A Fisher matrix code for population analysis of gravitational-wave events

Viola De Renzis

(University of Milano-Bicocca)

with: Francesco Iacovelli, Michele Mancarella , Davide Gerosa, Costantino Pacilio

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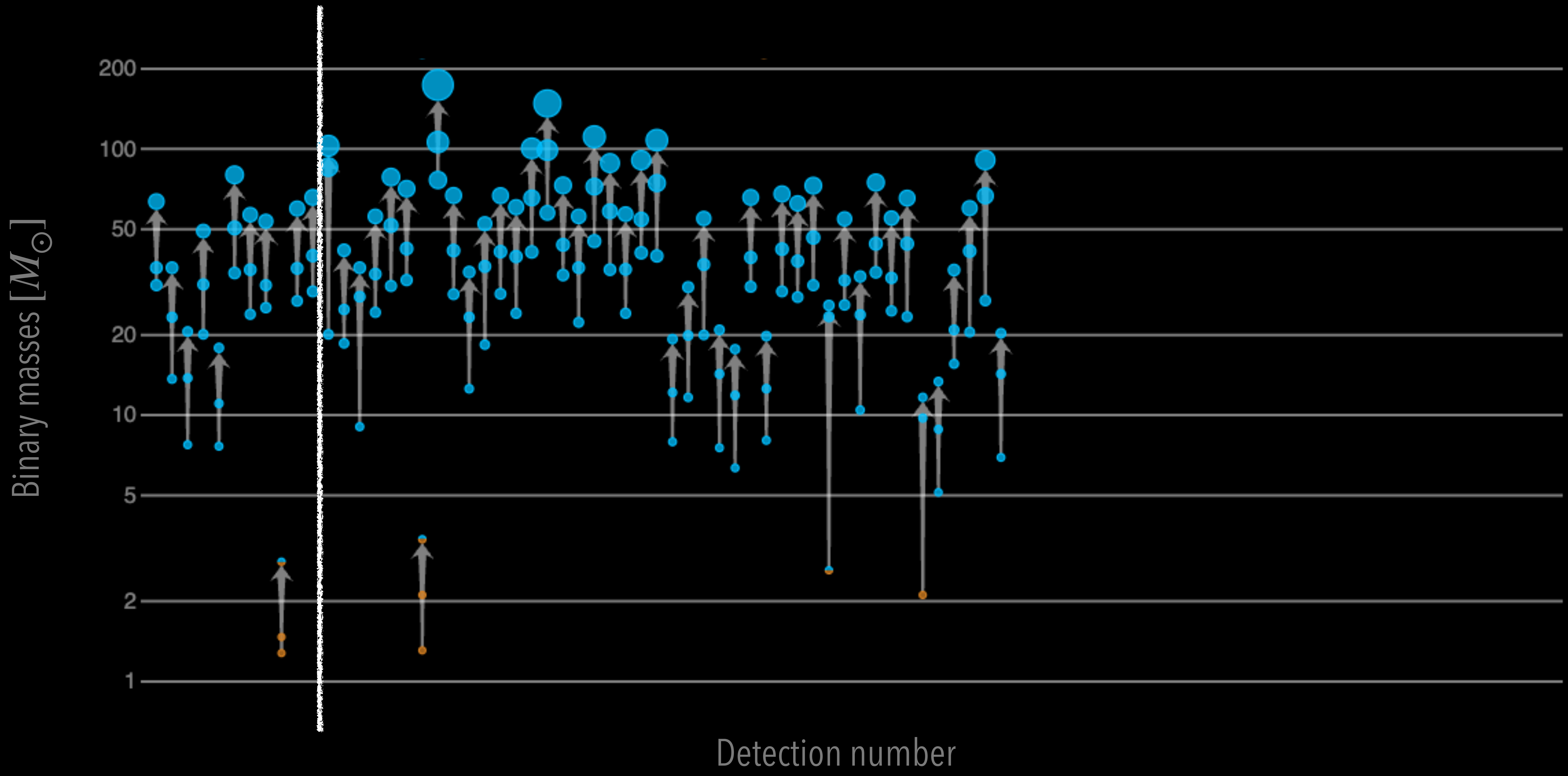
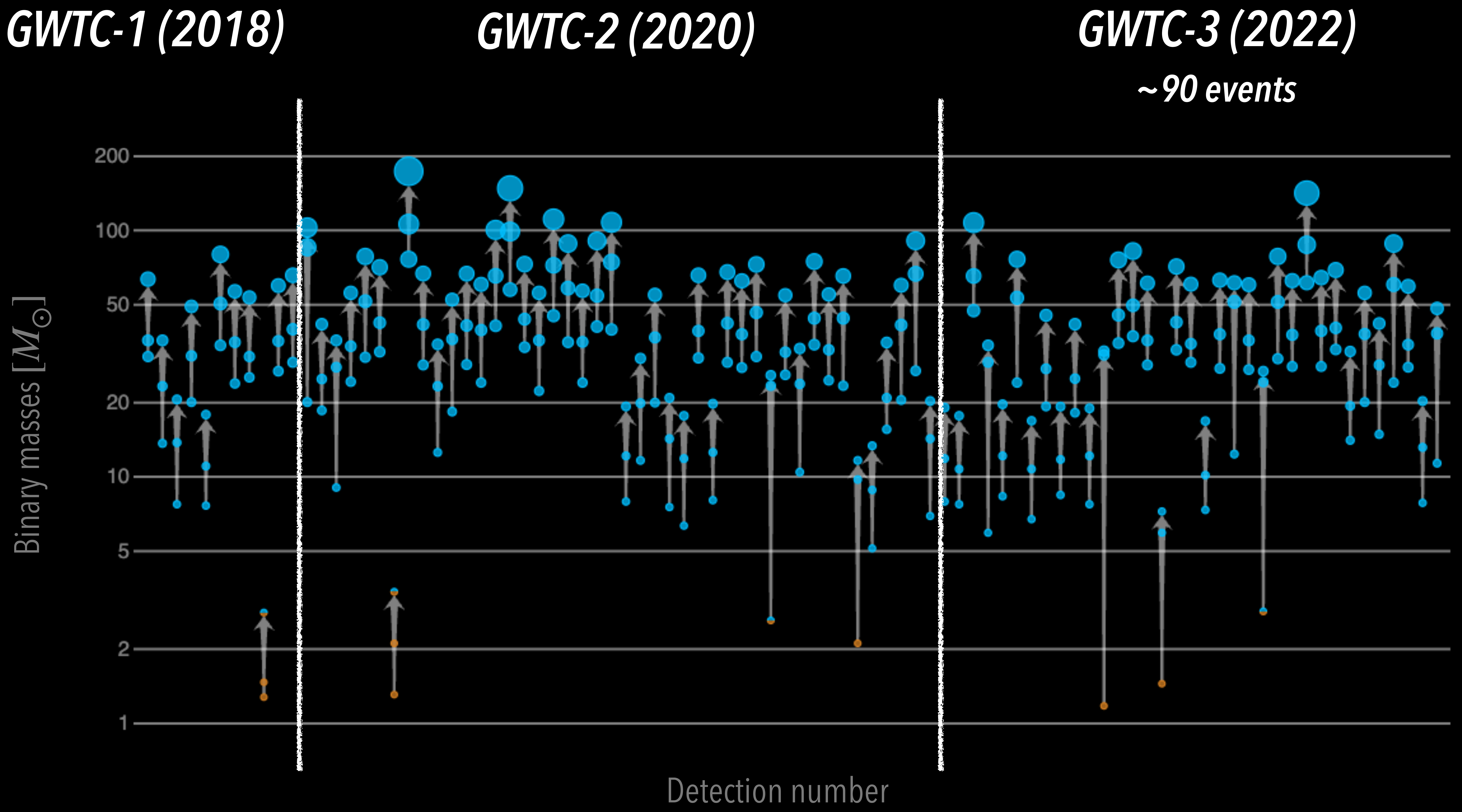
v.derenzis@campus.unimib.it

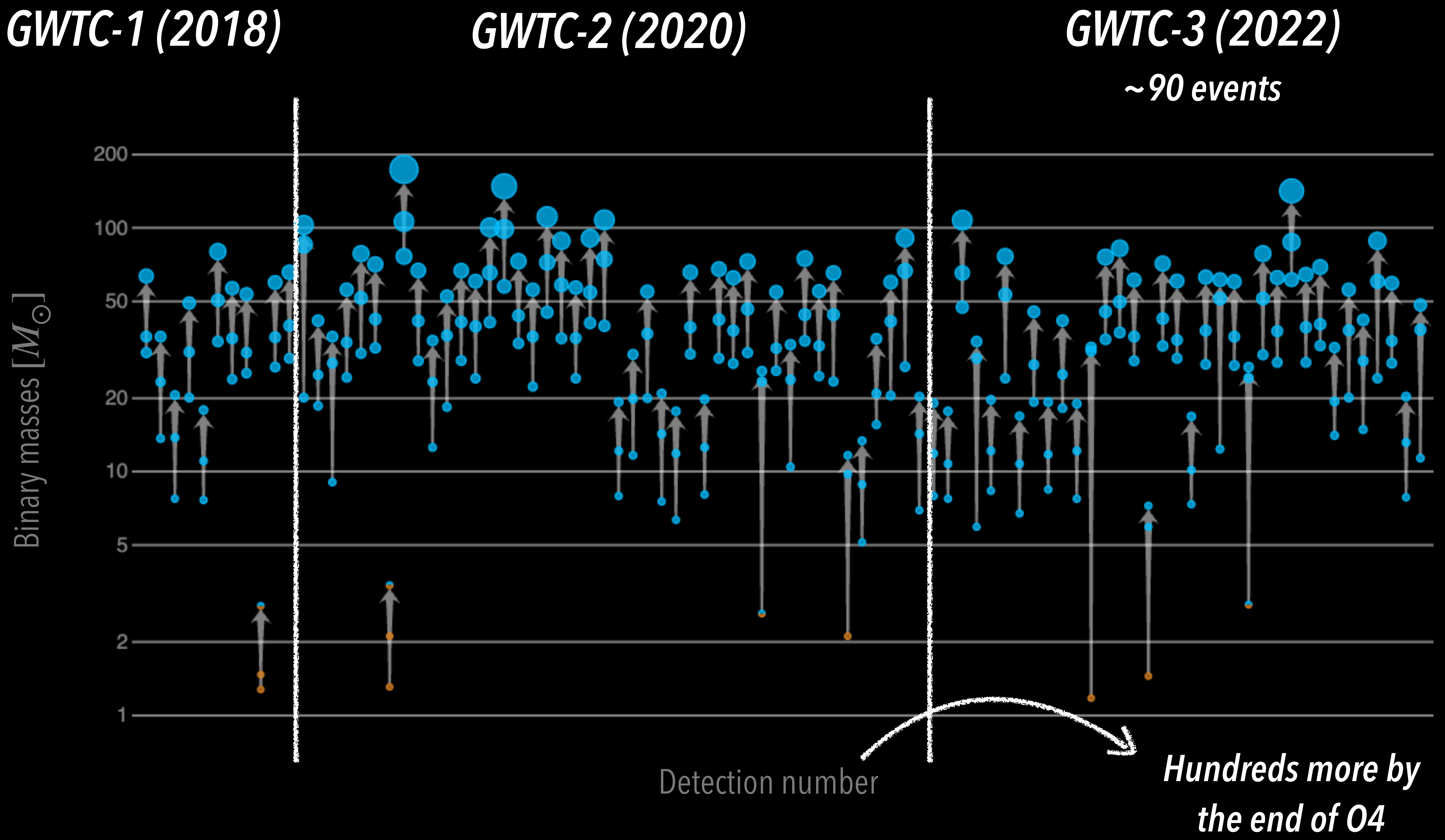
GWTC-1 (2018)

GWTC-1 (2018)

GWTC-2 (2020)




GWTC-1 (2018)
GWTC-2 (2020)
GWTC-3 (2022)
~90 events
200
100
50
20
10
5
2
1
Binary masses [$M_\odot$]
Detection number



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GWTC-2 (2020)
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~90 events
200
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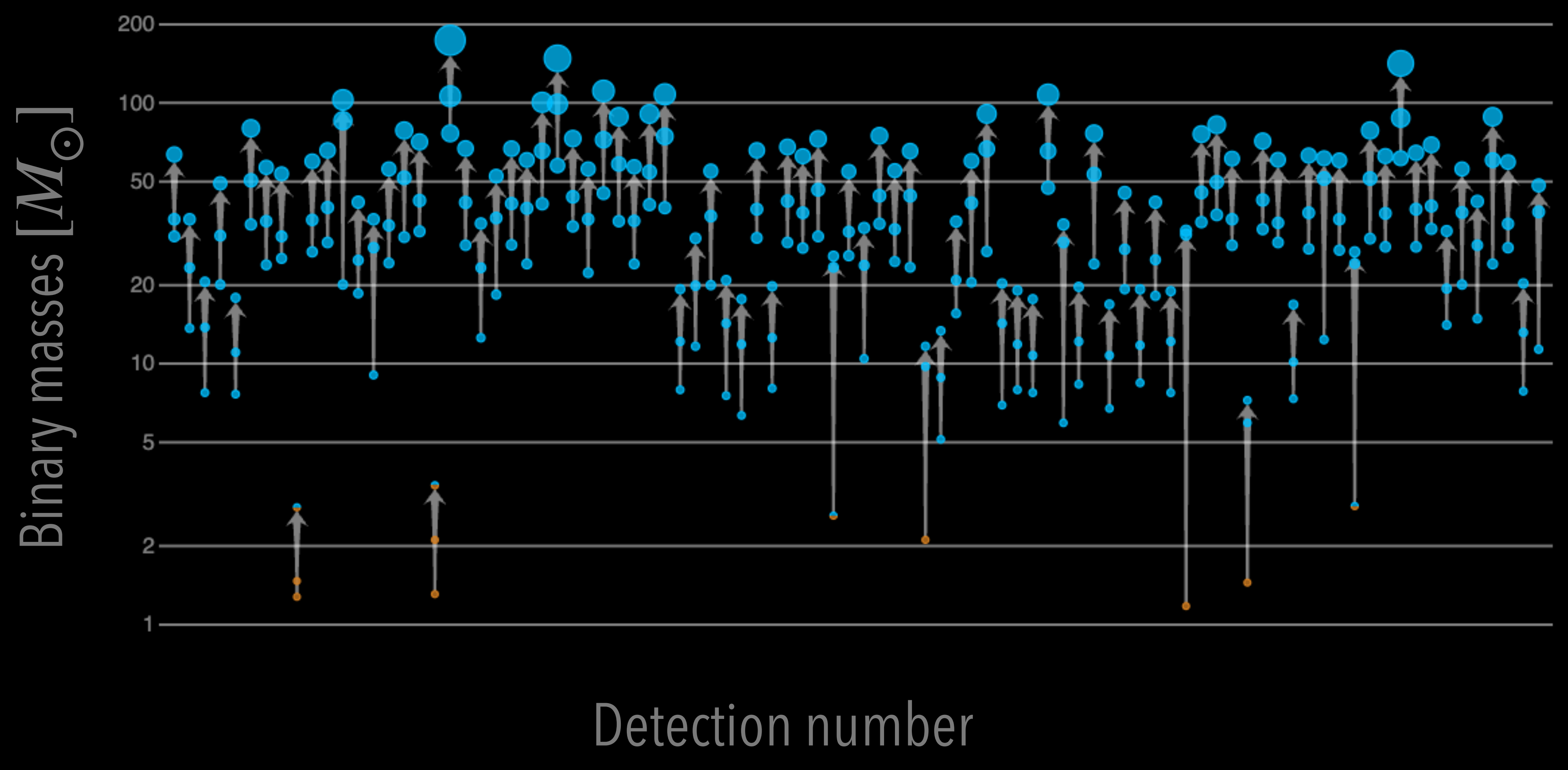
Binary masses $[M_\odot]$
Detection number
Hundreds more by the end of O4


GWTC-1 (2018)
GWTC-2 (2020)
GWTC-3 (2022)
~90 events
200
100
50
20
10
5
2
1
Binary masses $[M_\odot]$
Detection number
Millions more in ~20 years
with 3G detectors

Hierarchical Bayesian analysis for population studies



- **Lowest level**:
  What are the properties of **individual** GW sources?

[Thrane & Talbolt, 2018]

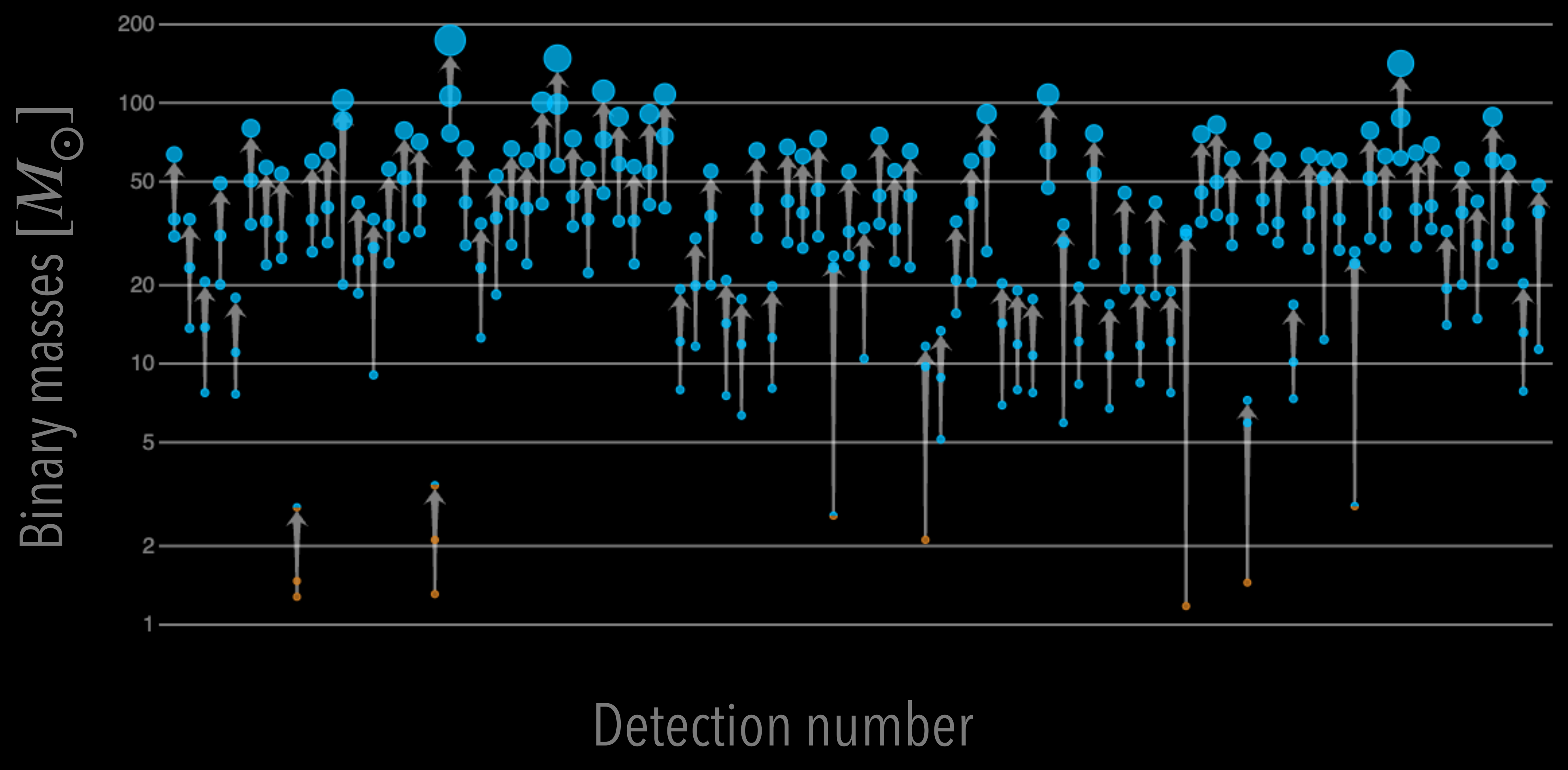
Hierarchical Bayesian analysis for population studies

- Lowest level:
  What are the properties of **individual** GW sources?

- Next level:
  What are the properties of the **underlying population** of GW sources?

[Thrane & Talbolt, 2018]

Hierarchical Bayesian analysis for population studies



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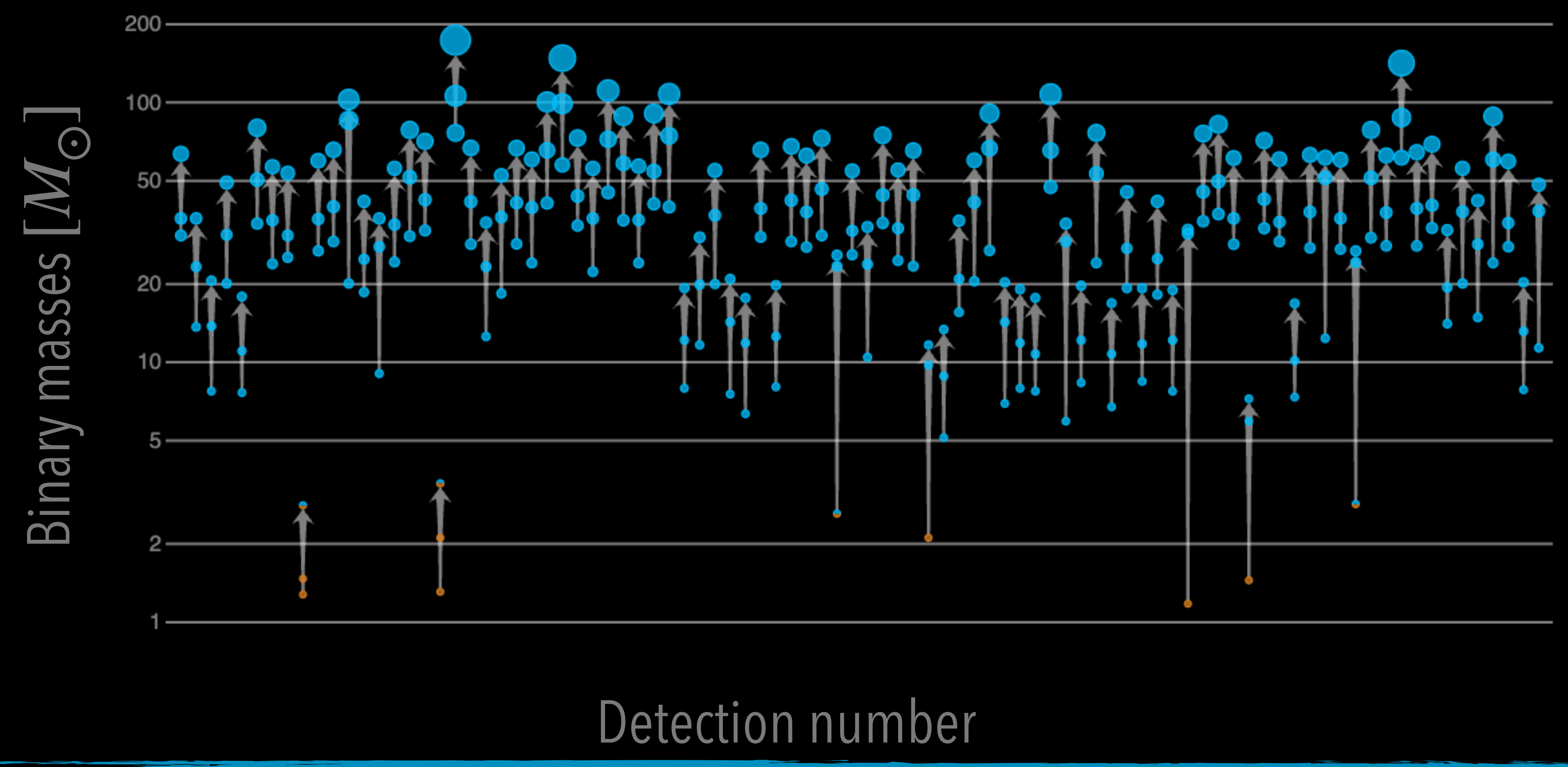
- **Ultimate goal**: Figure out the formation pathways and population properties of compact binary systems
  - how many formation channels?
  - properties of each individual channel (common envelope, kicks…)
  - merger rate for each channel
  - many more open questions

[Thrane & Talbolt, 2018]

Hierarchical Bayesian analysis for population studies



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Hierarchical Bayesian inference

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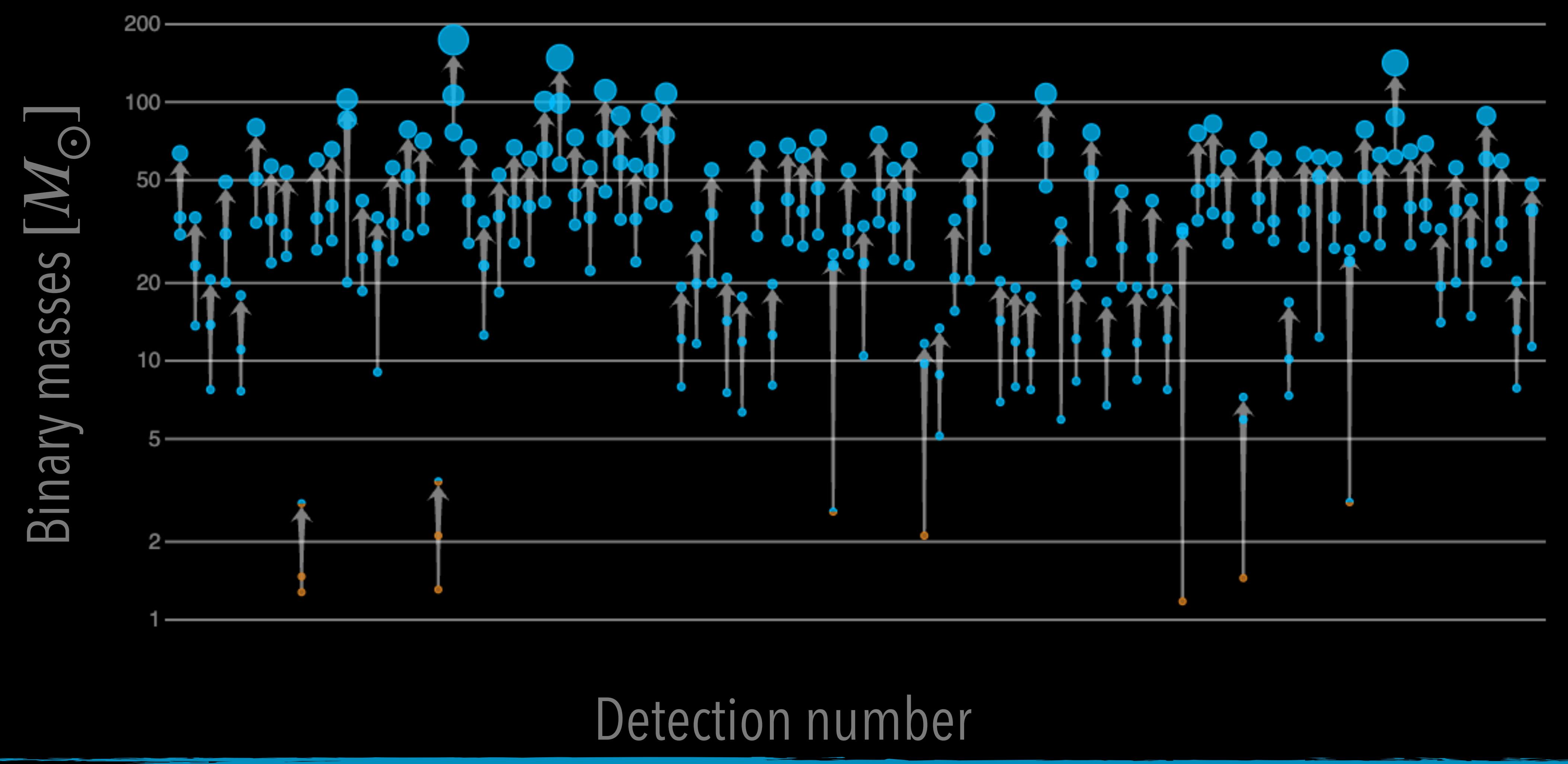

- Lowest level:

  What are the properties of individual GW sources?

$$p(\vec{\theta}\,|\,d) \propto \mathscr{L}(d\,|\,\vec{\theta})\pi(\vec{\theta})$$

$\vec{\theta}$ = individual source parameters

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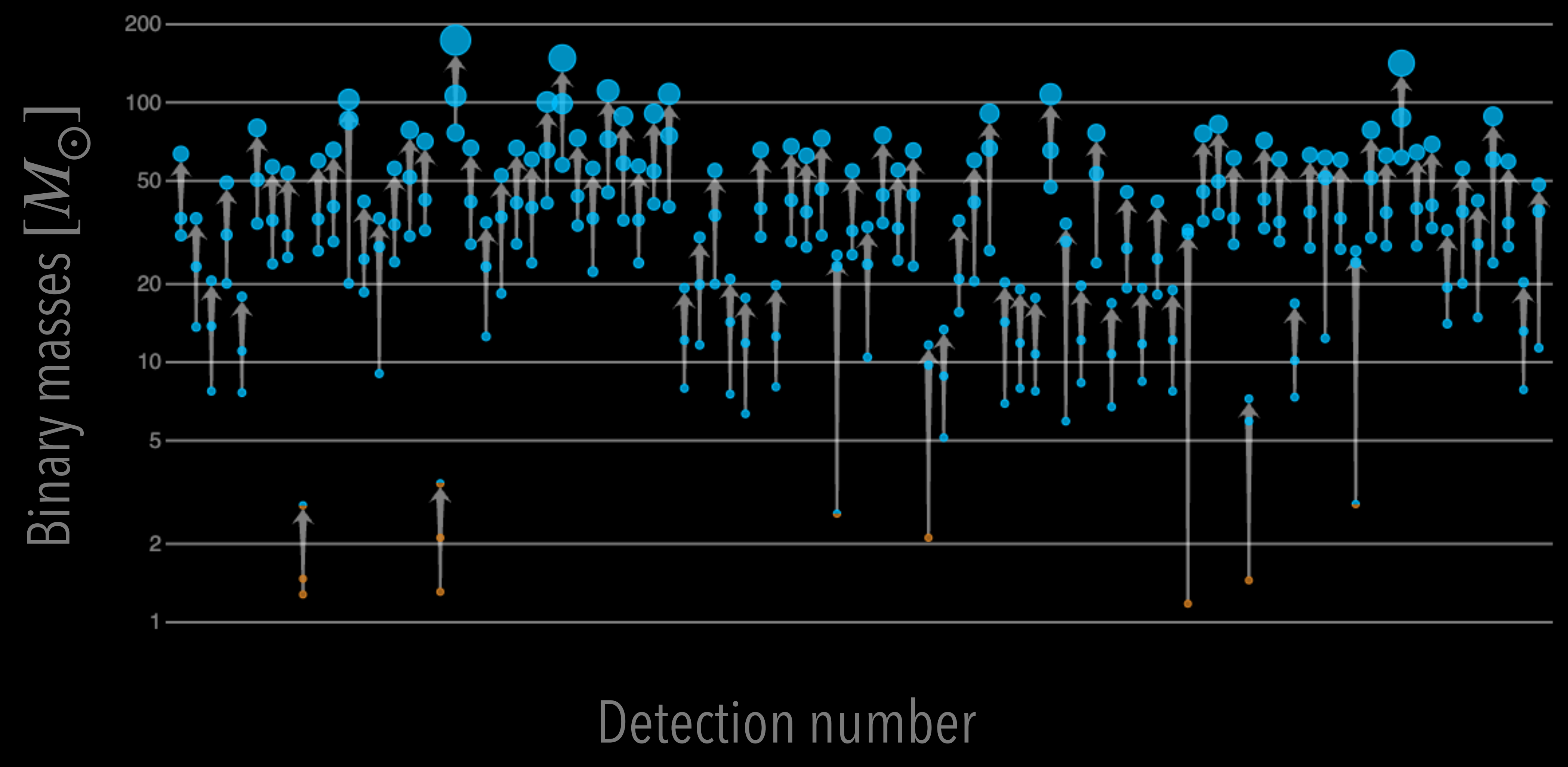
  What are the properties of the underlying population of GW sources?

