beta} + \mathcal{O}(\epsilon)$$

Field equations

- Field equations for each m-mode take the form:

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)}] = T^{(1)}_{\alpha\beta}$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)}] = T^{(2)}_{\alpha\beta} - \delta^2 G^{[0]}_{\alpha\beta}[h^{(1)}, h^{(1)}] - \delta G^{[1]}_{\alpha\beta}[h^{(1)}]$$

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- Self-force computed from these regular fields

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon f^\mu_{(1)}(h^{(1)R}) + \epsilon^2 f^\mu_{(2)}(h^{(1)R}, h^{(2)R}) + \mathcal{O}(\epsilon^3)$$

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MiSaTaQuWa equations

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Pound 2012
Gralla 2012

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- Non-compact
- Diverges on the worldline

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Pound 2012
Gralla 2012

- Non-compact
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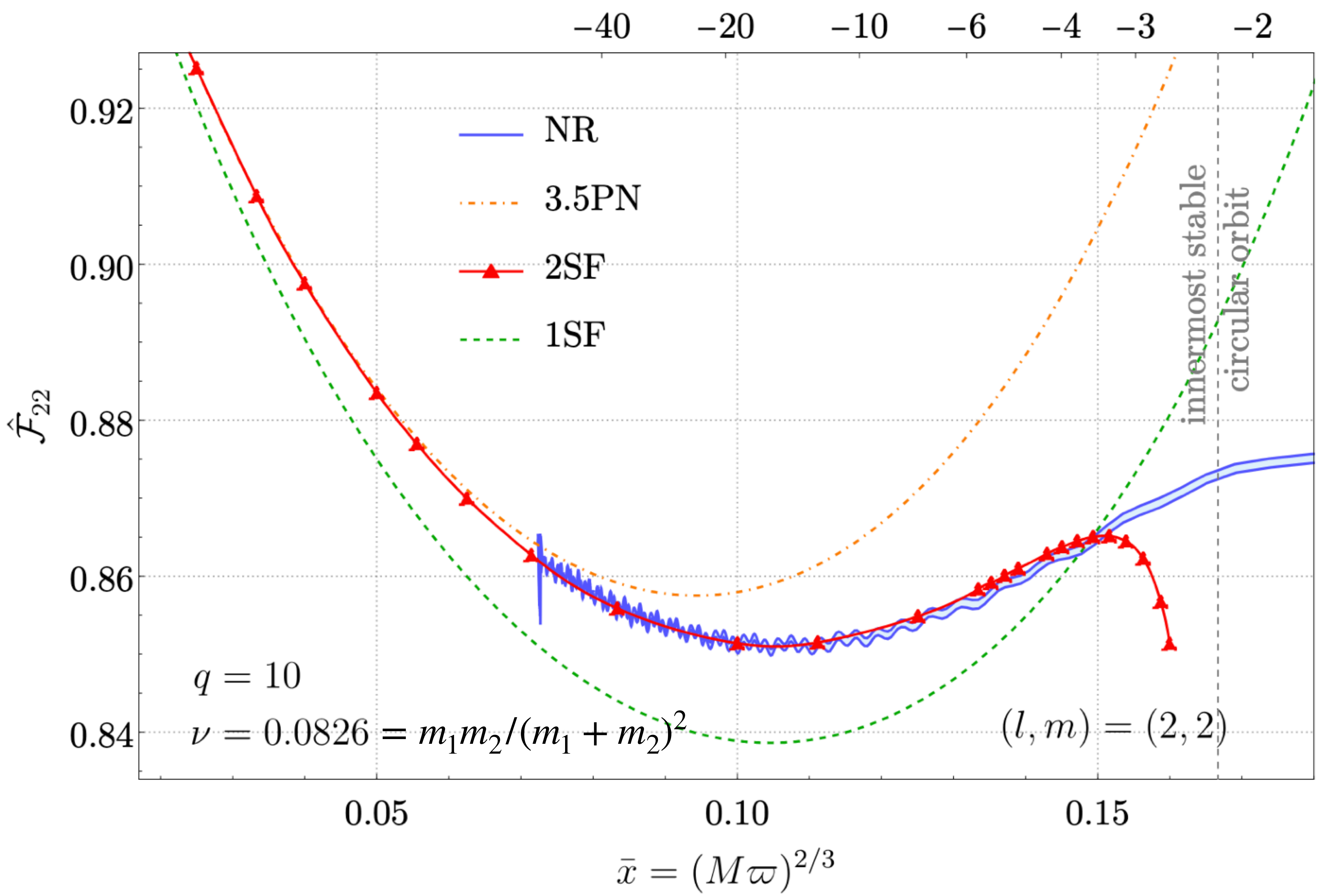
- Non-compact
- $\propto \dot{\Omega}\partial_\Omega h^{(1)}$

Flux: 2SF vs NR

NW, Pound, Wardell, Miller, Durkan
Phys. Rev. Lett. 127, 151102 (2021)
arXiv:2107.01298



Remarkable agreement with NR for mass ratios as small as $q = 10$

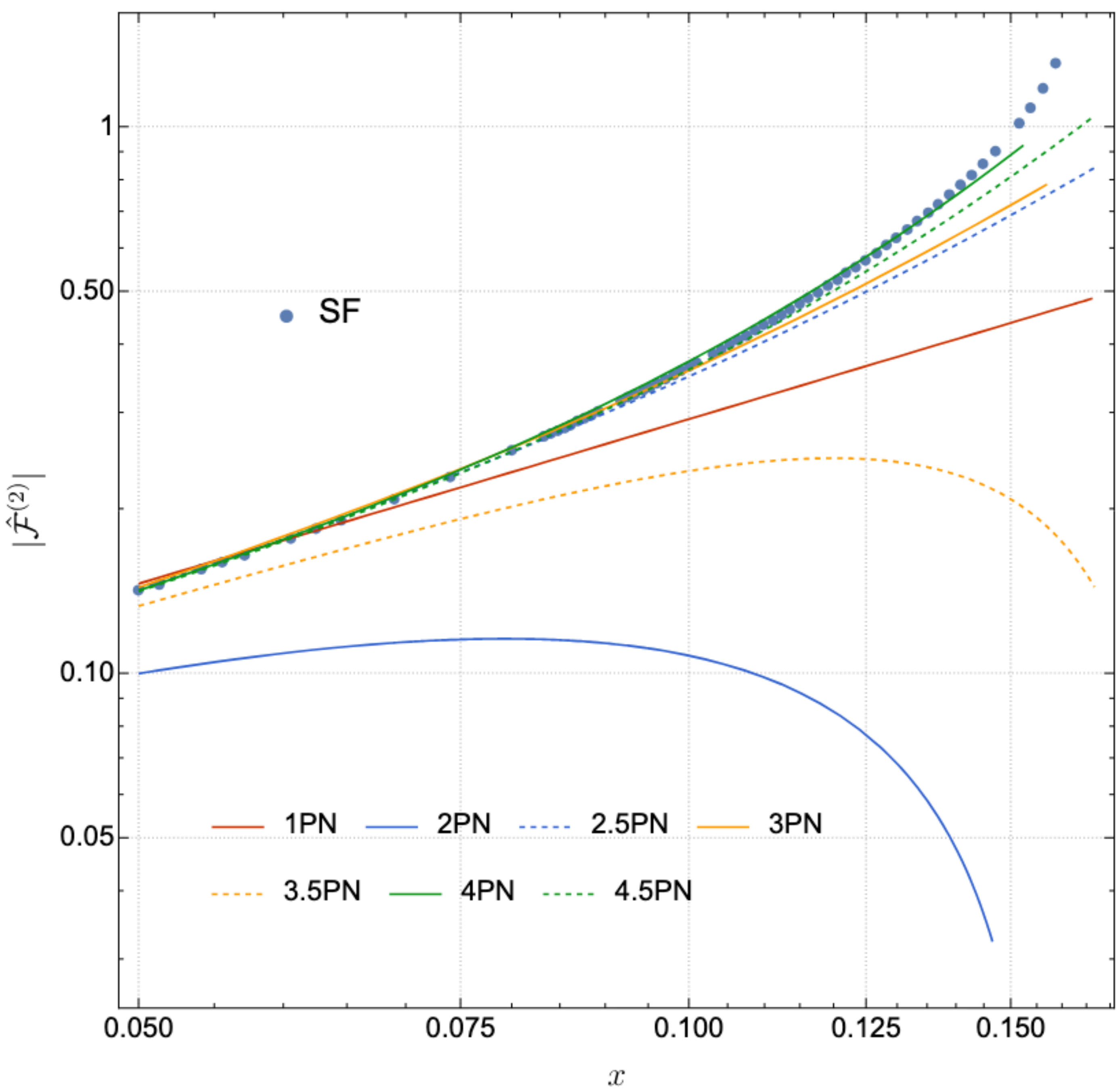
Flux: 2SF vs 4.5PN

NW et al.
arXiv:2407.00366

- Write waveform as
  $h_{lm} = A_{lm}(u)e^{-i\omega\psi(u)}$
  where the A_{lm} are real

