Stable and causal fluids with bulk viscosity

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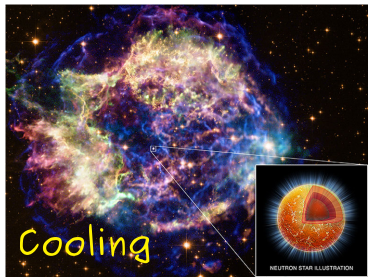
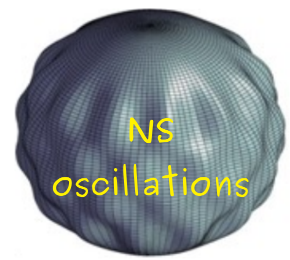
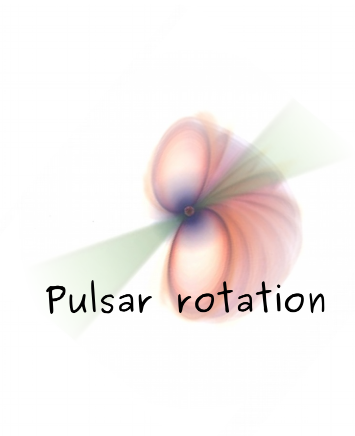
Dissipation in Neutron Stars







Define the R-modes
Instability window

Shear viscosity:
(out of equilibrium distribution)
(electron VS nuclei, protons, impurities)
(binary collisions of phonons)

For both cold NS
and proto-NS

According
to recent
estimates

Bulk viscosity:
(out of equilibrium distribution)
(nuclear reactions)
(phonon-phonon collisions)

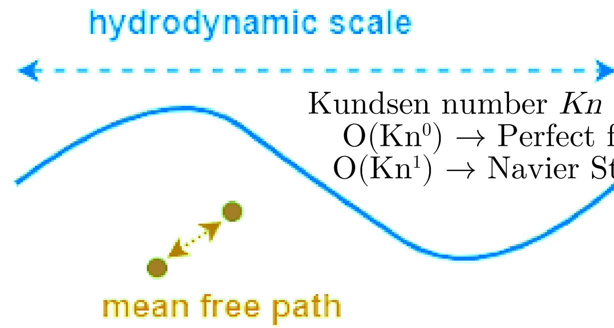
Late-stage
cooling

Vortex mediated friction:
(vortex motion in the superfluid)

Fundamental
for pulsar glitch
recoveries

Luminosity/radiation
(photon/neutrino emission)

What is hydrodynamics?

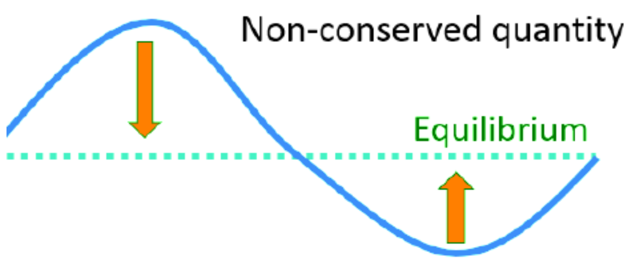


Kundsen number $Kn < 0.01$
$O(Kn^0) \to$ Perfect fluid
$O(Kn^1) \to$ Navier Stokes

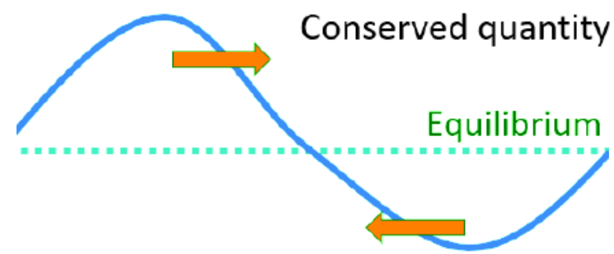
Emergent (large-scale) effective theory for **slow** processes

$\to$ "large-scale" wrt some relevant microscopic length
$\to$ "slow" wrt the microscopic timescales

Slow evolution characterised by those **few** DOF (**conserved** and **quasi-conserved** quantities) that equilibrate over macroscopic time-scales



Thermodynamic relaxation
Most DOF relax over the "micro" timescale
Local process (no need to "communicate")
$\to$ **fast** process ~ collision time



Conserved quantity out of equilibrium
A conserved charge can only be moved around
The only way to equilibrate is transfer across regions
$\to$ **slow** process for large systems and small gradients

The "slow" DOF play the role of effective fields $\to$ hydrodynamics is the low-frequency field theory for such DOF

Relativistic hydrodynamics: relativistic **thermodynamics** + relativistic **classical field theory**

Irreversible dynamics
Final equilibrium state must be **stable**

Causality & **well-posedness**
of the initial value problem

Three "steps"

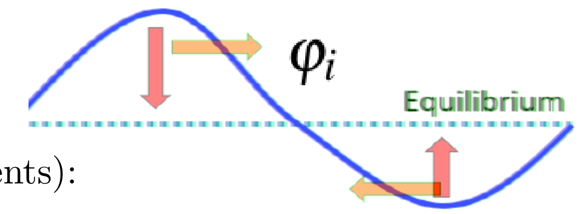
Review: Gavassino & MA, (2021)
Unified Extended Irreversible Thermodynamics and the stability of theories for dissipation

Defining a hydrodynamic model is a 3-step procedure:

1 – **Identify/choose the "slow" fields** φ (one for each **conservation** law)

Conservation of *energy-momentum* → e.g. velocity, temperature (4 quantities)

Conservation of *baryon number* → e.g. baryon chemical potential...



Additional field φ for each **quasi-conservation** law (relaxation due to "rare" events):

Chemical fractions in the presence of slow chemical reactions → reaction affinity

Stresses in the presence of friction/viscosity → strains...

2 – The fields φ locally characterize the state of the system → we have to provide the **constitutive relations**

$$T^{\nu\rho} = T^{\nu\rho}(\varphi_i, \nabla_\sigma \varphi_i, \nabla_\sigma \nabla_\lambda \varphi_i, \ldots)$$
$$n^{\nu} = n^{\nu}(\varphi_i, \nabla_\sigma \varphi_i, \nabla_\sigma \nabla_\lambda \varphi_i, \ldots)$$
$$s^{\nu} = s^{\nu}(\varphi_i, \nabla_\sigma \varphi_i, \nabla_\sigma \nabla_\lambda \varphi_i, \ldots)$$

Ideally provided by some microscopic theory
They define the physical meaning of the model
If only 1 conserved curent → "simple fluid"

3 – Prescribe some **equations of motion (EOM)**

$$\mathfrak{F}_h(\varphi_i, \nabla_\sigma \varphi_i, \nabla_\sigma \nabla_\lambda \varphi_i, \ldots) = 0$$

Ideally consistent with:
Causality & well-posedness
Stability of the equilibrium state

Steps 2 & 3: How? You decide... but they must be at least consistent with:

$$\nabla_\nu T^{\nu\rho} = 0$$

$$\nabla_\nu n^{\nu} = 0$$

For all the conserved quantities in (1)

$$\nabla_\nu s^{\nu} \geq 0$$

II Law of Thermodynamics:
"=" non-dissipative
">" dissipative

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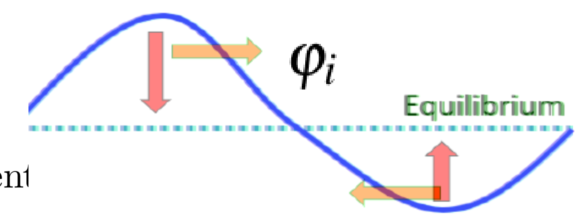

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3 – Prescribe some **equations of motion (EOM)**

$$\mathfrak{F}_h(\varphi_i, \nabla_\sigma \varphi_i, \nabla_\sigma \nabla_\lambda \varphi_i, \ldots) = 0$$

Each conservation law can be used as EOM...

... but if there are **quasi-conserved** currents then you **need to supply a “model”** for how the current is dissipated.

Example: for a current affected by chemical reactions: $\nabla_\mu J^\mu - \Xi \mathbb{A} = 0$

computed via chemical kinetics

Reaction “affinity”

The simplest example

Review: Gavassino & MA, (2021)
Unified Extended Irreversible Thermodynamics and the stability of theories for dissipation

Zero-order in the deviation from equilibrium → perfect fluid

$$\nabla_\alpha \mathcal{T}^\alpha_\beta = 0 \qquad \nabla_\alpha J^\alpha = 0 \qquad \text{(4+1 conservation equations)}$$

Energy-momentum tensor and baryon current:

$$J_\alpha = n u_\alpha \qquad \mathcal{T}_{\alpha\beta} = (p+\varrho) u_\alpha u_\beta + p g_{\alpha\beta} \qquad p = p(\varrho, n)$$

The "3 steps" are trivial: (1) choose your fields, (2) constitutive relations, (3) EOM

1) 4+1 conservation equations → need 5 DOF $\quad (\varphi_i) = (u^\sigma, \varrho, n)$

   0 quasi-conservation equations → need 0 "extra" DOF

2) Constitutive relations: $\quad J_\alpha = n u_\alpha \qquad \mathcal{T}_{\alpha\beta} = (p+\varrho) u_\alpha u_\beta + p g_{\alpha\beta} \qquad p = p(\varrho, n)$

3) EOM: the 4+1 conservation laws are enough

Neutron stars are "conductive" → many flows happen at the same time!