Hierarchical Bayesian inference

[Thrane & Talbolt, 2018]

Hierarchical Bayesian analysis for population studies



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  What are the properties of the **underlying population** of GW sources?

Single-event likelihood

Population model

$$p\left(\{d_i\}\,|\,\vec{\lambda}\right) = \frac{\int \mathscr{L}\left(d\,|\,\vec{\theta}\right)\, p\left(\vec{\theta}\,|\,\vec{\lambda}\right)\, d\vec{\theta}}{P_{\mathrm{det}}\left(\vec{\lambda}\right)}$$

Population Likelihood

Not all sources are equally easy to detect

[Thrane & Talbolt, 2018]

Hierarchical Bayesian analysis for population studies

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Population Likelihood

Selection effects

$$P_{\rm det}(\vec{\lambda}) = \int P_{\rm det}(\vec{\theta})p(\theta\,|\,\lambda)d\vec{\theta}$$

Detection probability:

$$P_{\rm det}(\vec{\theta}) = \int_{d>d_{th}} \mathscr{L}(d\,|\,\vec{\theta}){\rm d}d$$

Individual-event parameter estimation

- Bayesian inference codes (BILBY, lalsimulation, PyCBC, RIFT, DINGO…) [Ashton et al, 2018] [Henshaw et al, 2022] [Dax et al, 2021]
- Fisher codes (GWfast, GWbench, GWfish) [Iacovelli et al, 2022] [Borhanian, 2021] [Branchesi et al, 2023]

Population level parameter estimation

- Bayesian inference codes (GWpopulation, BILBY, deep learning codes, non-parametric models…)
- No fisher codes to make forecasts about the population properties that will be observed with 3G detectors

↓

We are developing a user-friendly python code to estimate the parameters that characterize a population of GW events including selection effects

Population Fisher Matrix

$$(\Gamma_\lambda)_{ij} \equiv N_{\rm det}\left[(\Gamma_{\rm I})_{ij} + (\Gamma_{\rm II})_{ij} + (\Gamma_{\rm III})_{ij} + (\Gamma_{\rm IV})_{ij} + (\Gamma_{\rm V})_{ij}\right]$$

$$(\Gamma_{\rm I})_{ij} = -\int \frac{\partial^2 \ln\left(p(\vec{\theta}\,|\,\vec{\lambda})/P_{\rm det}(\vec{\lambda})\right)}{\partial\lambda^i\partial\lambda^j}\frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})}p(\vec{\theta}\,|\,\vec{\lambda})\mathrm{d}\vec{\theta}$$

$$(\Gamma_{\rm II})_{ij} = \frac{1}{2}\int \frac{\partial^2 \ln\det\left(\Gamma + H\right)}{\partial\lambda^i\partial\lambda^j}\frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})}p(\vec{\theta}\,|\,\vec{\lambda})\mathrm{d}\vec{\theta}$$

$$(\Gamma_{\rm III})_{ij} = -\frac{1}{2}\int \frac{\partial^2}{\partial\lambda^i\partial\lambda^j}\left[(\Gamma + H)^{-1}_{kl}\right]\Gamma_{kl}\frac{p(\vec{\theta}\,|\,\vec{\lambda})}{P_{\rm det}(\vec{\lambda})}\mathrm{d}\vec{\theta}$$

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[Gair et al., 2023]

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1. Population model: $p(\vec\theta\,|\,\vec\lambda)$

[Gair et al., 2023]

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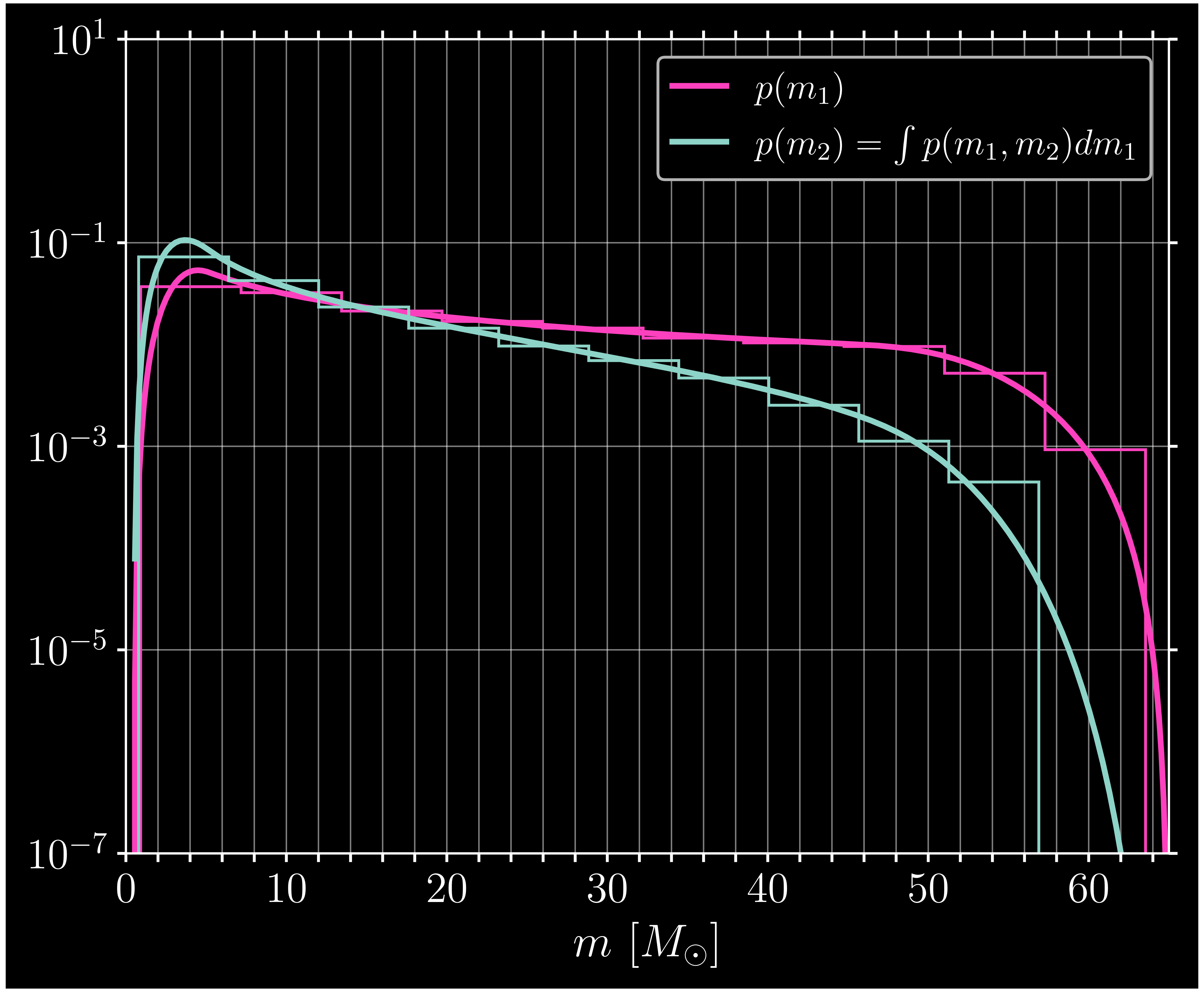
1. Masses: Power law?



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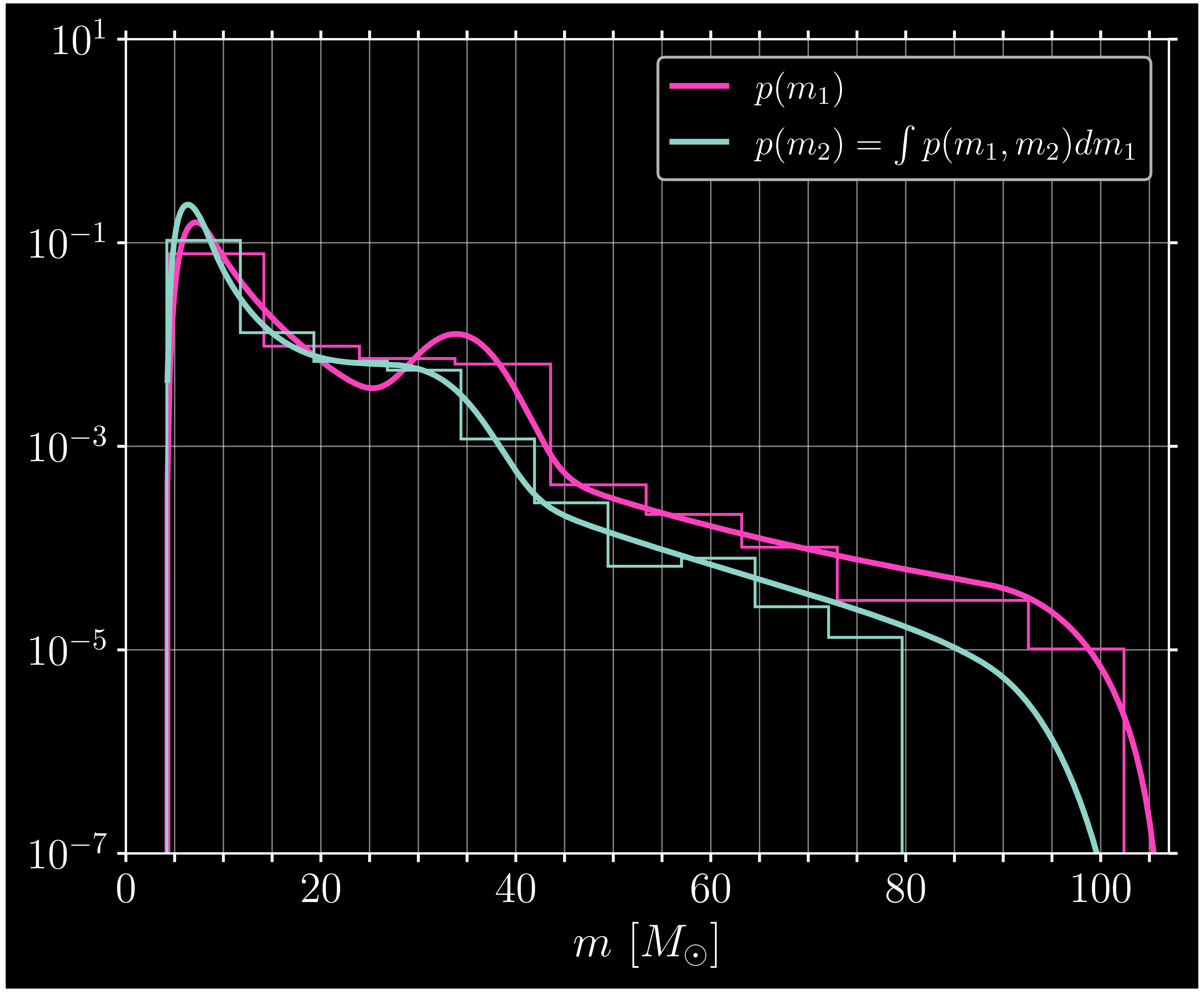
1. Masses: power law + peak?



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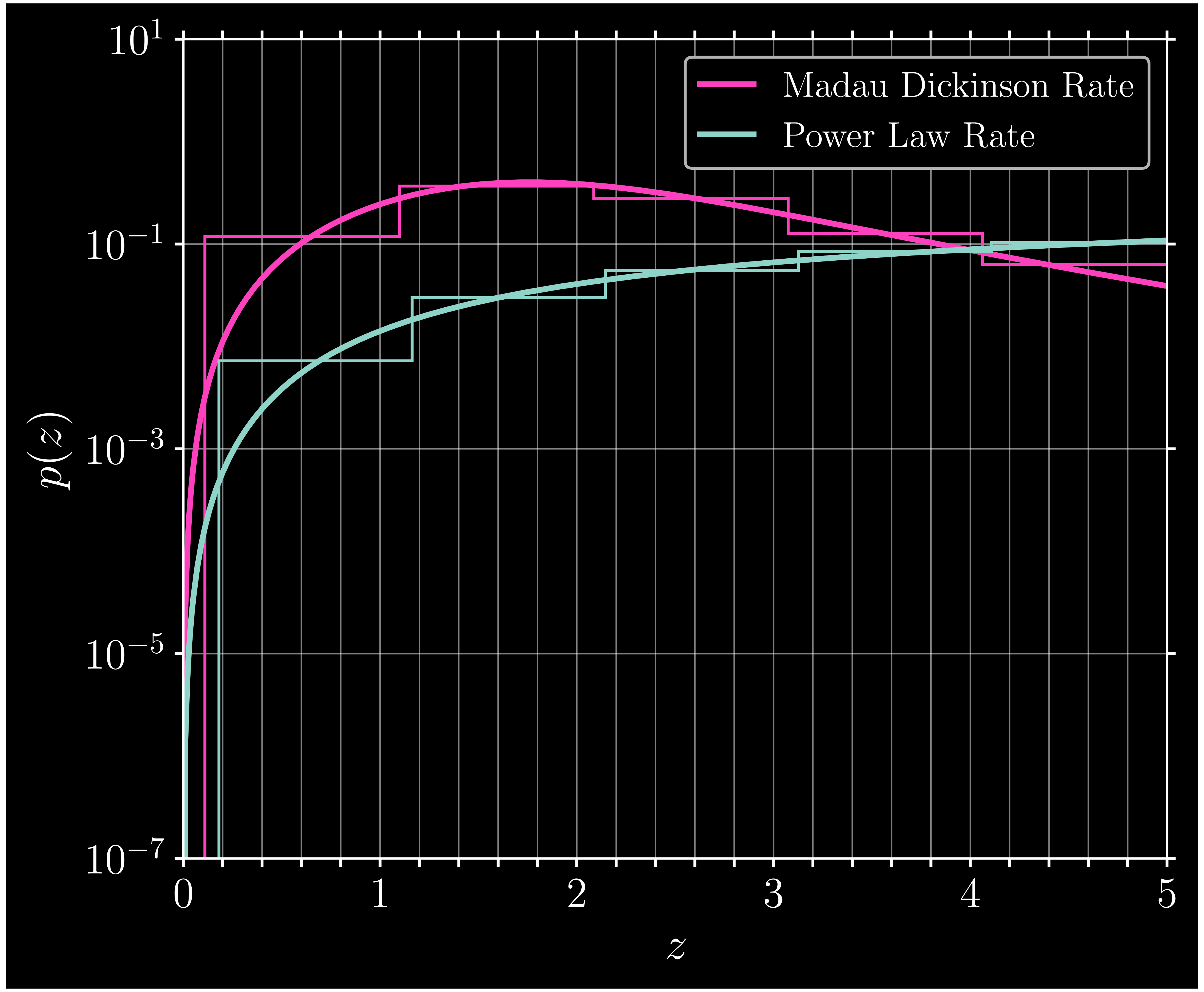
1.

Redshift:
PL/Madau Dickinson?



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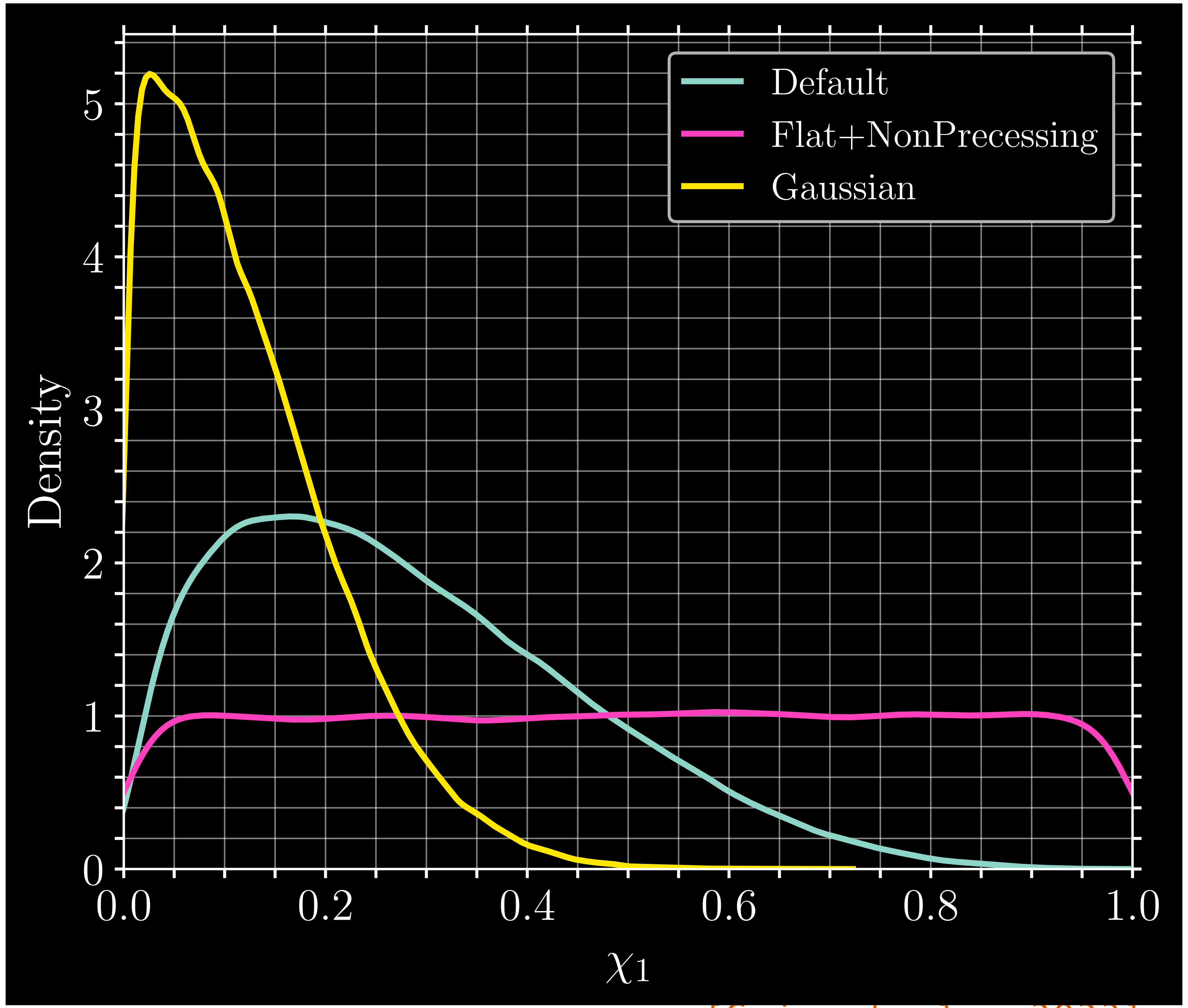
1.

Spins:
Default/Flat,Gaussian?



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Whatever functional form you like!
(as long as it is differentiable)

[Gair et al., 2023]

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1. Population model: $p(\vec\theta\,|\,\vec\lambda)$

with its first and second derivatives

$$P_i = \frac{\partial \ln p(\vec\theta\,|\,\vec\lambda)}{\partial\theta^i}$$

$$H_{ij} = -\frac{\partial^2 \ln p(\vec\theta\,|\,\vec\lambda)}{\partial\theta^i\partial\theta^j}$$

[Gair et al., 2023]

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1. Population model: $p(\vec{\theta}\,|\,\vec{\lambda})$

2. Selection effects:

$$P_{\rm det}(\vec{\lambda}) = \int P_{\rm det}(\vec{\theta})p(\theta|\lambda)d\vec{\theta}$$

[Gair et al., 2023]

Population Fisher Matrix

$$(\Gamma_\lambda)_{ij} \equiv N_{\rm det} \left[(\Gamma_{\rm I})_{ij} + (\Gamma_{\rm II})_{ij} + (\Gamma_{\rm III})_{ij} + (\Gamma_{\rm IV})_{ij} + (\Gamma_{\rm V})_{ij}\right]$$

$$(\Gamma_{\rm I})_{ij} = -\int \frac{\partial^2 \ln\left(p(\vec{\theta}\,|\,\vec{\lambda})\; P_{\rm det}(\vec{\lambda})\right)}{\partial\lambda^i\partial\lambda^j} \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})} p(\vec{\theta}\,|\,\vec{\lambda})$$

$$(\Gamma_{\rm II})_{ij} = \frac{1}{2}\int \frac{\partial^2 \ln\det(\Gamma + H)}{\partial\lambda^i\partial\lambda^j} \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})} p(\vec{\theta}\,|\,\vec{\lambda})$$

$$(\Gamma_{\rm III})_{ij} = -\frac{1}{2}\int \frac{\partial^2}{\partial\lambda^i\partial\lambda^j}\left[(\Gamma + H)^{-1}_{kl}\right]\Gamma_{kl} \frac{p(\vec{\theta}\,|\,\vec{\lambda})}{P_{\rm det}(\vec{\lambda})} {\rm d}\vec{\theta}$$

$$(\Gamma_{\rm IV})_{ij} = -\int \frac{\partial^2}{\partial\lambda^i\partial\lambda^j}\left[P_k(\Gamma + H)^{-1}_{kl}\right] D_l \frac{p(\vec{\theta}\,|\,\vec{\lambda})}{P_{\rm det}(\vec{\lambda})} {\rm d}\vec{\theta}$$

$$(\Gamma_{\rm V})_{ij} = -\frac{1}{2}\int \frac{\partial^2}{\partial\lambda^i\partial\lambda^j}\left[P_k(\Gamma + H)^{-1}_{kl} P_l\right] \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})} p(\vec{\theta}\,|\,\vec{\lambda})$$

1. Population model: $p(\vec{\theta}\,|\,\vec{\lambda})$

2. Selection effects:

$$P_{\rm det}(\vec{\lambda}) = \int P_{\rm det}(\vec{\theta}) p(\theta|\lambda) d\vec{\theta}$$

3. Detection probability:

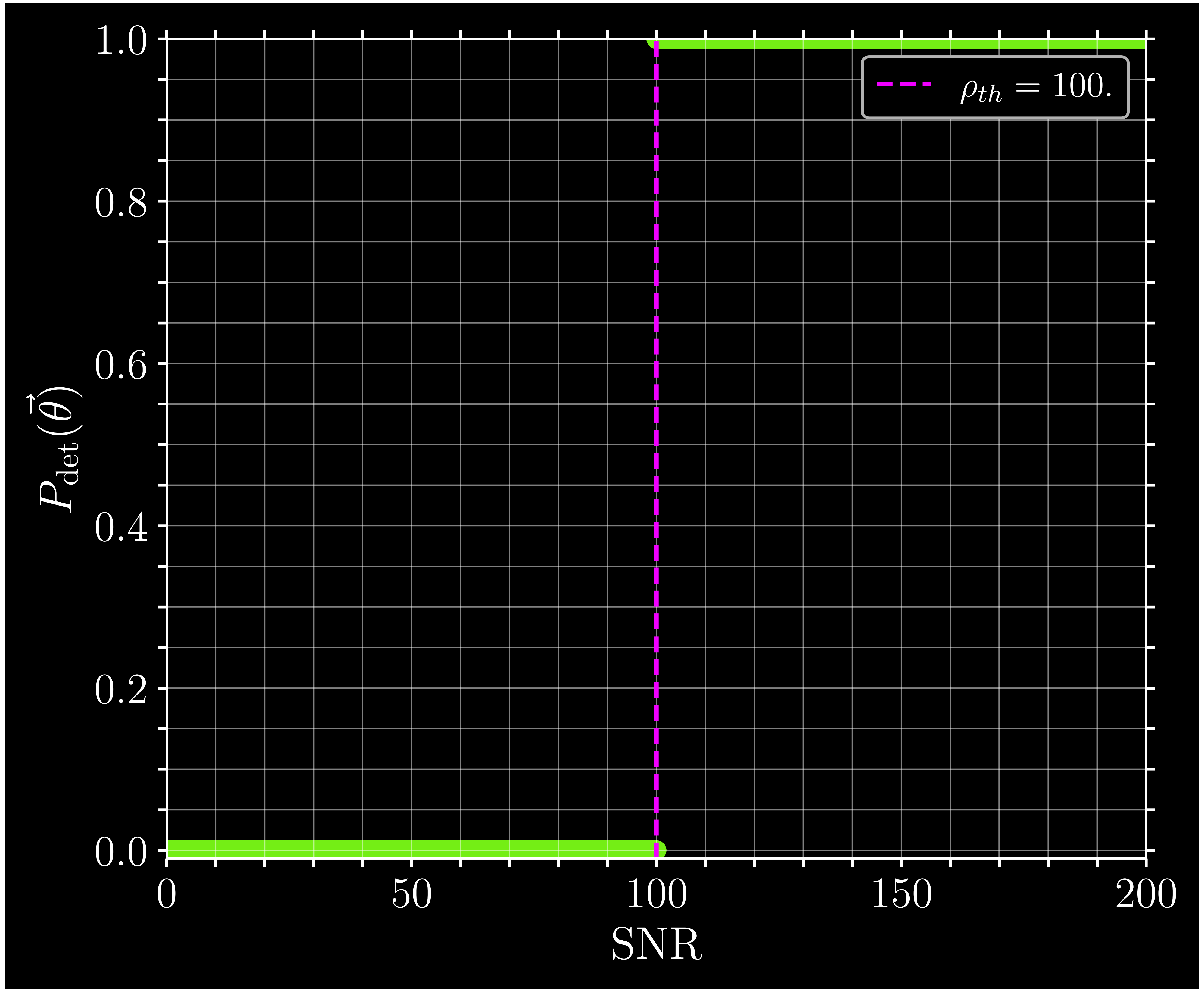
$$P_{\rm det}(\vec{\theta}) = \int_{d>d_{th}} p(d\,|\,\vec{\theta}){\rm d}\vec{\theta}$$

[Gair et al., 2023]

Population Fisher Matrix

$$(\Gamma_\lambda)_{ij} \equiv N_{\rm det}\left[(\Gamma_{\rm I})_{ij} + (\Gamma_{\rm II})_{ij} + (\Gamma_{\rm III})_{ij} + (\Gamma_{\rm IV})_{ij} + (\Gamma_{\rm V})_{ij}\right]$$

$P_{\rm det}(\vec{\theta})$ = Heaviside function



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[Gair et al., 2023]