- Define $x = (M\omega)^{2/3}$ and
  $\mathscr{F} = \nu^2\mathscr{F}^1 + \nu^3\mathscr{F}^2 + \mathcal{O}(\nu^4)$

- Compare total flux with new 4.5PN result from Blanchet et al. Phys. Rev. Lett. 131, 121402



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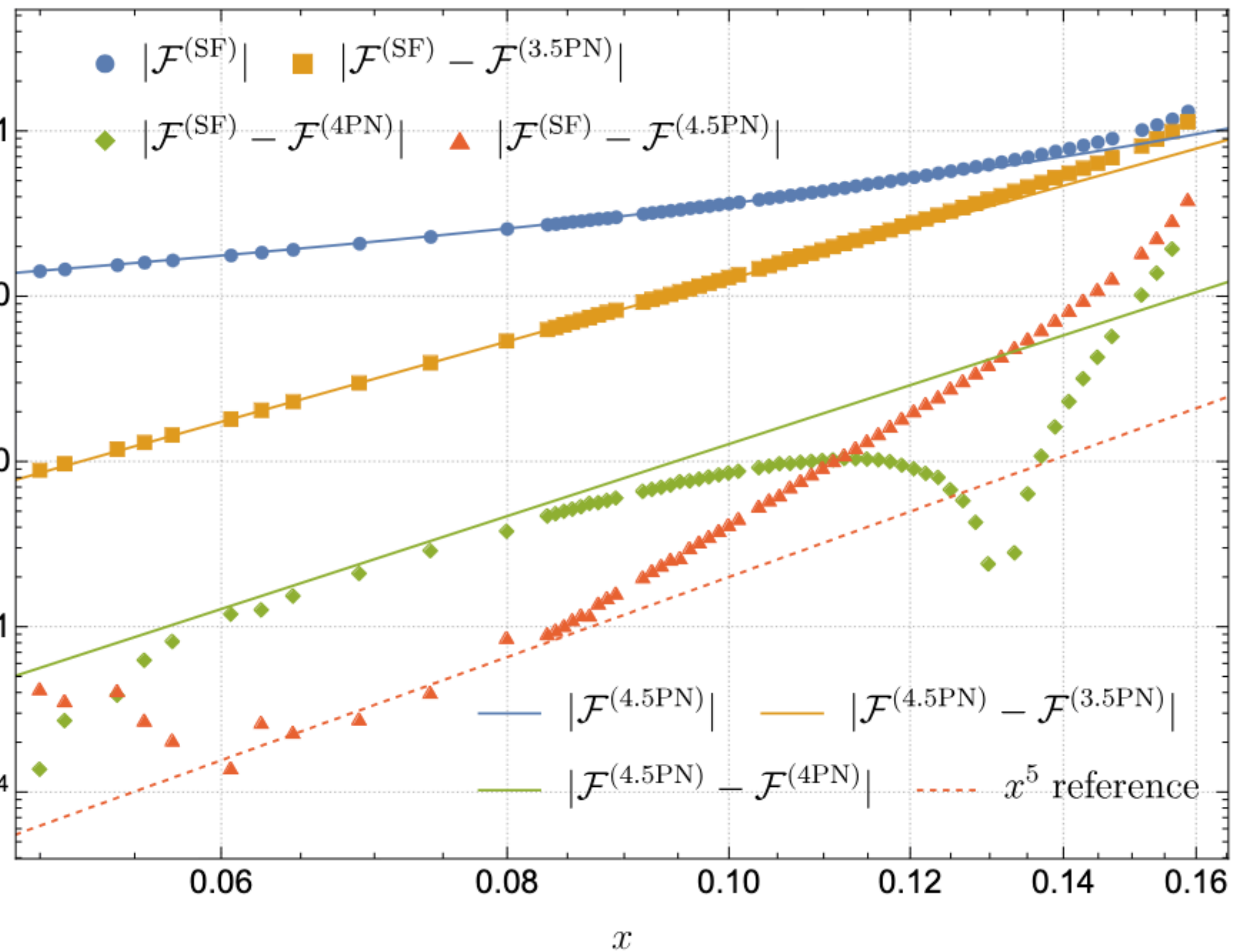
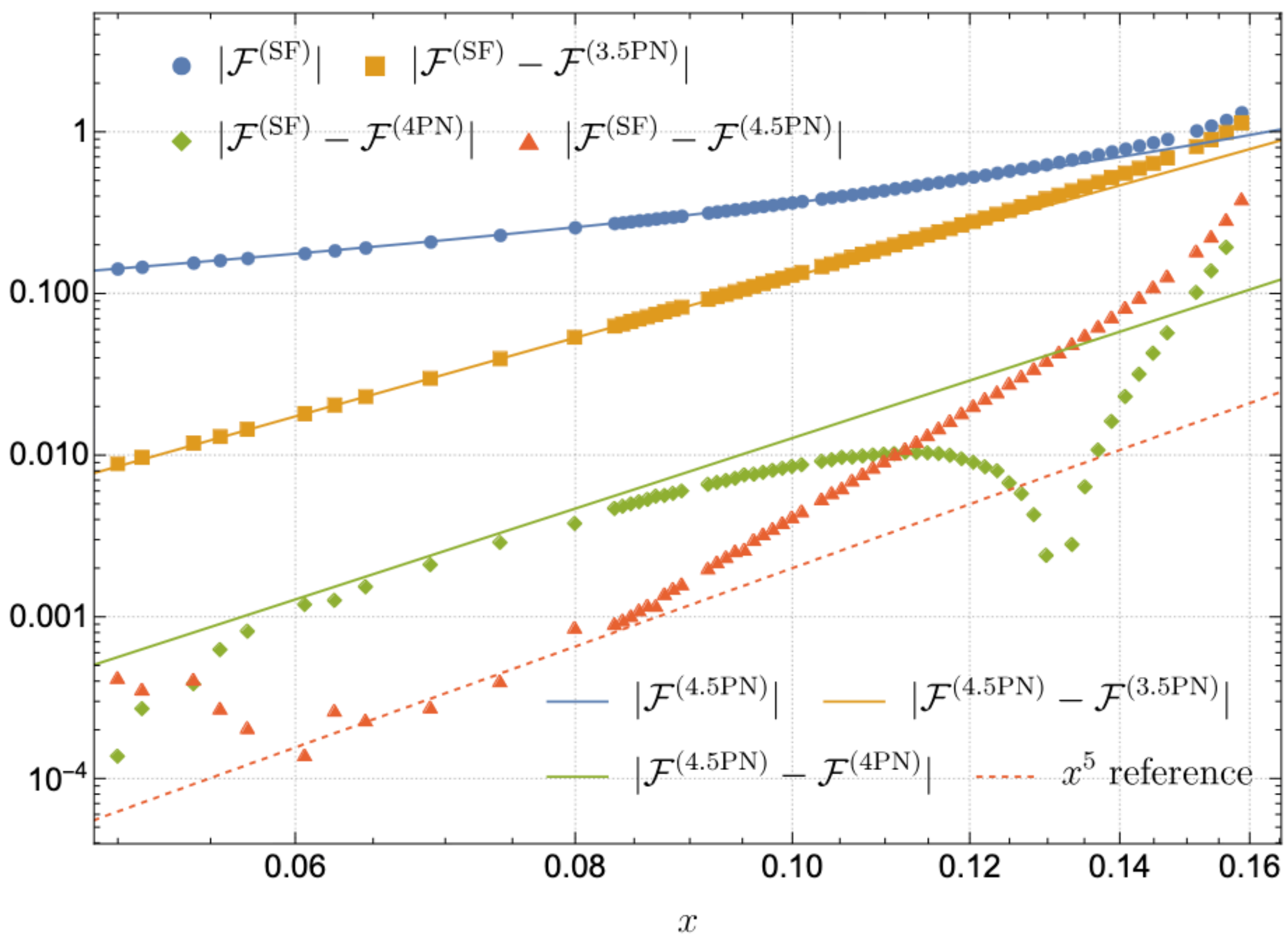
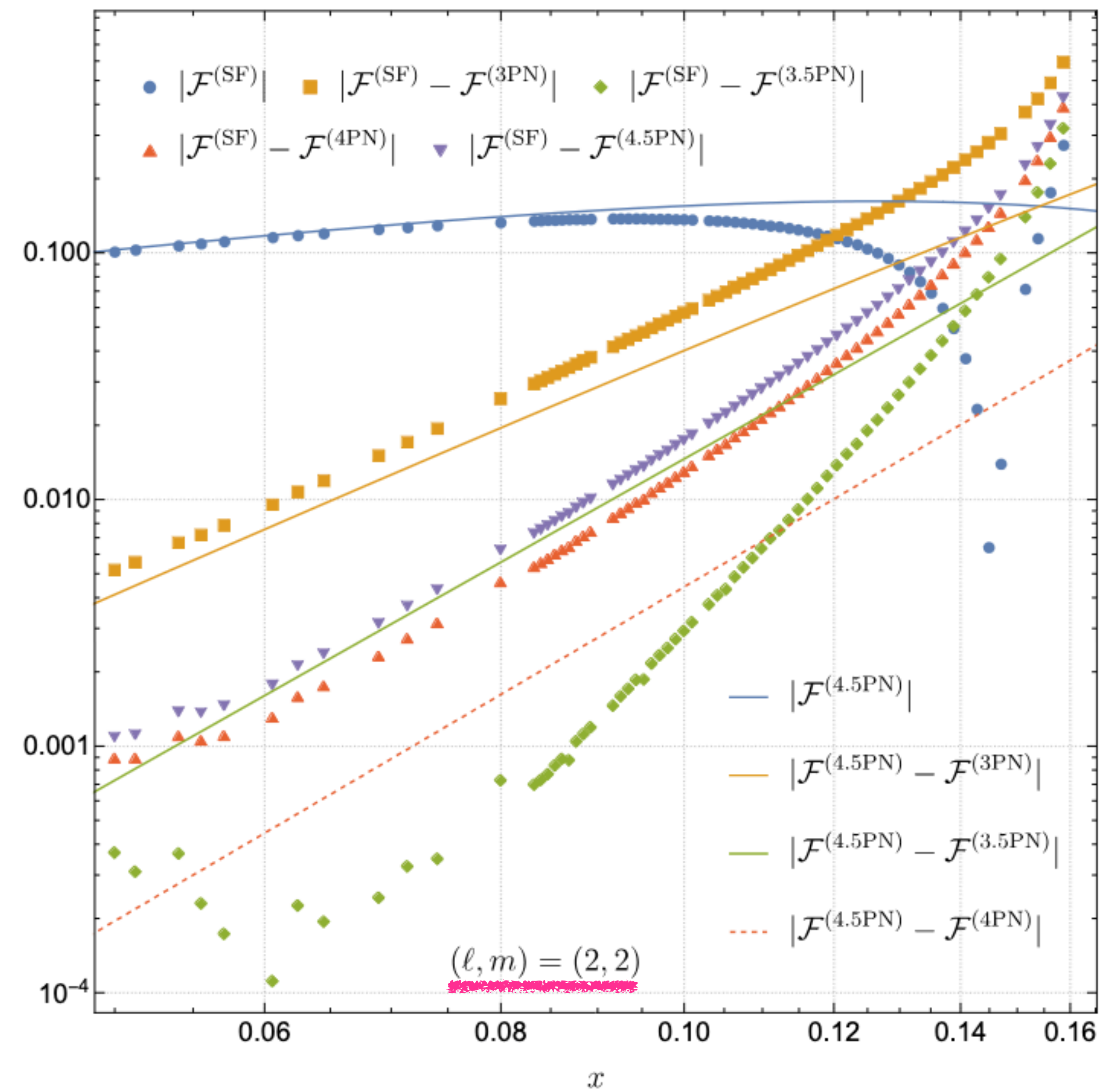


