Gravitational waves from quasielliptic compact binary systems in massless scalar-tensor theories

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Motivations

Why study Brans-Dicke-like theories despite stringent constraints from binary pulsar and solar system tests?

- easiest way to add a scalar field, helps develop techniques useful for more complicated theories (e.g. scalar Gauss Bonnet [Shiralilou+ '22])
- motivated by high-energy models (string theory, ...)
- cannot exclude that a subpopulation of LVK events are undetected because searched for using GR templates [Magee+ '23], so we want some alternative templates
- helps understand GR [scalar fields easier, toy model for GR]
- ...

What I'll be presenting

This is a spin-off of my PhD work [Bernard, Blanchet & Trestini '22], where I extend those results to the case of elliptic orbits.

Elliptic waveform models are important to avoid parameter bias: we already know that LVK has detected compact binaries with non-zero eccentricity!

Based on the two papers:

- [Trestini '24a] D. Trestini, "Quasi-Keplerian parametrization for eccentric compact binaries in scalar-tensor theories at second post-Newtonian order and applications," Phys. Rev. D **109**, 104003 (2024), arXiv:2401.06844
- [Trestini '24b] D. Trestini, "Gravitational waves from quasielliptic compact binaries in scalar-tensor theory to one-and-a-half post-Newtonian order," arXiv:2410.xxxxx — **to appear this week!**

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Post-Newtonian methods applied to scalar-tensor theory

Generalized Fierz-Pauli-Brans-Dicke theory

Action defined in Jordan frame : $S = S_{\mathrm{ST}}[g_{\alpha\beta}, \phi] + S_{\mathrm{m}}[g_{\alpha\beta}, \mathfrak{m}]$ where

$$S_{\mathrm{ST}} = \frac{c^3}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right]$$

The effective matter action for two point particles reads

$$S_{\mathrm{m}} = -c m_1(\phi) \sqrt{-(g_{\alpha\beta})_1 \mathrm{d}y_1^\alpha y_1^\alpha} + (1 \leftrightarrow 2)$$

The weak equivalence principle is broken: the inertial mass of a neutron star, when idealized as a point particle, depends on the local value of the scalar field [Eardley 1975].

Conformal transformation to Einstein frame

Perturbation around flat space and a constant scalar background

$$h^{\mu\nu} \equiv \sqrt{-\det[(\phi/\phi_0) g_{\alpha\beta}]} \times \frac{g^{\mu\nu}}{\phi/\phi_0} - \eta^{\mu\nu}$$

$$\psi \equiv \phi/\phi_0 - 1$$

Expansions of the functions appearing in ST theory

The ω function is expanded as

$$\omega(\phi) = \omega_0 + (\phi - \phi_0)\omega_0' + ...$$

For $A \in \{1, 2\}$, the mass function is expanded as

$$m_A(\psi) = m_A \left(1 + s_A\psi + \frac{s_A^2 - s_A + s'_A}{2}\psi^2 + ...\right)$$

where sensitivities are defined as

$$s_A = \frac{\mathrm{d}\ln m_A(\phi)}{\mathrm{d}\ln\phi}$$

$$s'_A = \frac{\mathrm{d}^2\ln m_A(\phi)}{\mathrm{d}\ln\phi^2}$$

$$...$$

Various parameters you might see are just complicated combinations of these parameters: $\tilde{G}, \alpha, \bar{\gamma}, \zeta, \lambda_A, \bar{\beta}_A, \bar{\chi}_A, \bar{\kappa}_A, \bar{\delta}_A, ...$

Equations of motion

The equations of motion follow from $\nabla_\mu T^{\mu\nu} = 0$ using the near-zone, post-Newtonian metric.

They are given in ST theory by [Mirshekari & Will '13][Bernard '18]

$$\boxed{\begin{aligned} a_1^i \equiv \frac{\mathrm{d}^2 y_1^i}{\mathrm{d}t^2} = -\frac{\tilde{G}\alpha m_2}{r_{12}} n_{12}^i + \frac{1}{c^2}(\text{1PN corrections in } \boldsymbol{y_1},\boldsymbol{y_2},\boldsymbol{v_1},\boldsymbol{v_1}) + \\ + \frac{1}{c^3}(\text{1.5PN radiation reactions terms}) + \dots. \end{aligned}}$$

where $\tilde{G}\alpha = \frac{G}{\phi_0}\left(1 + \frac{(1-2s_1)(1-2s_2)}{3+2\omega}\right)$ the gravitational coupling constant in this theory. One then obtains the relative equations of motion

$$a_{12}^i = a_1^i - a_2^i = -\frac{\tilde{G}\alpha(m_1+m_2)}{r_{12}} n_{12}^i + (\text{PN corrections})$$

Discarding radiation reaction terms, one can find a conserved energy E, angular momentum J^i, etc. [Bernard '19].

Quasi-Keplerian parametrization

The Kepler solution

Two-body problem in the context of *Newtonian gravity*, described by relative acceleration [in COM frame]:

$$a^i = a_1^i - a_2^i = -\frac{G_{12} m n^i}{r^2}$$

where G_{12} is the gravitational coupling constant.