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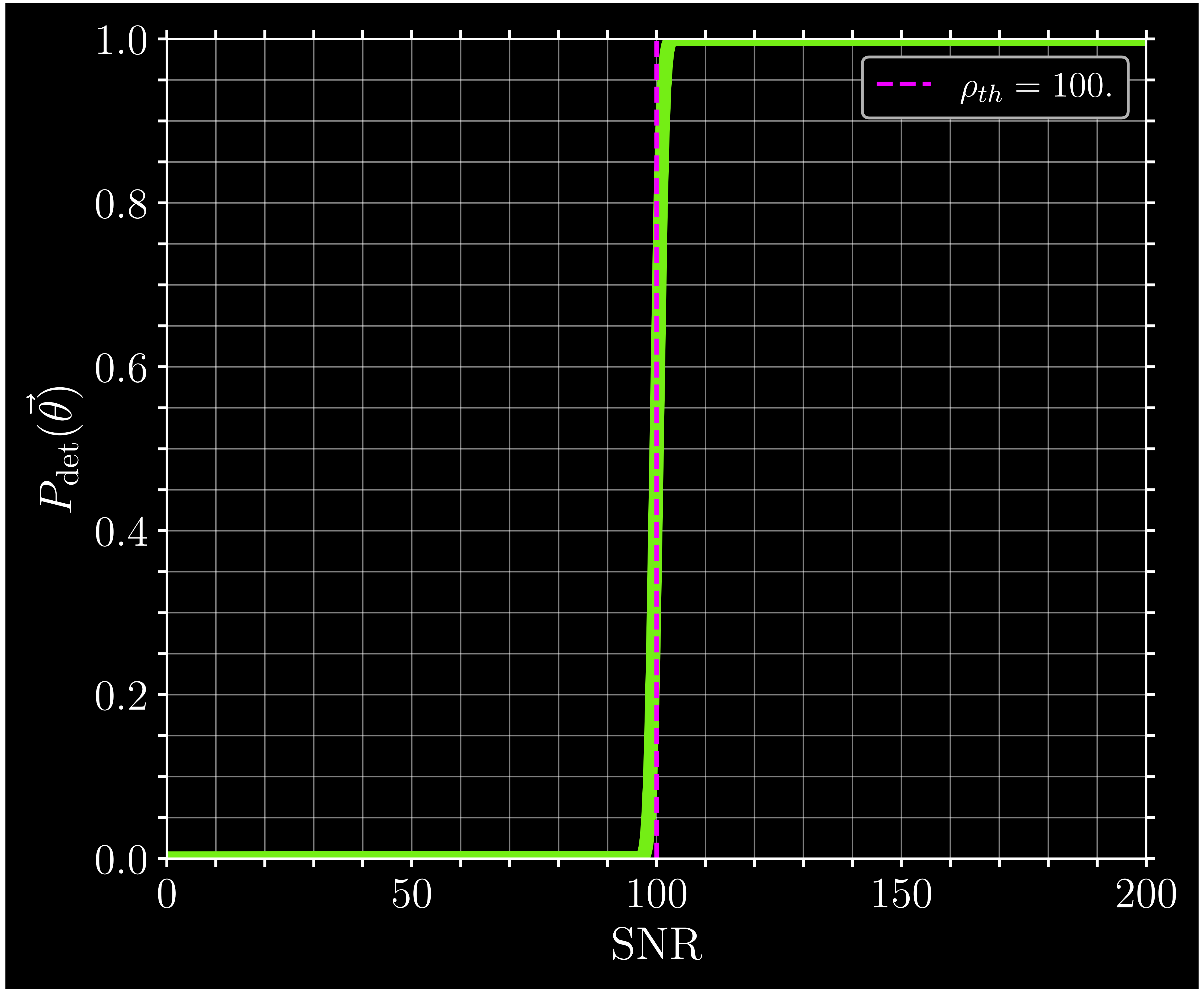
We opted for a more regular error function, i.e

$$P_{\rm det}(\vec{\theta}) = \frac{1}{2}{\rm erfc}\frac{(\rho - \rho_{\rm th})}{\sqrt{2}\sigma}$$



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[Gair et al., 2023]

Population Fisher Matrix

$$(\Gamma_\lambda)_{ij} \equiv N_{\rm det} \left[(\Gamma_{\rm I})_{ij} + (\Gamma_{\rm II})_{ij} + (\Gamma_{\rm III})_{ij} + (\Gamma_{\rm IV})_{ij} + (\Gamma_{\rm V})_{ij}\right]$$

$$(\Gamma_{\rm I})_{ij} = -\int \frac{\partial^2 \ln\left(p(\vec{\theta} \mid \vec{\lambda})\, P_{\rm det}(\vec{\lambda})\right)}{\partial\lambda^i \partial\lambda^j} \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})} p(\vec{\theta} \mid \vec{\lambda})$$

$$(\Gamma_{\rm II})_{ij} = \frac{1}{2}\int \frac{\partial^2 \ln \det\left(\Gamma + H\right)}{\partial\lambda^i \partial\lambda^j} \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})} p(\vec{\theta} \mid \vec{\lambda})$$

$$(\Gamma_{\rm III})_{ij} = -\frac{1}{2}\int \frac{\partial^2}{\partial\lambda^i \partial\lambda^j} \left[(\Gamma + H)^{-1}_{kl}\right] \Gamma_{kl} \frac{p(\vec{\theta} \mid \vec{\lambda})}{P_{\rm det}(\vec{\lambda})} {\rm d}\vec{\theta}$$

$$(\Gamma_{\rm IV})_{ij} = -\int \frac{\partial^2}{\partial\lambda^i \partial\lambda^j} \left[P_k (\Gamma + H)^{-1}_{kl}\right] D_l \frac{p(\vec{\theta} \mid \vec{\lambda})}{P_{\rm det}(\vec{\lambda})} {\rm d}\vec{\theta}$$

$$(\Gamma_{\rm V})_{ij} = -\frac{1}{2}\int \frac{\partial^2}{\partial\lambda^i \partial\lambda^j} \left[P_k (\Gamma + H)^{-1}_{kl} P_l\right] \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})} p(\vec{\theta} \mid \vec{\lambda})$$

1. Population model: $p(\vec{\theta} \mid \vec{\lambda})$

2. Selection effects:

$$P_{\rm det}(\vec{\lambda}) = \int P_{\rm det}(\vec{\theta}) p(\theta \mid \lambda) d\vec{\theta}$$

3. Detection probability:

$$P_{\rm det}(\vec{\theta}) = \int_{d > d_{th}} p(d \mid \vec{\theta}) {\rm d}\vec{\theta}$$

with its first derivative

$$D_l = \frac{\partial P_{\rm det}(\vec{\theta})}{\partial \theta^i}$$

[Gair et al., 2023]

Population Fisher Matrix

$$(\Gamma_\lambda)_{ij} \equiv N_{\rm det}\left[(\Gamma_{\rm I})_{ij} + (\Gamma_{\rm II})_{ij} + (\Gamma_{\rm III})_{ij} + (\Gamma_{\rm IV})_{ij} + (\Gamma_{\rm V})_{ij}\right]$$

$$(\Gamma_{\rm I})_{ij} = -\int \frac{\partial^2 \ln\left(p(\vec{\theta}\,|\,\vec{\lambda})\, P_{\rm det}(\vec{\lambda})\right)}{\partial\lambda^i\partial\lambda^j}\, \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})}\, p(\vec{\theta}\,|\,\vec{\lambda})$$

$$(\Gamma_{\rm II})_{ij} = \frac{1}{2}\int \frac{\partial^2 \ln\det\left(\Gamma + H\right)}{\partial\lambda^i\partial\lambda^j}\, \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})}\, p(\vec{\theta}\,|\,\vec{\lambda})\,{\rm d}\vec{\theta}$$

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$$(\Gamma_{\rm IV})_{ij} = -\int \frac{\partial^2}{\partial\lambda^i\partial\lambda^j}\left[P_k\left(\Gamma + H\right)^{-1}_{kl}\right] D_l\, \frac{p(\vec{\theta}\,|\,\vec{\lambda})}{P_{\rm det}(\vec{\lambda})}\,{\rm d}\vec{\theta}$$

$$(\Gamma_{\rm V})_{ij} = -\frac{1}{2}\int \frac{\partial^2}{\partial\lambda^i\partial\lambda^j}\left[P_k\left(\Gamma + H\right)^{-1}_{kl} P_l\right] \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})}\, p(\vec{\theta}\,|\,\vec{\lambda})\,{\rm d}\vec{\theta}$$

1. Population model: $p(\vec{\theta}\,|\,\vec{\lambda})$
2. Selection effects:
   $$P_{\rm det}(\vec{\lambda}) = \int P_{\rm det}(\vec{\theta})\, p(\theta|\lambda)\, d\vec{\theta}$$
3. Detection probability:
   $$P_{\rm det}(\vec{\theta}) = \int_{d>d_{th}} p(d\,|\,\vec{\theta})\,{\rm d}\vec{\theta}$$
4. Γ = Single event Fisher matrix with GWfast

[Iacovelli et al, 2022]
[Gair et al., 2023]

Population Fisher Matrix

$$(\Gamma_\lambda)_{ij} \equiv N_{\rm det}\,\left[(\Gamma_{\rm I})_{ij} + (\Gamma_{\rm II})_{ij} + (\Gamma_{\rm III})_{ij} + (\Gamma_{\rm IV})_{ij} + (\Gamma_{\rm V})_{ij}\right]$$

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$$(\Gamma_{\rm II})_{ij} = \int B(\vec\theta)\frac{P_{\rm det}(\vec\theta)}{P_{\rm det}(\vec\lambda)}p(\vec\theta\,|\,\vec\lambda)\mathrm{d}\vec\theta$$

$$(\Gamma_{\rm III})_{ij} = \int C(\vec\theta)\frac{p(\vec\theta\,|\,\vec\lambda)}{P_{\rm det}(\vec\lambda)}\mathrm{d}\vec\theta$$

$$(\Gamma_{\rm IV})_{ij} = \int D(\vec\theta)\frac{p(\vec\theta\,|\,\vec\lambda)}{P_{\rm det}(\vec\lambda)}\mathrm{d}\vec\theta$$

$$(\Gamma_{\rm V})_{ij} = \int E(\vec\theta)\frac{P_{\rm det}(\vec\theta)}{P_{\rm det}(\vec\lambda)}p(\vec\theta\,|\,\vec\lambda)\mathrm{d}\vec\theta$$

1. Population model: $p(\vec\theta\,|\,\vec\lambda)$

2. Selection effects:

$$P_{\rm det}(\vec\lambda) = \int P_{\rm det}(\vec\theta)p(\theta|\lambda)d\vec\theta$$

3. Detection probability:

$$P_{\rm det}(\vec\theta) = \int_{d>d_{th}} p(d\,|\,\vec\theta)\mathrm{d}\vec\theta$$

4. Γ = Single event Fisher matrix with GWfast

Errors on the single-event parameters $\vec\theta$ enter in the last four terms!

[Iacovelli et al, 2022]
[Gair et al., 2023]

Population Fisher Matrix

$$(\Gamma_\lambda)_{ij} \equiv N_{\rm det} \left[(\Gamma_{\rm I})_{ij} + (\Gamma_{\rm II})_{ij} + (\Gamma_{\rm III})_{ij} + (\Gamma_{\rm IV})_{ij} + (\Gamma_{\rm V})_{ij} \right]$$

$$(\Gamma_{\rm I})_{ij} = - \int \frac{\partial^2 \ln\left(p(\vec{\theta} \mid \vec{\lambda})\; P_{\rm det}(\vec{\lambda})\right)}{\partial \lambda^i \partial \lambda^j} \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})} p(\vec{\theta} \mid \vec{\lambda})$$

$$(\Gamma_{\rm II})_{ij} = \int B(\vec{\theta}) \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})} p(\vec{\theta} \mid \vec{\lambda}) {\rm d}\vec{\theta}$$

$$(\Gamma_{\rm III})_{ij} = \int C(\vec{\theta}) \frac{p(\vec{\theta} \mid \vec{\lambda})}{P_{\rm det}(\vec{\lambda})} {\rm d}\vec{\theta}$$

$$(\Gamma_{\rm IV})_{ij} = \int D(\vec{\theta}) \frac{p(\vec{\theta} \mid \vec{\lambda})}{P_{\rm det}(\vec{\lambda})} {\rm d}\vec{\theta}$$

$$(\Gamma_{\rm V})_{ij} = \int E(\vec{\theta}) \frac{P_{\rm det}(\vec{\theta})}{P_{\rm det}(\vec{\lambda})} p(\vec{\theta} \mid \vec{\lambda}) {\rm d}\vec{\theta}$$

1. Population model: $p(\vec{\theta} \mid \vec{\lambda})$

2. Selection effects:

$$P_{\rm det}(\vec{\lambda}) = \int P_{\rm det}(\vec{\theta}) p(\theta \mid \lambda) d\vec{\theta}$$

3. Detection probability:

$$P_{\rm det}(\vec{\theta}) = \int_{d>d_{th}} p(d \mid \vec{\theta}) {\rm d}\vec{\theta}$$

4. $\Gamma =$ Single event Fisher matrix with GWfast

If the single-event errors are small, $\Gamma_{\rm I}$ is the dominant term

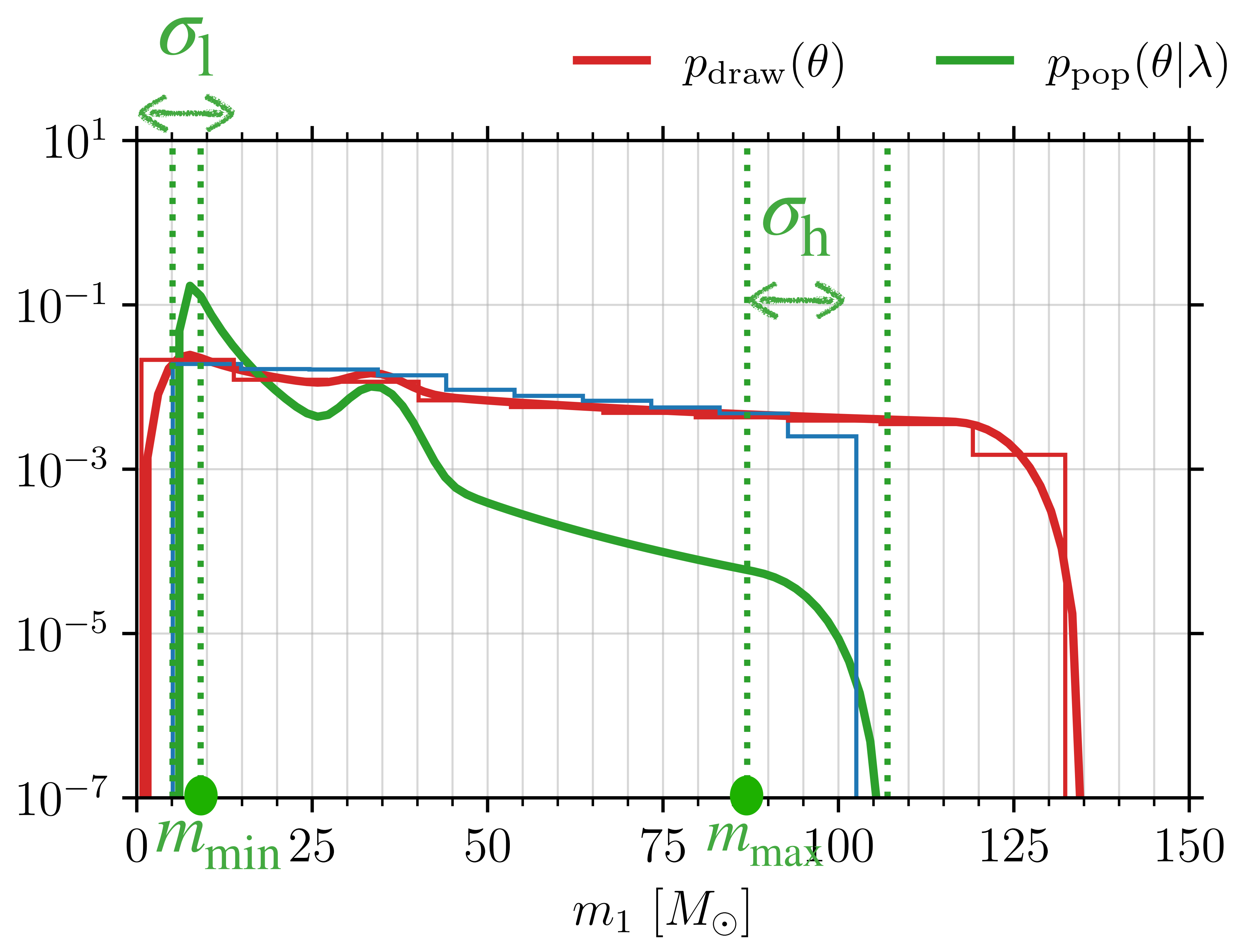
[Iacovelli et al, 2022]

[Gair et al., 2023]

3G forecasts

- **Mass distribution =Power Law Plus Peak**
- Redshift distribution = Madau Dickinson
- Spin= Default distribution

- Detectors: ET,ET+2CE
- SNR threshold=12
- Waveform: *IMRPhenomXPHM*



Parameter	Description	Fiducial Value
	Mass model: Power Law Plus Peak	
α_m	Spectral index for the power-law of the primary mass distribution.	3.4
β_q	Spectral index for the power-law of the mass ratio distribution.	1.1
m_{min}	Minimum mass of the power-law component of the primary mass distribution.	9.1 $M_\odot$
m_{max}	Maximum mass of the power-law component of the primary mass distribution.	87 $M_\odot$
λ_p	Fraction of binary BHs in the Gaussian component.	0.039
μ_m	Mean of the Gaussian component in the primary mass distribution.	34
σ_m	Width of the Gaussian component in the primary mass distribution.	3.6
σ_l	Width of mass smoothing at the lower end of the mass distribution.	4.0
σ_h	Width of mass smoothing at the upper end of the mass distribution.	0.5

3G forecasts

- Mass distribution =Power Law Plus Peak
- **Redshift distribution = Madau Dickinson**
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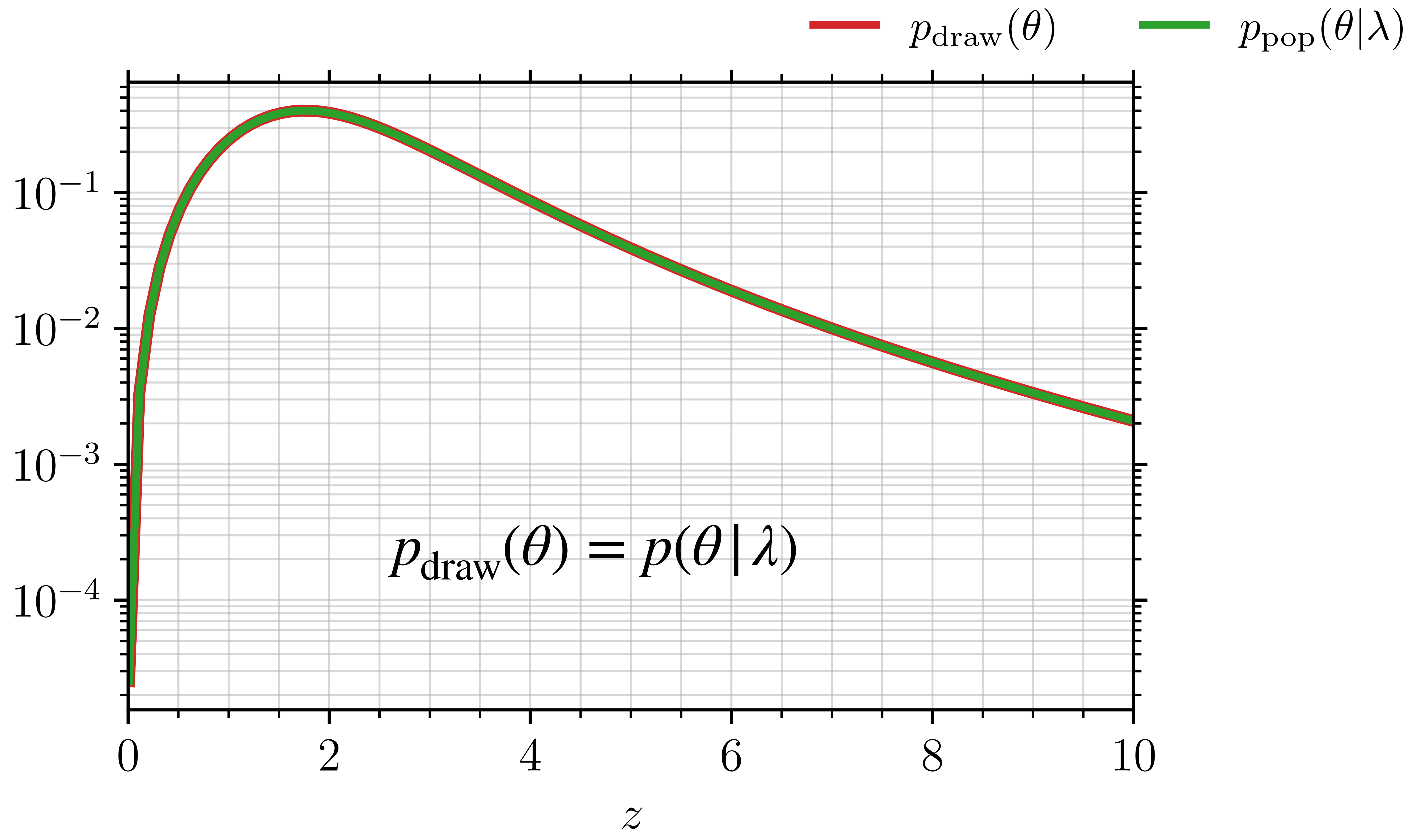
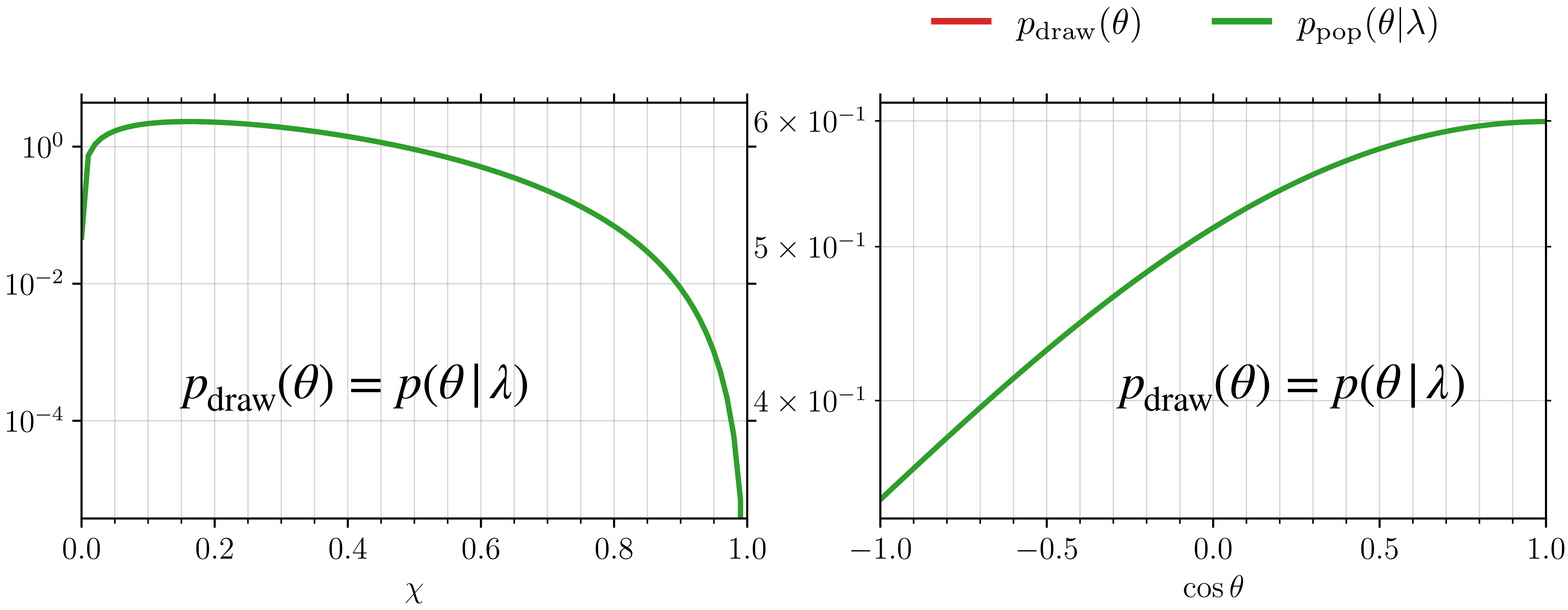


Parameter	Description	Fiducial Value
	Rate model: MADAU DICKINSON	
α_z	Power-law index governing the rise of the star formation rate at low redshift.	2.7
β_z	Power-law index governing the decline of the star formation rate at high redshift.	3.0
z_p	Redshift at which the star formation rate peaks.	2.0

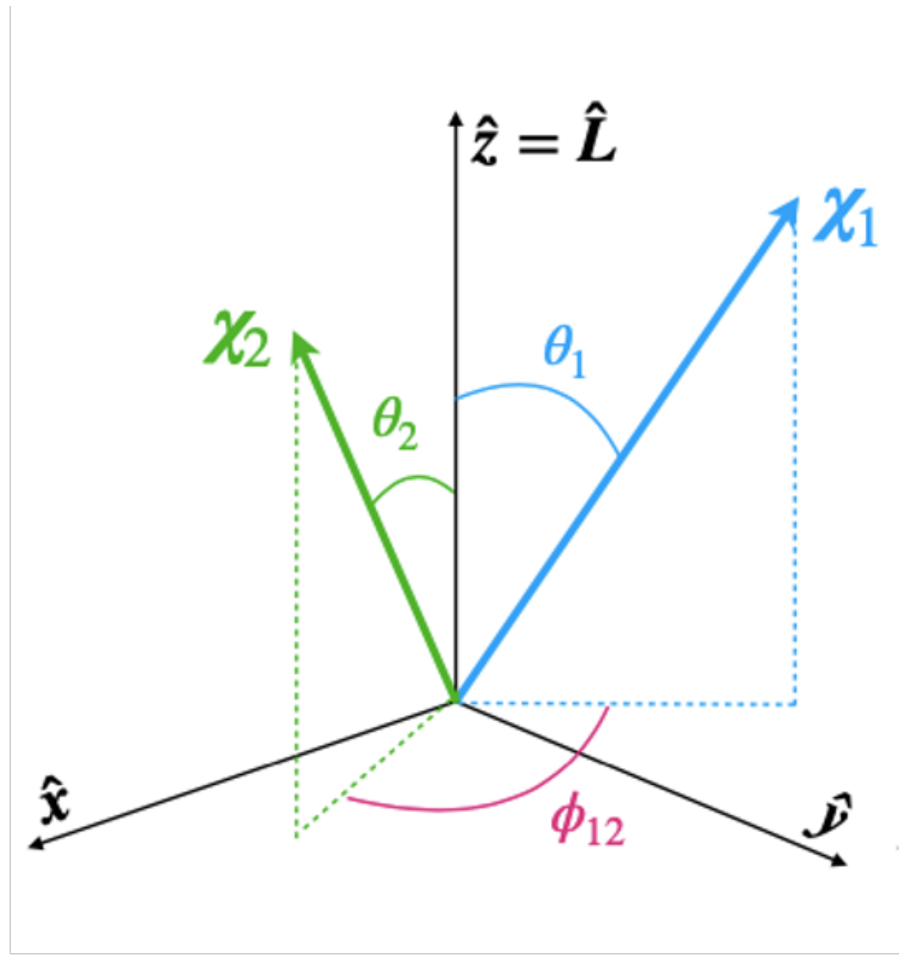
3G forecasts



- Mass distribution =Power Law Plus Peak
- Redshift distribution = Madau Dickinson
- **Spin= Default distribution**

- Detectors: ET, ET+2CE
- SNR threshold=12
- Waveform: *IMRPhenomXPHM*

Parameter	Description	Fiducial Value
	Spin model: DEFAULT	
α_χ	Shape parameter of the Beta distribution of the spin magnitudes.	1.6
β_χ	Shape parameter of the Beta distribution of the spin magnitudes.	4.12
ζ	Mixing fraction of mergers from the truncated Gaussian component for spin orientations.	0.66
σ_t	Width of the truncated Gaussian component for spin orientations, determining the typical spin misalignment.	1.5



Spin magnitudes $p(\chi_i|\alpha_\chi,\beta_\chi) \propto \chi_i^{\alpha-1}(1-\chi_i)^{\beta-1}$.

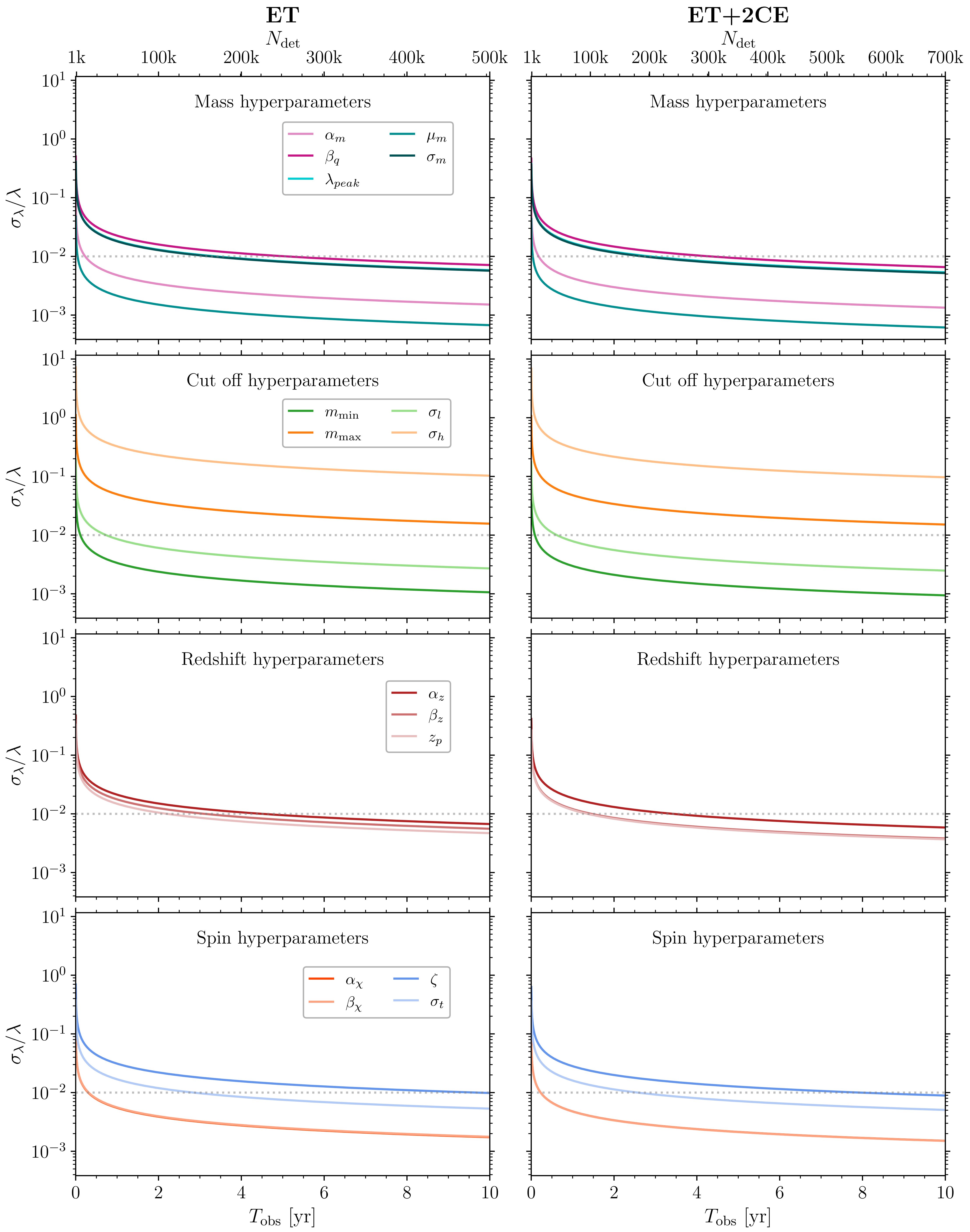
Spin tilts $p(\cos\theta_i|\zeta,\sigma_t) = \frac{1}{2}(1-\zeta) + \zeta\,\mathcal{N}_{[-1,1]}(\cos\theta_i;1,\sigma_t)$.