Electric current in MHD, superfluidity, heat conduction, neutrino and photon radiation...

...we typically need **more than 5 DOF**. Where do we get enough equations of motion?

Carter multifluid approach addresses thee three steps (1,2,3) for an arbitrary number of fluid "species"

Dissipative hydrodynamics

(Hiscock & Lindblom 1983)

$$\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$$

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha$$

Quantities as in the perfect fluid but there are additional "dissipative fluxes"

"First order" theories → the "dissipative fluxes" functions of the "perfect fluid variables" & derivatives

$$\pi = \pi(\varrho, u, \partial\varrho, \partial u, \dots) \text{ etc.}$$

Navier
Stokes
Fourier

→ Philosophy is that of the **gradient expansion**

→ 5 algebraic DOF (the same as the perfect fluid)

EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$

Israel-Stewart
hydrodynamics

"Second order" theories → the "dissipative fluxes" are new DOF

→ DOF: 5 (perfect fluid) + 1 (bulk) + 3 (heat flux) + 5 (traceless shear) = **14**

→ Philosophy is that of **moments method**

→ if only bulk & heat → 9 algebraic DOF

EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ $\qquad u^\mu \nabla_\mu \pi + \dots = 0$ etc.

Dissipative hydrodynamics: bulk viscosity

Bulk viscosity = dissipative response to compression and expansion (thermal/chemical re-equilibration)

No shear, ho heat, no superfluidity
$\rightarrow$ isotropy

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu}$$

"Expansion around equilibrium" approach $\rightarrow$ from kinetic theory up to a certain order ***O(f–f$_o$)***

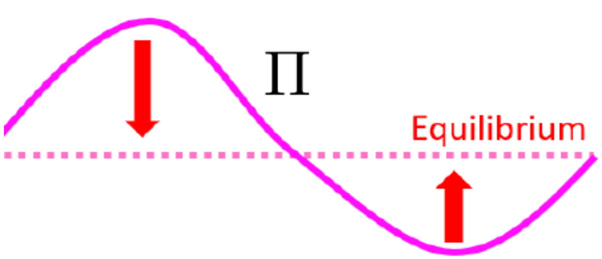
$$p = p^{\mathrm{eq}}(\rho, \epsilon) + \Pi$$

departure from perfect fluid

Zero order $\rightarrow$ relativistic perfect fluid: $\Pi = 0$

First order $\rightarrow$ Landau, Eckart (spurious gapped modes unstable, acausal) $\Pi = -\zeta \nabla_\lambda u^\lambda$

Second order $\rightarrow$ Israel-Stewart, Hiscock-Lindblom: stable and causal in the linear regime



$$\nabla_\mu (\Pi u^\mu) = -\frac{\Pi}{\tau} - \left(\frac{1}{\chi} - \frac{\Pi}{2}\right) \nabla_\mu u^\mu - \frac{\Pi}{2} u^\mu \nabla_\mu \left(\log \frac{\chi}{T^{\mathrm{eq}}}\right)$$

As time passes, full evolution becomes "first order".
Lindblom relaxation effect: tendency of dissipative fluids to lose DOF, by transforming dynamical equations into phenomenological constraints

Bulk viscosity (Navier-Stokes)

1 – Fields: 4-velocity (3 DOF)

s entropy density, n number density → equilibrium reference state

5 DOF of the Non-barotropic perfect fluid

2 - Constitutive relations: boil down to the perfect fluid for zero stress

$$T^{\mu\nu} = (e + p + \Pi) u^\mu u^\nu + (p + \Pi) g^{\mu\nu}$$

EOS ref. state: $p(n,s) \quad e(n,s)$

$$n^\mu = n u^\mu \qquad s^\mu = \left(s - \frac{\beta_0 \Pi^2}{2T} \right) u^\mu$$

Definition: $\Pi = -\zeta \nabla_\lambda u^\lambda$

3 - Hydrodynamic equations:

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\nabla_\mu n^\mu = 0$$

$$\nabla_\mu s^\mu = \frac{\Pi^2}{T\zeta} \qquad \textbf{(B)}$$

This theory is known to be **acausal** and **unstable** (Hiscock & Lindblom 1985)

→ homogeneous equilibrium state "tends to expolde"
→ PDE theory: the system is **not** hyperbolic
→ Impossible to set up an initial value on space surface and evolve in time
Why? Acausality → initial condition is influenced by the future

Bulk viscosity (Israel-Stewart)

(Hiscock & Lindblom 1983)

1 – Fields: 4-velocity

s entropy density, n number density → equilibrium reference state

Additional field: bulk stress → genuine additional DOF (deviation from equilibrium)

5 + 1 DOF
Non-barotropic perfect fluid
+ bulk stress

2 - Constitutive relations: boil down to the perfect fluid for zero stress

$$T^{\mu\nu} = (e + p + \Pi) u^\mu u^\nu + (p + \Pi) g^{\mu\nu}$$

$$n^\mu = n u^\mu \qquad s^\mu = \left(s - \frac{\beta_0 \Pi^2}{2T} \right) u^\mu$$

(A)

EOS of reference eq. state:

$p(n,s)$

$e(n,s)$

3 - Hydrodynamic equations:

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\nabla_\mu n^\mu = 0$$

$$\nabla_\mu s^\mu = \frac{\Pi^2}{T\zeta}$$

(B)

(A) & (B) → "Telegraph" equation for the bulk stress

Equation for the evolution of the bulk stress

$$\boxed{\Pi = -\zeta \left[\nabla_\mu u^\mu + \beta_0 \, u^\mu \nabla_\mu \Pi \right.} + \left. \frac{1}{2} \Pi \, T \, \nabla_\mu \left(\frac{\beta_0 u^\mu}{T} \right) \right]$$

Navier-Stokes

Maxwell-Cattaneo

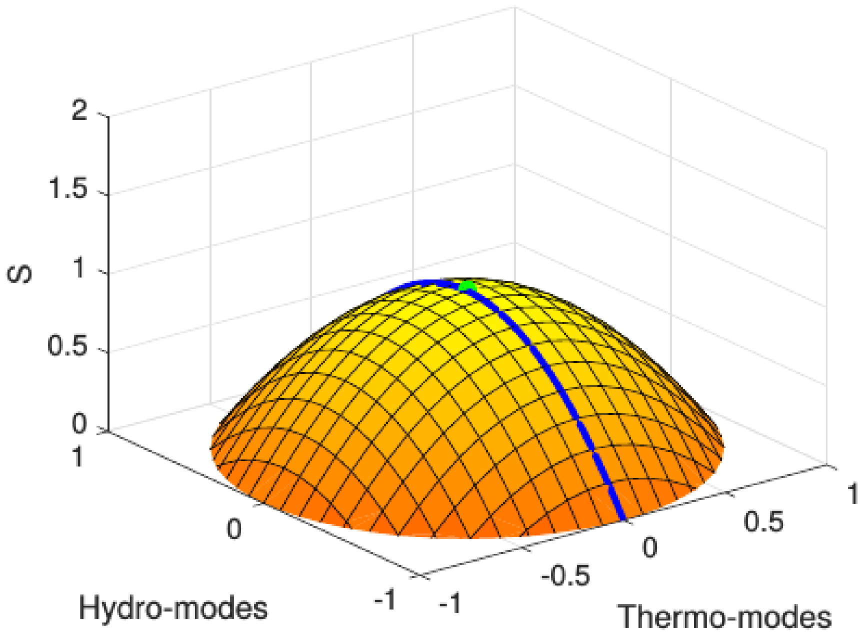
Relaxation timescale: $\zeta \, \beta_0$

new "thermo" modes

(relaxation transients, no need for gradients to have dynamics)

Stability (entropy interpretation)

Review: Gavassino & MA, (2021)
Unified Extended Irreversible Thermodynamics and the stability of theories for dissipation



Entropy of a stable fluid

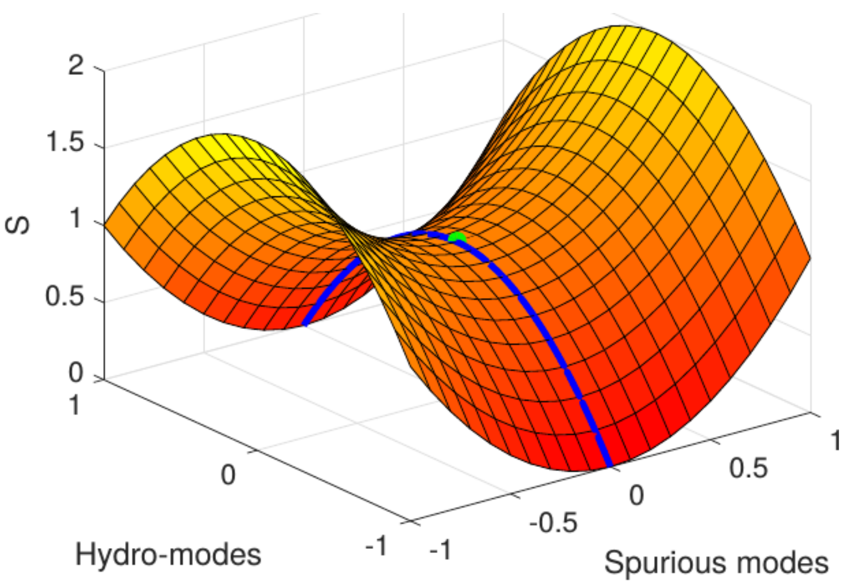
Maximum = equilibrium state (homogeneous perfect fluid state)

Blue line = states accessible in the relaxed limit (Navier-Stokes)

Any deviation from equilibrium reduces the entropy and therefore must decay when the second law is imposed.