Bound orbits are ellipses:

$$r = \frac{a(1-e^2)}{1 + e\cos(\phi - \phi_{\mathrm{peri}})}$$

where a is the semimajor axis and e the eccentricity ($e < 1$ for bound orbits), given in terms of the energy ($E < 0$) and angular momentum J:

$$a = -\frac{Gm}{2E} \qquad \text{and} \qquad e = \sqrt{1 + \frac{2EJ}{G^2 m^2}}$$

The Kepler solution

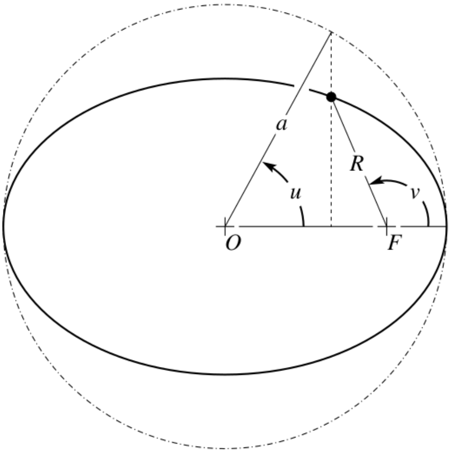
To describe the time evolution, it is however more practical to use the following set of three equations

$$
\begin{aligned}
r &= a(1 - e\cos u) \\
\ell = n(t - t_0) &= u - e\sin(u) \\
\phi - \phi_0 &= v(u)
\end{aligned}
$$

where we have introduced

- the *eccentric anomaly* u, which acts as an affine parameter
- the *true anomaly* $v(u) \equiv 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{u}{2}\right)\right]$
- the *mean motion* $n \equiv 2\pi/P$, where P is a time period
- the *mean anomaly* $\ell = n(t - t_0)$, which increases linearly with time and goes from 0 to 2π over one orbit

The Kepler solution



$$r = a(1 - e\cos u)$$
$$\ell = n(t - t_0) = u - e\sin(u)$$
$$\phi - \phi_0 = v$$
$$v = 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{u}{2}\right)\right]$$

Figure from [gr-qc/0407049]

The quasi-Keplerian solution at 2PN order

What happens if we now want to solve the equations of motion for the 2PN acceleration ?

$$a^i = -\frac{G_{12} m n^i}{r^2} + \frac{1}{c^2}\,(\text{many terms})^i + \frac{1}{c^4}\,(\text{many terms})^i$$

The solution to the equations of motion is the quasi-Keplerian (QK) parametrization [Damour & Deruelle '85] [Damour & Schäfer '88] at 2PN

$$\begin{aligned} r &= a_r(1 - e_r \cos u) \\ \phi - \phi_0 &= K\Big[v + f_\phi \sin(2v) + g_\phi \sin(3v)\Big] \\ n(t - t_0) &= u - e_t \sin(u) + f_t \sin(v) + g_t(v - u) \\ v(u) &= 2 \arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan\left(\frac{u}{2}\right)\right] \end{aligned}$$

In [Trestini '24a], I determined the QK parameters in ST theory at 2PN order. This entirely characterizes the quasielliptic motion.

Example of a quasi-Keplerian parameter: periastron advance

The periastron advance is given by:

$$K = 1 + \frac{\varepsilon}{j}\left[3 - \bar{\beta}_+ + 2\bar{\gamma} + \bar{\beta}_-\delta\right] + \frac{\varepsilon^2}{j^2}\left[... + \delta(...) + \nu(...) + \nu\delta(...) + j\left(... + (...)\delta + \nu(...) + \nu\delta(...)\right)\right]$$

where the symmetric mass ratio read $\nu = m_1 m_2/m^2$, the relative mass difference is $\delta = (m_1 - m_2)/m$, and the energy and angular momentum have been normalized as

$$\varepsilon = -\frac{2E}{\mu c^2} \qquad \text{and} \qquad j = -\frac{2J^2 E}{\mu^3(\tilde{G}\alpha m)^2}$$

Fluxes at infinity and evolution of orbital elements

The fluxes at infinity

Using standard PN methods, asymptotic fluxes obtained at 1.5PN order [2.5PN beyond -1PN dipolar radiation] in terms of $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{v}_1, \boldsymbol{v}_2)$

- in [Bernard, Blanchet & Trestini '22] for the energy flux
- in [Jain & Rettegno '24] [Trestini '24b] for the angular momentum flux

These are divided into:

- an instantaneous part
- a tail part [non-local integral in time]
- a memory part, which vanishes after orbit averaging
- a nonlocal term arising from the passage to the CM frame [see Luc's talk for the GR equivalent], which vanishes after orbit averaging

In [Trestini '24b], I use the QK parametrization and obtain fluxes for eccentric orbits at 1.5PN [2.5PN beyond LO] in terms of (x, e_t).

The secular evolution of orbital elements

Thanks to the fluxes just computed and the conserved energy and angular momentum, one can rewrite the balance equations

$$\left\langle \frac{\mathrm{d}\xi}{\mathrm{d}t} \right\rangle = \frac{\partial \xi}{\partial E} \left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle + \frac{\partial \xi}{\partial J} \left\langle \frac{\mathrm{d}J}{\mathrm{d}t} \right\rangle$$

where we use the flux balance equations

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = -\langle \mathcal{F} \rangle - \langle \mathcal{F}^s \rangle \quad \text{and} \quad \left\langle \frac{\mathrm{d}J}{\mathrm{d}t} \right\rangle = -\langle \mathcal{G} \rangle - \langle \mathcal{G}^s \rangle$$

In [Trestini '24b], I obtain the evolution of $\xi \in \{x, e_t\}$ at 1.5PN order [2.5PN beyond LO]. This is the main observable of the GW signal, equivalent to the "chirp" in the case of circular orbits!

Example: evolution of the frequency x

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t} \right\rangle = \frac{2c^3\zeta\nu x^4}{3\tilde{G}\alpha m}\Bigg\{ \frac{4\mathcal{S}_-^2\left(1+\frac{1}{2}e_t^2\right)}{(1-e_t^2)^{5/2}} + \frac{x}{15(1-e_t^2)^{7/2}}\Big(\mathfrak{X}_1 + e_t^2\mathfrak{X}_2 + e_t^4\mathfrak{X}_3\Big) + 8\pi\left(1+\tfrac{1}{2}\bar{\gamma}\right)\mathcal{S}_-^2\varphi_1^s(e_t)\, x^{3/2} + \text{(higher order terms)}\Bigg\}$$

where $\varphi_1^s(e_t)$ is an enhancement function. It arises from tails and can be computed numerically from an infinite sum, but can also be Taylor-expanded in small eccentricities:

$$\varphi_1^s(e_t) = 1 + 7e_t^2 + \frac{717}{32}e_t^4 + \mathcal{O}(e_t^6)$$

Peters' formula for ST theory

In general relativity, the secular evolution equations can be solved analytically at 0PN order, giving the relation

$$a_{\mathrm{GR}}(e) = \frac{c_0^{\mathrm{GR}} e^{12/19}}{1 - e^2} \left(1 + \frac{121}{304} e^2\right)^{870/2299}$$