Population forecasts with 3G detectors

$$\frac{\sigma_\lambda}{\lambda} \propto \sigma_\lambda \propto \frac{1}{\sqrt{N}} \propto \frac{1}{\sqrt{N_{\rm det}}} \propto \frac{1}{\sqrt{T_{\rm obs}}}$$

Relative errors after 10 years of observation

σ_λ/λ	**ET**	$T_{0.01}^{ET}$	**ET+2CE**	$T_{0.01}^{ET+2CE}$	**Fiducial Values**
α_m	1.51×10^{-3}	0.2	1.34×10^{-3}	0.2	3.4
β_q	7.12×10^{-3}	5.1	6.57×10^{-3}	4.3	1.1
$m_{\rm min}$	1.06×10^{-3}	0.1	9.41×10^{-4}	0.1	$9.1\ M_\odot$
$m_{\rm max}$	1.57×10^{-2}	>10	1.51×10^{-2}	>10	$87\ M_\odot$
$\lambda_{\rm peak}$	5.82×10^{-3}	3.4	5.35×10^{-3}	2.9	0.039
σ_l	2.71×10^{-3}	0.7	2.49×10^{-3}	0.6	4.0
σ_h	1.03×10^{-1}	>10	9.67×10^{-2}	>10	0.5
μ_m	6.74×10^{-4}	0.1	6.15×10^{-4}	0.04	34
σ_m	5.70×10^{-3}	3.3	5.19×10^{-3}	1.5	3.6
α_z	6.71×10^{-3}	4.5	5.85×10^{-3}	1.4	2.7
β_z	5.57×10^{-3}	3.1	3.81×10^{-3}	0.2	3.0
z_p	4.69×10^{-3}	2.2	3.68×10^{-3}	0.2	2.0
α_χ	1.74×10^{-3}	0.3	1.51×10^{-3}	0.2	1.6
β_χ	1.77×10^{-3}	0.3	1.50×10^{-3}	0.2	4.12
ζ_χ	9.83×10^{-3}	9.7	8.90×10^{-3}	7.9	0.66
σ_t	5.32×10^{-3}	2.8	5.06×10^{-3}	2.6	1.5



De Renzis+ in prep

Population forecasts with 3G detectors

No big difference between ET and ET+2CE :

- Both detectors will constrain instrinsic parameters with incredible accuracy
- The scale of the problem is the number of detection
- Adding 2CE to ET introduces more distant events with lower SNR near the detection threshold

Relative errors after 10 years of observation

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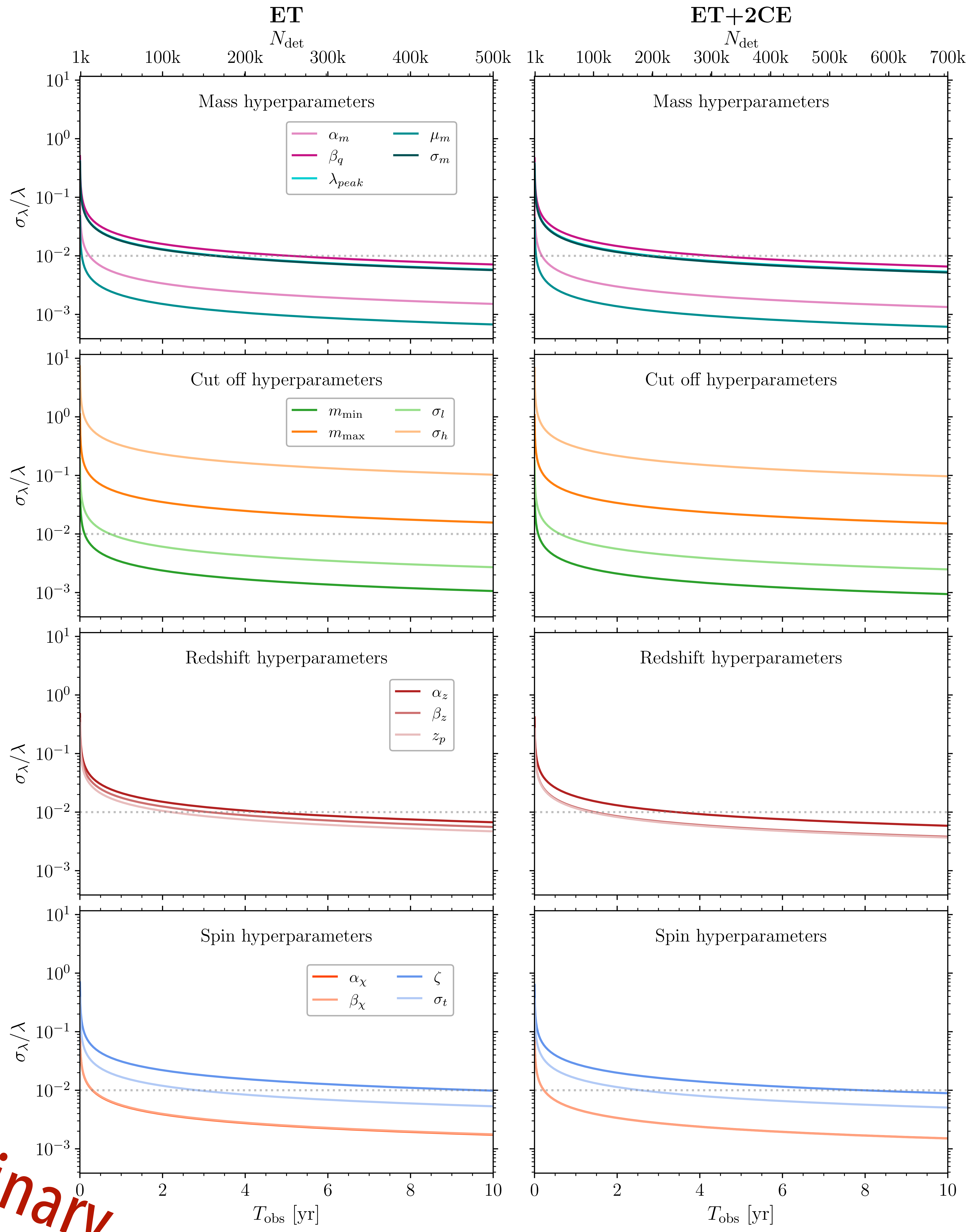

Preliminary

Mass hyperparameters

- α_m, $m_{\min}$ and μ_m are the best measured parameters
- β_q measured with less accuracy w.r.t. α_m
- The worst measured are σ_h and $m_{\max}$

Relative errors after 10 years of observation

σ_λ/λ	**ET**	$T_{0.01}^{ET}$	**ET+2CE**	$T_{0.01}^{ET+2CE}$	**Fiducial Values**
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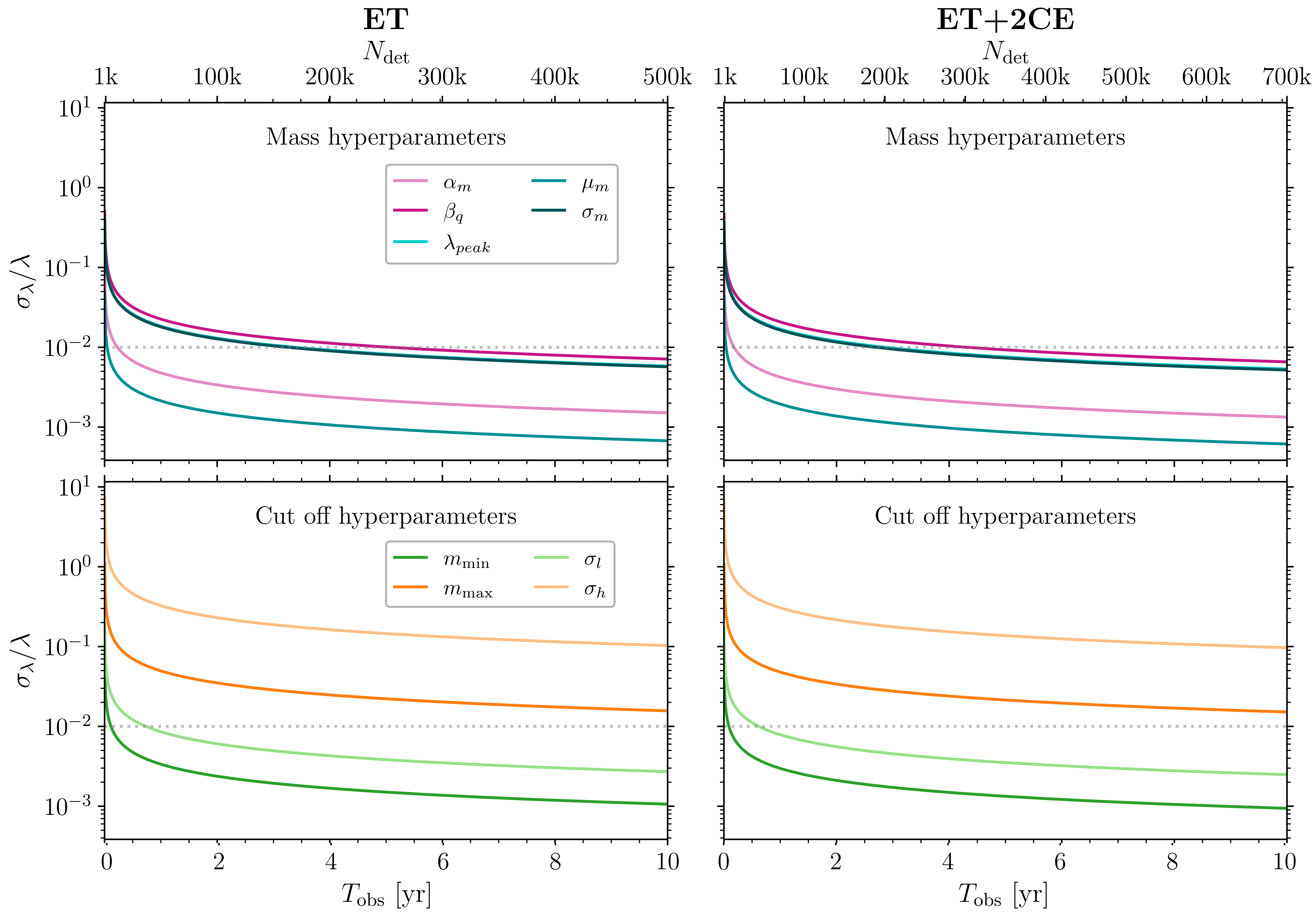
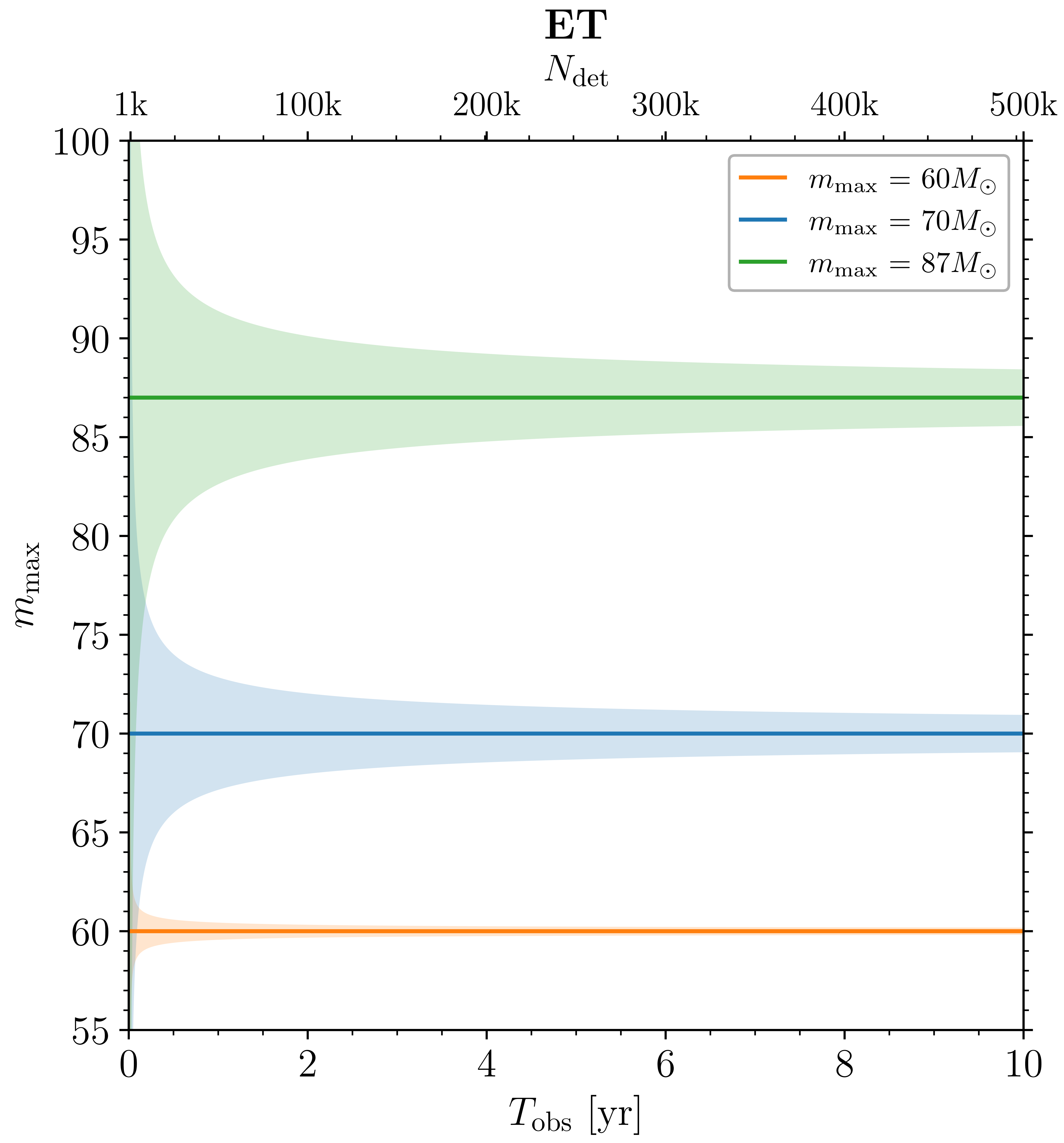
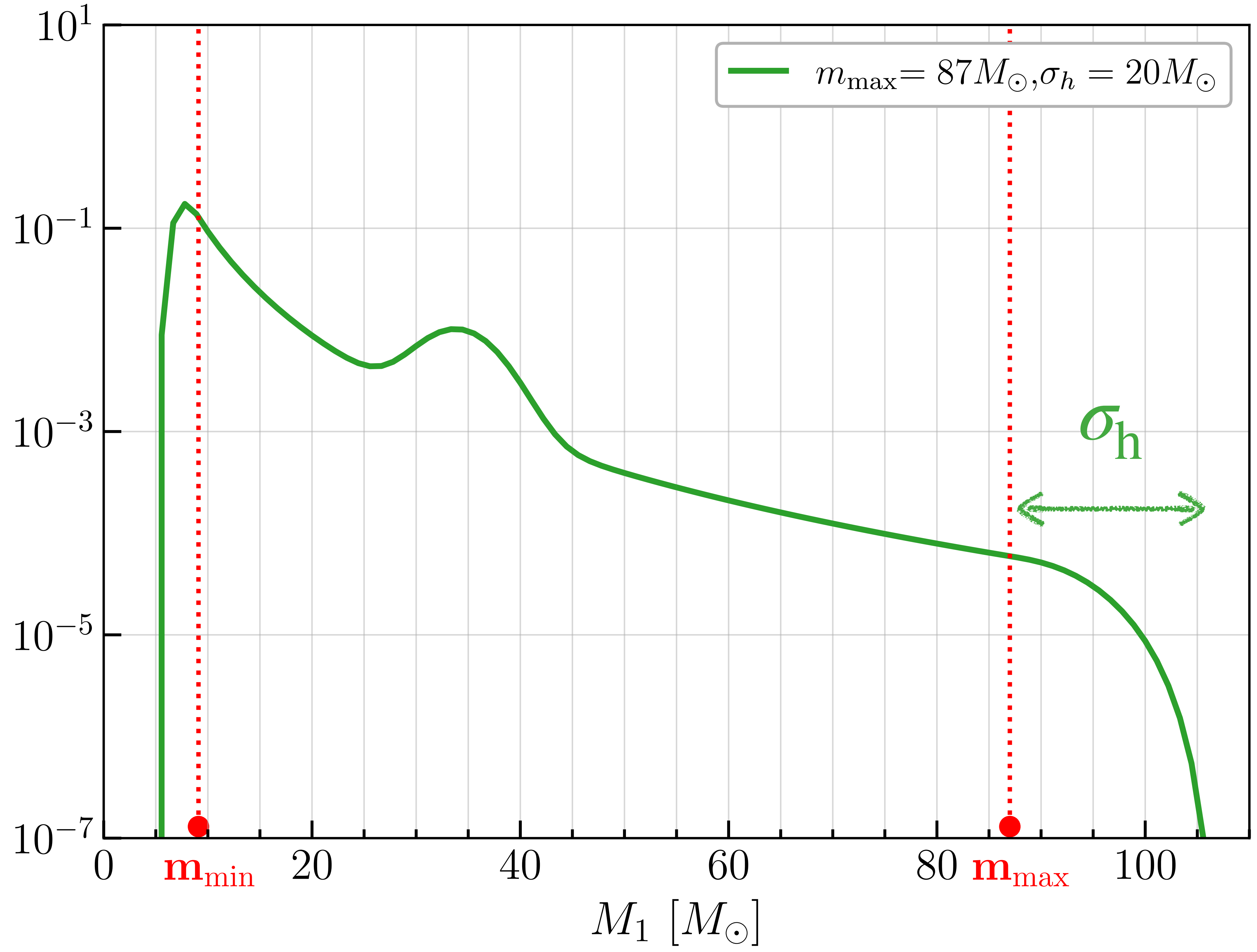


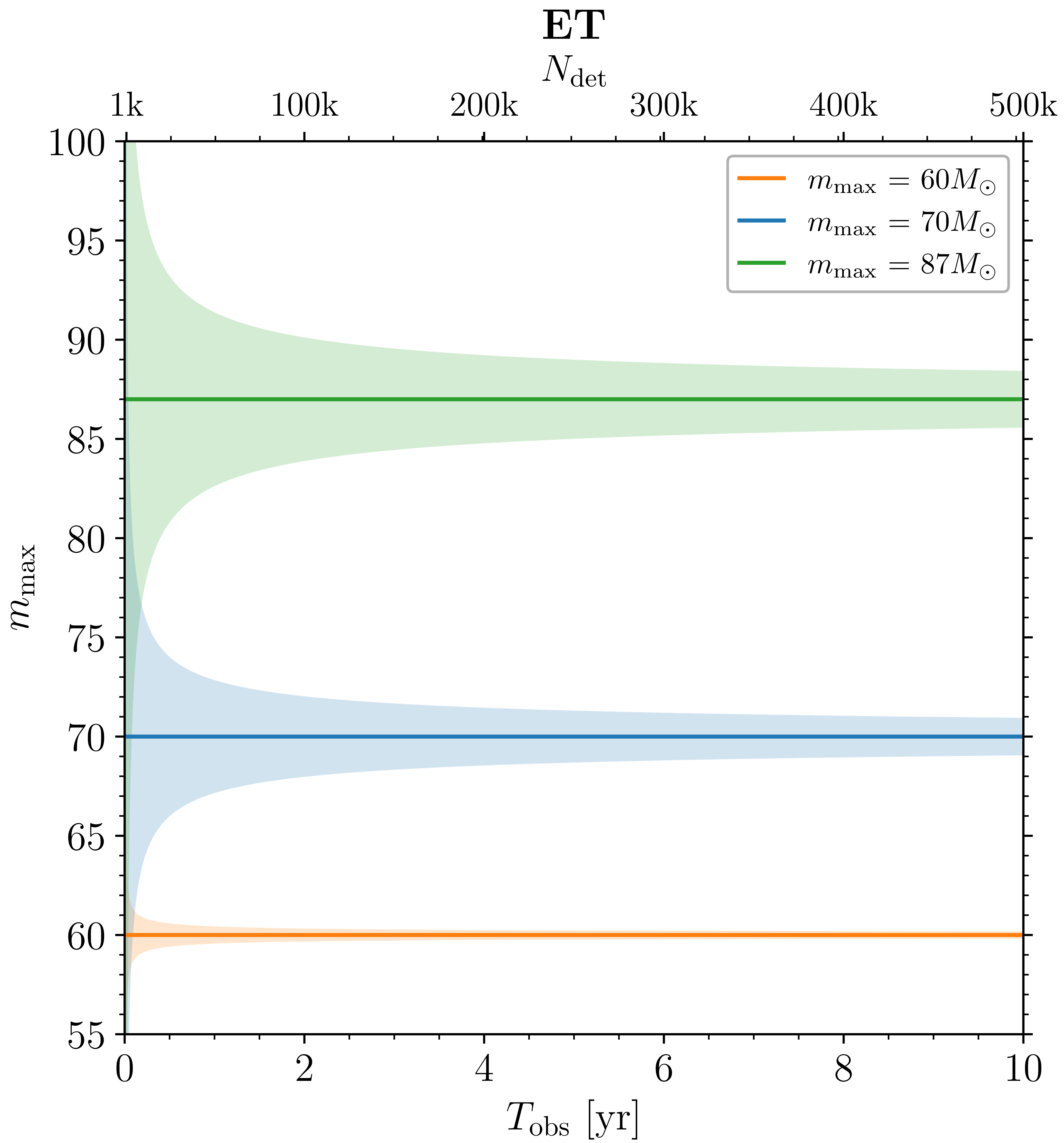
Preliminary

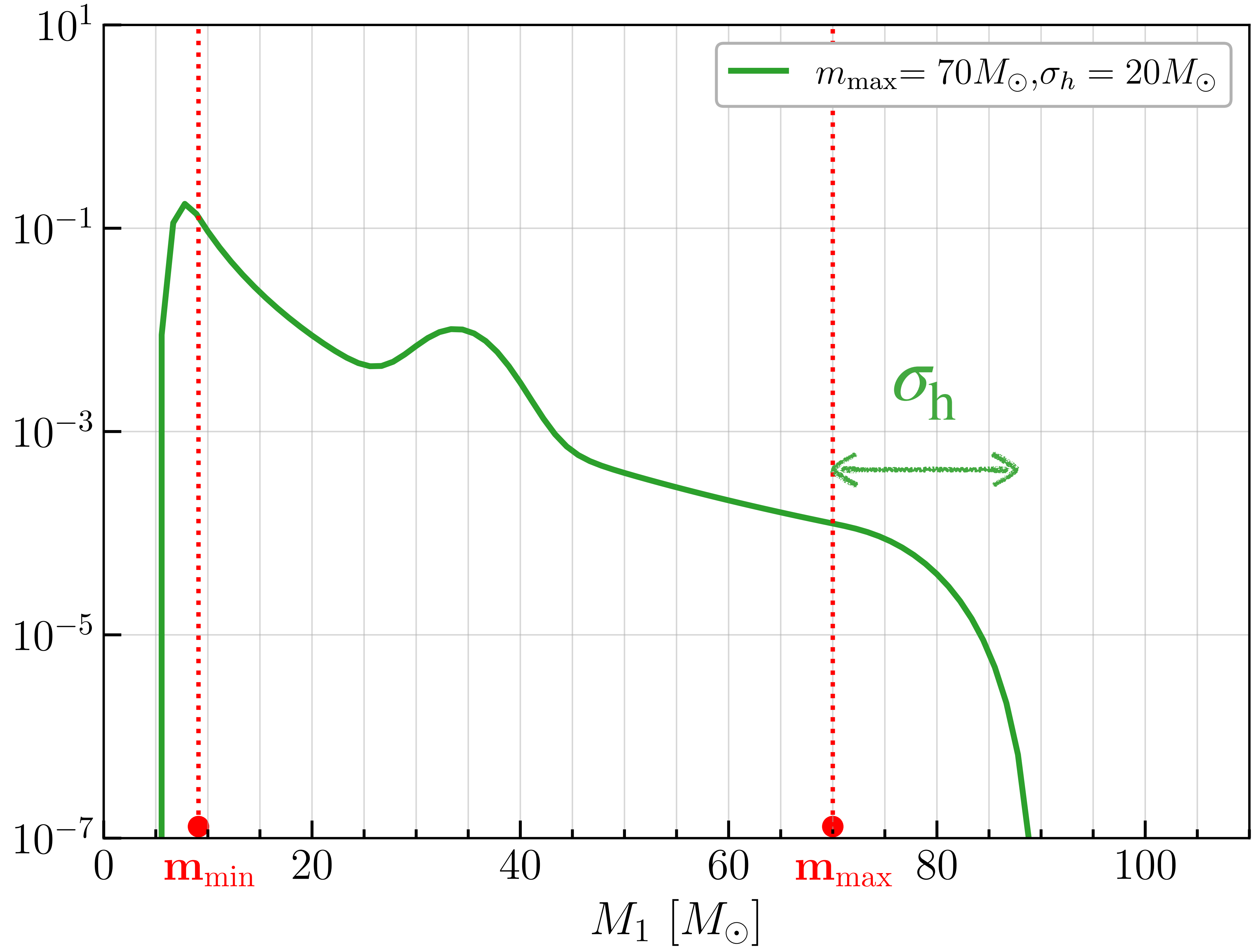
Mass hyperparameters: Variation of $m_{\max}$



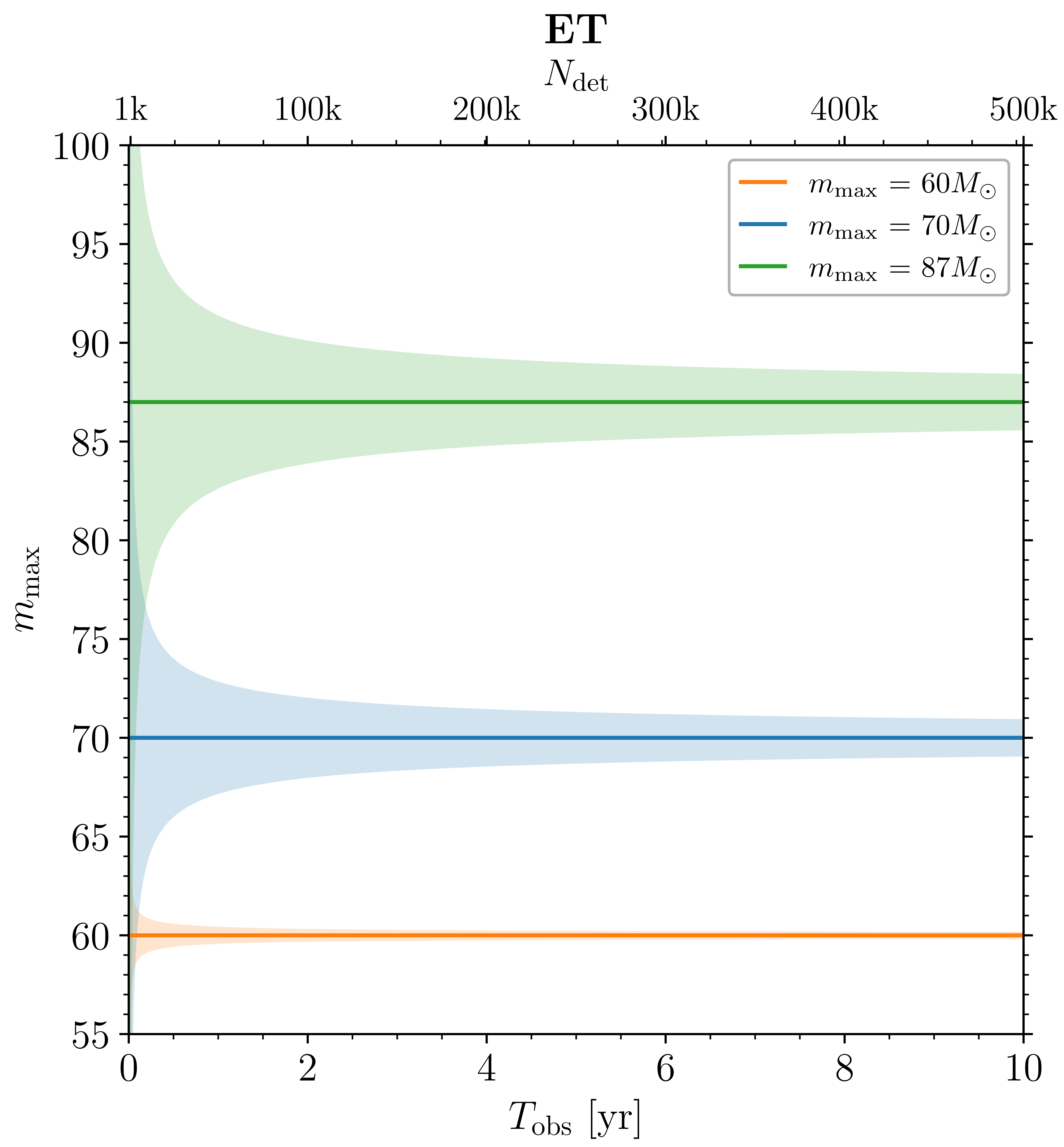


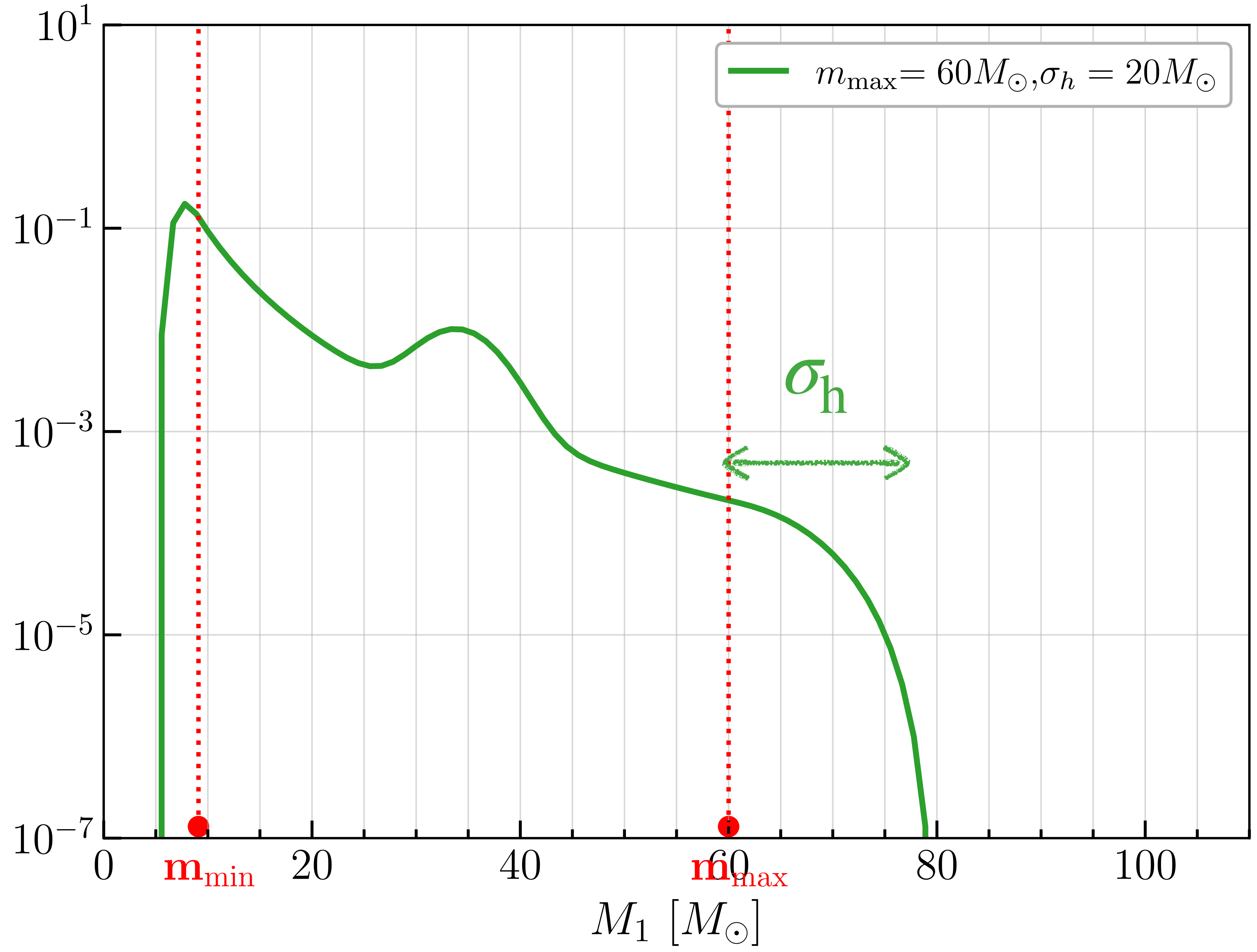
Mass hyperparameters: Variation of $m_{\rm max}$