Thermo-modes (gapped) $\rightarrow$ Non-equilibrium thermodynamics

Hydro-modes (gapless) $\rightarrow$ Navier-Stokes-Fourier approach



Entropy of relativistic Navier-Stokes

Saddle point = equilibrium state (homogeneous perfect fluid state)

Blue line = states accessible by non-rel limit

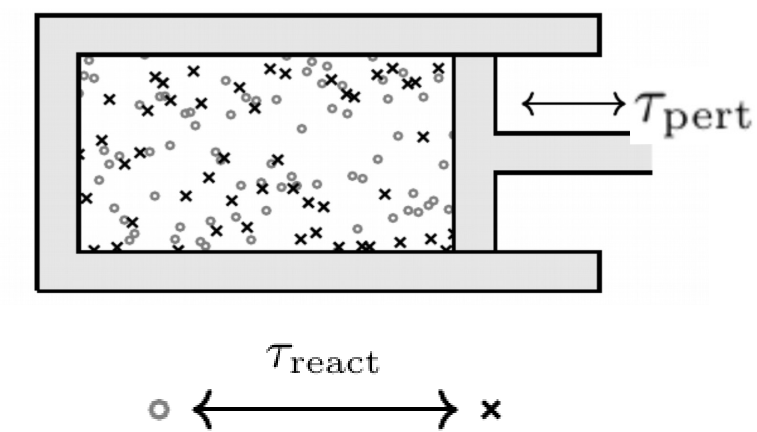
Hydro-modes $\rightarrow$ damped if we impose the validity of the II Law

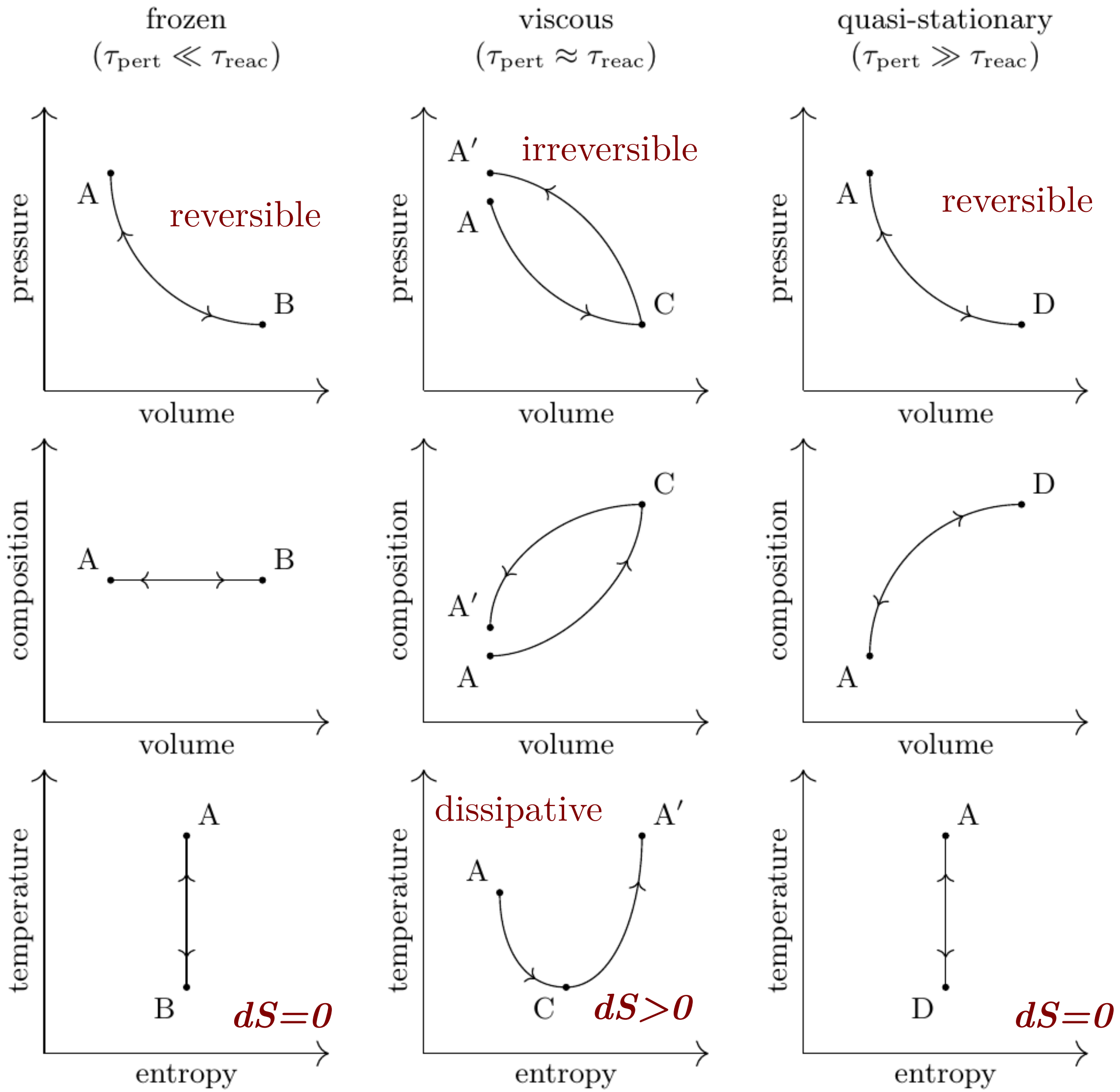
Spurious (gapped) modes $\rightarrow$ the II Law forces them to grow indefinitely, originating the instability.

Bulk viscosity

Bulk viscosity = dissipative response to compression and expansion (thermal/chemical re-equilibration)

Maximum dissipation when:
reaction time ~ perturbation period





Gavassino+ 2021 CQG

1) Every bulk viscous fluid can be mapped into a reacting mixture (not vice-versa)
2) It is easy to formulate causal & stable hydro for reacting mixtures
3) Simpler to find stability-causality criteria for the mixture rather than Israel-Stewart
(see Gabriele's talk)

Bulk viscosity: theoretical approaches

Camelio+
arXiv:2204.11809
arXiv:2204.11810

Bulk viscosity = dissipative response to compression and expansion (thermal/chemical re-equilibration)

No shear, ho heat, no superfluidity
$\rightarrow$ isotropy

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu}$$

"Expansion around equilibrium" approach $\rightarrow$ full "Israel-Stewart" by Hiscock-Lindblom (1983)

$$p = p^{\mathrm{eq}}(\rho, \epsilon) + \Pi$$

$$\nabla_\mu (\Pi u^\mu) = -\frac{\Pi}{\tau} - \left(\frac{1}{\chi} - \frac{\Pi}{2}\right) \nabla_\mu u^\mu - \frac{\Pi}{2} u^\mu \nabla_\mu \left(\log \frac{\chi}{T^{\mathrm{eq}}}\right)$$

"Multifluid" approach $\rightarrow$ no "expansion", only assumes "separation of timescales"

Meaning: each independent "reaction coordinate" that evolves slowly goes into the EOS

$$p = p(\rho, \epsilon, \{Y_i\}_i)$$

$$\mathrm{d}u = \frac{p}{\rho^2} \mathrm{d}\rho + \frac{T}{m_\mathrm{n}} \mathrm{d}s - \sum_i \frac{\mathbb{A}^i}{m_\mathrm{n}} \mathrm{d}Y_i$$

$$\nabla_\mu (\rho u^\mu) = 0$$
$$\nabla_\mu (T^{\mu\nu}) = -\mathcal{Q} u^\nu$$
$$\nabla_\mu (\rho Y_i u^\mu) = m_\mathrm{n} \mathcal{R}_i$$

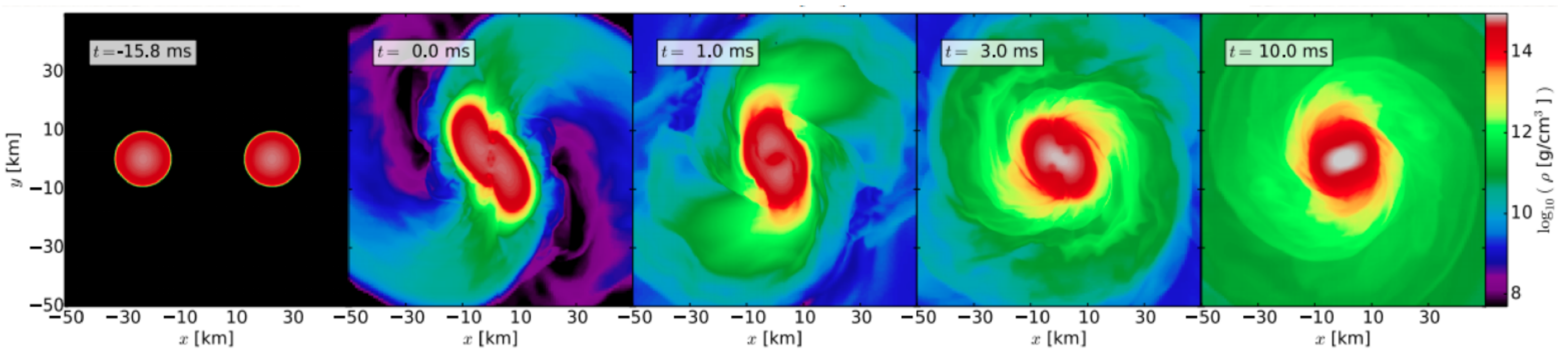
$\mathbb{A}^i = 0$ Chemical equilibrium

Viscous effects in neutron star mergers?