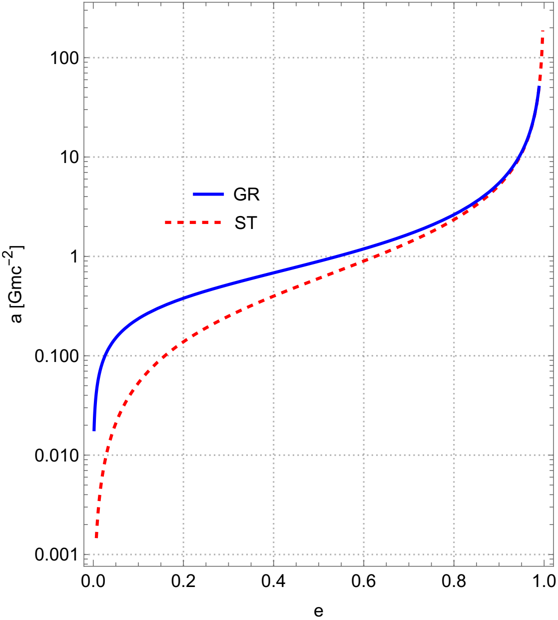
where c_0 is a constant as a and e evolve secularly.

In [Trestini '24a], I find an analogous formula in ST theory at -1PN order:

$$a(e) = \frac{c_0 e^{4/3}}{1 - e^2}$$

Note that it does not depend on any ST parameter, it arises purely from the existence of dipolar radiation!

The "Peters and Mathews" formula for ST theories



GW amplitudes at Newtonian order

A common description of the GW waveform is to perform a mode decomposition.

$$h_+ - \mathrm{i}h_\times = \frac{2\tilde{G}(1-\zeta)m\nu x}{Rc^2}\sqrt{\frac{16\pi}{5}}\sum_{\ell=2}^{+\infty}\sum_{m=-\ell}^{\ell}\hat{H}^{\ell m}e^{-\mathrm{i}m\phi}\,{}_{-2}Y^{\ell m}(\Theta,\Phi)$$

$$\psi = \frac{2\mathrm{i}\tilde{G}\zeta\sqrt{\alpha}\mathcal{S}_- m\nu\sqrt{x}}{Rc^2}\sqrt{\frac{8\pi}{3}}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell}\hat{\Psi}^{\ell m}e^{-\mathrm{i}m\phi}\,Y^{\ell m}(\Theta,\Phi)$$

Completing mode decomposition at Newtonian order [1PN beyond LO], the waveform amplitudes read e.g. [Trestini '24a]:

$$\hat{\Psi}^{11} = \frac{1-e_t^2-\mathrm{i}e_t\sqrt{1-e_t^2}\sin u}{\sqrt{1-e_t^2}(1-e_t\cos u)} + \mathcal{O}(x)$$

Inversing the Kepler equation $\ell = u - e_t\sin(u)$ numerically or using:

$$u = \ell + 2\sum_{n=1}^{\infty}\frac{1}{n}J_n(ne)\sin(n\ell)$$

Conclusion

In this work, I

- solve the ST equations of motion using the 2PN quasi-Keplerian parametrization
- express the fluxes of energy and angular momentum at 1.5PN [2.5PN beyond LO] using the QK parametrization
- obtain the secular evolution of orbital elements at 1.5PN order [2.5PN beyond LO]
- obtain Newtonian amplitudes [1PN beyond LO]

This extends for the first time our GW template bank for alternative theories to eccentric orbits!

Backup slides

Linearized metric in exterior vacuum

N.B. I will focus on the scalar field for pedagogy

In the exterior vacuum zone, we formally perform a multipolar post-Minkowskian expansion $\psi = G\psi_1 + G^2\psi_2 + \ldots$

At linear level, the scalar field equation reads $\Box\psi_1 = 0$, so we can express it as a multipolar expansion [Thorne 1980]:

$$\psi_1 = -\frac{2}{c^2}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left[r^{-1}\mathrm{I}^s_L\right]$$

The "source moments" can be matched to a near-zone, post-Newtonian ($v \ll c$) computation involving the matter, such that they can be expressed as functions of the phase space variable of the compact binary system

$$\mathrm{I}^s_L[\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{v}_1, \boldsymbol{v}_2]$$

Multipolar moments: an example

For example, we have [2201.10924]

$$I_i^s = -\frac{m_1(1-2s_1)y_1^i}{\phi_0(3+\omega_0)} - \frac{m_2(1-2s_2)y_2^i}{\phi_0(3+\omega_0)} + \mathcal{O}\left(\frac{1}{c^2}\right)$$

where various ST parameters come from

$$\omega(\phi) = \omega_0 + (\phi - \phi_0)\omega_0' + ...$$

and [for $A \in \{1, 2\}$] :

$$m_A(\psi) = m_A\left(m_A + s_A\psi + ...\right)$$

Note that the weak equivalence principle is broken so the inertial mass of a star (seen as a point-particle) can depend on the local value of the scalar field, hence the need to introduce sensitivities, e.g.

$$s_A = \frac{\mathrm{d}\ln m_A(\phi)}{\mathrm{d}\ln \phi}$$

MPM metric in exterior vacuum

Now that the linear metric is entirely determined, we go back to the MPM expansion: $\psi = G\psi_1 + G^2\psi_2 + ...$ and inject it into our full vacuum field equation

$$\Box\psi = \Lambda_s[h, \psi]$$

where $\Lambda_s[h, \psi]$ is at least quadratic in the fields. Thus, we contruct the MPM metric by iterating:

$$\Box\psi_n = \Lambda_s^{(n)}[h_1, ..., h_{n-1}; \psi_1, ..., \psi_{n-1}]$$

This generates nonlocal effects such as tail, the quadratic memory, etc. !