Detailed comparison shows agreement for total flux

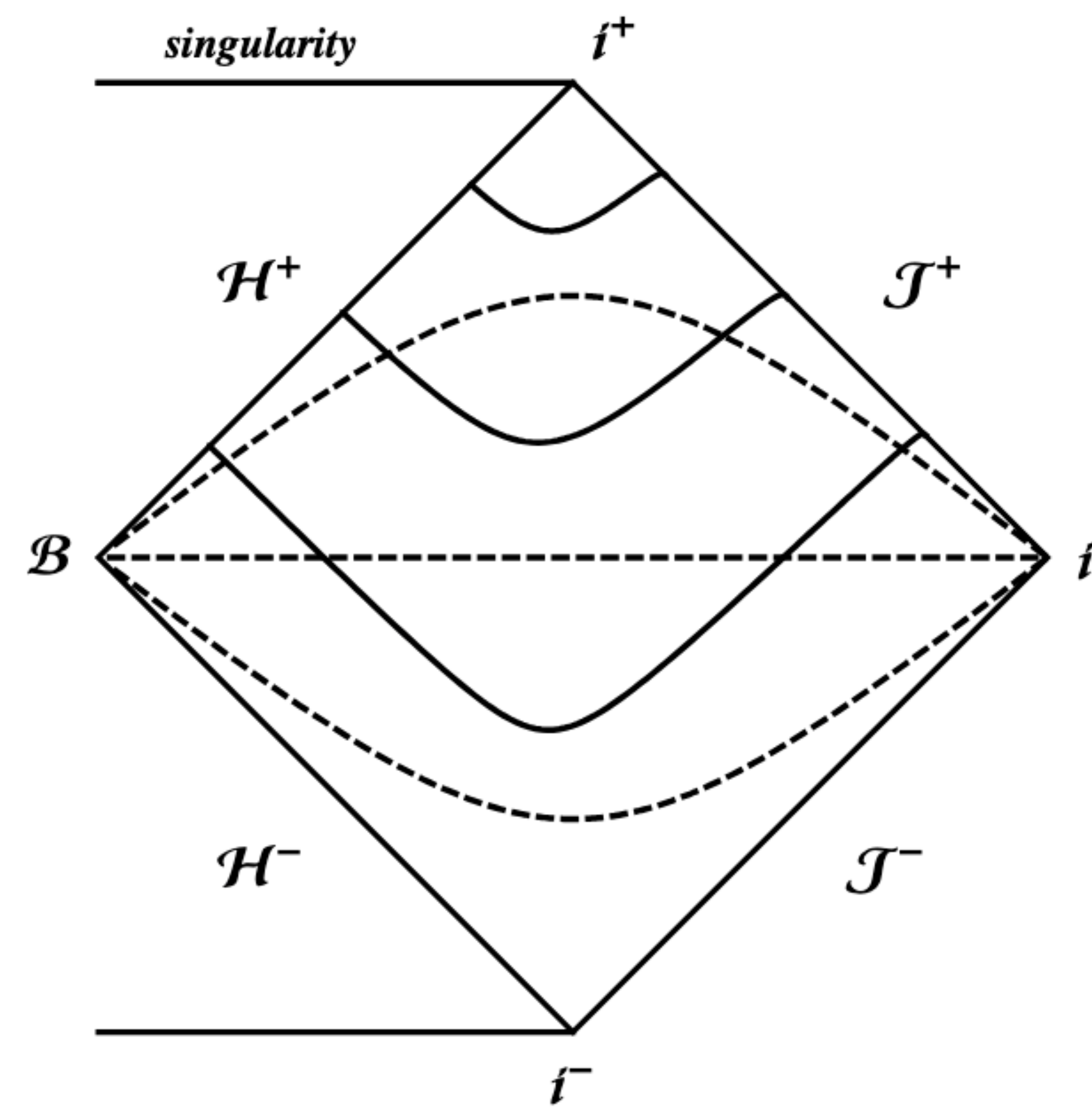




Find agreement with 4.5PN for the total flux but not for the individual modes. Suggestions the calculations are in different frames