Duez+ PRD (2004), Shibata+ PRD (2017), Most+ PRL 2019, Hammond+ PRD 2021, Celora+ CQG 2022...

Previous understanding → viscous effects negligible

Based on the simulations/knowledge at that time: temperatures not so large, system very smooth, gradients too small



Example: Alford+ PRL (2018) → rough estimates of the importance of dissipation channels

Simulations (**ideal fluid!**): estimates for macroscopic scale L of fluid variables gradients

From microscopic arguments: estimate for the characteristic microscopic scales l in the system

Knudsen number $\sim l/L$ may not be small in some cases (viscosity may affect the GW signal)

Shear → Relevant for trapped neutrinos if T > 10 MeV and gradients at small scales ~ 10 m (turbulence)

Heat → Relevant for trapped electron neutrinos if T > 10 MeV and gradients at scales ~ 100 m

Bulk → Should affect **density oscillations after merger!** Alford, Harris, PRC (2019)

*"Effects of **bulk viscosity** should be consistently included in merger simulations. This has not been attempted before and requires a formulation of the relativistic-hydrodynamic equations that is **hyperbolic** and **stable**"*

First attempt (postmerger): Camelio+ **arXiv:2204.11809** and **arXiv:2204.11810** (both PRD, 2023)

Hydro-bulk-1D

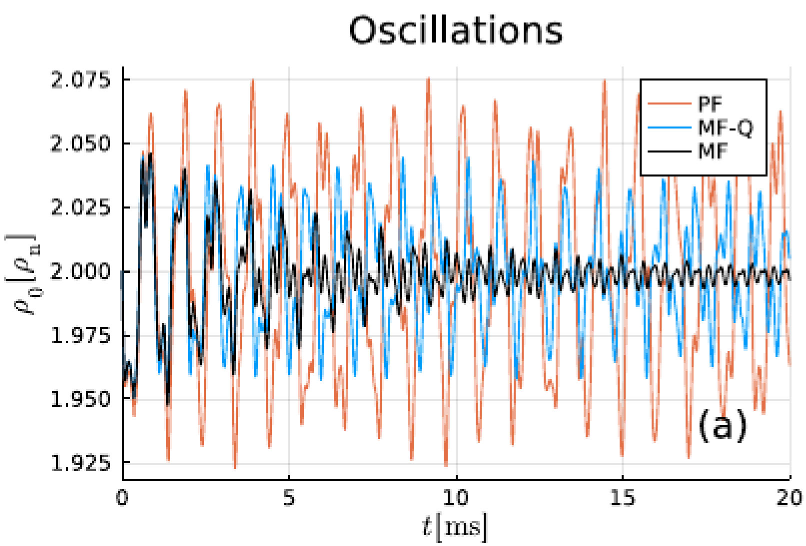
Camelio+
arXiv:2204.11809
arXiv:2204.11810

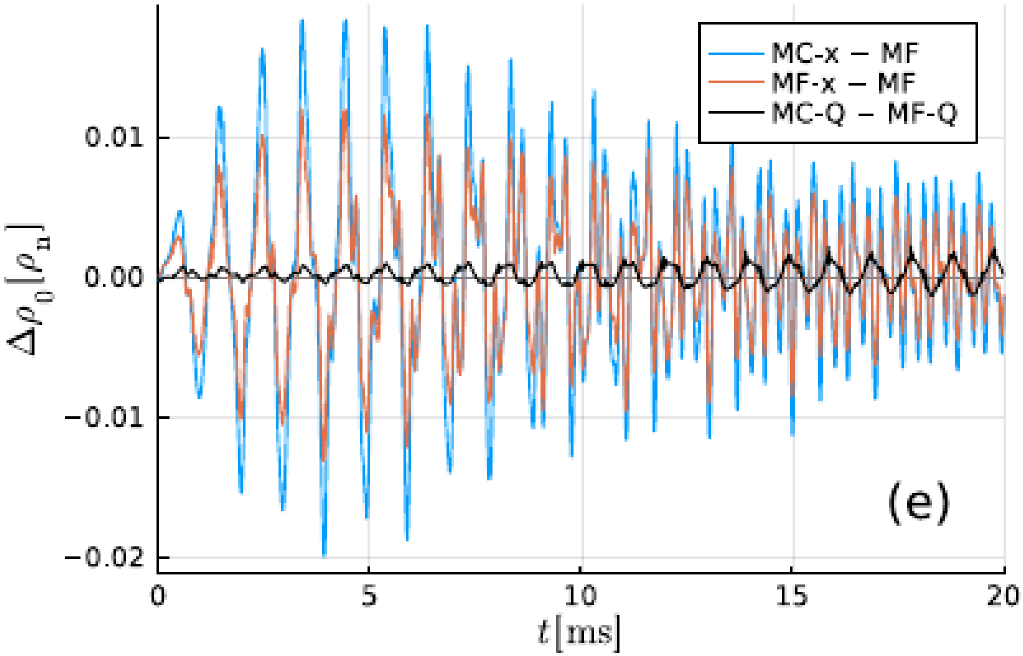
First simulation of a NS with the **complete Hiscock-Lindblom model** of bulk viscosity.

One-dimensional, GR code, publicly available: Giovanni Camelio, `hydro-bulk-1D` (2022)

We include the **energy loss due to the luminosity of the reactions** in the bulk stress formulation.

Bulk & luminosity should be consistent (the same reactions are responsible for both)





PF = perfect fluid

MF = multifluid out of beta equilibrium (npe matter)

MF-Q = multifluid out of beta equilibrium (npe matter) + consistent neutrino luminosity

MC = “equivalent” Maxwell-Cattaneo

mURCA & dURCA effect in NS-NS mergers

Most+ PRL 2024 (arXiv: 2207.00442v2)

First simulation of NS-NS merger to quantify the effect of mURCA & dURCA mediated bulk viscosity:

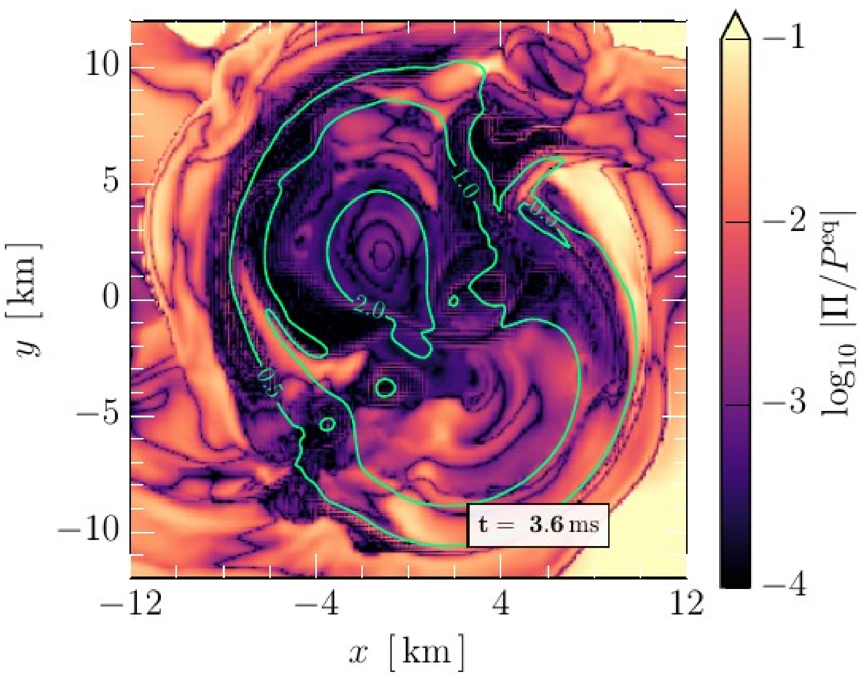
→ Same theoretical "multicomponent" scheme of Camelio+ 2023 (i.e. same equations but different EOS & rates)