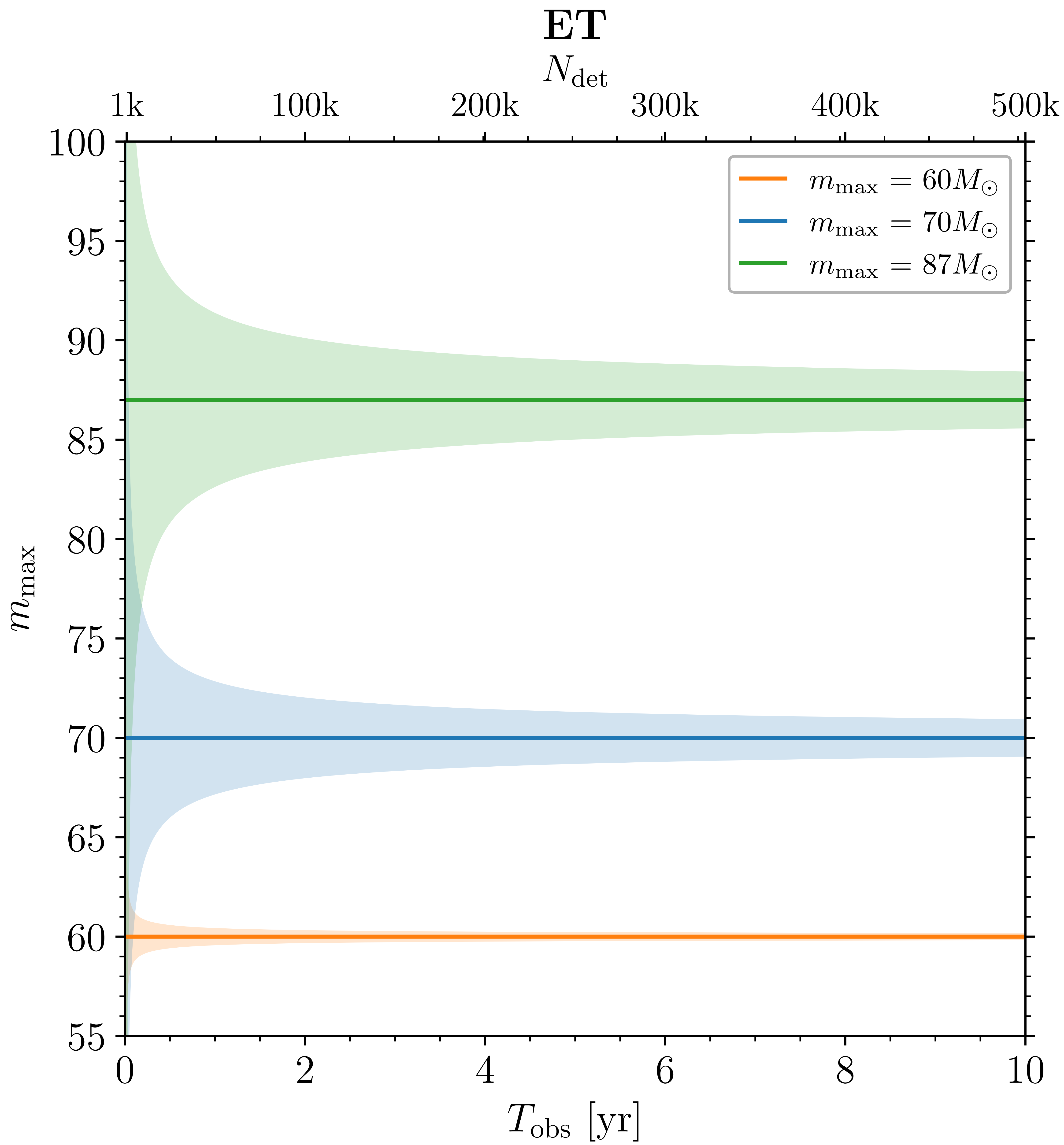


Mass hyperparameters: Variation of $m_{\rm max}$





Mass hyperparameters: Variation of $m_{\rm max}$



Absolute errors after 10 years of observation

σ_λ	$m_{\rm max} = 60 M_\odot$	$m_{\rm max} = 70 M_\odot$	$m_{\rm max} = 87 M_\odot$
α_m	2.79×10^{-3}	2.49×10^{-3}	5.14×10^{-3}
β_q	7.13×10^{-3}	8.04×10^{-3}	7.83×10^{-3}
$m_{\rm min}$	1.02×10^{-2}	9.48×10^{-3}	9.66×10^{-3}
$m_{\rm max}$	1.16×10^{-1}	8.78×10^{-1}	1.36
$\lambda_{\rm peak}$	2.05×10^{-4}	1.88×10^{-4}	2.27×10^{-4}
σ_l	1.15×10^{-2}	1.07×10^{-2}	1.08×10^{-2}
σ_h	1.18×10^{-1}	1.31	2.06
μ_m	2.47×10^{-2}	2.51×10^{-2}	2.29×10^{-2}
σ_m	2.48×10^{-2}	2.99×10^{-2}	2.05×10^{-2}

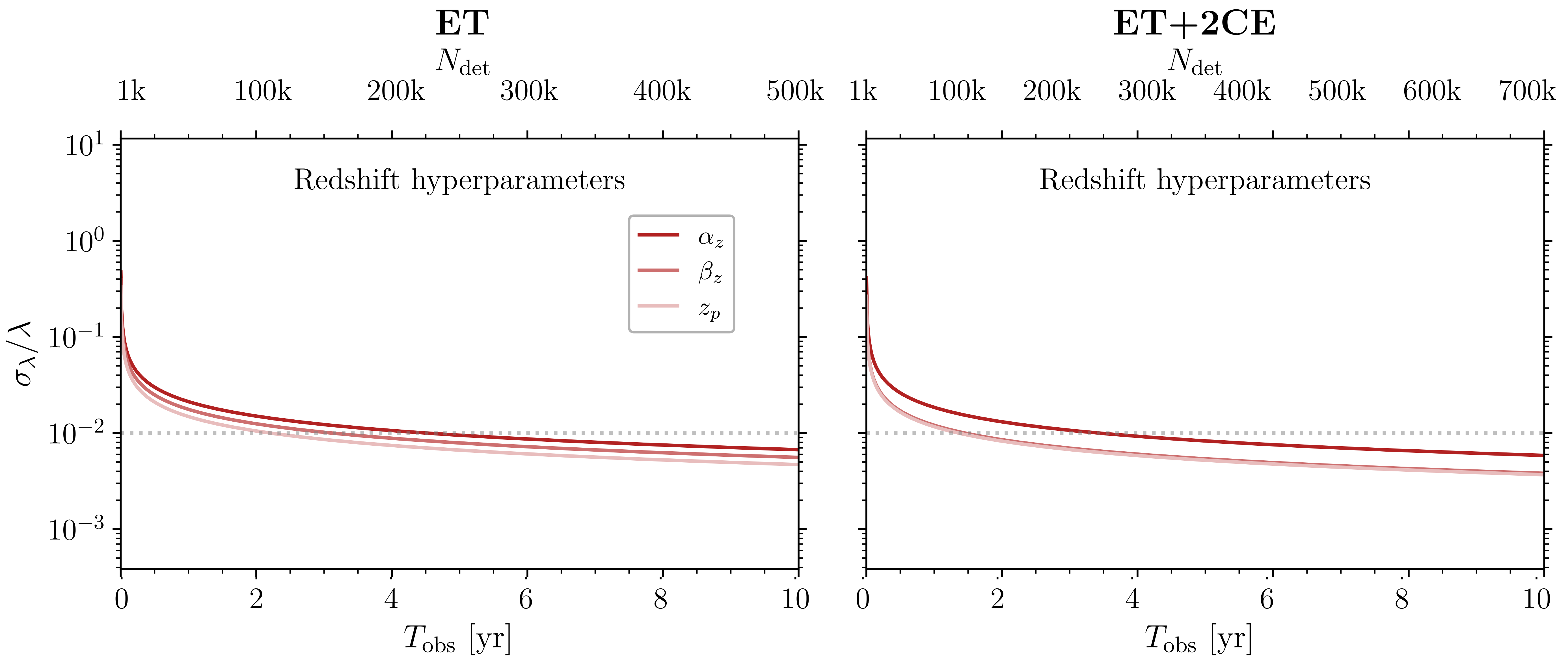
- The errors decrease for smaller values of $m_{\rm max}$
- Correlation between σ_h and $m_{\rm max}$
- The errors on the other mass parameters remain approximately stable using different $m_{\rm max}$

Redshift hyperparameters

- No big difference between ET and ET+2CE (the scale of the problem is the number of detections!)

Relative errors after 10 years of observation

σ_λ/λ	**ET**	$T_{0.01}^{ET}$	**ET+2CE**	$T_{0.01}^{ET+2CE}$
α_z	6.71×10^{-3}	4.5	5.85×10^{-3}	1.4
β_z	5.57×10^{-3}	3.1	3.81×10^{-3}	0.2
z_p	4.69×10^{-3}	2.2	3.68×10^{-3}	0.2



A network of well-separated interferomenters has better sky localization capabilities, resulting in an improved distance (and thus redshift) determination

Preliminary

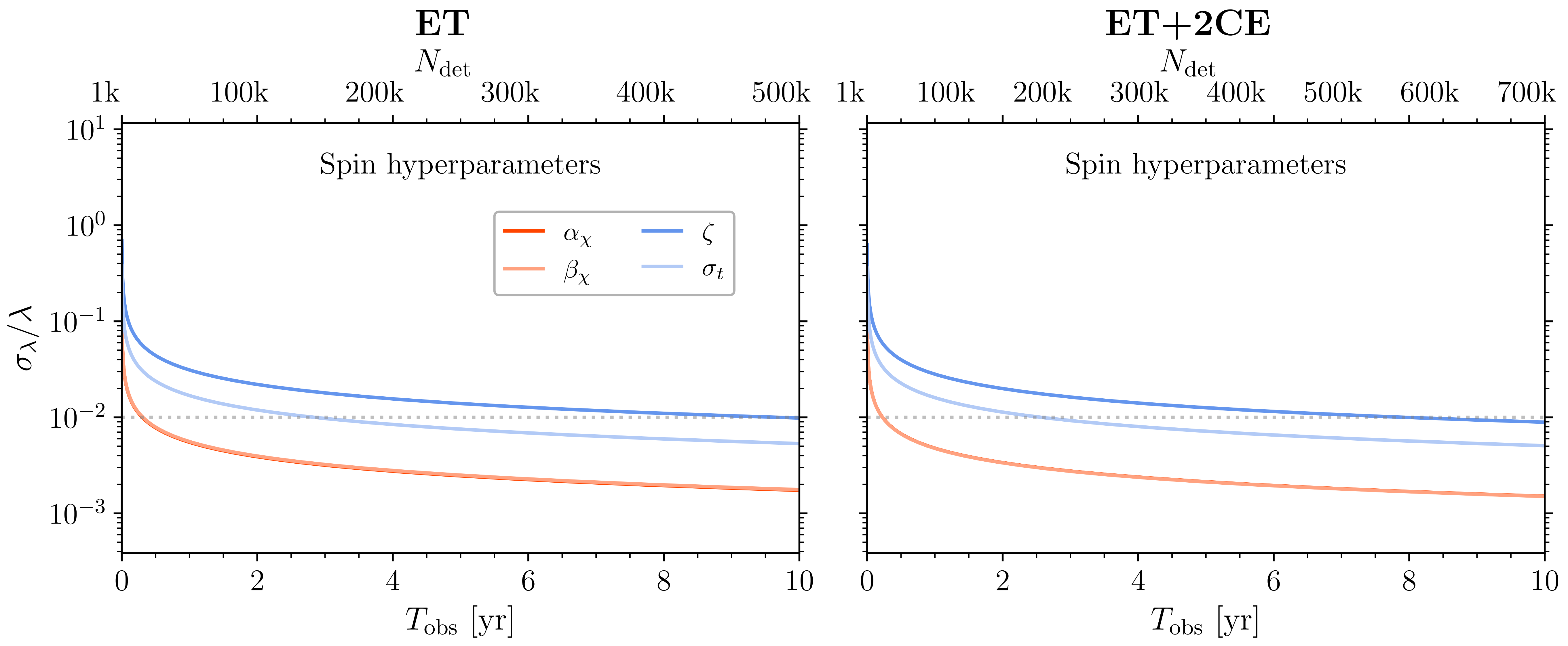
Spin hyperparameters

Preliminary

- No big difference between ET and ET+2CE (the scale of the problem is the number of detections!)

Relative errors after 10 years of observation

σ_λ/λ	**ET**	$T_{0.01}^{ET}$	**ET+2CE**	$T_{0.01}^{ET+2CE}$
α_χ	1.74×10^{-3}	0.3	1.51×10^{-3}	0.2
β_χ	1.77×10^{-3}	0.3	1.50×10^{-3}	0.2
ζ_χ	9.83×10^{-3}	9.7	8.90×10^{-3}	7.9
σ_t	5.32×10^{-3}	2.8	5.06×10^{-3}	2.6



- Difficulty in reconstructing spin directions even with the accuracy of 3G instruments.
- Single event errors are no more irrelevant for spins

Spin magnitudes $p(\chi_i|\alpha_\chi,\beta_\chi)\propto\chi_i^{\alpha-1}(1-\chi_i)^{\beta-1}$.

Spin tilts $p(\cos\theta_i|\zeta,\sigma_t)=\frac{1}{2}(1-\zeta)+\zeta\,\mathcal{N}_{[-1,1]}(\cos\theta_i;1,\sigma_t)$.

Conclusions

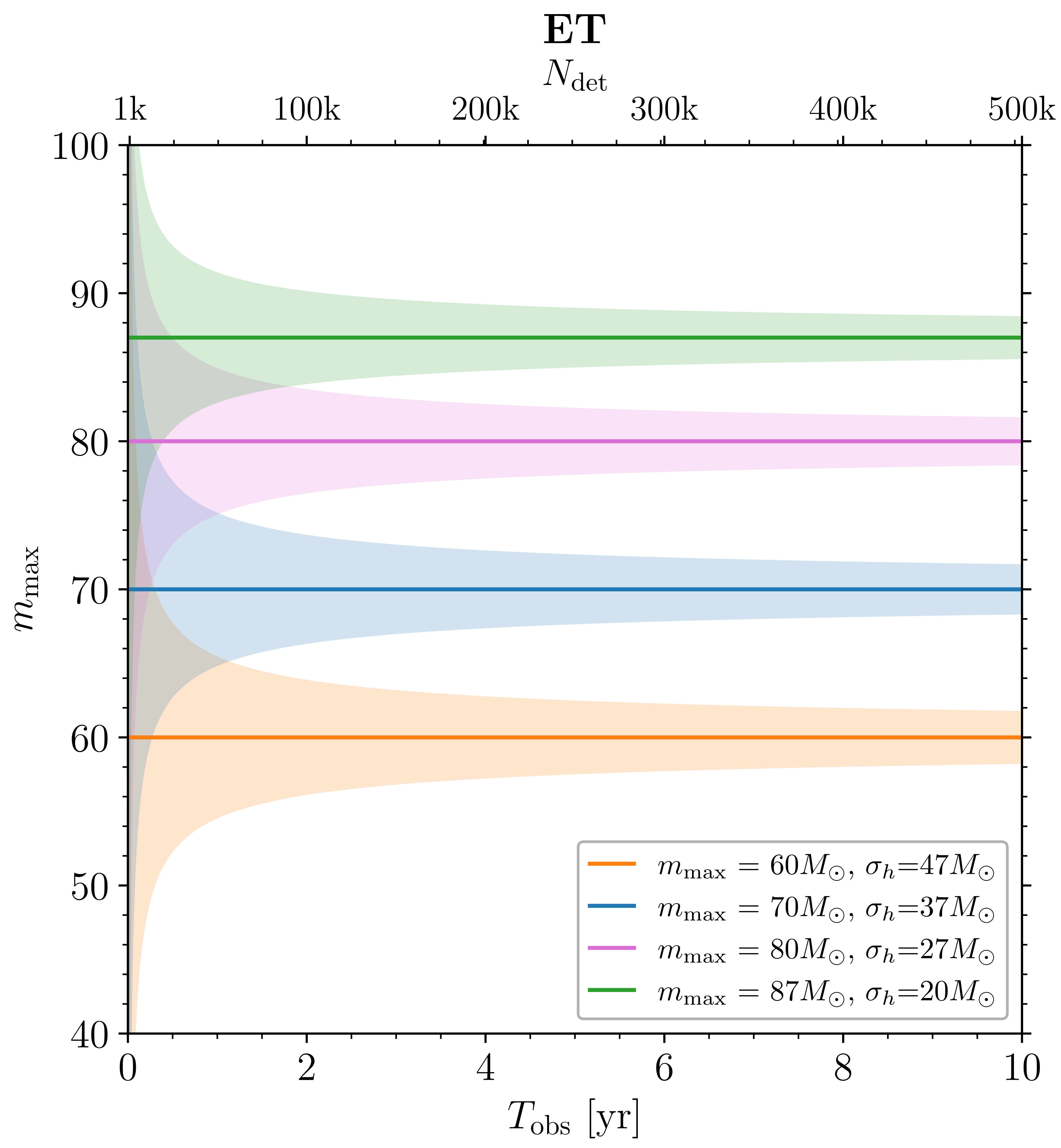
- We developed a (not yet public) Fisher code for population analysis with selection effects that enable the use of parametric (differentiable) models.
- Our forecast show the outstanding constraining power of 3G detectors
- After just a few years of observation, these detectors are projected to constrain hyperparameters with percent-level accuracy

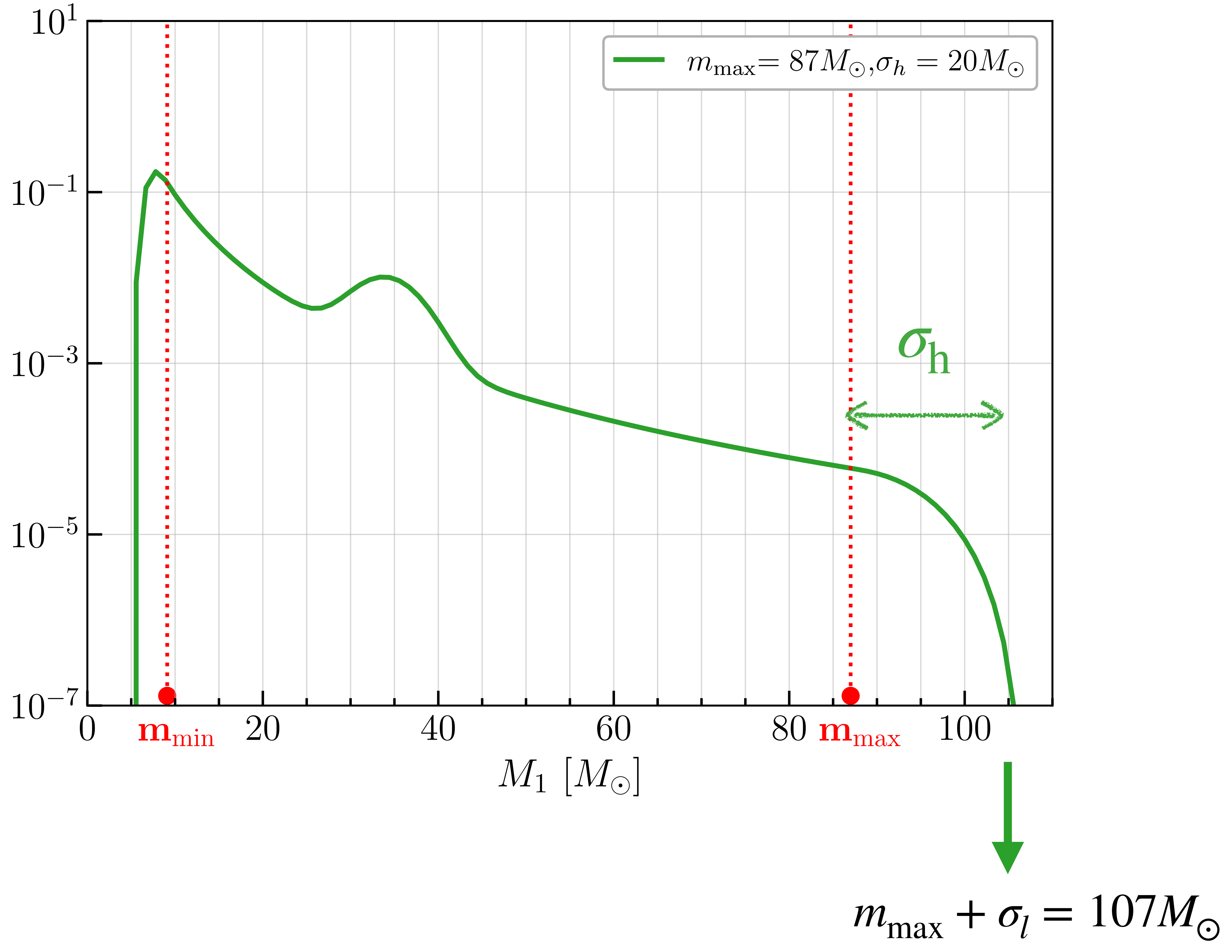
Future work

- Exploring the correlations between parameters
- Adding cosmology
- Perform hierarchical test of GR
- Other ideas?

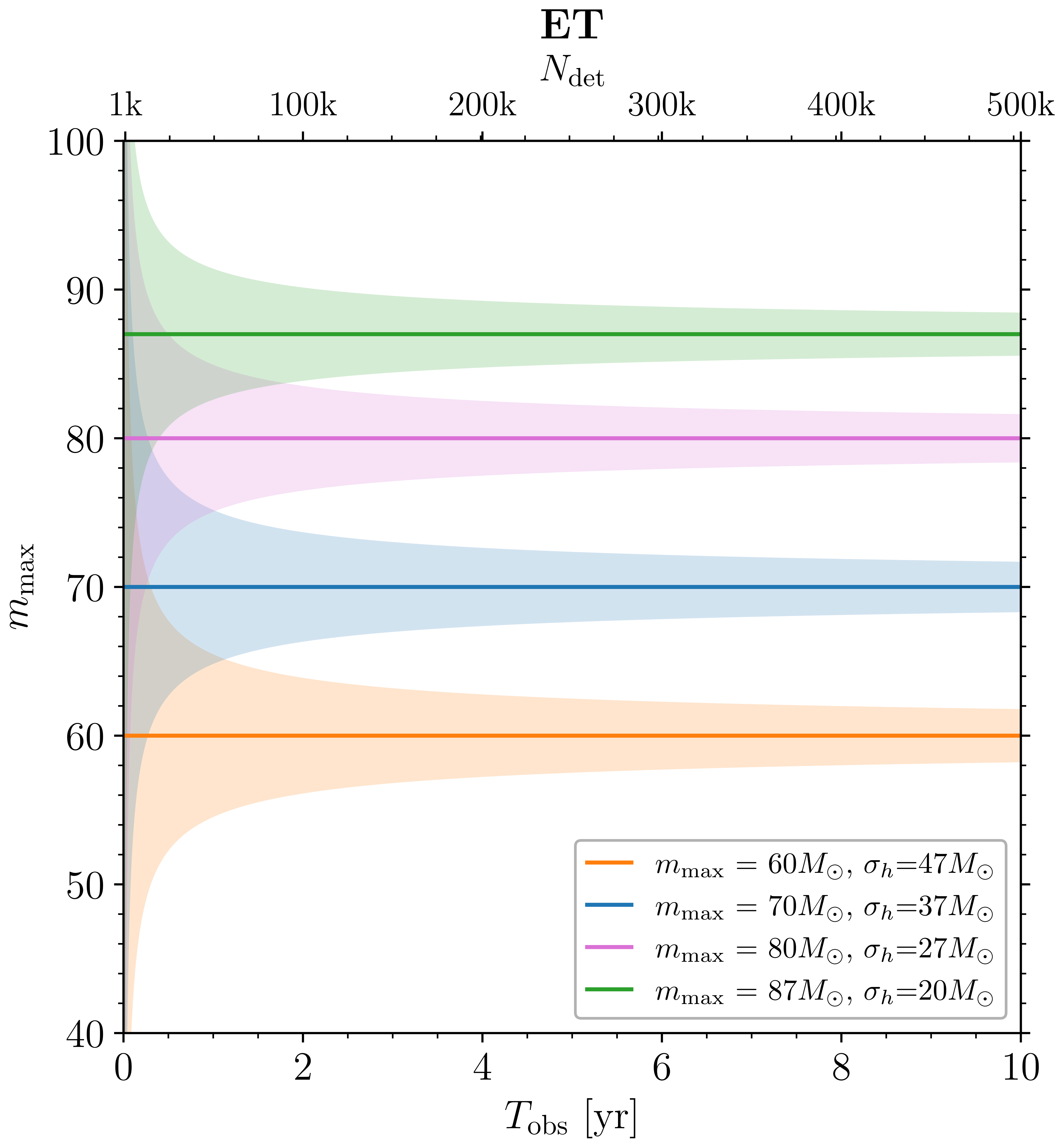
Back up slides

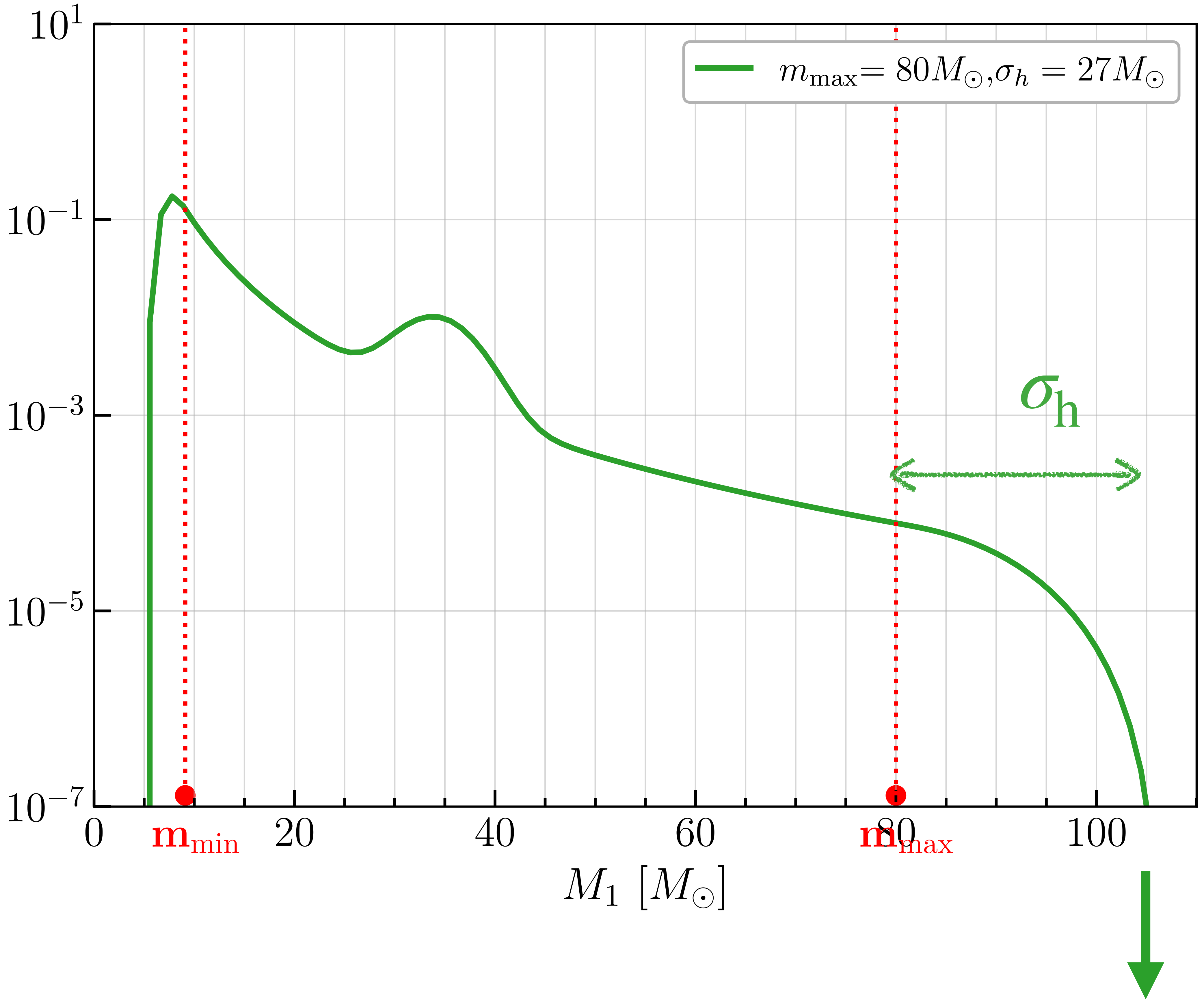
Mass hyperparameters: Variation of $m_{\max}$ and σ_h