Future improvements to 2SF calculations

- Current 2SF calculation is carried out on t-slicing
- Must place outer boundary at a finite radius with $r_{\text{out}}\omega \gg 1$



[image credit: A. Zenginğlu]

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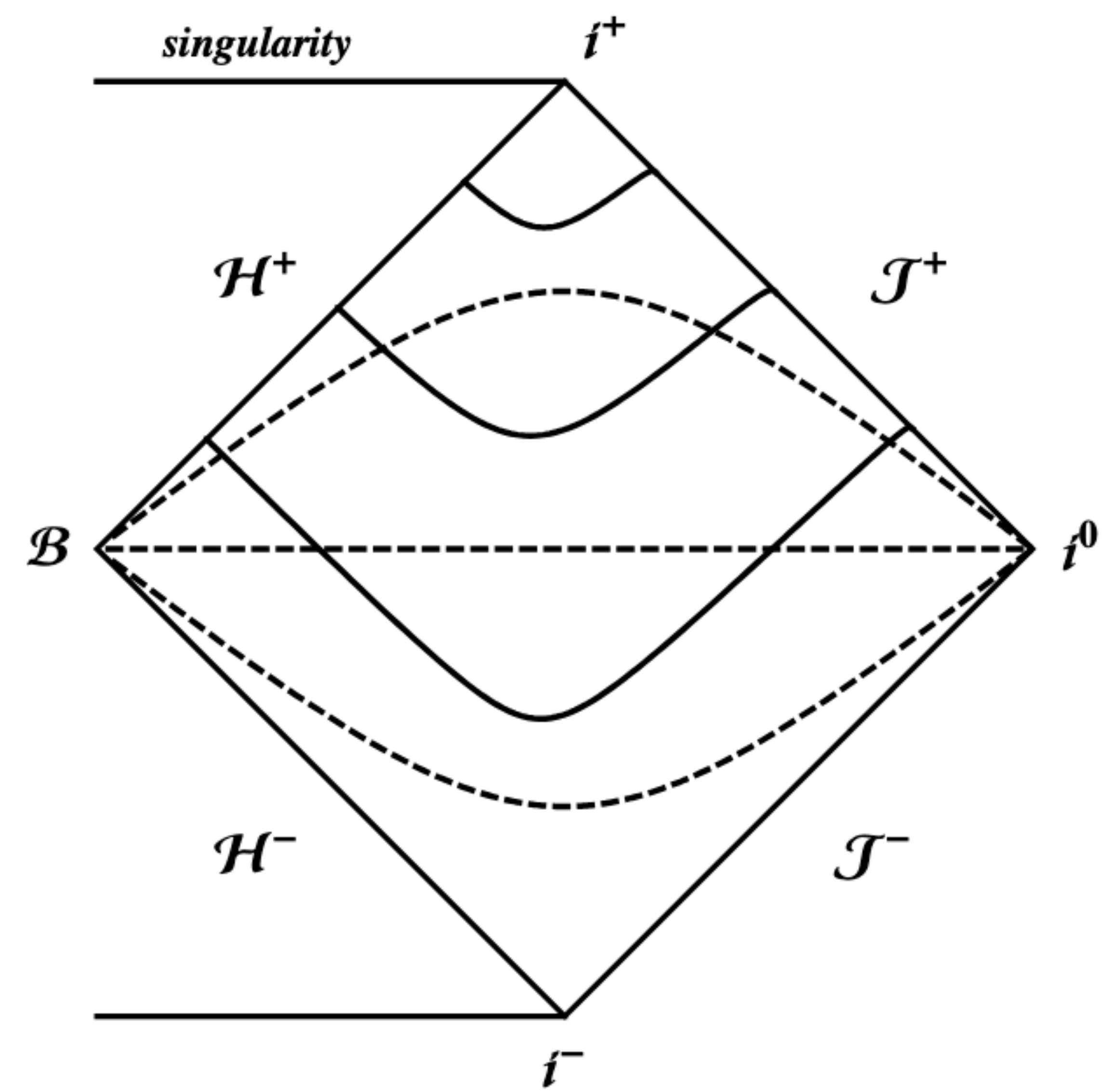
$$\psi(r) = C^{+}(r)\psi^{+}(r) + C^{-}(r)\psi^{-}(r)$$

$$C^{\pm} = \int \frac{\psi^{\mp} S}{W}$$



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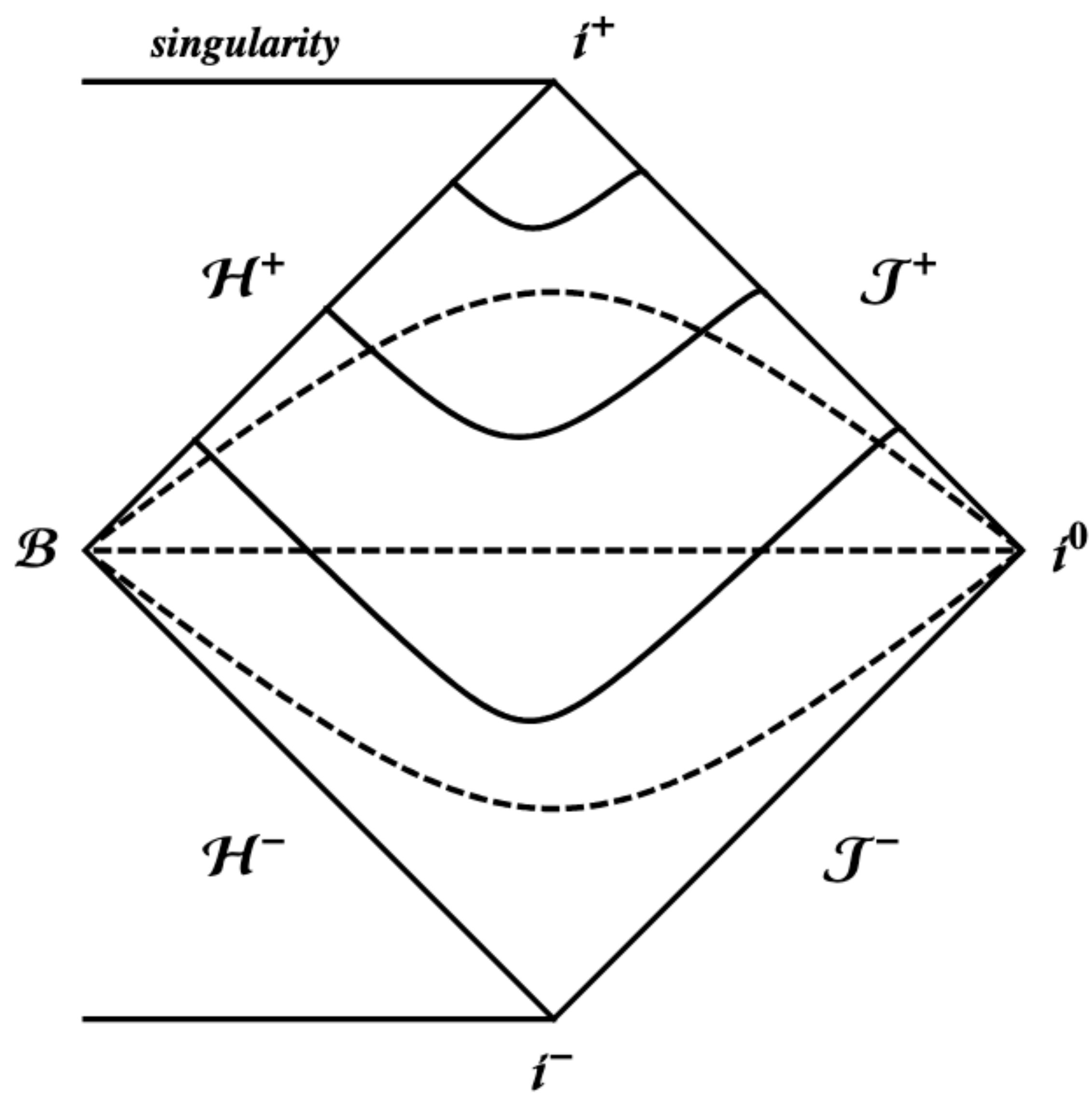
- Switch to hyperboloidal slices. This allows compactification. Boundary conditions become regularity conditions.