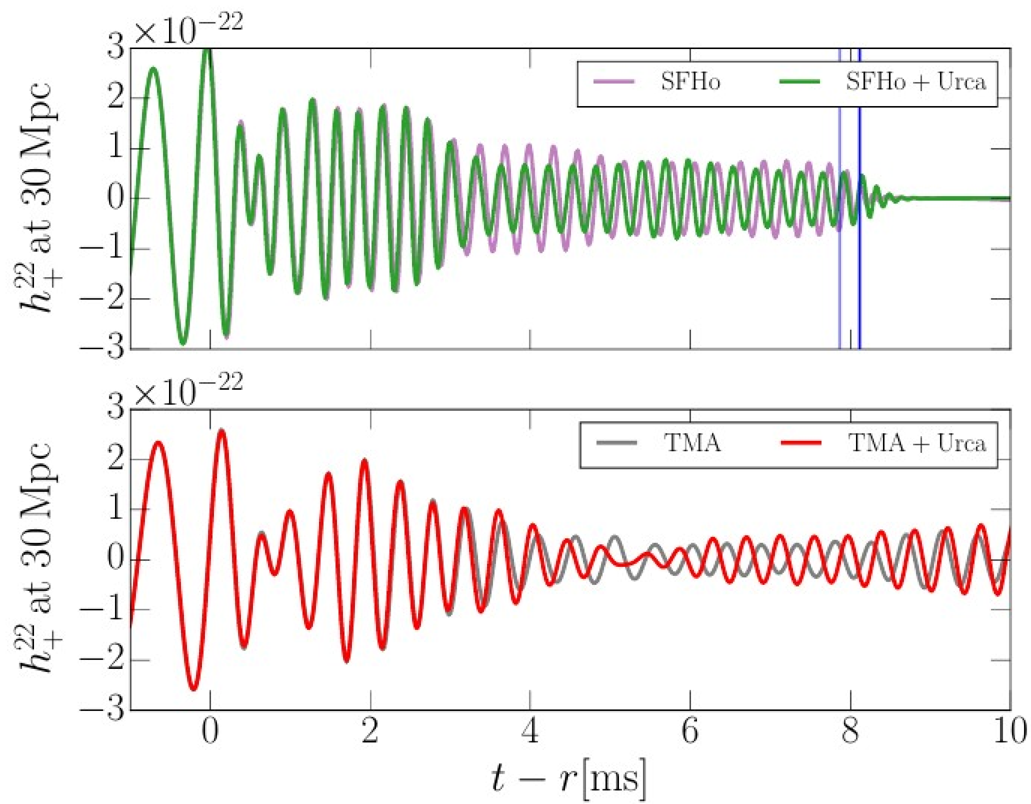
→ Gravitational wave strain extracted for two EOSs

The shift in the dominant GW frequency depends on the assumptions on neutrino transparency:

→ Neutrino transparent regime (Urca vs. frozen composition) feature characteristic shifts of Δf ≃ 40 Hz

→ When including neutrino trapping above T > 1 MeV, overall shift of about 50 Hz





Three "steps": the multifluid approach

Review: Gavassino & MA, (2021)
Unified Extended Irreversible Thermodynamics and the stability of theories for dissipation

Carter's multifluid framework provides a simple solution when the #DOF > 5

1 – **Identify/choose the "slow" fields** φ

Assume that there is a set of currents which completely specify the macrostate of the system

$$(\varphi_i) = (n_i^\sigma)$$

Two can be taken to be: n^σ s^σ

...the remaining ones depend on the non-equilibrium thermodynamic properties of the system.
Some currents may be "locked" together, others can flow independently.

2 – **Constitutive relations:** the only non-trivial one is the energy-momentum

$$T^{\nu\rho} = T^{\nu\rho}(n_i^\sigma)$$

...Carter assumes that it can be derived from a "master function" with constitutive relation:

$$\Lambda = \Lambda(n_i^\sigma, g_{\sigma\lambda})$$

3 – Prescribe some **equations of motion (EOM)**

...again, derived from the "master function" $\Lambda = \Lambda(n_i^\sigma, g_{\sigma\lambda})$

What is needed:

- Physically motivated identification of the set of currents (1)
- Constitutive relation for a single scalar function (2)

$$\Lambda = \Lambda(n_i^\sigma, g_{\sigma\lambda})$$

This has great practical value:

Energy-momentum is "derived"

Full set of EOMs is "derived"

Carter's multifluid (no dissipation)

Carter, *Covariant Theory of Conductivity in Ideal Fluids* (1987) → Relativistic
Prix, *Variational description of multi-fluid hydrodynamics* (2002) → Newtonian

Variational approach based on Einstein-Hilbert+Matter action

$$I_{EH} = \int_{\mathcal{M}} \frac{R}{16\pi}\sqrt{-g}\, d_4x \qquad I_m = \int_{\mathcal{M}} \Lambda\sqrt{-g}\, d_4x$$

$$T_{\nu\rho} = -\frac{2}{\sqrt{-g}}\frac{\delta I_m}{\delta g^{\nu\rho}}$$

Seems reasonable to try with a tentative "Lagrangian" of the kind

$$\Lambda(n_x^\nu) := \Lambda(-g_{\rho\nu} n_x^\rho n_y^\nu)$$

→ simpler to prescribe a Lagrangian than the equations of motion or the energy-momoentum tensor
→ easy to add extra macroscopic fields (e.g. MHD)
→ straightforward to incorporate additional fluid components (useful for mixtures)
→ suitable for **conduction**

Basic requirements:

→ Should reduce to the usual **perfect fluid** if 1 current
→ Simple extensions of the perfect fluid (many constituents: "perfect multifluid") → $\nabla_\nu n_x^\nu = 0$
→ Connection with superfluid thermodynamics: Gavassino & MA, CQG (2020)
The "hydro" of the homogeneous state is just "thermo"
Equilibrium state with relativistic **persistent currents**

Carter's multifluid (no dissipation)

Tentative: proceed as in usual field theory (**unconstrained** variation)

→ Lagrangian: $\Lambda(n^\nu_x) := \Lambda(-g_{\rho\nu} n^\rho_x n^\nu_y)$

→ Canonical momenta: $\mu^x_\nu := \dfrac{\partial \Lambda}{\partial n^\nu_x} = \mathcal{B}^x n_{x\nu} + \sum_{y \neq x} \mathcal{A}^{xy} n_{y\nu}$

→ Entrainment: $\mathcal{B}^x := -2 \dfrac{\partial \Lambda}{\partial n^2_{xx}} \qquad \mathcal{A}^{xy} := - \dfrac{\partial \Lambda}{\partial n^2_{xy}}$

→ Energy-momentum:

$$T^\nu{}_\rho = \Psi \delta^\nu{}_\rho + \sum_x n^\nu_x \mu^x_\rho$$

$$\Psi = \Lambda - \sum_x n^\rho_x \mu^x_\rho$$

In short:

the usual variation of the action does not work: the action is minimized if there is no fluid at all (zero density everywhere)

Problem #1! Equations of motion (ignore surface terms in the action):

This goes with the "gravity" part

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left[\sum_{\mathrm{x}} \mu^{\mathrm{x}}_a \delta n^a_{\mathrm{x}} + \frac{1}{2}\left(\Lambda g^{ab} + \sum_{\mathrm{x}} n^a_{\mathrm{x}} \mu^b_{\mathrm{x}}\right) \delta g_{ab}\right]$$

Trivial & useless dynamics!

Problem #2!

No conservation laws!

$$\nabla_\nu n^\nu_x = 0$$

Where is this? Nowhere!

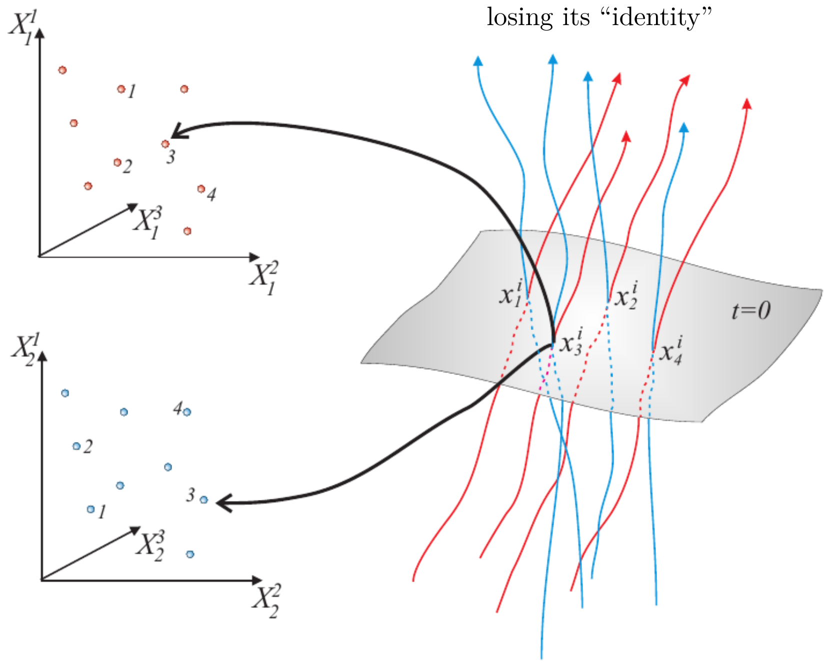
Not surprising since we used unconstrained variations of the currents

Carter's multifluid (constrained)

Solution: we have to guarantee the identity of each fluid element's worldline!