Radiative moments

Once the MPM metric constructed, we can discard all subdominant terms in the $R \to \infty$ limit. We thus recover an (asymptotically) multipolar structure:

$$\psi \sim \frac{1}{r} \sum \hat{n}_L \mathcal{U}_L^s$$

We recover the tail terms of GR , but also find new ST tail terms and a new ST memory term:

$$\begin{aligned}\mathcal{U}_{ij} &= \overset{(2)}{\mathrm{I}}_{ij} + \frac{2GM}{\phi_0 c^3} \int_0^{+\infty} \mathrm{d}\tau \, \overset{(4)}{\mathrm{I}}_{ij}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{12}\right] \\ &\quad + \frac{G(3+2\omega_0)}{3c^3} \int_0^{+\infty} \mathrm{d}\tau \left[\overset{(2)}{\mathrm{I}}{}^s_{\langle i} \overset{(2)}{\mathrm{I}}{}^s_{j\rangle}\right](u-\tau) + (\mathrm{inst}) + \mathcal{O}\left(\frac{1}{c^4}\right)\end{aligned}$$

$$\mathcal{U}_i^s = \overset{(1)}{\mathrm{I}}{}_i^s + \frac{2GM}{\phi_0 c^3} \int_0^{+\infty} \mathrm{d}\tau \, \overset{(3)}{\mathrm{I}}{}_i^s (u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + 1\right] + (\mathrm{inst}) + \mathcal{O}\left(\frac{1}{c^6}\right)$$

Fluxes at infinity

Now that we know that asymptotic structure of the scalar waves [idem for GWs] at $\mathcal{I}^+$, we can deduce the fluxes of energy and angular momentum that they carry [2401.06844]:

$$\begin{aligned}\mathcal{F}^s &= \frac{c^3 R^2 (3+2\omega_0)\phi_0}{16\pi G}\int \mathrm{d}^2\Omega \dot{\psi}^2 \\ &= \sum_{\ell=0}^{\infty} \frac{G\phi_0(3+2\omega_0)}{c^{2\ell+1}\ell!(2\ell+1)!!}\dot{\mathcal{U}}^s_L \dot{\mathcal{U}}^s_L\end{aligned}$$

$$\begin{aligned}\mathcal{G}^s_i &= \frac{c^3 R^3 (3+2\omega_0)\phi_0}{16\pi G}\int \mathrm{d}^2\Omega \dot{\psi}\epsilon_{iab} n_a \partial_b \psi \\ &= \sum_{\ell=1}^{\infty} \frac{G\phi_0(3+2\omega_0)}{c^{2\ell+1}(\ell-1)!(2\ell+1)!!}\epsilon_{iab}\mathcal{U}^s_{aL-1}\dot{\mathcal{U}}^s_{bL-1}\end{aligned}$$

where we have used $\psi \sim \frac{1}{r}\sum \hat{n}_L \mathcal{U}^s_L(t-r/c)$.

Quasicircular orbits

Why are we interested in the fluxes ? Consider the case of a quasicircular orbit. First, the angular momentum flux is related to the energy flux by $\mathcal{F} = \omega \mathcal{G}$, so we only consider the energy balance law:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} - \mathcal{F}^s$$

In the COM frame, the only dynamical variables are $r = |\boldsymbol{y}_1 - \boldsymbol{y}_2|$, $\boldsymbol{n} = (\boldsymbol{y}_1 - \boldsymbol{y}_2)/r$ and $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$.

The fluxes depend on them only through $r \approx (Gm/\omega^2)^{2/3}$, $v^2 \approx (Gm\omega)^{2/3}$ and $\boldsymbol{n} \cdot \boldsymbol{v} \approx 0$, where ω is the orbital frequency.

Thus, the energy balance equation reduces to an equation of the type:

$$\boxed{\frac{\mathrm{d}\omega}{\mathrm{d}t} = f(\omega)}$$

This immediately yields the phase and frequency evolution !

The phase at 1.5PN for quasi-circular orbits

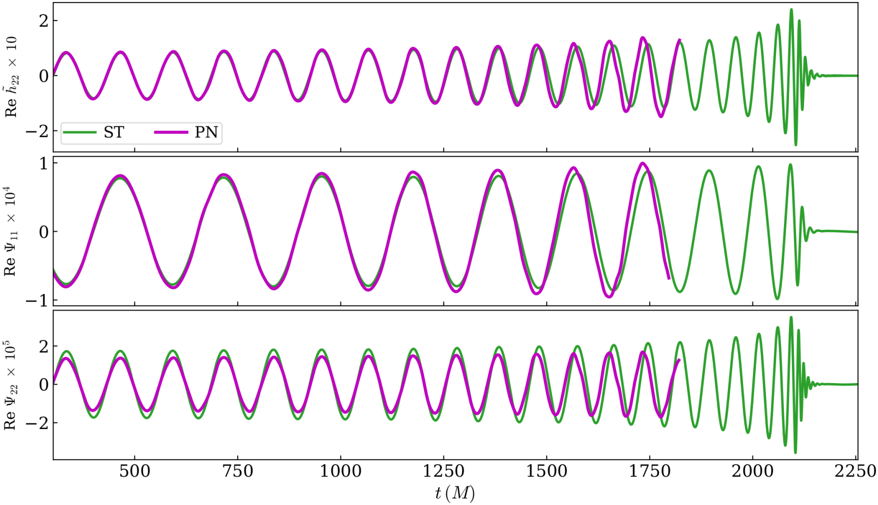
In [2201.10924], we found:

$$
\begin{aligned}
\phi_{\text{circ}} = -\frac{1}{4\zeta\mathcal{S}_-^2\nu x^{1/2}}\Bigg[& x^{-1} \\
& + \frac{3}{2} + 8\bar{\beta}_+ - 2\bar{\gamma} - 12\bar{\beta}_+\bar{\gamma}^{-1} - \frac{72}{5}\zeta^{-1}\mathcal{S}_-^{-2} \\
& - 6\zeta^{-1}\bar{\gamma}\mathcal{S}_-^{-2} - 12\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+ \\
& + \delta\Big[- 8\bar{\beta}_- + 12\bar{\beta}_-\bar{\gamma}^{-1} + 12\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+\Big] + \frac{7}{2}\nu \\
& + 3\pi x^{1/2}\log(x)\left(1 + \frac{\bar{\gamma}}{2}\right) \\
& + x\left\{\text{complicated expression}\right\} \\
& + \frac{\pi x^{3/2}}{1-\zeta}\left\{\text{complicated expression}\right\}\Bigg].
\end{aligned}
$$

This is the main observable in a GW !

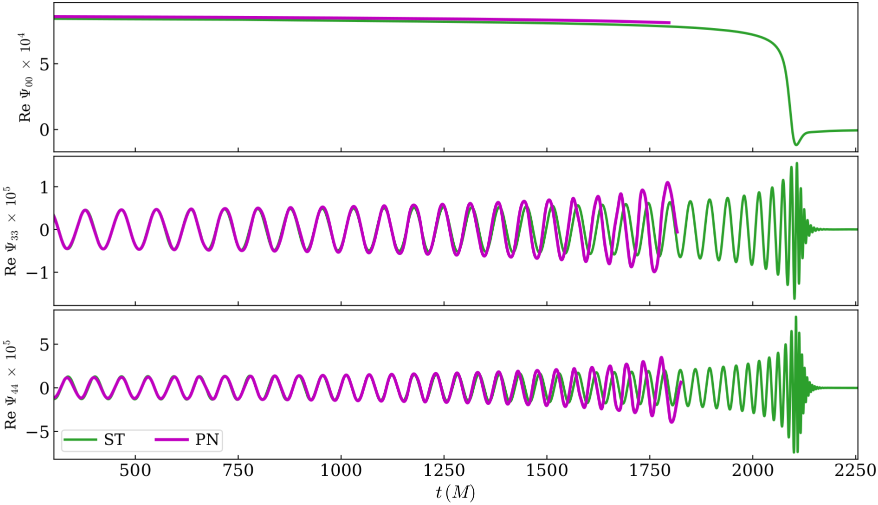
Comparison to NR

Numerical simulation [2304.11836] found good agreement with our results:



Comparison to NR (cont'd)

even for the DC memory effect !



(a) Scalar modes

Determining the QK parameters

Assume we are working in *some theory of gravity* [e.g. GR or ST theory], and that we have determined (in a PN sense):

$$E = f(r, \dot{r}, \dot{\phi}) \qquad \text{and} \qquad J = g(r, \dot{r}, \dot{\phi})$$

For many theories of gravity, we can invert this as

$$\dot{r}^2 = A + \frac{B}{r} + \frac{C}{r^2} + \frac{D_1}{r^3} + \frac{D_2}{r^4} + \frac{D_3}{r^5} + \mathcal{O}\left(\frac{1}{c^6}\right)$$
$$\dot{\phi} = \frac{F}{r^2} + \frac{I_1}{r^3} + \frac{I_2}{r^4} + \frac{I_3}{r^5} + \mathcal{O}\left(\frac{1}{c^6}\right)$$

where A, B, C and F are of order 1, but D_1 and D_2 are 1PN and the others 2PN. **All these parameters are functions of E and J.**

Determining the QK parameters (cont'd)

Solving these EOM [technical !] directly yields the 2PN QK representation, and one reads off the expressions of the PK parameters (a_r, e_t, g_t, ...) in terms of A, B, D_1, ... For example [2401.06844]:

$$a_r = -\frac{B}{A} + \frac{D_1}{2C} + \frac{2BD_1^2 - 2BCD_2 + 4B^2D_3 - ACD_3}{2C^3} + \mathcal{O}\left(\frac{1}{c^4}\right)$$

Expression of A, B, ... depends of the theory. For example, in ST theory:

$$B = \tilde{G}\alpha m \left\{1 + \varepsilon\left[3 + \bar{\gamma} - \frac{7}{2}\nu\right] + \varepsilon^2\left[\frac{9}{4} + \frac{3}{4}\bar{\gamma} + \nu\Big(-12 - \frac{15}{4}\bar{\gamma}\Big) + \frac{21}{4}\nu^2\right]\right\}$$

where $\varepsilon = -2E/(m\nu c^2) > 0$ and $\varepsilon = \mathcal{O}(1/c^2)$.

To get the GR results, replace $\tilde{G}\alpha \to G$ and $\bar{\gamma} \to 0$.

If (i) you have $E = f(r, \dot{r}, \dot{\phi})$ and $J = g(r, \dot{r}, \dot{\phi})$ for your favorite theory; (ii) it has some nice properties
$\Rightarrow$ use these results to immediately obtain the QK representation

Fluxes at Newtonian order

At Newtonian order [reminder: the leading order is -1PN], the flux is *instantaneous*, i.e. no tails or memory. The QK representation allows us to write the fluxes only in terms of the eccentric anomaly:

$$\mathcal{F} = f[r, \phi, \dot{r}, \dot{\phi}] = g[r, \phi] = h[u]$$

After some trigonometry, we find that the structure is in fact

$$\mathcal{F} = \sum_k \left[\frac{\alpha_k}{[1 - e_t \cos(u)]^k} + \frac{\beta_k \sin(u)}{[1 - e_t \cos(u)]^k} \right]$$