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Future improvements to 2SF calculation

- In the full 2SF problem we have a non-compact source. One piece comes from the parametric derivatives $\partial_\Omega h^{(1)}_{\alpha\beta}$
- We can consider a toy problem of a scalar field sourced by a particle on a circular orbit of radius r_p

$$\Box_\omega \phi = 4\pi\rho \qquad \Box_\omega = \partial^2_{r*} + \omega^2 + \ldots \qquad \omega = m\sqrt{m_1/r_p^3}$$

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- Taking an r_p derivative we get:

$$\begin{aligned}\Box_\omega(\partial_{r_p}\phi) &= 4\pi\partial_{r_p}\rho - (\partial_{r_p}\Box_\omega)\phi \\ &= 4\pi\partial_{r_p}\rho - (2\omega\partial_{r_p}\omega)\phi\end{aligned}$$

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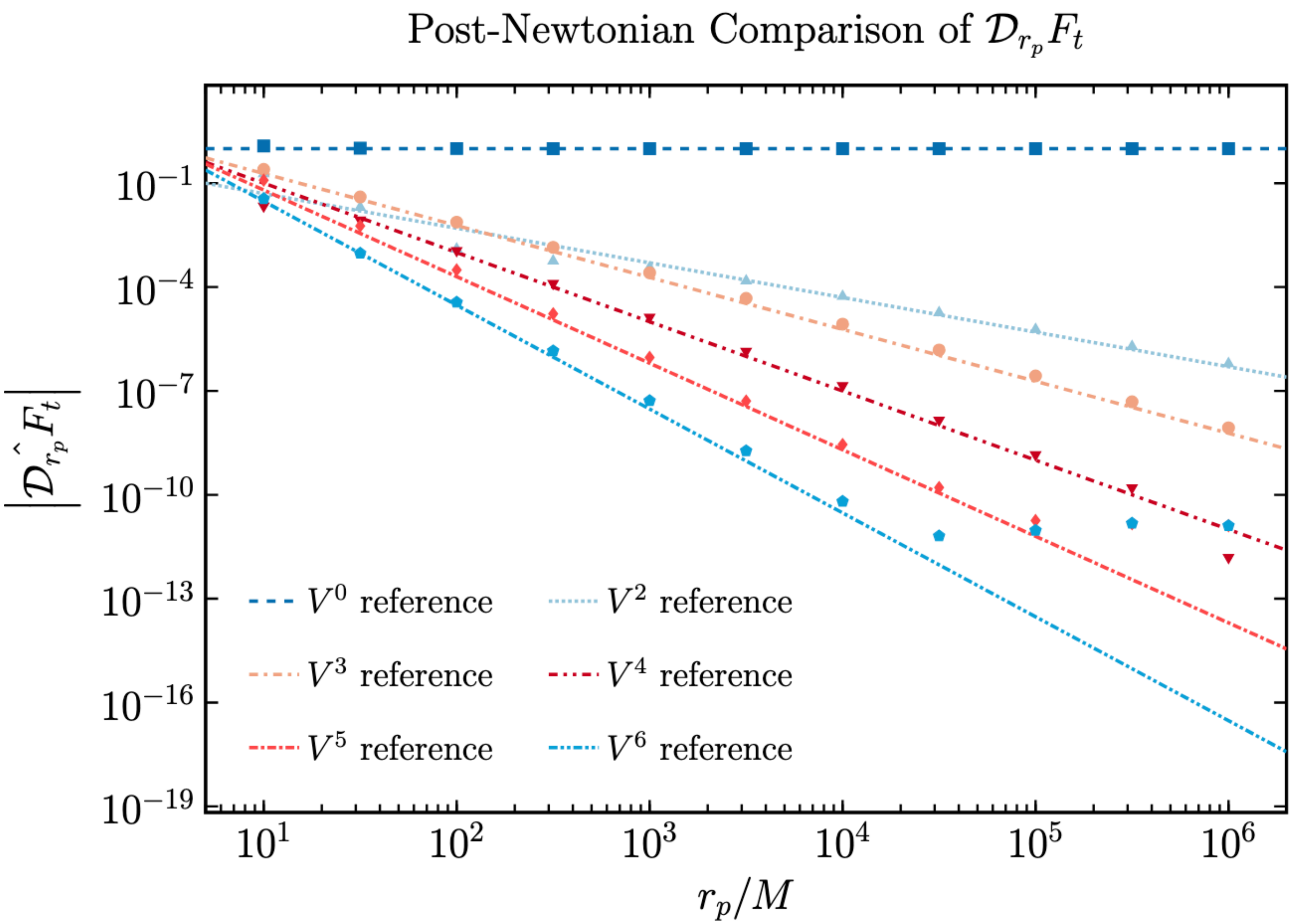
- Numerically solve this problem using hyperboloidal, compactified coordinates with a pseudo-spectral method

Future improvements to 2SF calculation

R. Macedo et al.
arXiv:2202.01794



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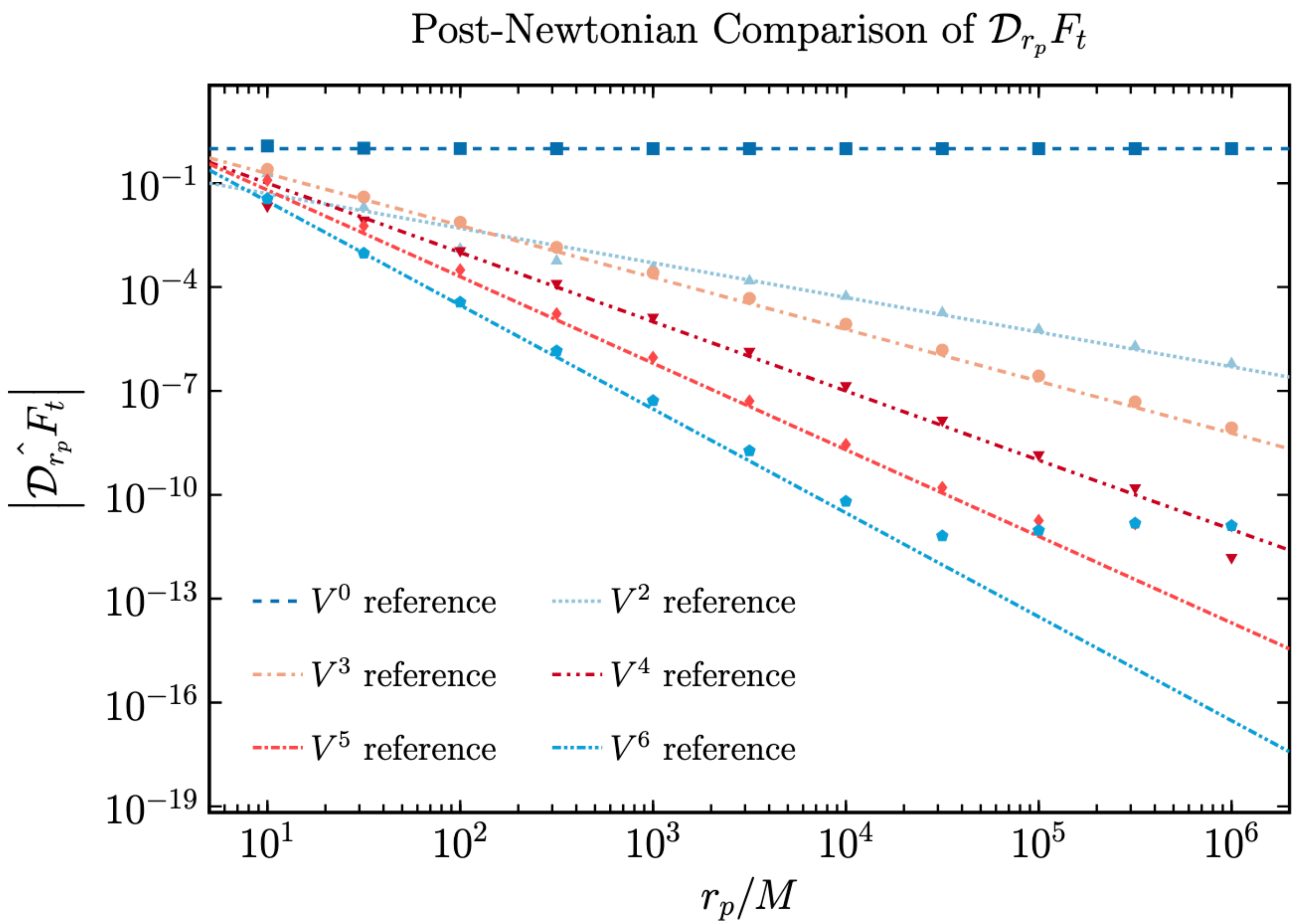