...keep the **definitions**:

Each component has its own set of worldlines that it follows without losing its "identity"



→ **Not a Lagrangian** but "master function":

$$\Lambda(n^\nu_x) := \Lambda(-g_{\rho\nu} n^\rho_x n^\nu_y)$$

→ Canonical momenta: $\mu^x_\nu := \dfrac{\partial \Lambda}{\partial n^\nu_x} = \mathcal{B}^x n_{x\nu} + \sum_{y \neq x} \mathcal{A}^{xy} n_{y\nu}$

→ Entrainment: $\mathcal{B}^x := -2 \dfrac{\partial \Lambda}{\partial n^2_{xx}} \qquad \mathcal{A}^{xy} := -\dfrac{\partial \Lambda}{\partial n^2_{xy}}$

→ Energy-momentum: $T^\nu{}_\rho = \Psi \delta^\nu{}_\rho + \sum_x n^\nu_x \mu^x_\rho$

$$\Psi = \Lambda - \sum_x n^\rho_x \mu^x_\rho$$

...but modify the variation procedure (variations of the currents constrained to keep identity of worldlines)

The "real" Lagrangian is in terms of the "trajectories" (like for the point particle)

$$\mathcal{L}[g, X^\alpha_s, X^\alpha_i] = \Lambda(-g_{\rho\nu}\, n^\rho_x[g, X^\alpha_x]\, n^\nu_y[g, X^\alpha_y])$$

$$\nabla_\nu n^\nu_x = 0$$

Where is this?

The **domain of the action is restricted** so that conservation is ensured both on-shell and off-shell

Carter's multifluid: non-dissipative dynamics

Carter, *Covariant Theory of Conductivity in Ideal Fluids* (1987)

In a nutshell: $\Lambda = \Lambda(n^2_{xy})$ $\quad n^2_{xy} := -n^\nu_x n_{y\nu}$ $\quad I = \int \left(\frac{R}{16\pi} + \Lambda \right) \sqrt{-g}\, d^4x$

1) Variational procedure to ensure the conservation of the number density currents

Taub, PhysRev 94 (1954), Comer & Langlois CQG 10 (1993), Andersson & Comer, LRR (2007)

→ domain of the action restricted by imposing that $\nabla_\nu n^\nu_x = 0$ both on-shell and off-shell: variations of the currents are taken in the "Taub form"

$$\delta n^\nu_x = \xi^\rho_x \nabla_\rho n^\nu_x - n^\rho_x \nabla_\rho \xi^\nu_x + n^\nu_x \left(\nabla_\rho \xi^\rho_x - \frac{1}{2} g^{\rho\sigma} \delta g_{\rho\sigma} \right)$$

ξ_x = trajectory displacements

→ Variation of the action produced by the ξ_x (ignoring the boundary terms)

$$\delta I = \int \left(\sum_x f^x_\nu \xi^\nu_x \right) \sqrt{-g}\, d^4x \qquad f^x_\nu := 2 n^\rho_x \nabla_{[\rho} \mu^x_{\nu]}$$

Equations of motion: $n^\rho_x \nabla_{[\rho} \mu^x_{\nu]} = 0$ for each constituent (all coupled by entrainment!)

$$\mu^x_\nu := \frac{\partial \Lambda}{\partial n^\nu_x} = \mathcal{B}^x n_{x\nu} + \sum_{y \neq x} \mathcal{A}^{xy} n_{y\nu}$$

If all EOM satisfied → $\nabla_\nu T^{\nu\rho} = 0$

(so this may replace 1 EOM)

Kelvin's theorem

Perfect multifluid → EOM are: $n^a_{\mathrm{x}} \nabla_{[a}\mu^{\mathrm{x}}_{b]} = 0$

...what's their meaning?

Consider the usual 1-component perfect fluid (at T=0 or "barotropic")

Take $\nabla_\alpha T^{\alpha\beta} = 0$ and project orthogonally to the 4-velocity with $\perp_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$

$$2n^\mu \nabla_{[\mu}\mu_{\nu]} + (\nabla_\mu n^\mu)\,\mu_\nu = 0 \qquad \mu_\mu = \mu u_\mu$$

4-momentum per baryon

relativistic Kelvin theorem: vorticity is transported by the 4-velocity

... ofc this is useful to model cold neutron stars interiors (you'd like to know how vortices move)

→ **dissipation in superfluids:** when vortices do **NOT** flow with the current!

→ This also tells us how to extend Carter's perfect multifluid to include dissipation...

...the Lagrangian becomes a "generating function"

Carter's multifluid: "generating function"

Dissipation $\rightarrow$ entropy is not conserved... we have to break currents's conservation without falling into the useless "unconstrained" model

Keep the central postulate: there is a function $\Lambda = \Lambda(n_i^\sigma, g_{\sigma\lambda})$

Energy-momentum obtained as $T^{\nu\rho} = \frac{2}{\sqrt{|g|}} \frac{\partial(\sqrt{|g|}\Lambda)}{\partial g_{\nu\rho}}\Big|_{\sqrt{|g|}\, n_i^\sigma}$

The Lagrangian is downgraded to be just a "generating function" for the energy-momentum and the canonical momenta

$$\boxed{\begin{aligned} T^\mu{}_\nu &= \Psi \delta^\mu{}_\nu + \sum n_{\rm x}^\mu \mu_\nu^{\rm x} \\ \Psi &= \Lambda - \sum \mu_\mu^{\rm x} n_{\rm x}^\mu \end{aligned}}$$

$$\mu_\nu^h := \frac{\partial \Lambda}{\partial n_h^\nu}\Big|_{n_i^\sigma, g_{\sigma\lambda}}$$

Equations of motion: just take the divergence of the energy-momentum and see... $\Lambda = \Lambda(n_i^\sigma, g_{\sigma\lambda})$ does not give the EOM anymore!

$$\nabla_\nu T^\nu{}_\rho = \sum_h \left(\mu_\rho^h \nabla_\nu n_h^\nu + 2 n_h^\nu \nabla_{[\nu}\mu_{\rho]}^h \right)$$

$$\mu_\rho^h \nabla_\nu n_h^\nu + 2 n_h^\nu \nabla_{[\nu}\mu_{\rho]}^h = \mathfrak{R}_\rho^h$$

$$\sum_h \mathfrak{R}_\rho^h = 0 \qquad \mathfrak{R}_\rho^n n^\rho = 0 \qquad \frac{\mathfrak{R}_\rho^s s^\rho}{\mu_\lambda^s s^\lambda} \geq 0$$

Baryon conservation

II Law

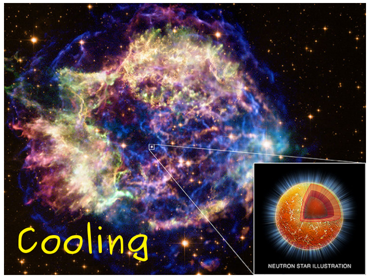
Is this a viable scheme for dissipation in relativity?

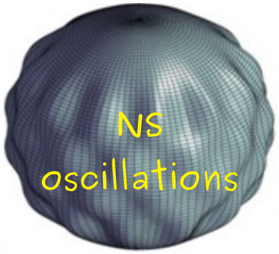
Is it more or less universal than "Israel-Stewart"?

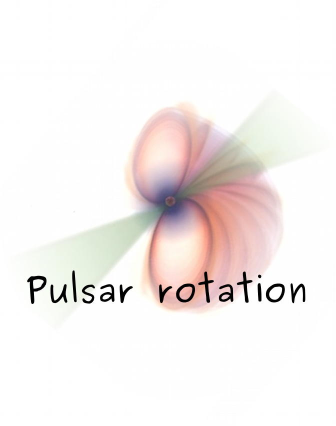
Causality, stability? $\rightarrow$ difficult question but linearly stable & causal for "simple" forces

Dissipation in Neutron Stars

"Generating function" approach good for... ?