The orbit averaged flux reads:

$$\langle \mathcal{F} \rangle = \frac{1}{P} \int_0^P \mathrm{d}t \mathcal{F} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\ell \, \mathcal{F} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}u \frac{\mathrm{d}\ell}{\mathrm{d}u} \mathcal{F}$$

where $\mathrm{d}\ell / \mathrm{d}u = 1 - e_t \cos(u)$. We can then use:

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}u}{[1 - e_t \cos(u)]^n} = \frac{P_{n-1}(1/\sqrt{1 - e_t^2})}{(1 - e_t^2)^{n/2}}$$

Averaged fluxes at Newtonian order

I find at Newtonian (relative 1PN) order [2401.06844]

$$\langle \mathcal{F} \rangle = \frac{32 c^5 x^5 \nu^2 (1+\bar{\gamma}/2)}{5 \tilde{G} \alpha} \cdot \frac{1 + \frac{73}{24} e_t^2 + \frac{37}{96} e_t^4}{(1-e_t^2)^{7/2}}$$

$$\langle \mathcal{G} \rangle = \frac{32 c^2 m x^{7/2} \nu^2 (1+\bar{\gamma}/2)}{5} \cdot \frac{1 + \frac{7}{8} e_t^2}{(1-e_t^2)^2}$$

$$\langle \mathcal{F}^s \rangle = \frac{c^5 x^4 \nu^2 \zeta}{3 \tilde{G} \alpha} \left[4 \mathcal{S}_-^2 \frac{1 + e_t^2/2}{(1-e_t^2)^{5/2}} + \frac{x}{(1-e_t^2)^{7/2}} \left(\mathcal{C}_0 + \mathcal{C}_2 e_t^2 + \mathcal{C}_4 e_t^4 \right) \right]$$

$$\langle \mathcal{G}^s \rangle = \frac{c^2 m x^{7/2} \nu^2 \zeta}{3} \left[\frac{4 \mathcal{S}_-^2}{1-e_t^2} + \frac{x}{(1-e_t^2)^{7/2}} \left(\mathcal{D}_0 + \mathcal{D}_2 e_t^2 \right) \right]$$

where we introduce the dimensionless PN parameter

$$x = \left(\frac{\tilde{G} \alpha m \omega}{c^3} \right)^{2/3}$$

Here, $\omega = nK$ is the *angular* frequency (and n is the *radial* frequency)

Orbital evolution at leading Newtonian order

At leading order, we consider: (i) a Keplerian orbit; (ii) the Newtonian E and J; (iii) the -1PN fluxes $\mathcal{F}^s$ and $\mathcal{G}^s$. The balance equations $\mathrm{d}E/\mathrm{d}t = -\mathcal{F}^s$ and $\mathrm{d}J/\mathrm{d}t = -\mathcal{G}^s$ can be rewritten as [2401.06844]

$$\left\langle \frac{\mathrm{d}a}{\mathrm{d}t} \right\rangle = -\frac{8}{3}\,\frac{(\tilde{G}\alpha m)^2 \zeta \mathcal{S}_-^2 \nu}{c^3} \cdot \frac{1+e^2/2}{a^2(1-e^2)^{5/2}}$$

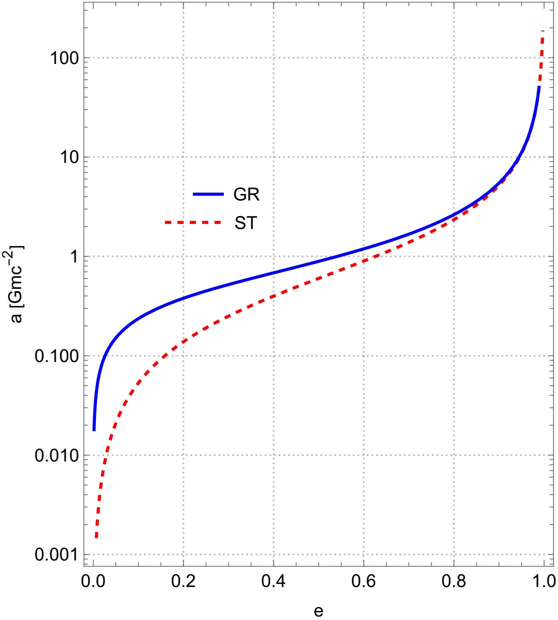
$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle = -\frac{(\tilde{G}\alpha m)^2 \zeta \mathcal{S}_-^2 \nu}{c^3} \cdot \frac{2e}{a^3(1-e^2)^{3/2}}.$$

Eliminating the time dependency and solving: $\boxed{a = \dfrac{c_0 e^{4/3}}{1-e^2}}$

In GR, the equivalent formula (due to Peters [PhysRev.136.B1224]) reads

$$a = \frac{c_0' e^{12/19}}{1-e^2}\left(1 + \frac{121}{304} e^2\right)^{870/2299}$$

The “Peters and Mathews” formula for ST theories



Evolution of orbital parameters at 1PN order

This time, we work at 1PN order in the QK parametrization and Newtonian order in the fluxes. THis time, we rewrite the balance equations $\mathrm{d}E/\mathrm{d}t = -\mathcal{F} - \mathcal{F}^s$ and $\mathrm{d}J/\mathrm{d}t = -\mathcal{G} - \mathcal{G}^s$ in terms of x and e_t [2401.06844]

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t} \right\rangle = \frac{2c^3\zeta x^4\nu}{3\tilde{G}\alpha m}\left\{ \frac{4\mathcal{S}_-^2(1+\frac{1}{2}e_t^2)}{(1-e_t^2)^{5/2}} + \frac{x}{15(1-e_t^2)^{7/2}}\Big(\mathcal{C}_1 + e_t^2\mathcal{C}_2 + e_t^4\mathcal{C}_3\Big)\right\}$$

$$\left\langle \frac{\mathrm{d}e_t}{\mathrm{d}t} \right\rangle = -\frac{c^3\zeta x^3\nu}{\tilde{G}\alpha m}\left\{ \frac{2\mathcal{S}_-^2 e_t}{(1-e_t^2)^{3/2}} + \frac{x\, e_t}{15(1-e_t^2)^{5/2}}\Big(\mathcal{C}_4 + e_t^2\mathcal{C}_5\Big)\right\}$$

Since all other QK parameters (e.g. a_r, e_r, f_t, ...) can be expressed in terms of the pair (x, e_t), we have entirely characterized the motion!