We get good numerical results out to $r_p = 10^6 M$. This makes comparison with PN much easier

Conclusions

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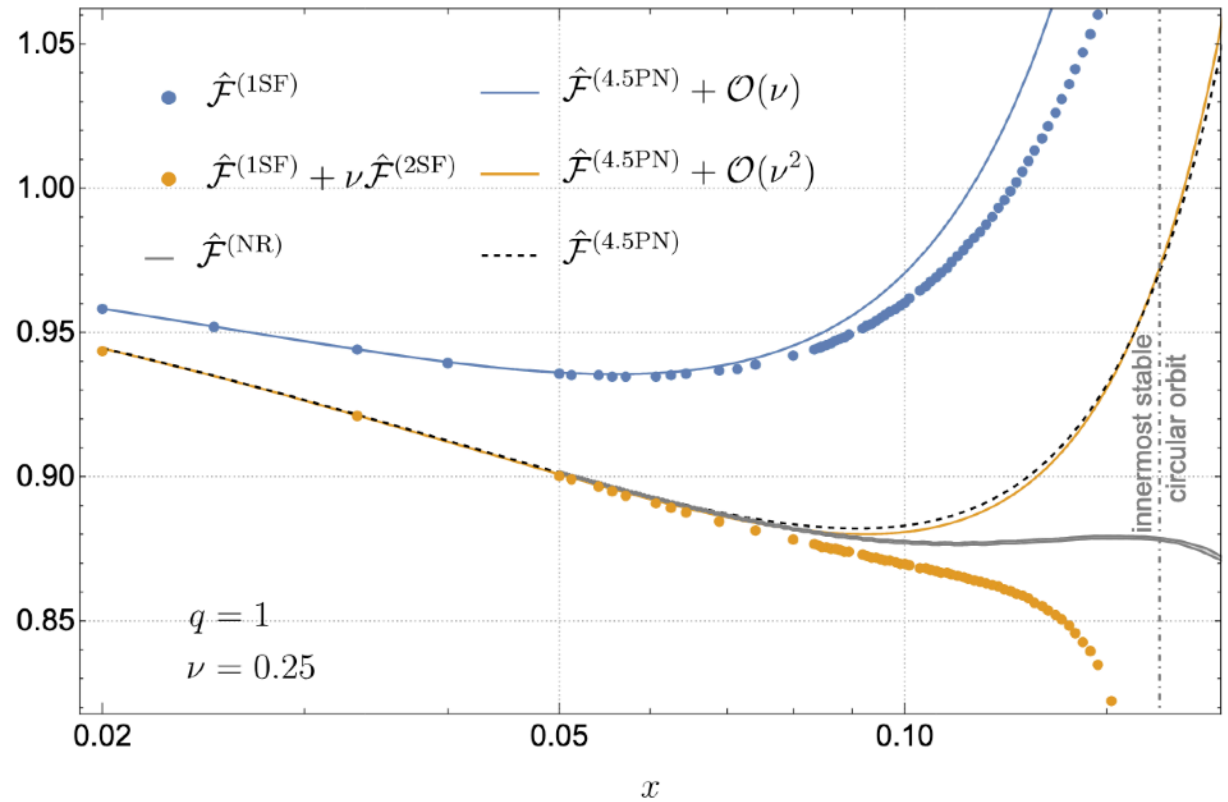
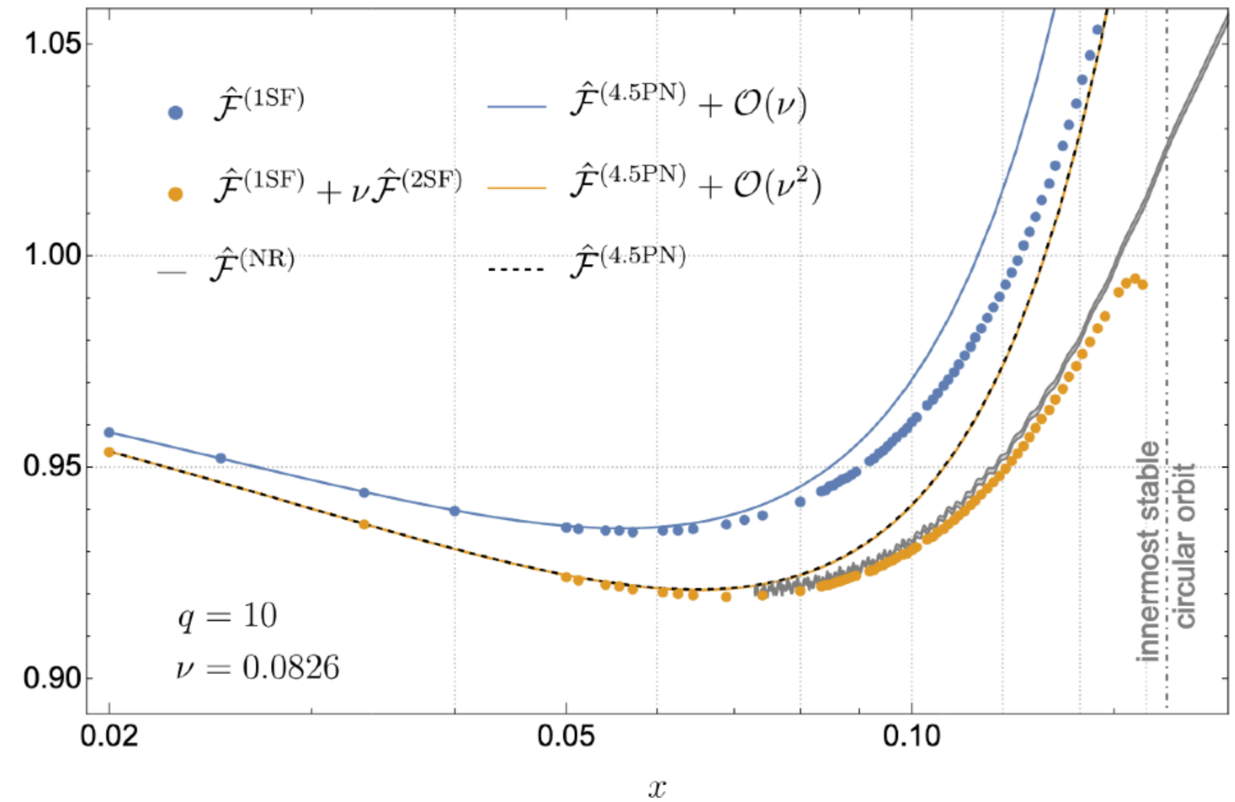
- Future work: hybridise PN and SF results to create 2SF waveform model that works for all frequencies