Shear viscosity:
(out of equilibrium distribution)
(electron VS nuclei, protons, impurities)
(binary collisions of phonons)

Maybe yes but NO
(need to introduce a "flux of a flux" together with the other currents)

Bulk viscosity:
(out of equilibrium distribution)
(nuclear reactions)
(phonon-phonon collisions)

Can be "better" than Israel-Stewart
→ can evolve multiple independent chemical fractions **(arXiv:2003.04609)**
→ upgrade Israel-Stewart to superfluid matter **(arXiv:2110.05546)**

Vortex mediated friction:
(vortex motion in the superfluid)

This is what Carter's multifluid does and others can not **(arXiv:2012.10288)**

Luminosity/radiation
(photon/neutrino emission)

Simple fluid + non-conserved ultra-relativistic fluid
→ **M1 radiation hydrodynamics** (**arXiv:2007.09481**)

ArXiv: 2110.05546

Superfluid + bulk viscosity + heat

How to combine **superfluidity** with **dissipation**? Two different languages are used:

Relativistic theory for **superfluidity** → **Carter's perfect multifluid**

Relativistic theory for **dissipation** → "Israel-Stewart" / "second-order"

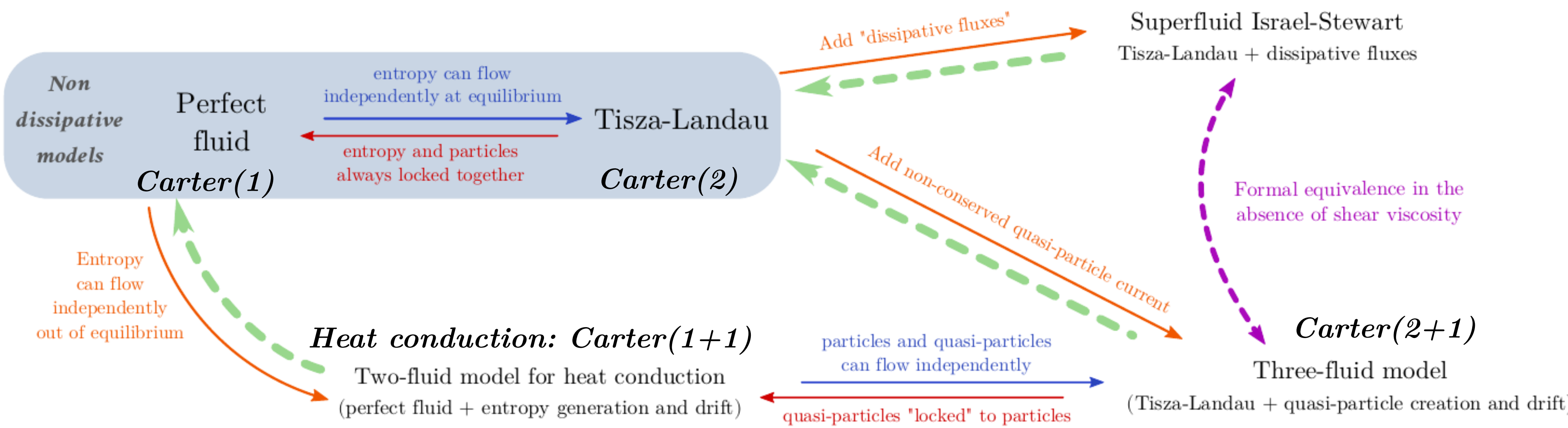
They do not seem to have anything in common → how to merge the two consistently?

Close to equilibrium: ***Carter(N+1)* = *Carter(N)* + Israel-Stewart dissipation (bulk viscosity & heat)**

→ If ***Carter(N)*** is non-dissipative, in ***Carter(N+1)*** we have to unlock the II Law

→ Advantage: easy to derive causality & stability conditions for *Carter(N+1)*

→ II Law valid both on and off-shell: the equations can be solved!



Message: How to include **bulk & heat dissipation** in a **superfluid?** → **Add 1 non-conserved current!**

Which current depends on microphysics (phonons, rotons, photons, non-superfluid baryons in NS crust)

Superfluid + bulk viscosity + heat

Carter's dissipative multifluid with 3 currents: $\Lambda = \Lambda(n^2, s^2, z^2, n^2_{ns}, n^2_{nz}, n^2_{sz})$

$$\nabla_\nu n^\nu = 0 \qquad \nabla_\nu s^\nu \geq 0 \qquad \nabla_\nu z^\nu \neq 0$$

Non-conserved quasiparticle number

$z + z \rightleftharpoons z + z + z$

Equilibrium: non-dissipative 2-fluid of Tisza-Landau (persistent current of entropy wrt particles)

We only need density and entropy to define the state $\rightarrow z^\nu = z^\nu_{\text{eq}}(n^\rho, s^\rho)$

Out-of-equilibrium: $z^\nu = z^\nu(n^\rho, s^\rho, \Pi, Q^\rho)$

Israel-Stewart is "perturbative": dissipative fluxes *(Π, Q)* are defined as deviations from equilibrium.

Expand around $z^\nu(n^\rho, s^\rho, 0, 0) = z^\nu_{\text{eq}}(n^\rho, s^\rho)$ and find: $Q^\nu = Q^\nu(n^\rho, s^\rho, z^\rho)$ $\quad \Pi = \Pi(n^\rho, s^\rho, z^\rho)$

12 algebraic DOF = 9 (I&S heat+bulk) + 3 (superflow) $\quad (n^\rho, s^\rho, \Pi, Q^\rho) \longleftrightarrow (n^\rho, s^\rho, z^\rho)$

Generating function formalism:

$$T^\nu{}_\rho = \Psi \delta^\nu{}_\rho + n^\nu \mu_\rho + s^\nu \Theta_\rho - z^\nu \mathbb{A}_\rho$$

$$\Psi = \Lambda - n^\nu \mu_\nu - s^\nu \Theta_\nu + z^\nu \mathbb{A}_\nu$$

Equations of motion

$$\mathcal{R}^n_\rho = 2n^\nu \nabla_{[\nu} \mu_{\rho]} = 0$$

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$$\mathcal{R}^z_\rho = -2z^\nu \nabla_{[\nu} \mathbb{A}_{\rho]} - \mathbb{A}_\rho \nabla_\nu z^\nu$$

Need to find $\mathcal{R}^z_\rho = -\mathcal{R}^s_\rho$ from quasiparticle kinetics $\rightarrow$ $\nabla_\nu T^\nu{}_\rho = \mathcal{R}^n_\rho + \mathcal{R}^s_\rho + \mathcal{R}^z_\rho = 0$

Final considerations

From the general considerations in **arXiv:2003.04609** (also, arXiv:2110.05546)

→ Dissipative Carter's multifluid can encore heat and bulk viscosity

→ Theoretically identical to Israel-Stewart close to equilibrium

→ Far from equilibrium: Israel-Stewart is pertubative, Carter is not

→ "Carter" = non perturbative generalization of "Israel-Stewart" without shear

Hydro-Bulk-1D: First simulation of proto-NS with the **complete Hiscock-Lindblom model** of bulk viscosity and neutrino luminosity. Comparison with Carter's formalism for bulk viscosity.

One-dimensional, GR code, publicly available: Giovanni Camelio, `hydro-bulk-1D` (2022)

Numerical check of the theoretical result (arXiv:2204.11810):

Israel-Stewart is a good approximations of the multi-component fluid when:

→ small perturbations

→ the equation of state of the fluid depends on only one independent particle fraction

For more than one independent particle fraction and for large perturbations (e.g. muons)

→ the bulk stress approximation is still valid but less accurate

Message: in mergers, isolated NS (cold and hot), supernovae... just use "Carter" for bulk viscosity!

→ arXiv:2003.04609 (general theory), arXiv:2204.11810 (comparison of 3 approaches to bulk and numerics)

Superfluid + bulk viscosity + heat

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$$\boxed{\text{12 algebraic DOF} \qquad (n^\rho, s^\rho, \Pi, Q^\rho) \longleftrightarrow (n^\rho, s^\rho, z^\rho)}$$

We have "Carter" from the "generating functional" $\rightarrow$ **expand to find "superfluid Israel-Stewart"**

Eckart frame of the "excitation gas" $u^\nu := z^\nu / \sqrt{-z^\rho z_\rho}$

Dissipation is mediated by collisions between quasiparticles: local thermodynamic equilibrium = collinearity between $\boldsymbol{s}$ and $\boldsymbol{z}$

Heat: $s^\nu = s^E u^\nu + \dfrac{Q^\nu}{\Theta_E} \qquad Q^\nu u_\nu = 0$

Bulk: $\Psi = \Psi_{\mathrm{eq}} + \Pi \quad \rightarrow \quad \Pi = \mathbb{A}_E \dfrac{\partial \Psi}{\partial \mathbb{A}_E}$ where $\mathbb{A}_E = -\mathbb{A}_\nu u^\nu$

Telegraph-type evolution for heat & bulk $\rightarrow$ consistent with non relativistic hydrodynamics of Khalatnikov

Entropy and dissipative force ($\boldsymbol{k}$ = heat conduction coefficient):

$$[\nabla_\nu s^\nu]_{\mathrm{bulk}} = \frac{\Xi \mathbb{A}_E^2}{\Theta_E} \qquad [\nabla_\nu s^\nu]_{\mathrm{heat}} = \frac{Q_\nu Q^\nu}{k\, \Theta_E^2} \qquad \mathcal{R}^s_\rho = \Xi\, \mathbb{A}_E^2\, u_\rho - \frac{s^E}{k} Q_\rho$$