How to treat nonlocal terms ?

Starting at 0.5PN order, nonlocal tail terms appear, such as:

$$\mathcal{F}^{s,\mathrm{tail}} = \frac{4G^2\mathrm{M}(3+2\omega_0)}{3c^6} \overset{(2)}{\mathrm{I}}{}^s_i \int_0^\infty \mathrm{d}\tau \, \overset{(4)}{\mathrm{I}}{}^s_i(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + 1\right]$$

For a quasi-circular orbit,

$$I^i_s(t) = \sum_{k=-k_{\mathrm{max}}}^{k_{\mathrm{max}}} \alpha_k e^{ik\omega t}$$

where $k \neq 0$, so we trivially use the formula

$$\int_0^{+\infty} \mathrm{d}\tau \ln\left(\frac{c\tau}{2b_0}\right) e^{-ik\omega\tau} = \frac{\mathrm{i}}{k\omega}\Big[\ln\Big(\frac{2|n|b_0}{c}\Big) + \gamma_{\mathrm{E}} + \mathrm{i}\frac{\pi}{2}\mathrm{sg}(n)\Big]$$

to compute the flux. **However, for an (quasi-)elliptic orbit, the time dependence of the dipole is much more complicated!**

$\Rightarrow$ the solution is to compute expand the moments in *Fourier series*

Fourier expansion of moments at Newtonian order

We consider as an example de dipole momnet I_i^s. At Newtonian order it is periodic in $\ell = n(t - t_0)$, so we can decompose it as a Fourier *series* as:

$$\mathrm{I}_L^s(t) = \sum_{p \in \mathbb{Z}} {}_p\widetilde{\mathrm{I}}_L^s \, e^{\mathrm{i}p\ell}$$

The coefficients are given by

$${}_p\widetilde{\mathrm{I}}_L^s = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\ell \, \mathrm{I}_L^s(t) e^{-\mathrm{i}p\ell}$$

Changing variables to the eccentric anomaly u [using Kepler's equation $\ell = u - e_t \sin(u)$], we find that all integrals reduce to:

$$J_p(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}u \, e^{-\mathrm{i}(pu - x \sin u)}$$

which is simply a representation of the Bessel function. Thus:

$${}_p\widetilde{\mathrm{I}}_x^s \propto \frac{1}{p} J_p(ep) \qquad {}_p\widetilde{\mathrm{I}}_y^s \propto -\frac{\mathrm{i}\sqrt{1-e^2}}{ep} J_p(ep) \qquad {}_p\widetilde{\mathrm{I}}_z^s = 0$$

Fourier expansion of the dipole at 1PN order

At 1PN order, this Fourier expansion becomes more complicated, as the dipole is *not* periodic in ℓ anymore. We exploit the *doubly periodic* structure [recall $K = 1 + k$] to write the dipole as

$$\mathrm{I}_i^s = \sum_{m\in\{-1,1\}} {}^{m}\widetilde{\mathrm{I}}_i^s e^{imk\ell} = \sum_{m\in\{-1,1\}} \sum_{p\in\mathbb{Z}} {}^{m}_{p}\widetilde{\mathrm{I}}_i^s e^{i(mk+p)\ell}$$

where the decomposition in $e^{imk\ell}$ is trivial to obtain, and the decomposition in $e^{ip\ell}$ is computing using

$$ {}^{m}_{p}\widetilde{\mathrm{I}}_i^s = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\ell \; {}^{m}\widetilde{\mathrm{I}}_i^s \, e^{-\mathrm{i}p\ell} \tag{1}$$

This integrand is perform by PN-expanding $\widetilde{\mathrm{I}}_i^s$ and some trigonometry, but ultimately, we will nonetheless need a new master integral [1607.05409]:

$$\Theta_p^q(e_t) \equiv \frac{i}{\pi}\int_0^{2\pi} \mathrm{d}u \, e^{-i[(p+q)u - pe_t \sin u]} \arctan\left[\frac{\sin(u)}{\frac{e_t}{1-\sqrt{1-e_t^2}} - \cos(u)}\right]$$

Fluxes associated to tails

Injecting the Fourier expansion into the tail integrals that enter the flux, using standard integrals and orbit averaging, I find for example:

$$\begin{aligned}\left\langle \mathcal{F}^{s,\mathrm{tail}} \right\rangle &= \frac{4\pi G^2 \mathrm{M}(3+2\omega_0)}{c^6} \\ &\times \Bigg\{ \frac{n^5}{3} \sum_{p=1}^{\infty} \Bigg[p^5 \sum_{\substack{m\in\{-1,1\} \\ s\in\{-1,1\}}} \left({}^{m}_{p}\widetilde{\mathrm{I}}^{s}_{i} \right) \left({}^{m}_{p}\widetilde{\mathrm{I}}^{s}_{i} \right)^* + 5kp^4 \sum_{m\in\{-1,1\}} m \left({}^{m}_{p}\widetilde{\mathrm{I}}^{s}_{i} \right) \left({}^{m}_{p}\widetilde{\mathrm{I}}^{s}_{i} \right)^* \Bigg] \\ &\qquad + \frac{n^7}{30c^2} \sum_{p=1}^{\infty} p^7 \left({}_{p}\widetilde{\mathrm{I}}^{s}_{ij} \right) \left({}_{p}\widetilde{\mathrm{I}}^{s}_{ij} \right)^* + \frac{n^3}{\phi_0^2 c^2} \sum_{p=1}^{\infty} p^3 \left({}_{p}\widetilde{E}^{s} \right) \left({}_{p}\widetilde{E}^{s} \right)^* \Bigg\}\end{aligned}$$

where the Fourier coefficients are given as functions of x et e_t.