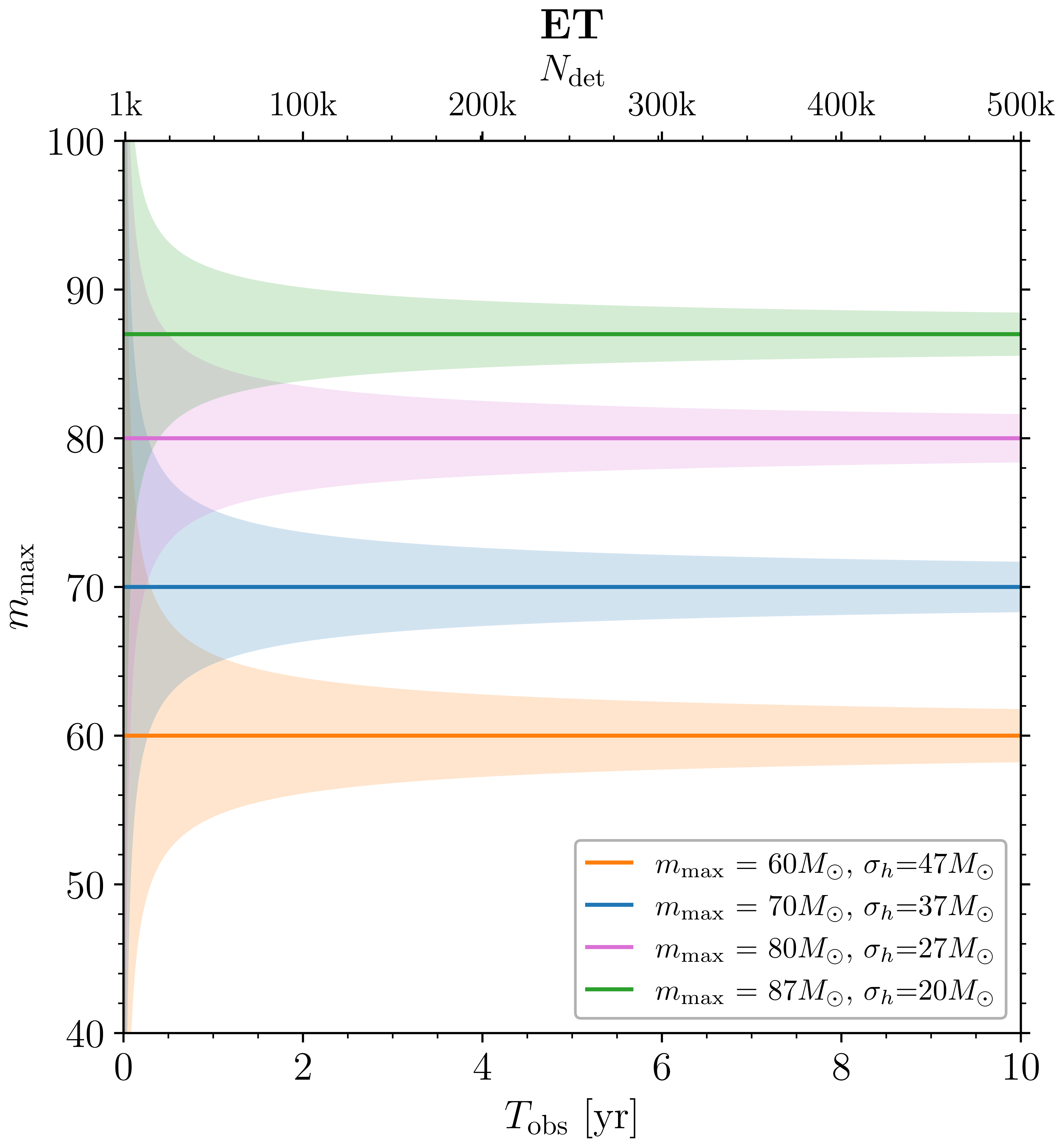
Mass hyperparameters: Variation of $m_{\max}$ and σ_h

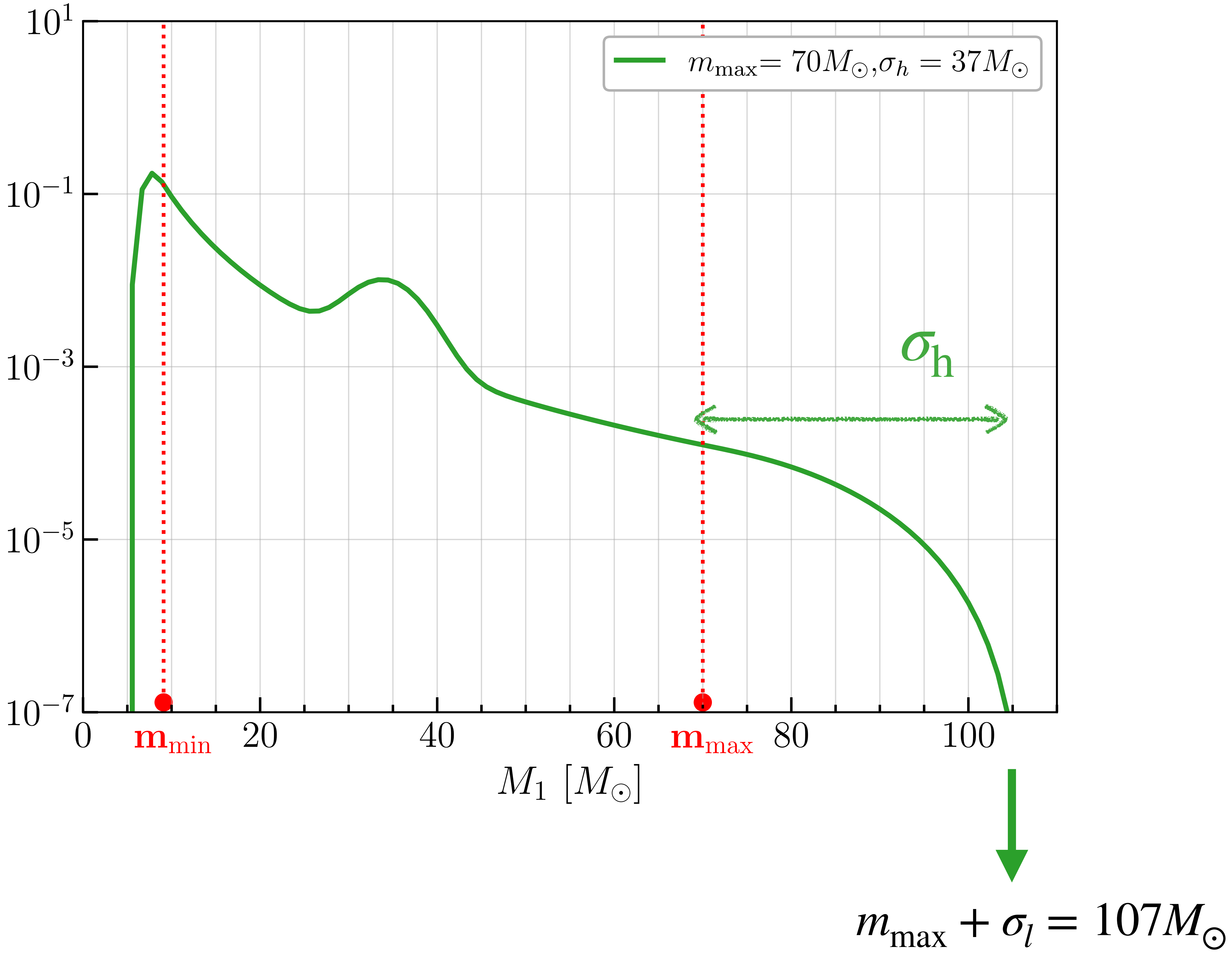




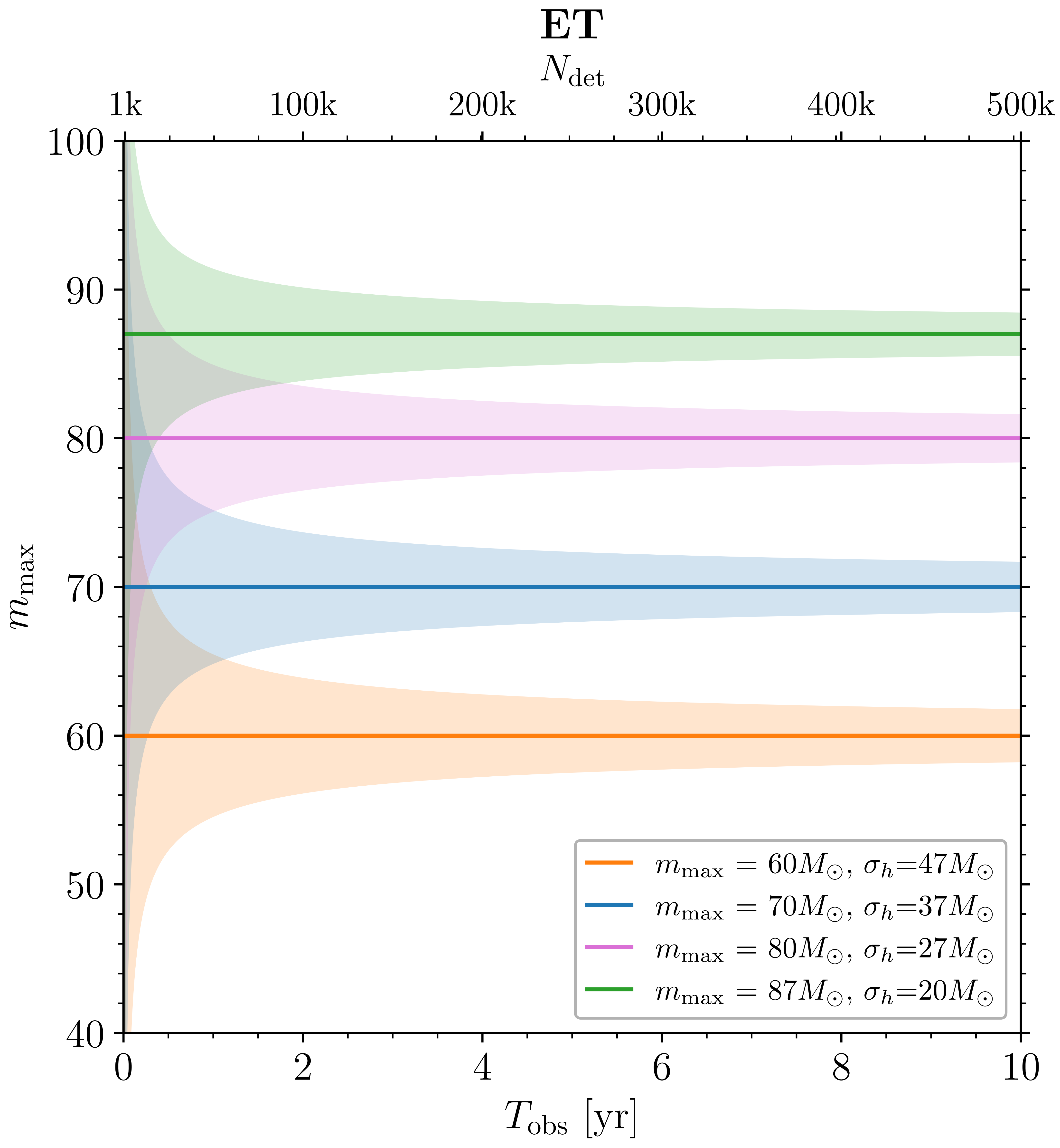
$m_{\max} + \sigma_l = 107M_\odot$

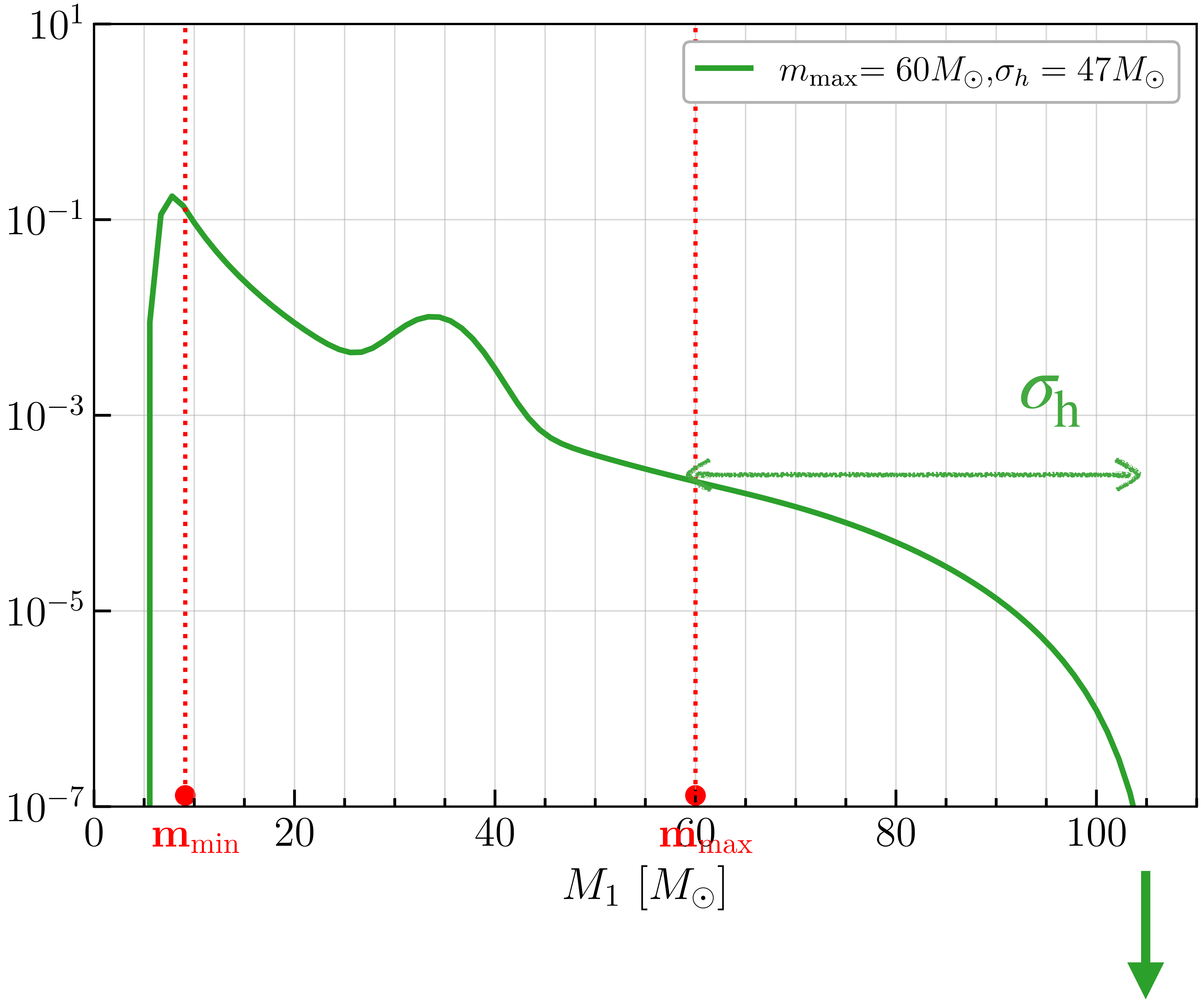
Mass hyperparameters: Variation of $m_{\max}$ and σ_h





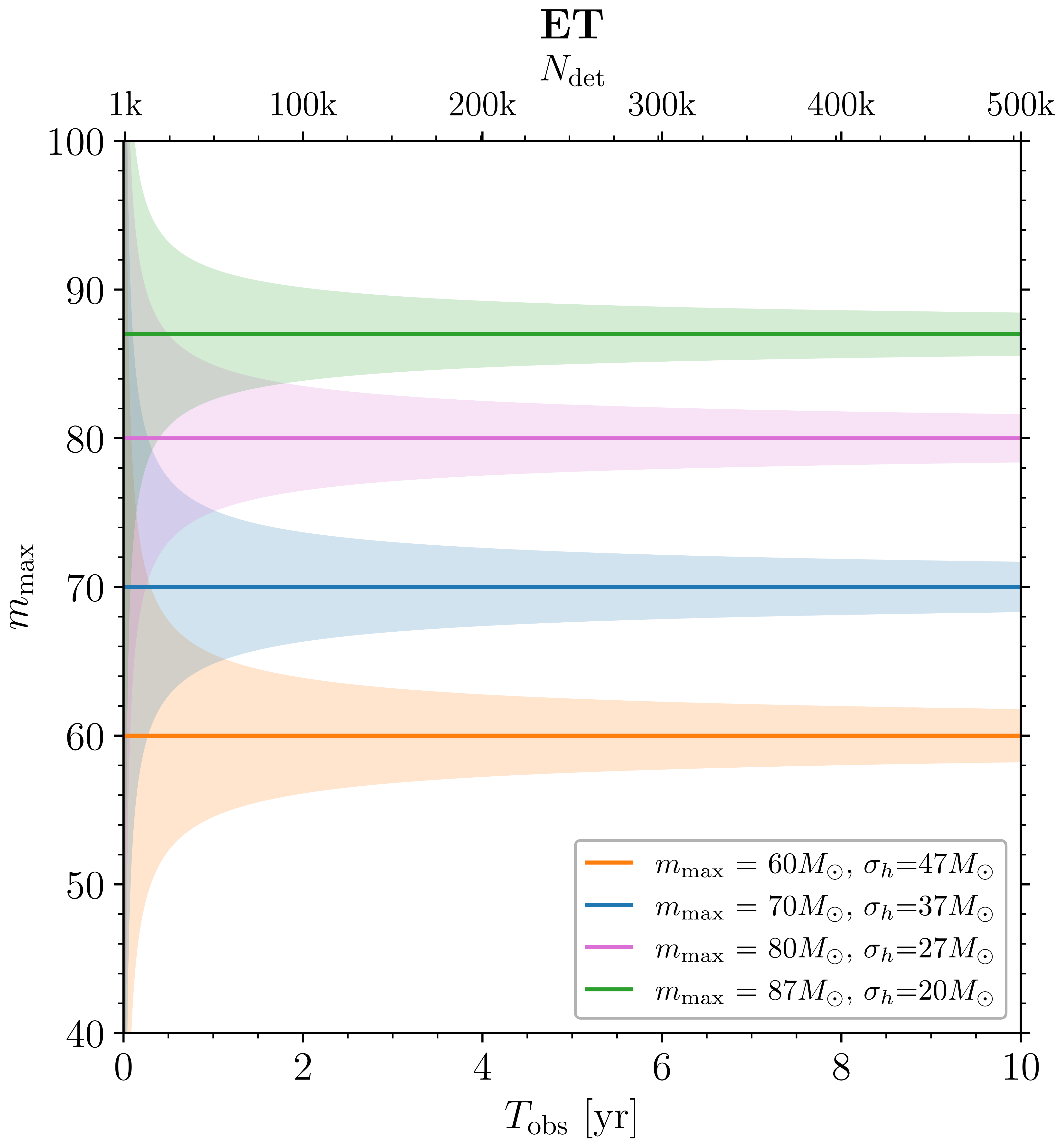
Mass hyperparameters: Variation of $m_{\max}$ and σ_h

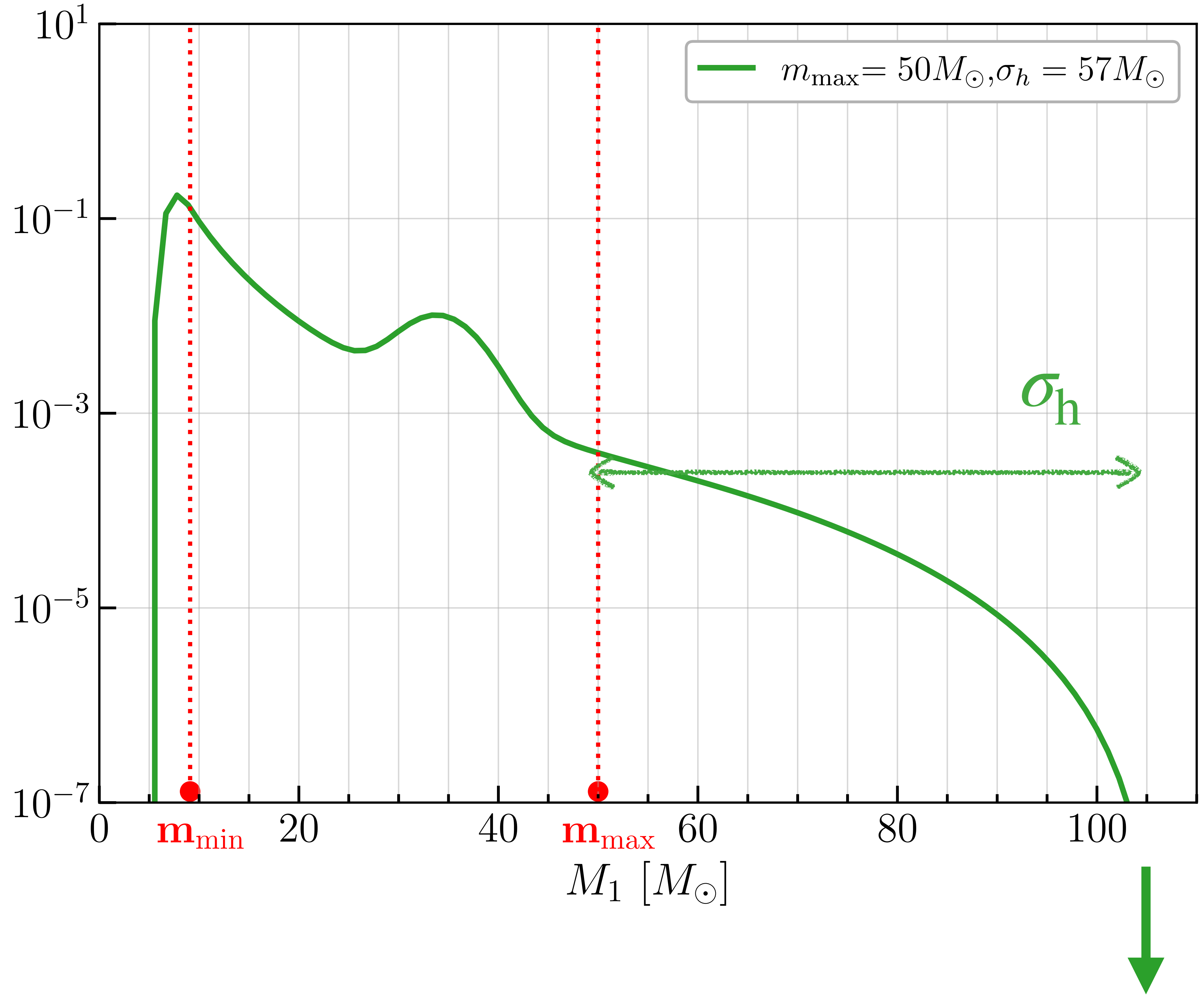




$m_{\max} + \sigma_l = 107M_\odot$

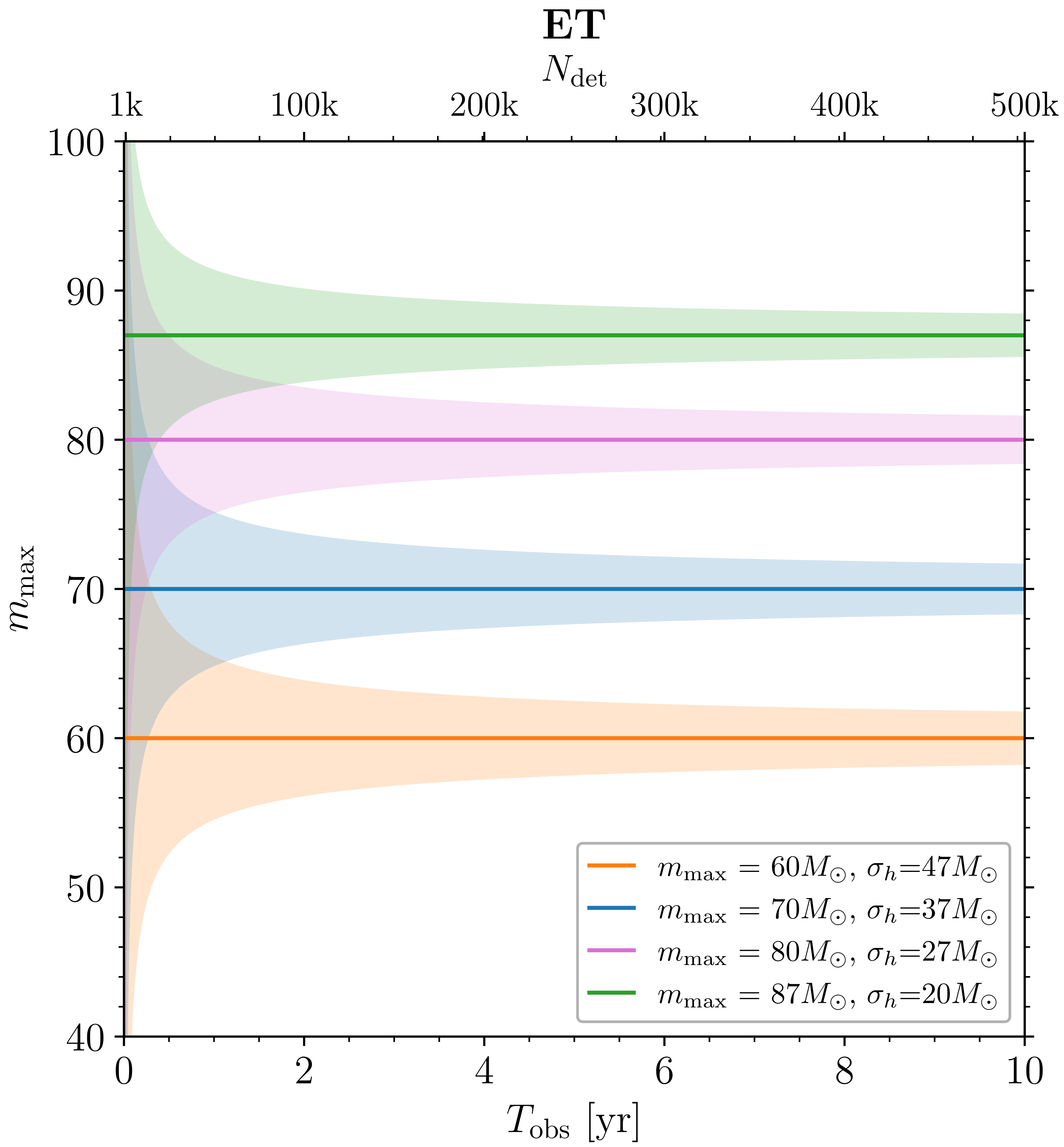
Mass hyperparameters: Variation of $m_{\max}$ and σ_h





$m_{\max} + \sigma_l = 107 M_\odot$

Mass hyperparameters: Variation of $m_{\max}$ and σ_h



σ_λ	$m_{\max} = 60M_\odot$	$m_{\max} = 70M_\odot$	$m_{\max} = 80M_\odot$	$m_{\max} = 87M_\odot$
α_m	5.59×10^{-3}	5.40×10^{-3}	5.27×10^{-3}	5.14×10^{-3}
β_q	8.15×10^{-3}	8.12×10^{-3}	8.11×10^{-3}	7.83×10^{-3}
$m_{\min}$	9.38×10^{-3}	9.29×10^{-3}	9.23×10^{-3}	9.66×10^{-3}
$m_{\max}$	1.70	1.61	1.53	1.36
$\lambda_{\rm peak}$	2.35×10^{-4}	2.35×10^{-4}	2.35×10^{-4}	2.27×10^{-4}
σ_l	1.03×10^{-2}	1.02×10^{-2}	1.02×10^{-2}	1.08×10^{-2}
σ_h	2.64	2.45	2.30	2.06
μ_m	2.40×10^{-2}	2.40×10^{-2}	2.39×10^{-2}	2.40×10^{-2}
σ_m	2.15×10^{-2}	2.14×10^{-2}	2.13×10^{-2}	2.05×10^{-2}