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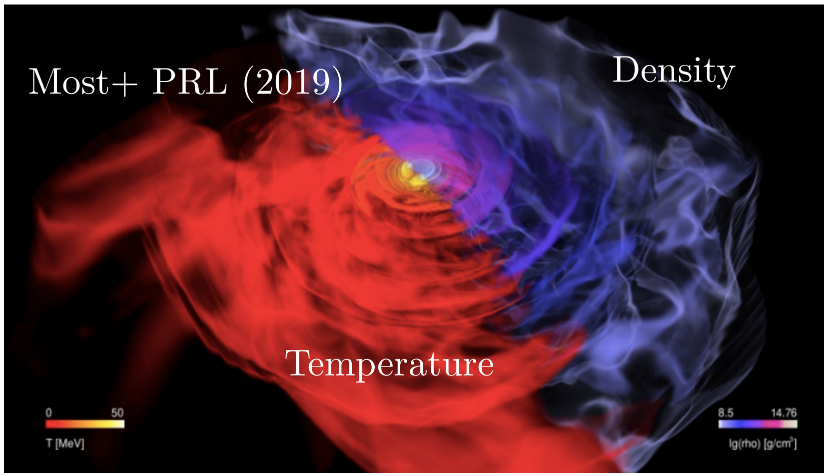
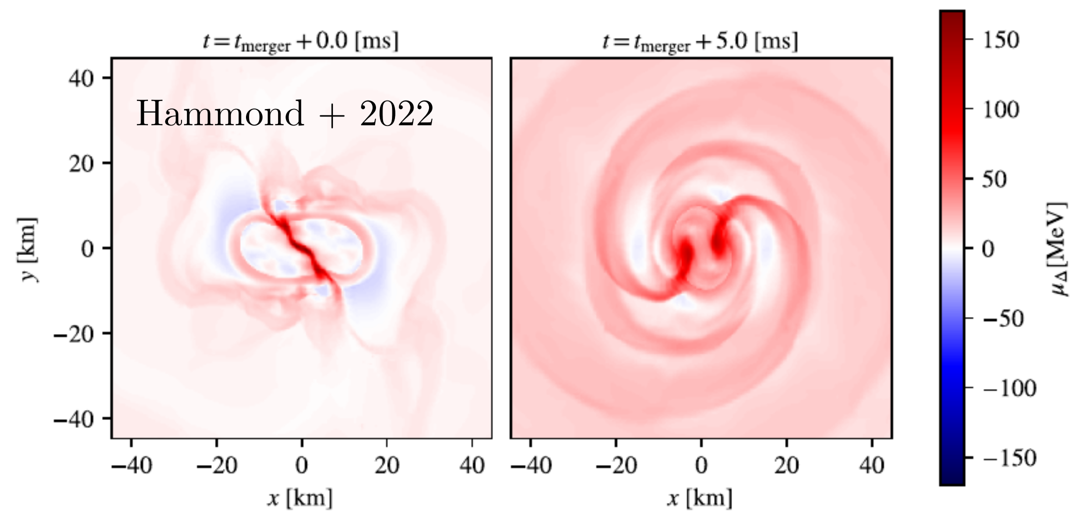
Viscous effects in neutron star mergers?

Duez et al PRD (2004), Shibata et al. PRD (2017), Alford et al. PRL (2018)

Rarefaction/compression of the fluid elements → Chemical abundances are pushed out of chemical equilibrium.
Possibly relevant for: CC superovae, NS mergers, NS oscillations





*"The effects of **bulk viscosity** should be consistently included in future merger simulations. This has not been attempted before and requires a formulation of the relativistic-hydrodynamic equations that is hyperbolic and stable"*
Alford et al. PRL (2018)

$$n \to p + e^- + \bar{\nu}_e$$
$$p + e^- \to n + \nu_e$$

$$n + X \to p + X + e^- + \bar{\nu}_e$$
$$p + e^- + X \to n + X + \nu_e$$

Perfect multifluid in relativity

We have seen the perfect fluid:
- local equilibrium (equilibrium thermodynamic variables defined in the local rest frame)
- 5 DOF: $\nabla_\nu n_n^\nu = 0 \qquad \nabla_\rho T^\rho{}_\nu = 0$ are enough to define the dynamics

Neutron stars are conductive!
Many flows are possible at the same time (electric current in MHD, superfluidity, heat conduction...)
...we typically need **more DOF**. Where do we get enough equations of motion?

Carter multifluid approach solves the problem of deriving the **equations of motion** for a **conductive mixture** of an arbitrary number of fluid species ("species": abstract concept, e.g. "entropy").

Important: *Carter's approach gives the equations of motion in the* ***inviscid*** *limit* (non dissipative).
Why? It is a variational approach → Liouville theorem is incompatible with relaxation to equilibrium.
Dissipative variational approaches exist but are of different nature (often they need a "DOF doubling").

Equations of motion

$$\mu^h_\rho \nabla_\nu n^\nu_h + 2 n^\nu_h \nabla_{[\nu} \mu^h_{\rho]} = \mathfrak{R}^h_\rho$$

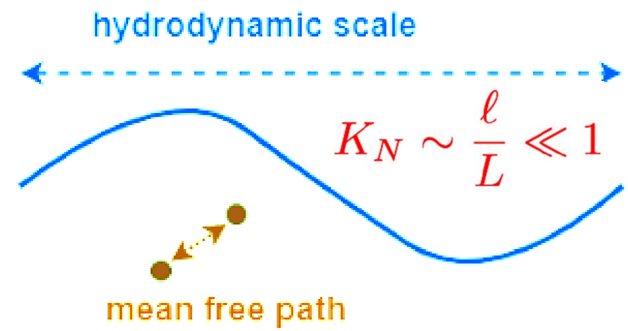
Transfusion between species gives rise to a force

Single perfect fluid part

Hydro force (e.g, **friction**, not specidied by the model, must be supplied)

Hydrodynamics as a derivative expansion

Hydrodynamics may also be seen as a macroscopic treatment based on:

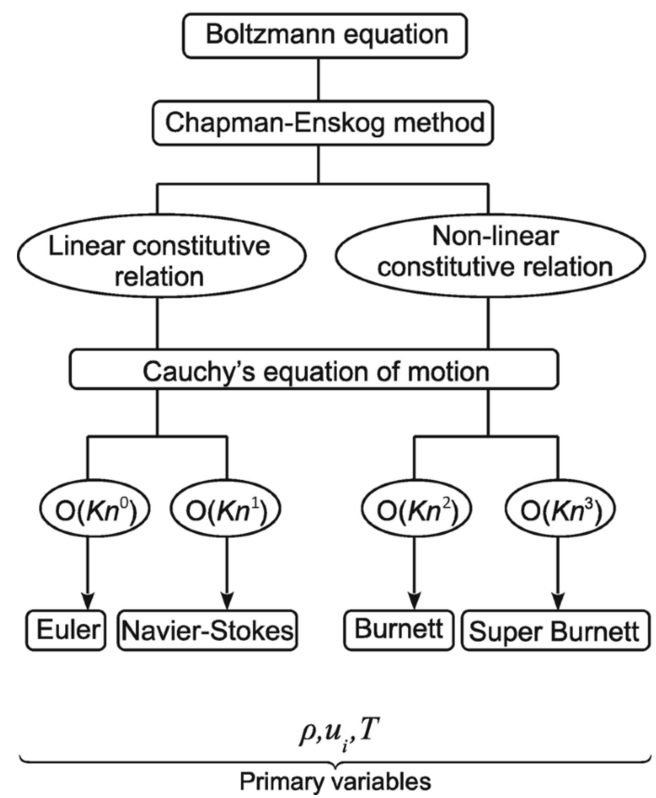


- **Separation of length scales** (Kundsen number)

  Kn ~ "mean free path"/"system length scale" ~ Mach/Reynolds
  Kn ~ 0.01 or smaller → continuum approximation

- **Conservation laws**

  energy-momentum, charges + possibly external symmetries



Example: Navier Stokes equation for a viscous fluid
→ conservation of mass + Newton's II law + local isotropy

$$\underbrace{\partial_t \vec{V} + (\vec{V}\cdot\vec{\nabla})\vec{V} + \frac{\vec{\nabla} P}{\rho_0}}_{\text{Ideal fluid } \sim \mathcal{O}(K_n^0)} = \underbrace{\frac{\eta}{\rho_0}\nabla^2\vec{V}}_{\sim \mathcal{O}(K_n)} + \underbrace{\mathcal{O}(K_N^2)}_{\text{Higher order}}$$

Note: at every order you always have the 5 DOF of the perfect fluid!

References

Work done with: L. Gavassino (Vanderbilt University), G. Camelio, B. Haskell (CAMK, Warsaw)

General theory:

Equilibrium thermodynamics of a multifluid → arXiv:1906.03140
Stability and causality of Carter's multifluid → arXiv:2202.06760
Multicomponent fluid with bulk viscosity → arXiv:2003.04609
Dissipation in superfluids → arXiv:2012.10288 (vortices), arXiv:2110.05546 (heat & bulk viscosity)
Radiation hydrodynamics (M1) as a Carter's multifluid → arXiv:2007.09481

Some applications:

Glitches in pulsars → arXiv:1710.05879 (glitch amplitude), arXiv:2001.08951 (glitch timescale)
Effect of bulk viscosity due to chemical reactions in neutron star oscillations
→ arXiv:2204.11809 (formalism)
→ arXiv:2204.11810 (simualtions)

Reviews:

Andersson & Comer "Relativistic Fluid Dynamics" arXiv:gr-qc/0605010
Gavassino & Antonelli "Unified Extended Irreversible Thermodynamics and the stability of theories for dissipation"