Fluxes associated to tails in terms of enhancement functions

Finally we can express the orbit average flux as

$$\left\langle \mathcal{F}^{s,\mathrm{tail}} \right\rangle = \frac{c^5 x^5 \nu^2 \zeta}{3\tilde{G}\alpha} \times 4\pi(1+\bar{\gamma}/2)\sqrt{x}\Bigg\{ 2\mathcal{S}_-^2 \varphi_1^s(e_t) + x\left[\mathcal{C}_1 \varphi_2^s(e_t) + \frac{41}{15}\mathcal{S}_-^2 \nu \theta_1^s(e_t) + \mathcal{C}_2 \alpha_1^s(e_t) + \mathcal{C}_3 x \varphi_0^s(e_t) \right] \Bigg\}$$

$$\left\langle \mathcal{G}^{s,\mathrm{tail}} \right\rangle = \frac{c^2 x^{7/2} \nu^2 \zeta}{3} \times 4\pi(1+\bar{\gamma}/2)\sqrt{x}\Bigg\{ 2\mathcal{S}_-^2 \tilde{\varphi}_1^s(e_t) + x\left[\mathcal{D}_1 \tilde{\varphi}_2^s(e_t) + \frac{41}{30}\mathcal{S}_-^2 \nu \tilde{\theta}_1^s(e_t) + \mathcal{D}_2 \tilde{\alpha}_1^s(e_t) \right] \Bigg\}$$

where $\varphi_s^s(e_t)$, etc., are *enhancement functions* of the eccentricity whose limit as $e_t \to 1$ is 1 [except for $\alpha_1^s(e_t)$ and $\alpha_2^s(e_t)$, for which it is zero]. Thus, we explicitly recover the expression for circular orbits of [2201.10924].

Expression of the enhancement functions

Typically, an enhancement function is exactly defined in terms of the Fourier coefficients, e.g.

$$\varphi_1^s(e_t) = 2\sum_{p=1}^{\infty} \sum_{\substack{m\in\{-1,1\}\\ s\in\{-1,1\}}} p^5 \left({}^{m}_{p}\widehat{\mathrm{I}}^{s,00}_{i}\right) \left({}^{m}_{p}\widehat{\mathrm{I}}^{s,00}_{i}\right)^*$$

This is simply a function over $e_t \in [0,1]$ which can be computed numerically, but we can also perform a $e_t \to 0$ expansion:

$$\varphi_1^s(e_t) = 1 + 7e_t^2 + \frac{717}{32}e_t^4 + \frac{7435}{144}e_t^6 + \frac{7305575}{73728}e_t^8 + \frac{103947697}{614400}e_t^{10} + \mathcal{O}(e_t^{12})$$

For increased accuracy, it is possible to resum these by factorizing by $(1-e_t^2)^{-n/2}$ for some n [2308.13606]. Other more complex resummation methods exist as well [1607.05409].

Memory contribution to the flux

The angular momentum flux also has a *memory-like* nonlocal term:

$$\mathcal{G}_i^{\text{mem}} = -\frac{2G^2\phi_0(3+2\omega_0)}{15c^8}\epsilon_{ik(j}\overset{(3)}{\mathrm{I}}_{a)k}\int_0^\infty \mathrm{d}\tau \left[\overset{(2)}{\mathrm{I}}{}^s_a \overset{(2)}{\mathrm{I}}{}^s_j\right](u-\tau)$$

Replacing the moments by their Fourier decomposition leads to an integrand of the type $\sum_k \alpha_k e^{ik\ell}$.

The $k \neq 0$ terms correspond to the "AC' contribution". They are trivial to integrate and can to be shown to vanish upon orbit averaging.

The $k = 0$ is the "DC term", and reads:

$$\mathcal{G}_{\text{DC}}^{\text{mem}} = \frac{4G^2G^3m^5\nu^2}{105c^{10}}\mathrm{I}_{xy}^{(3)}(u)\int_0^\infty \mathrm{d}\tau \left[\frac{e^2(13+2e^2)}{a^5(1-e^2)^{7/2}}\right](t_{\text{ret}}-\tau)$$

The DC term is finite and essentially constant, so the DC flux vanishes upon orbit averaging.

Thus, $\boxed{\langle \mathcal{G}^{\text{mem}} \rangle = 0}$

Instantaneous' flux at 1.5PN

Finally, we can compute the intanteneous flux at 1.5PN with the 2PN QK parametrization in the same way as before ... but there is a complicating when trying to go the the CoM frame !

Indeed, the linear momentum P^i and CoM position G^i of the matter content satisfy the balance equations:

$$\frac{\mathrm{d}G_i}{\mathrm{d}t} = P_i - \mathcal{F}^i_{s,\boldsymbol{G}} - \mathcal{F}^i_{\boldsymbol{G}} \qquad \text{and} \qquad \frac{\mathrm{d}P_i}{\mathrm{d}t} = -\mathcal{F}^i_{s,\boldsymbol{P}} - \mathcal{F}^i_{\boldsymbol{P}}$$

where the fluxes enter at 2.5PN order. But COM frame is defined not only for the matter content, but also for the GWs ! Thus is is defined by

$$G_i + \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t'' \left[\mathcal{F}^i_{s,\boldsymbol{G}} + \mathcal{F}^i_{\boldsymbol{G}}\right](t) + \int_{-\infty}^{t} \mathrm{d}t' \left[\mathcal{F}^i_{s,\boldsymbol{P}} + \mathcal{F}^i_{\boldsymbol{P}}\right](t') = 0$$

which reduces to $G_i = 0$ only at 2PN order. Thus, we have an extra nonlocal term to deal with!