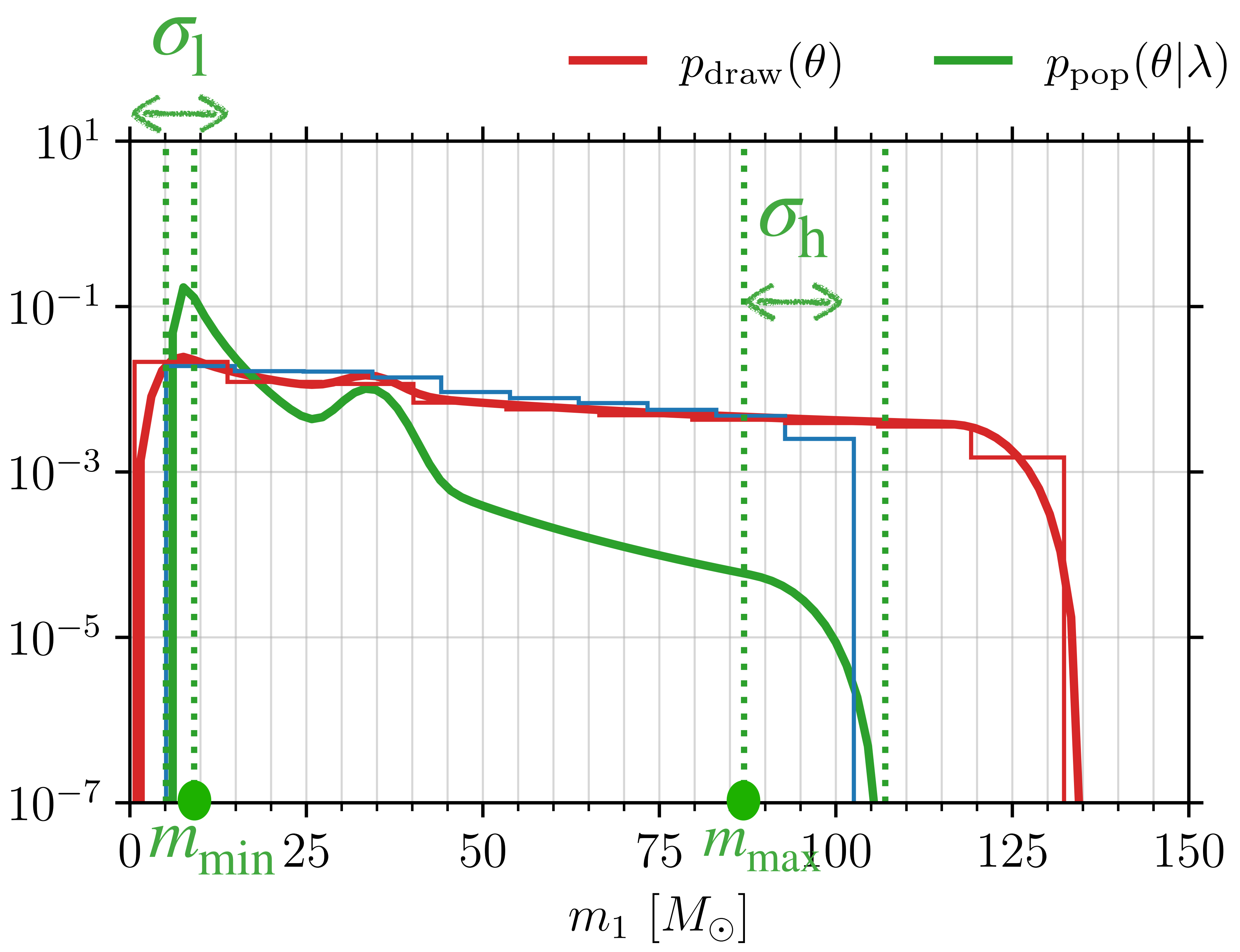
- The errors increase for smaller values of $m_{\max}$

Monte Carlo integration

We approximate the equation for $\Gamma_{\mathrm{I}}, \Gamma_{\mathrm{II}}, \Gamma_{\mathrm{III}}, \Gamma_{\mathrm{IV}}, \Gamma_{\mathrm{V}}$ with Monte Carlo integrals

$$I(\lambda) = \int X(\theta, \lambda) p_{\mathrm{draw}}(\theta) d\theta \simeq \frac{1}{N_{\mathrm{draw}}} \sum_{i}^{N_{\mathrm{draw}}} X(\theta_i, \lambda)$$

Mass distribution $p_{\mathrm{pop}}(\theta \mid \lambda)$=Power Law Plus Peak



Fiducial values for $p_{\mathrm{draw}}(\theta)$

$\alpha_m = \beta_q = 0.7$

$\sigma_{\mathrm{l}} = 8.7 M_\odot \quad \sigma_{\mathrm{h}} = 20 M_\odot$

$m_{\mathrm{min}} = 9.1 M_\odot \quad m_{\mathrm{max}} = 115 M_\odot$

Fiducial values for $p_{\mathrm{pop}}(\theta \mid \lambda)$

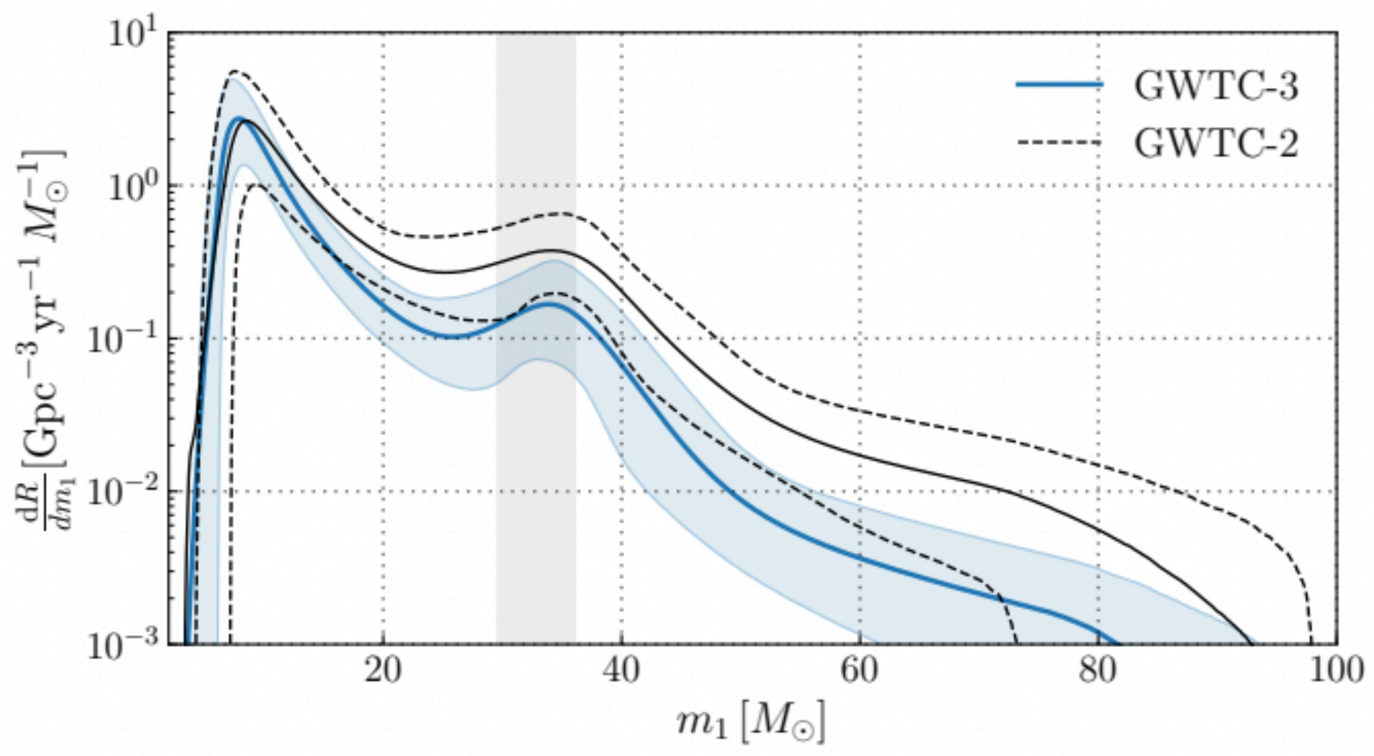
$\alpha_m = 3.4 \quad \beta_q = 1.1$

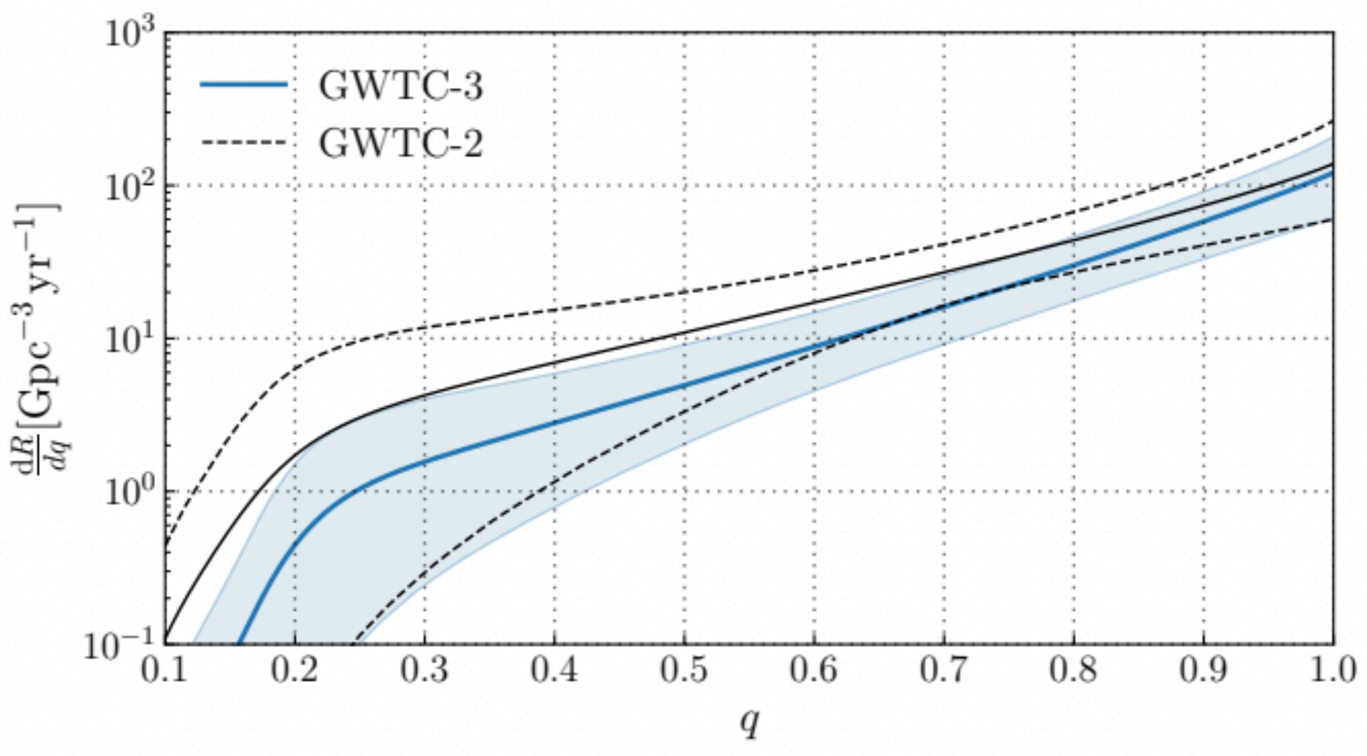
$\sigma_{\mathrm{l}} = 5 M_\odot \quad \sigma_{\mathrm{h}} = 20 M_\odot$

$m_{\mathrm{min}} = 9.1 M_\odot \quad m_{\mathrm{max}} = 87 M_\odot$

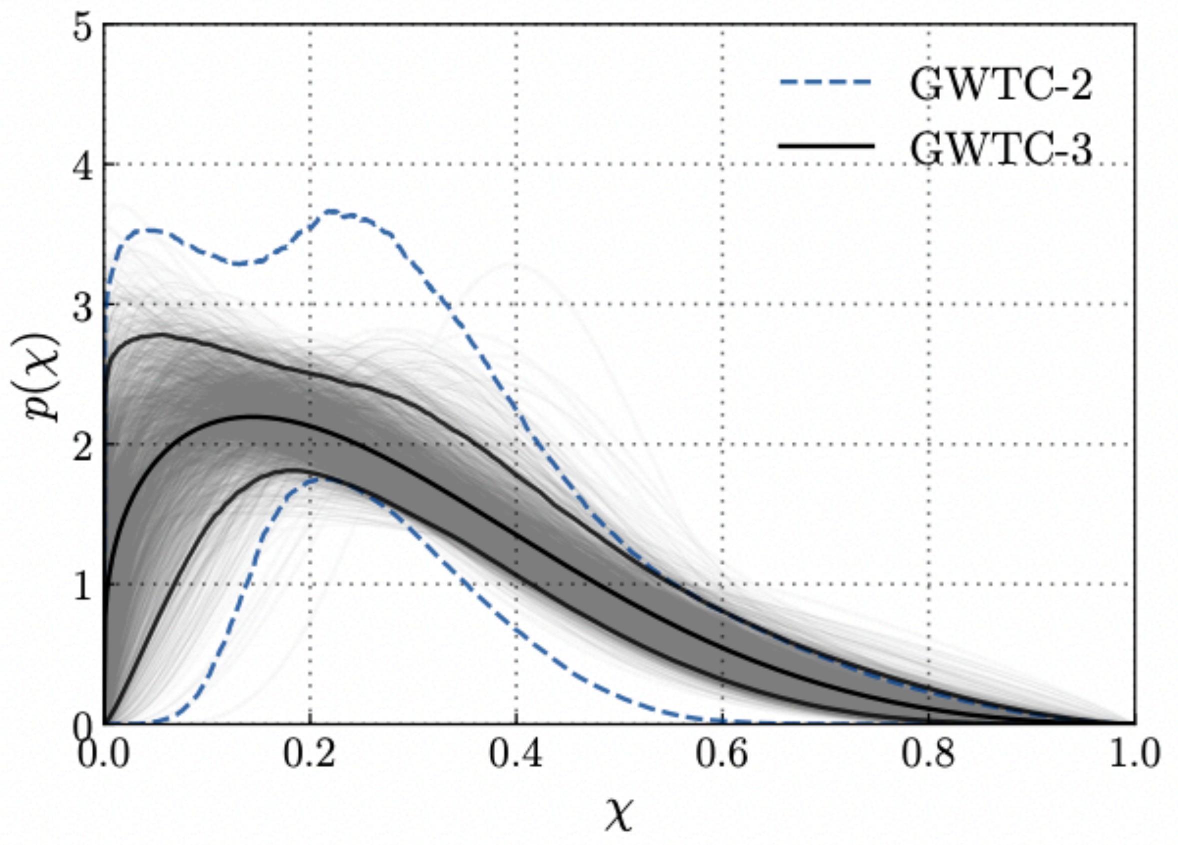
Astrophysical implications

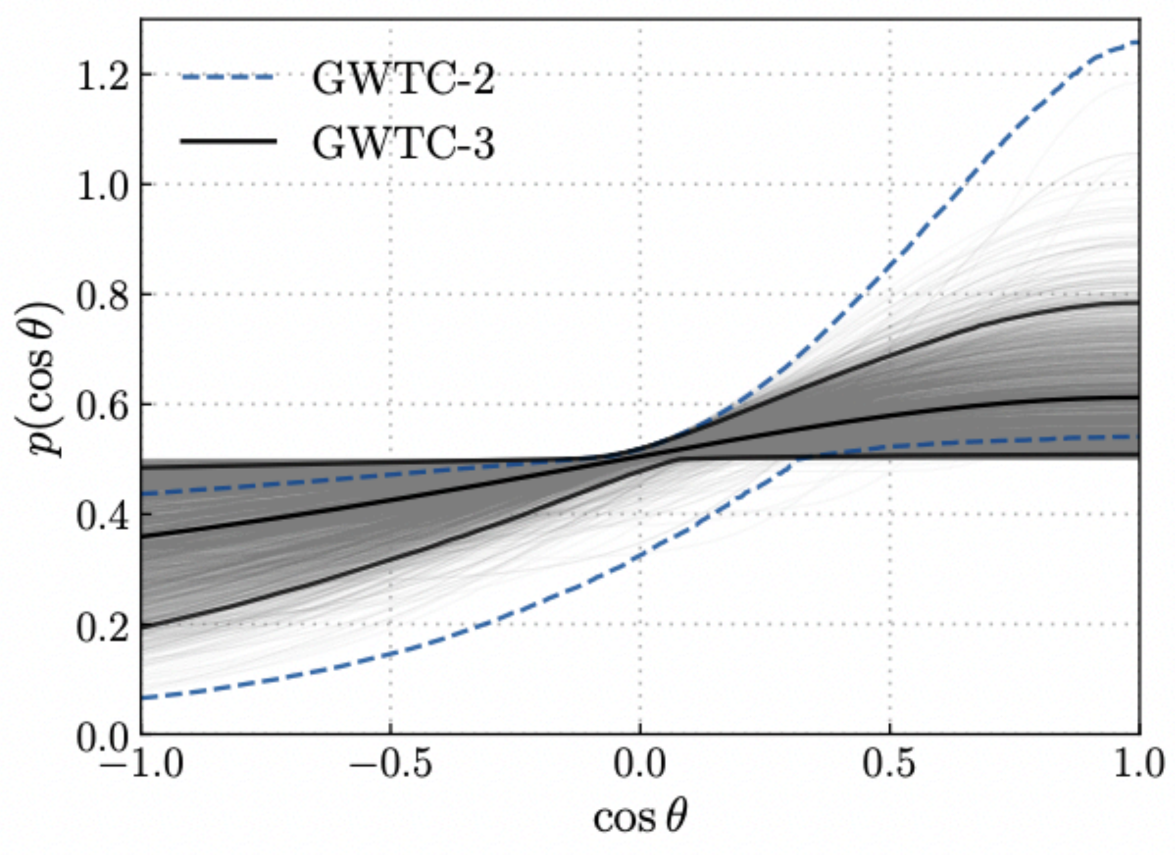
What's the formation, evolution, and distribution of BBHs in the universe?





- Lower mass end at ~ $10M_\odot$ → Field binaries
- $m_{max} \simeq 90M_\odot$ → Hierarchical mergers in dense environments
- Peak at $\sim 34M_\odot$ → Support for PISN mass gap between $\sim[40,60]M_\odot$





- Low (but non-zero) spin magnitudes
- Mixture of aligned and misaligned spins
- Weak precession

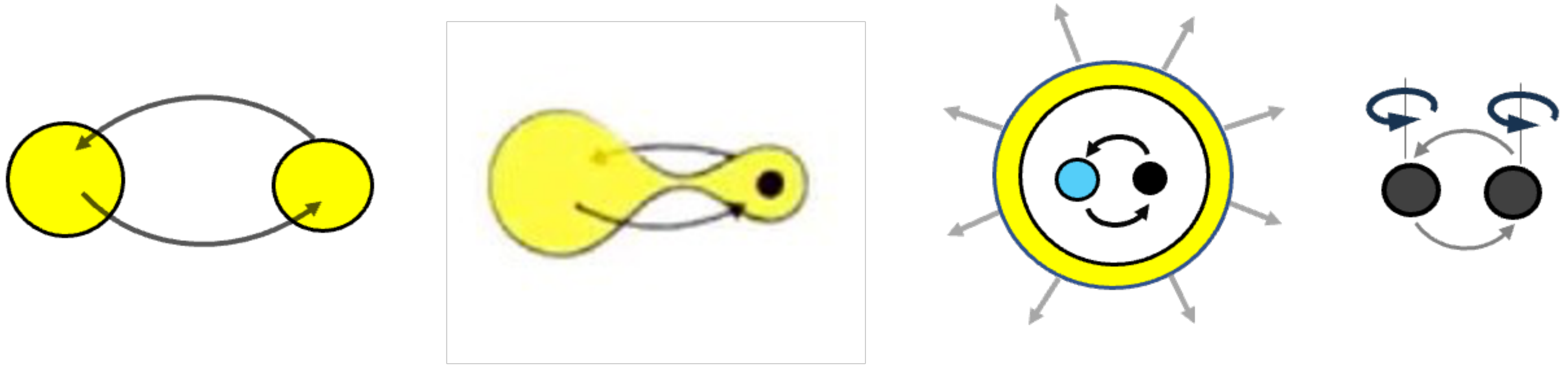
Mix of BBH formation channels (dynamical and isolated)

[LVK, 2022]

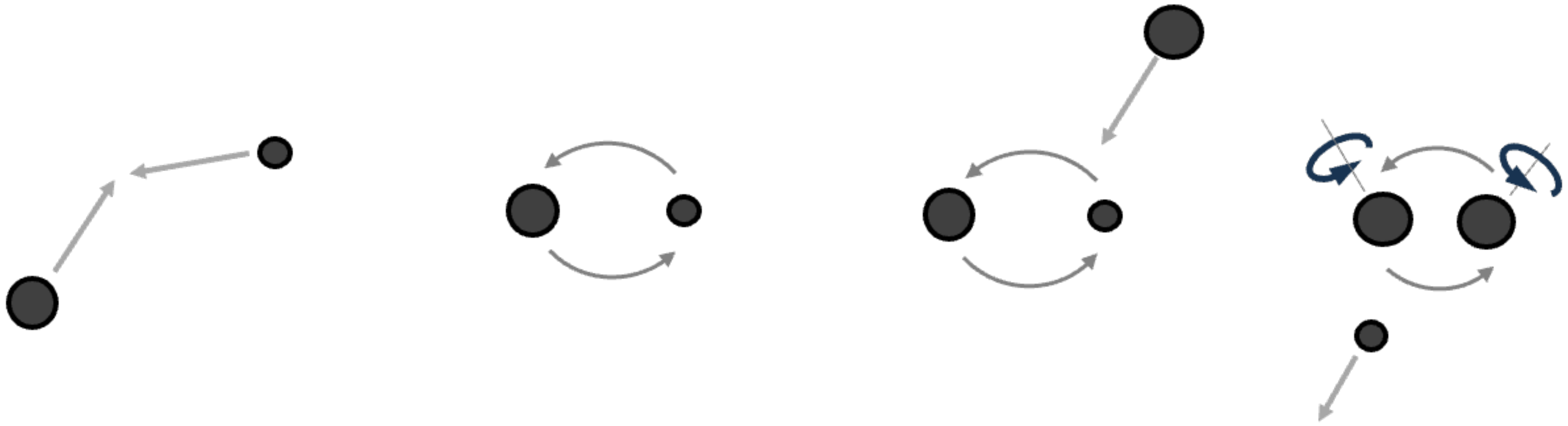
Astrophysical implications: spin distribution

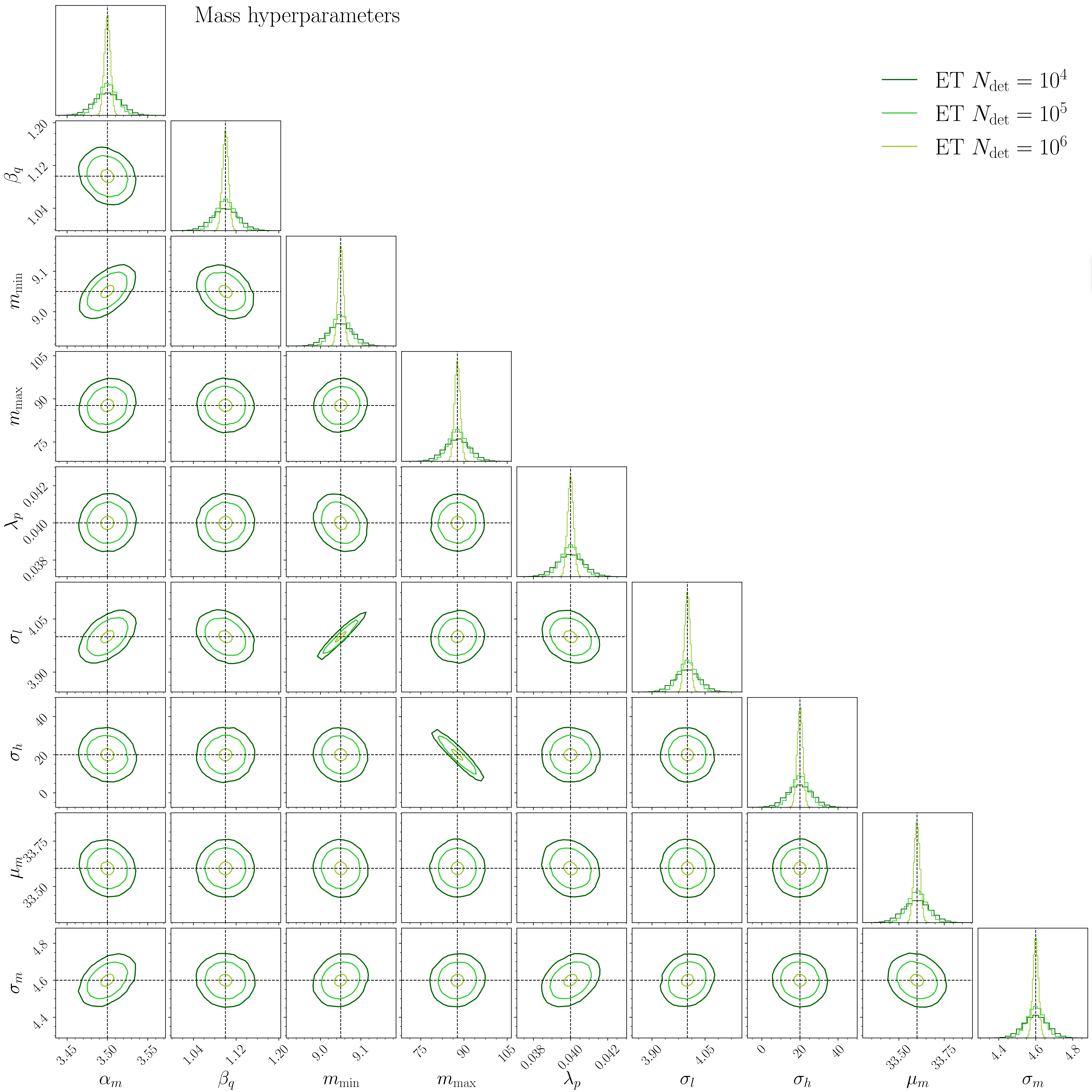
The directions of the BH spins are believed to be clean tracers of the astrophysical formation pathway

Isolated formation channel:
preferentially aligned spins
(non precessing binaries)



Dynamical formation channel:
isotropically oriented spins
(misalignment → spin
precession)





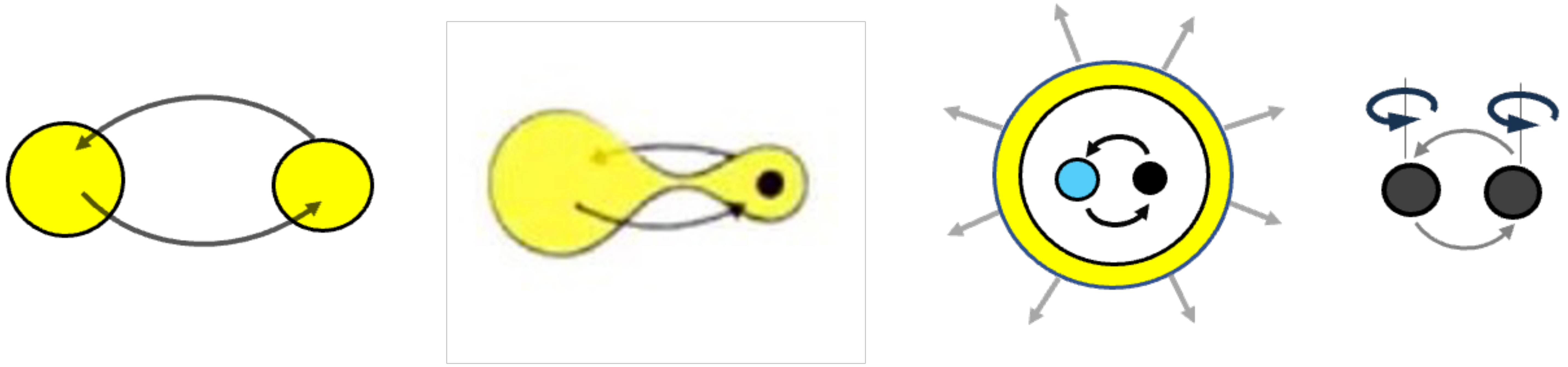
Mass hyperparameters: correlations

- Positive correlation between σ_l and $m_{\rm min}$
- Positive correlation between $m_{\rm min}$ and α_m
- Positive correlation between $m_{\rm min}$ and $\lambda_{\rm peak}$
- Negative correlation between σ_h and $m_{\rm max}$