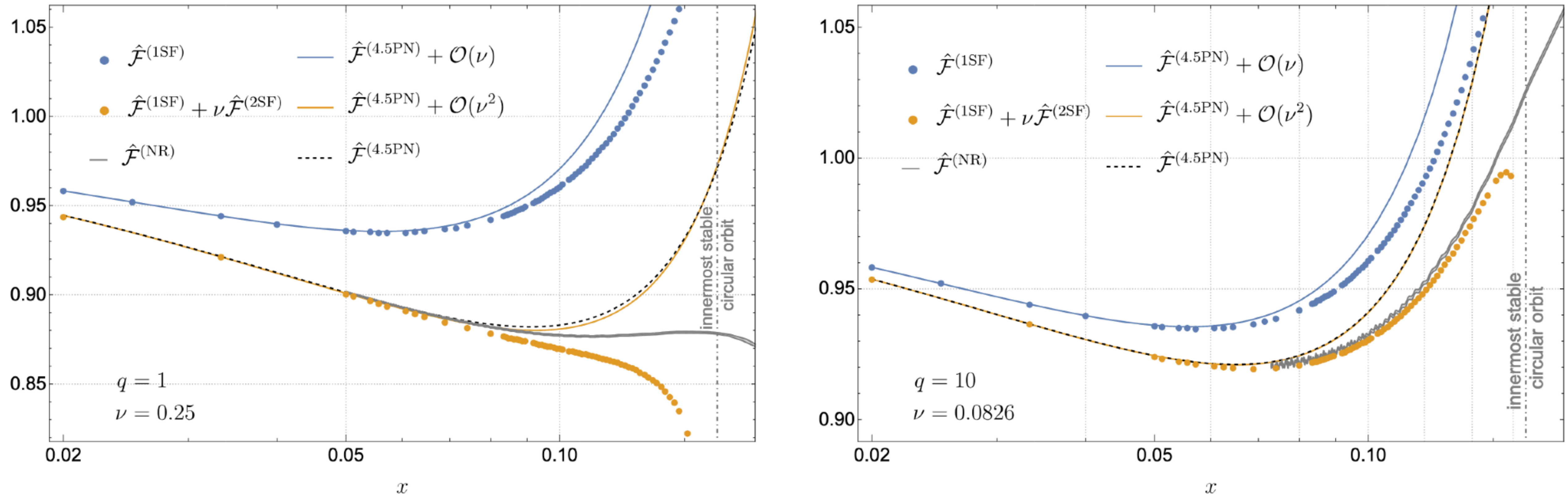
Extra slides

Flux: NR vs 4.5PN vs 2SF





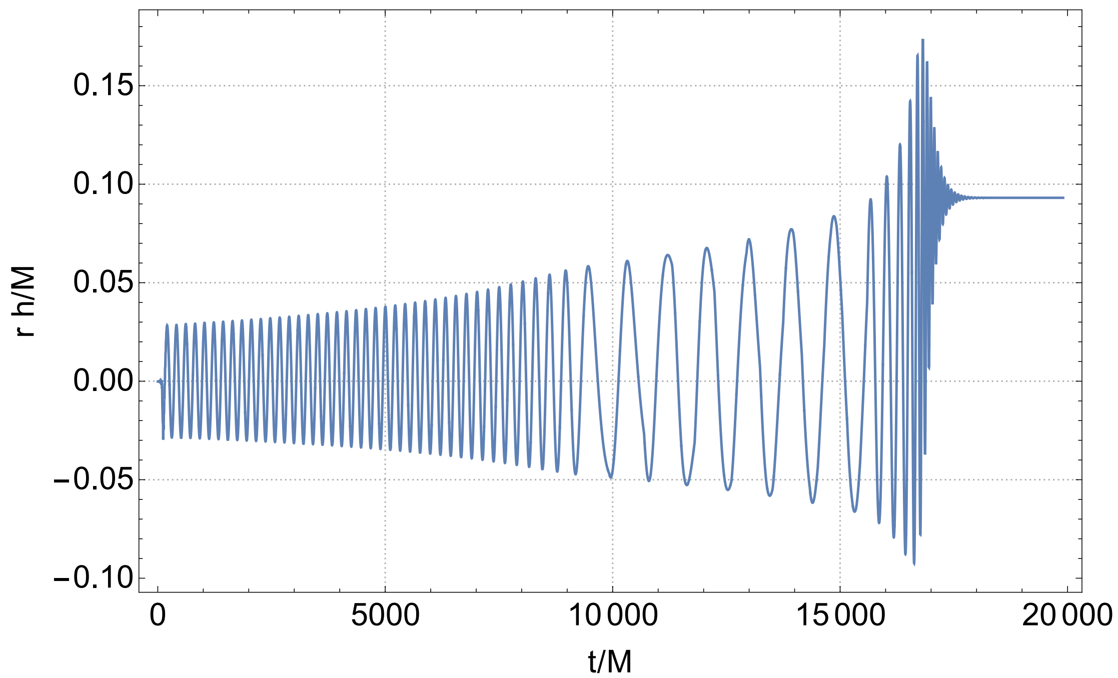
Flux: NR vs 4.5PN vs 2SF



- Truncating at $\mathcal{O}(\nu^3)$ appears to capture almost all of the PN result
- Suggests if one can compute PN flux through $\mathcal{O}(\nu^3)$ to high order this will be very effective for $q \geq 10$ binaries
- Chris Kavanagh, Adam Pound are working on this

Gravitational wave memory

Cunningham, Kavanagh, Trestini, Pound, NW

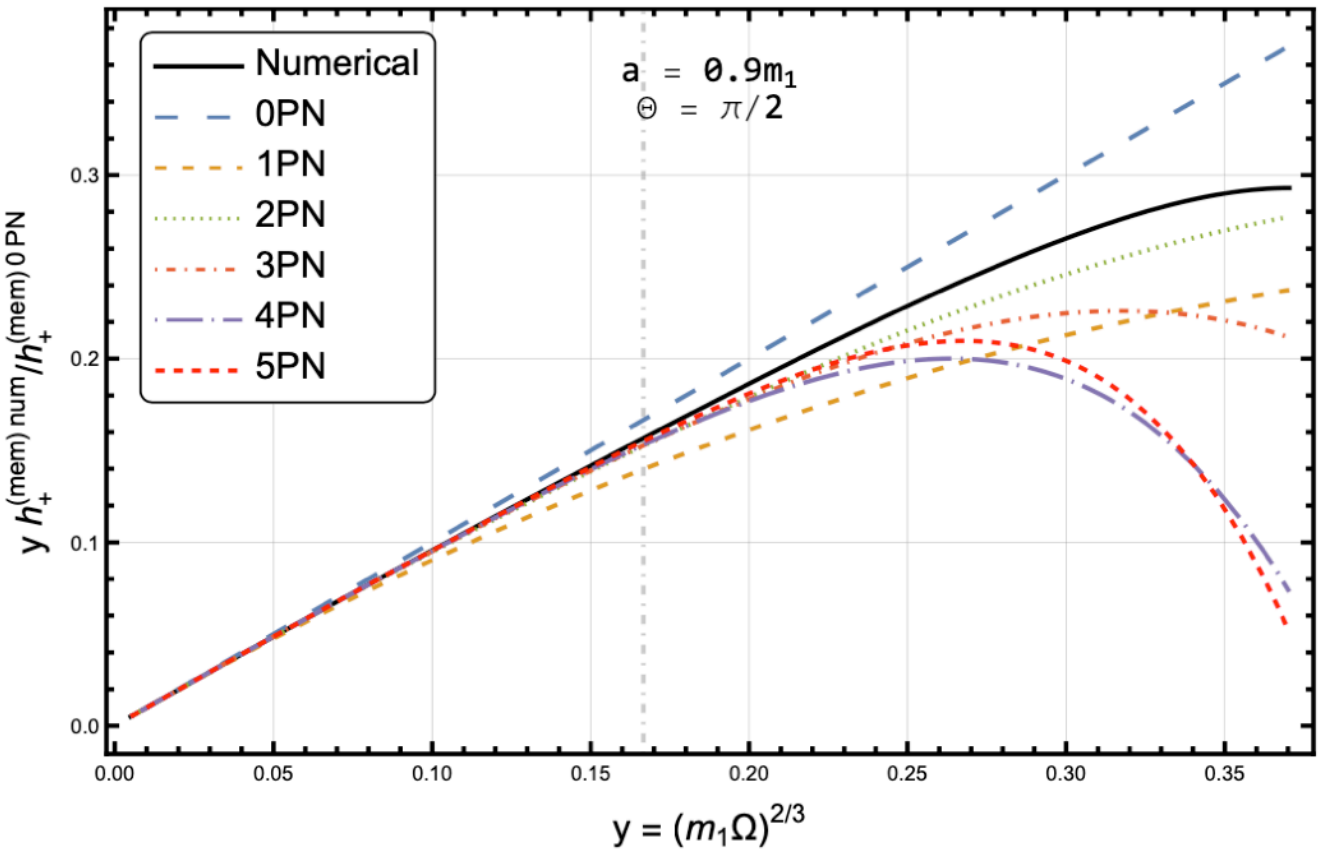


Memory in SXS:BBH:1124 (q=1)

- GW memory leads to a permanent displacement of the test masses after the GW has passed

In a forthcoming paper we:

- Calculate the memory from $h^{(1)}_{\alpha\beta}$ during inspiral for a quasi-circular orbit into a Kerr BH
- Numerical and 5PN-SF results



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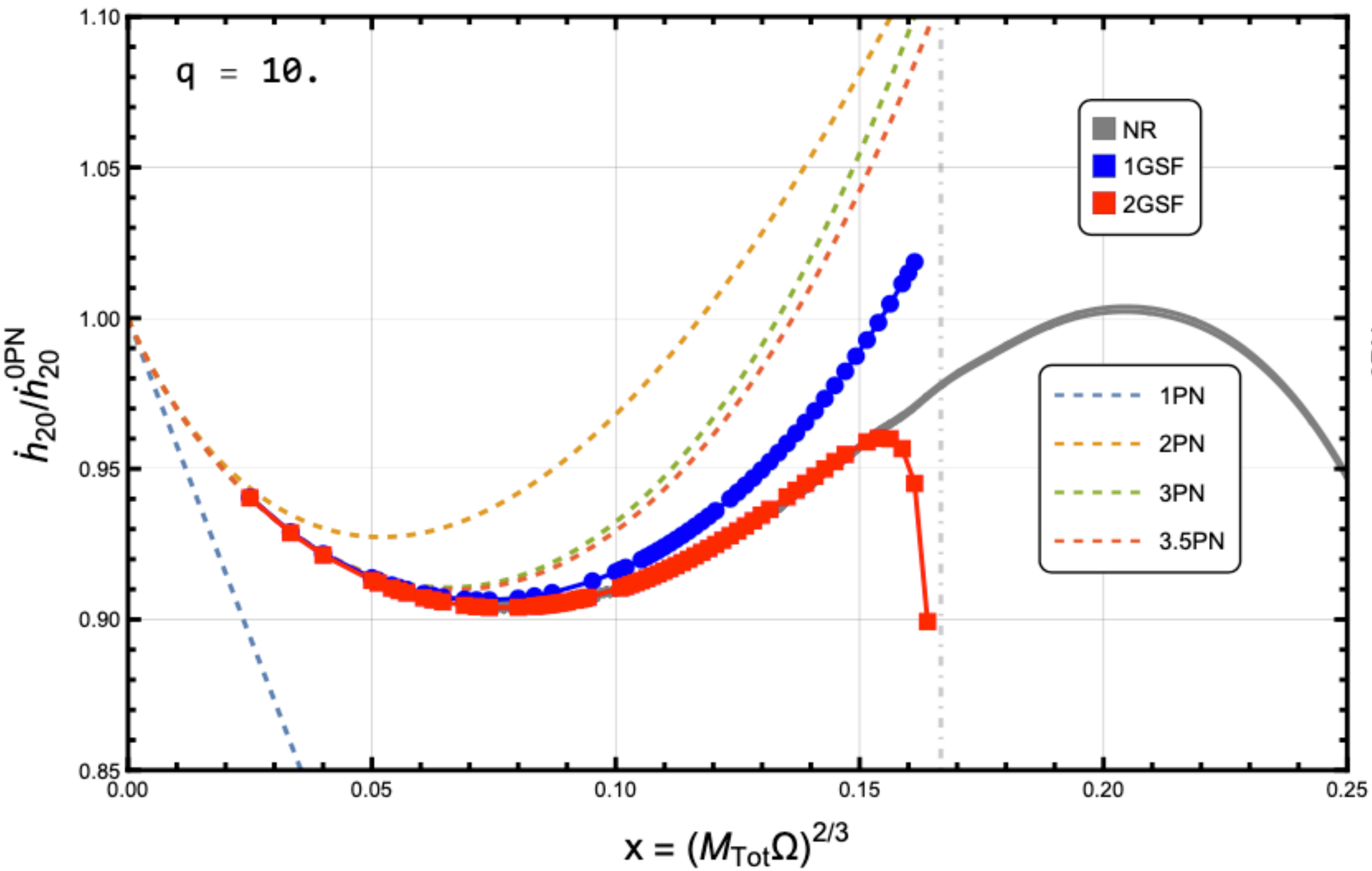
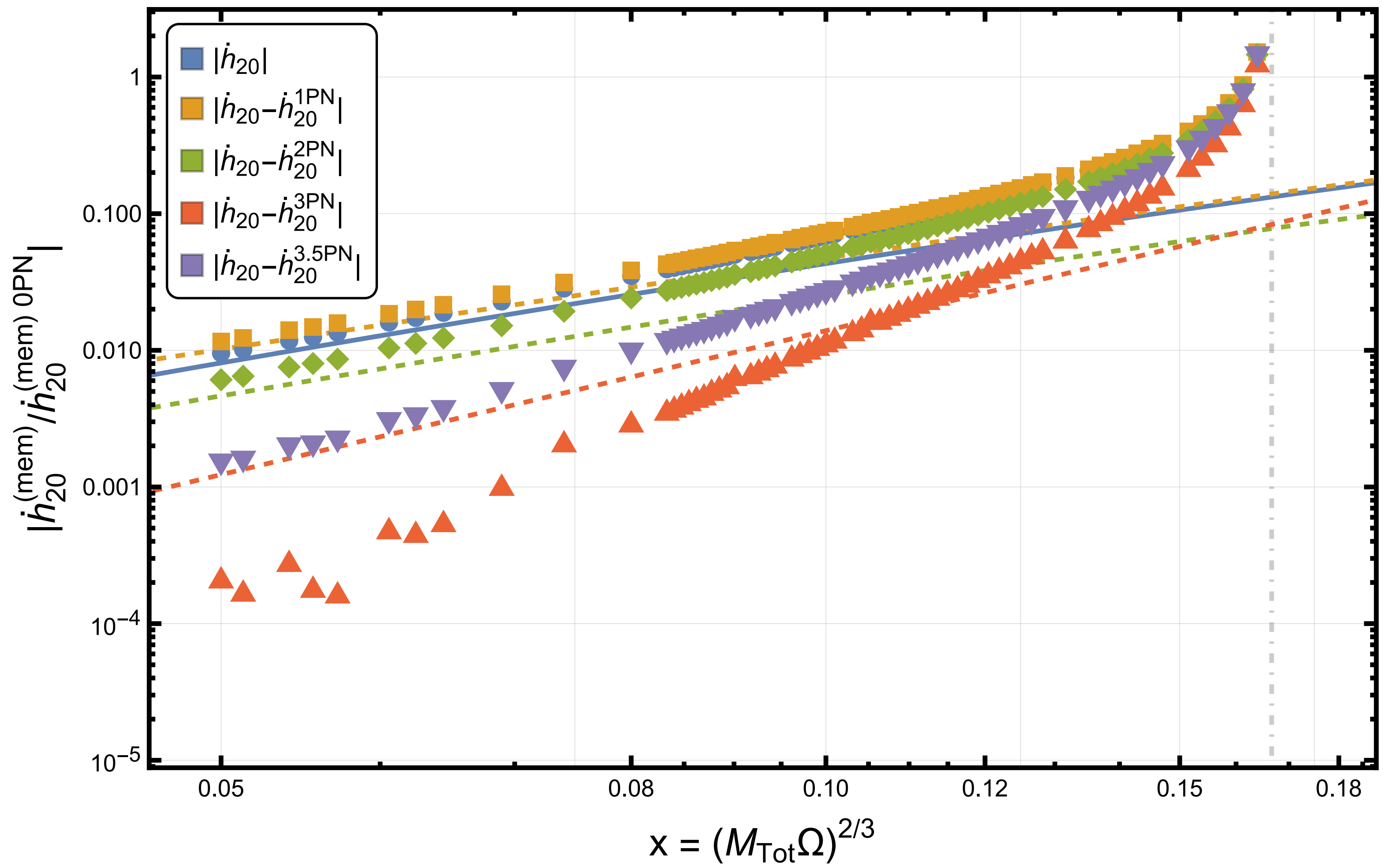


We also make the computation including $h^{(2)}_{\alpha\beta}$ and find good agreement with NR at, e.g., $q = 10$

Gravitational wave memory

Cunningham, Kavanagh, Trestini, Pound, NW



- Do not yet have 4.5PN memory (requires full waveform to 4.5PN)
- Comparison with new 3.5PN results show similar signs of mode mixing issue