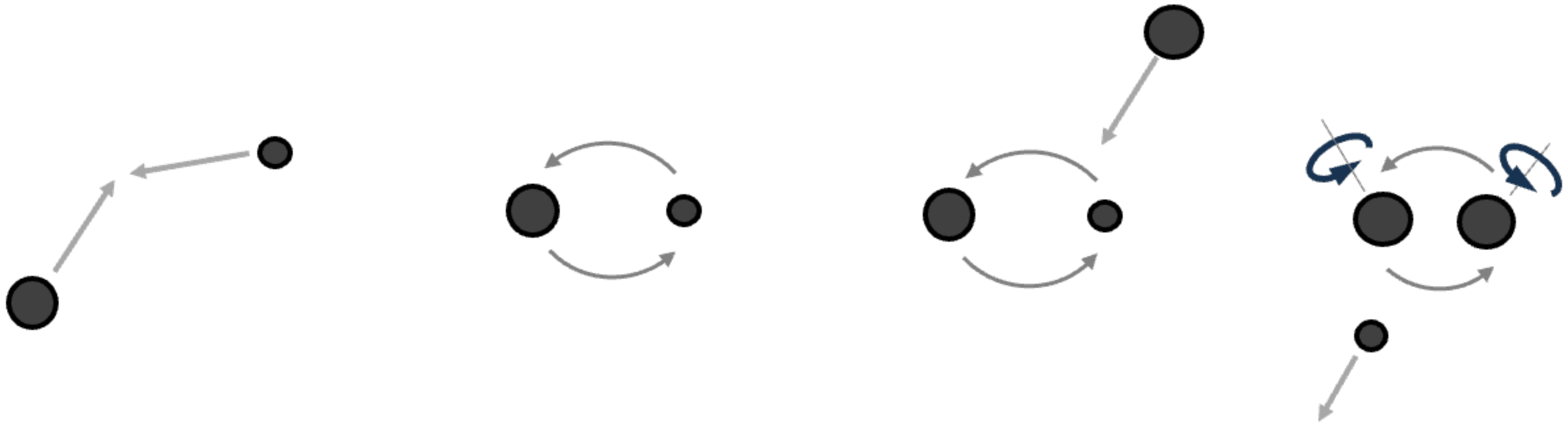
Astrophysical implications: spin distribution

The directions of the BH spins are believed to be clean tracers of the astrophysical formation pathway

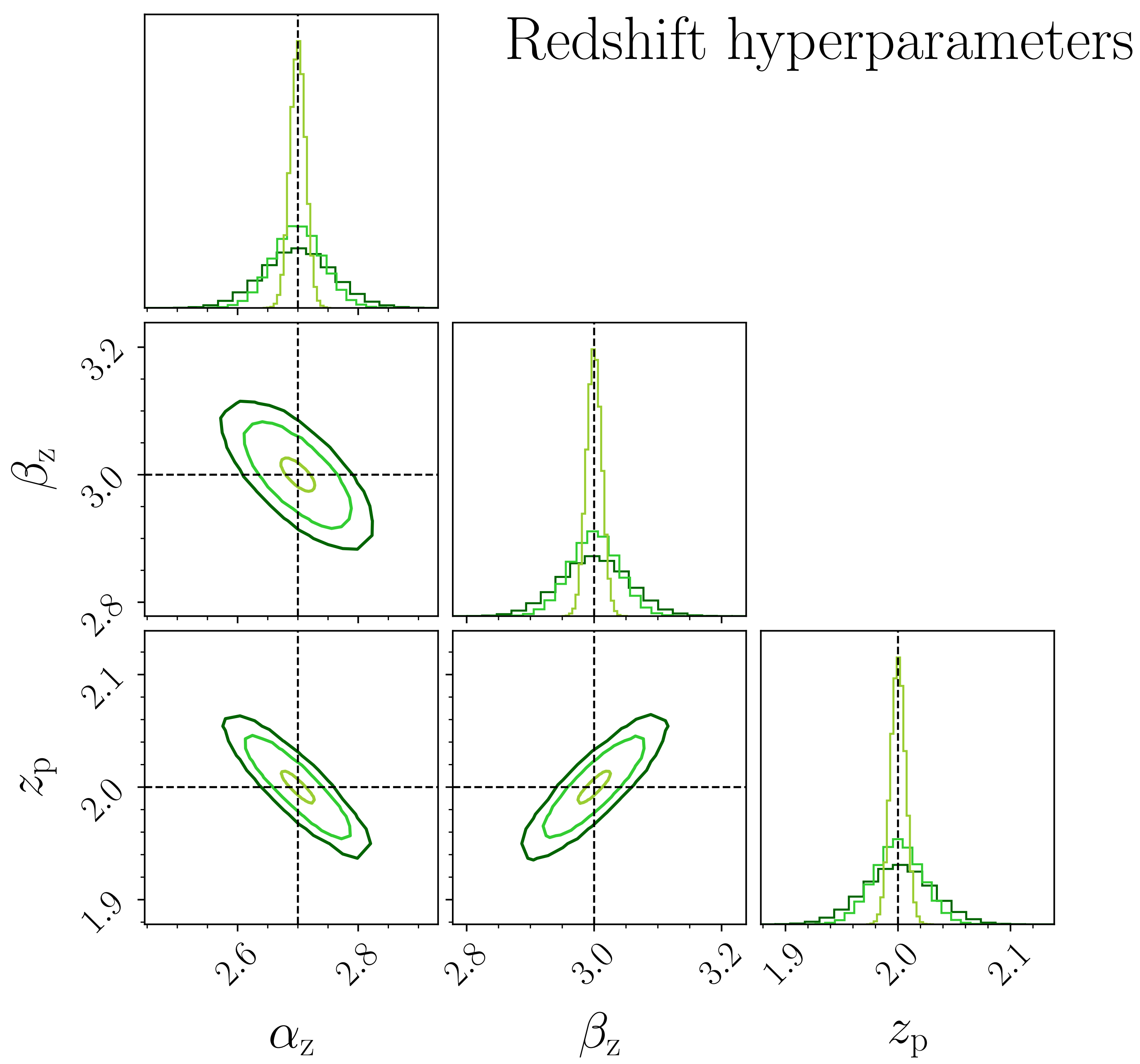
Isolated formation channel:
preferentially aligned spins
(non precessing binaries)



Dynamical formation channel:
isotropically oriented spins
(misalignment → spin
precession)

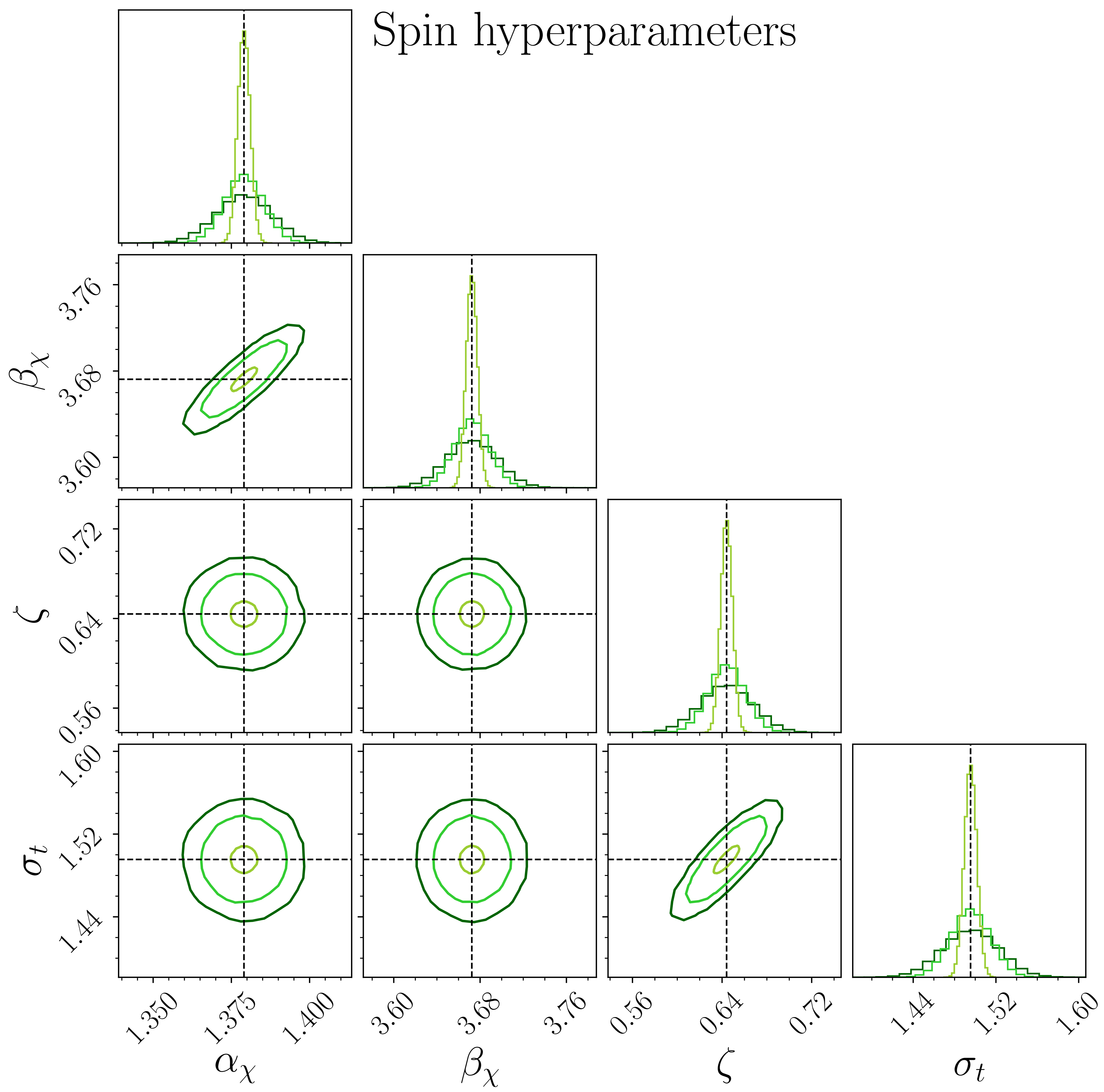


Redshift hyperparameters: correlations



Preliminary

Spin hyperparameters: correlations



Preliminary

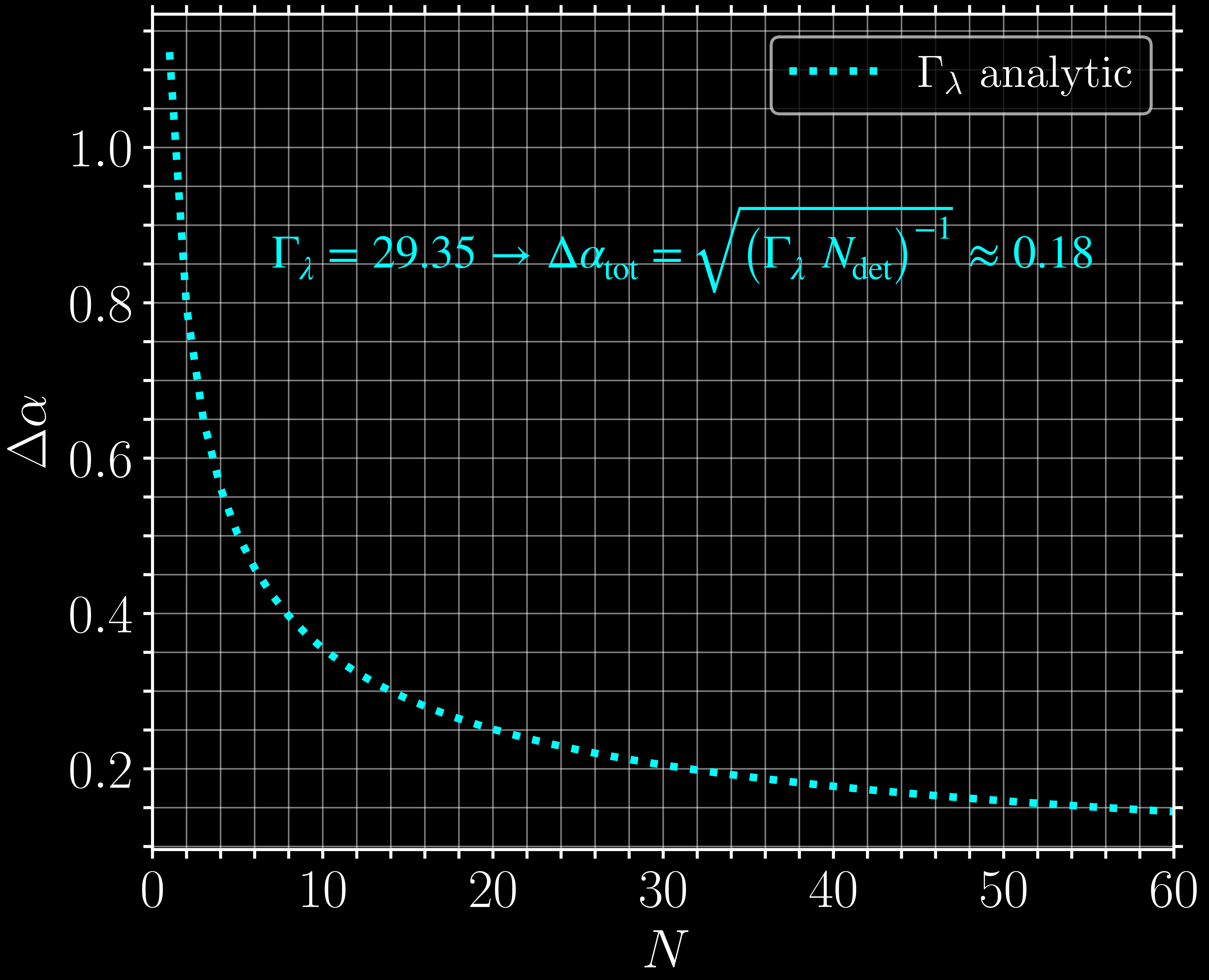
Power law of SMBHs with 1 parameter and 1 hyperparameter

[Gair et al. (2023) MNRAS 519 2736]

$$p(M_1 \,|\, \alpha) = \frac{\alpha}{M_{\max}^{\alpha} - M_{\min}^{\alpha}} M_1^{\alpha-1}$$

- ★ 1 parameter $\theta = M_1$
- ★ 1 hyperparameter $\lambda = \alpha$ with $\alpha_{\rm true} \approx 0$
- ★ $M_{\min} = 10^4 M_\odot$
- ★ $M_{\max} = 10^7 M_\odot$

- ★ $d_{th} = 5 \times 10^5 M_\odot$
- ★ $\Gamma = \frac{1}{\sigma^2}$ with $\sigma = 0.1$
- ★ $N_{\rm obs} = 100$
- ★ $N_{\rm det} = 39$



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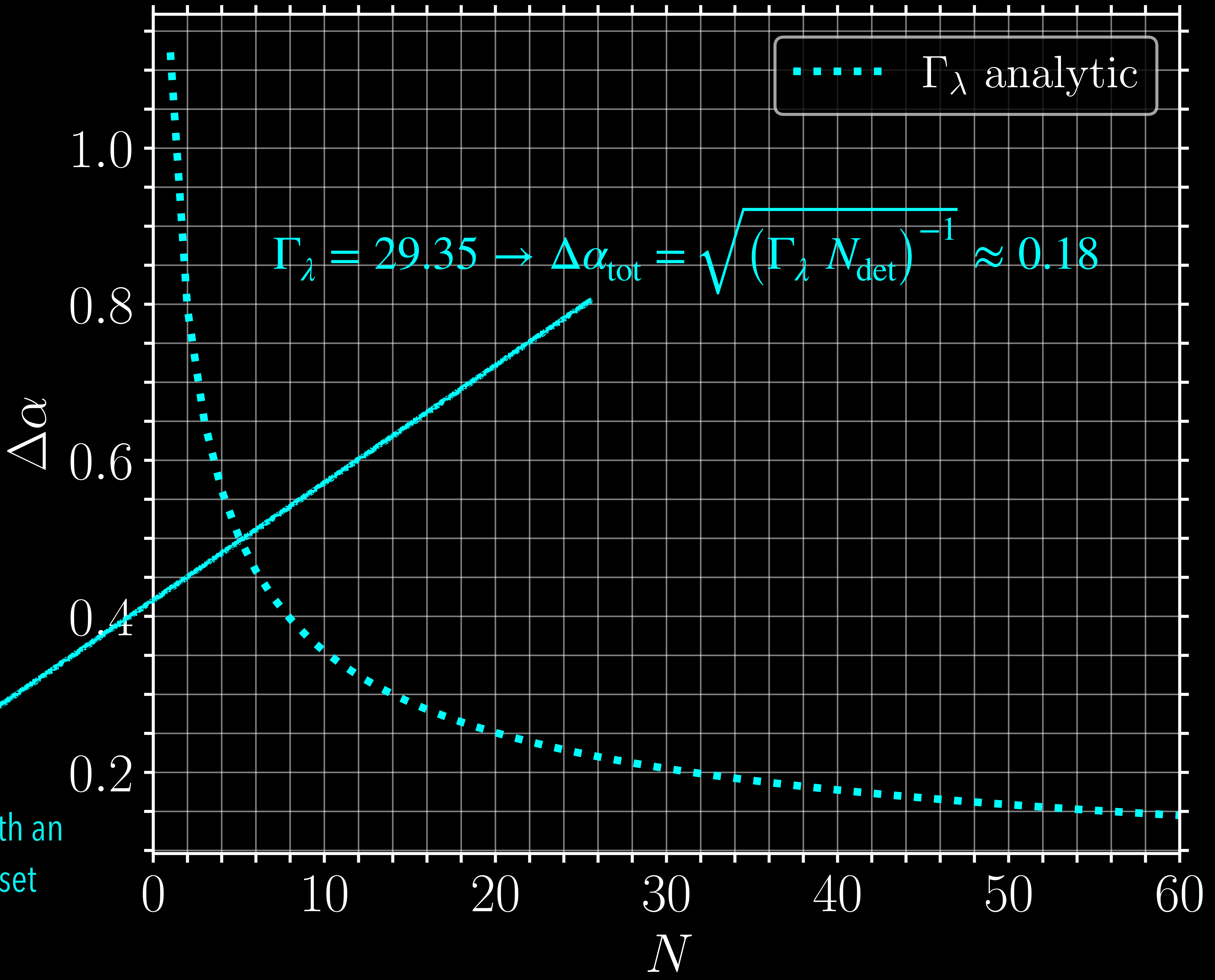

Fisher predictions were validated with an MCMC analysis for the same data set

Power law of SMBHs with 1 parameter and 1 hyperparameter

[Gair et al. (2023) MNRAS 519 2736]

$$p(M_1 | \alpha) = \frac{\alpha}{M_{\max}^{\alpha} - M_{\min}^{\alpha}} M_1^{\alpha-1}$$

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- ★ 1 hyperparameter $\lambda = \alpha$ with $\alpha_{\text{true}} \approx 0$
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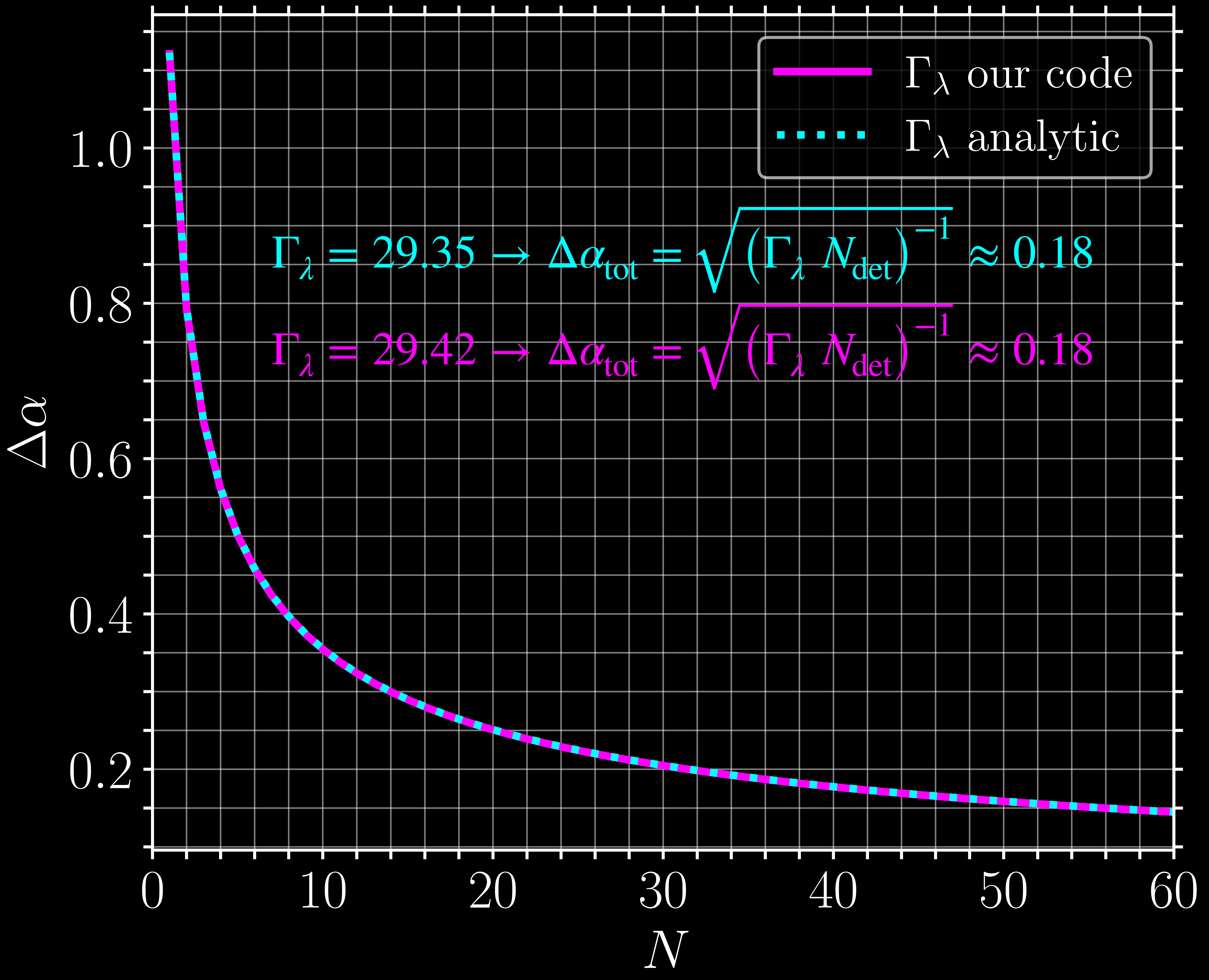


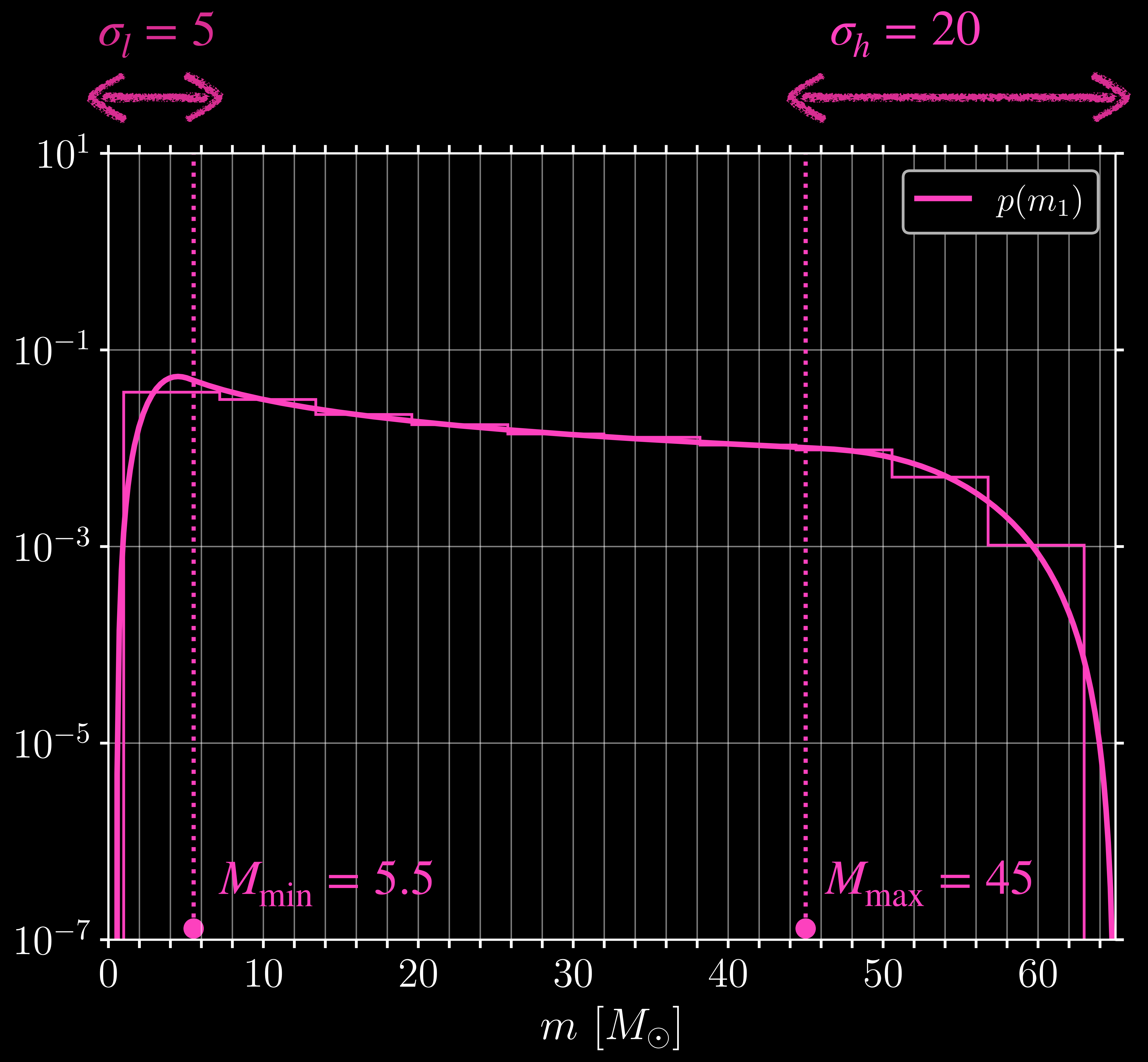
Power Law (PL) mass function with smoothing

$$p\left(M_1 \,|\, \overrightarrow{\lambda}\right) \propto M_1^{-\alpha}\; S(M_1, M_{\min}, \sigma_l, M_{\max}, \sigma_h)$$

$$p\left(M_2 \,|\, M_1,\, \overrightarrow{\lambda}\right) \propto M_2^{\beta}\; S(M_2, M_{\min}, \sigma_l, M_{\max}, \sigma_h)$$

6 hyperparameter $\overrightarrow{\lambda}$:

- α : Spectral index for the PL of the primary mass distribution ($\alpha_{true} = 0.75$)
- β: Spectral index for the PL of the mass ratio distribution ($\beta_{true} = 0.1$)
- $M_{\max}$, $M_{\min}$: Maximum and minimum mass of the PL component of the primary mass distribution
- σ_h , σ_l : Width of the smoothing component at the upper and lower edge of the mass distribution

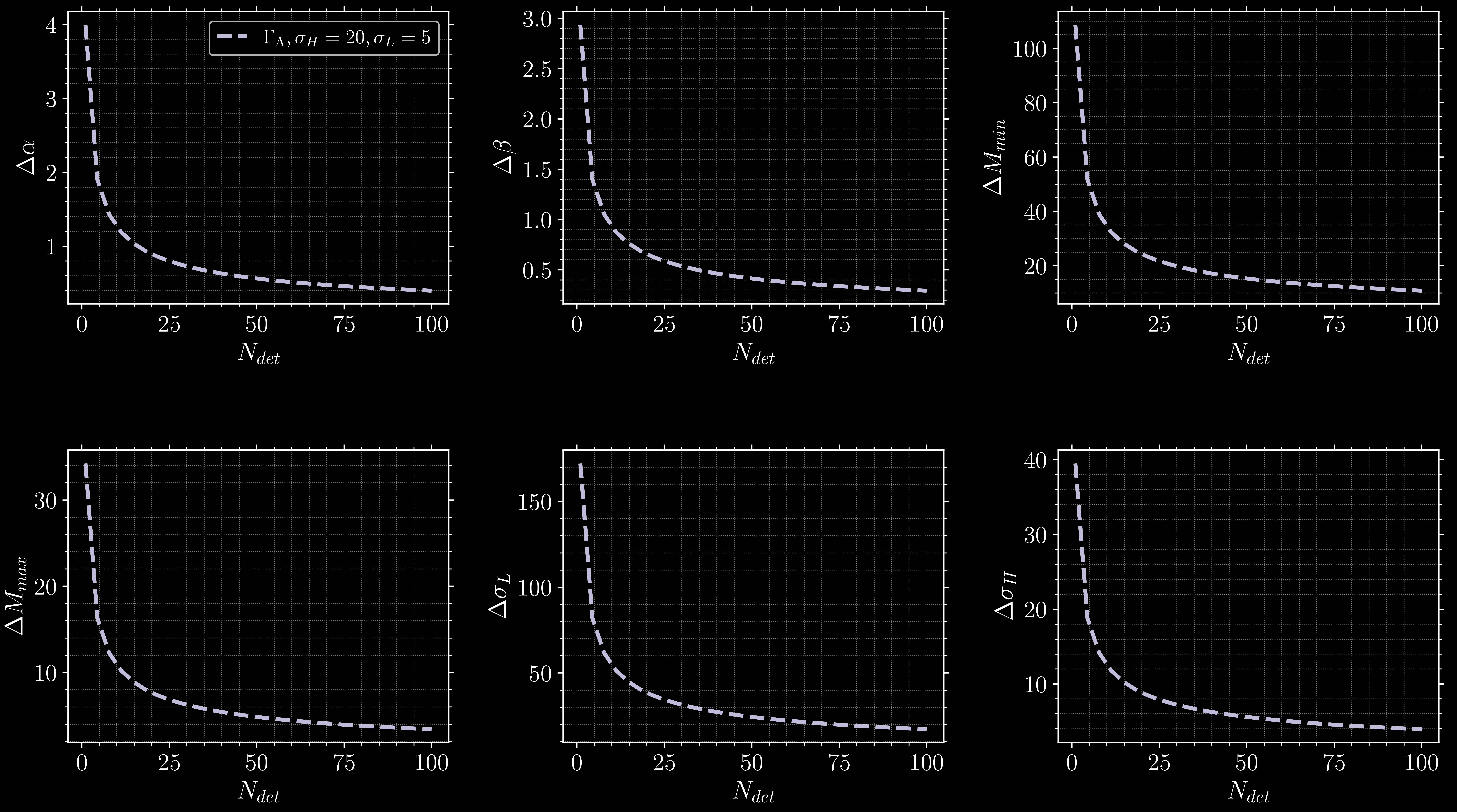


O4 sentivity curve

IMRPhenomHM waveform model

$\rho_{th=20}$

PowerLaw mass function with smoothing



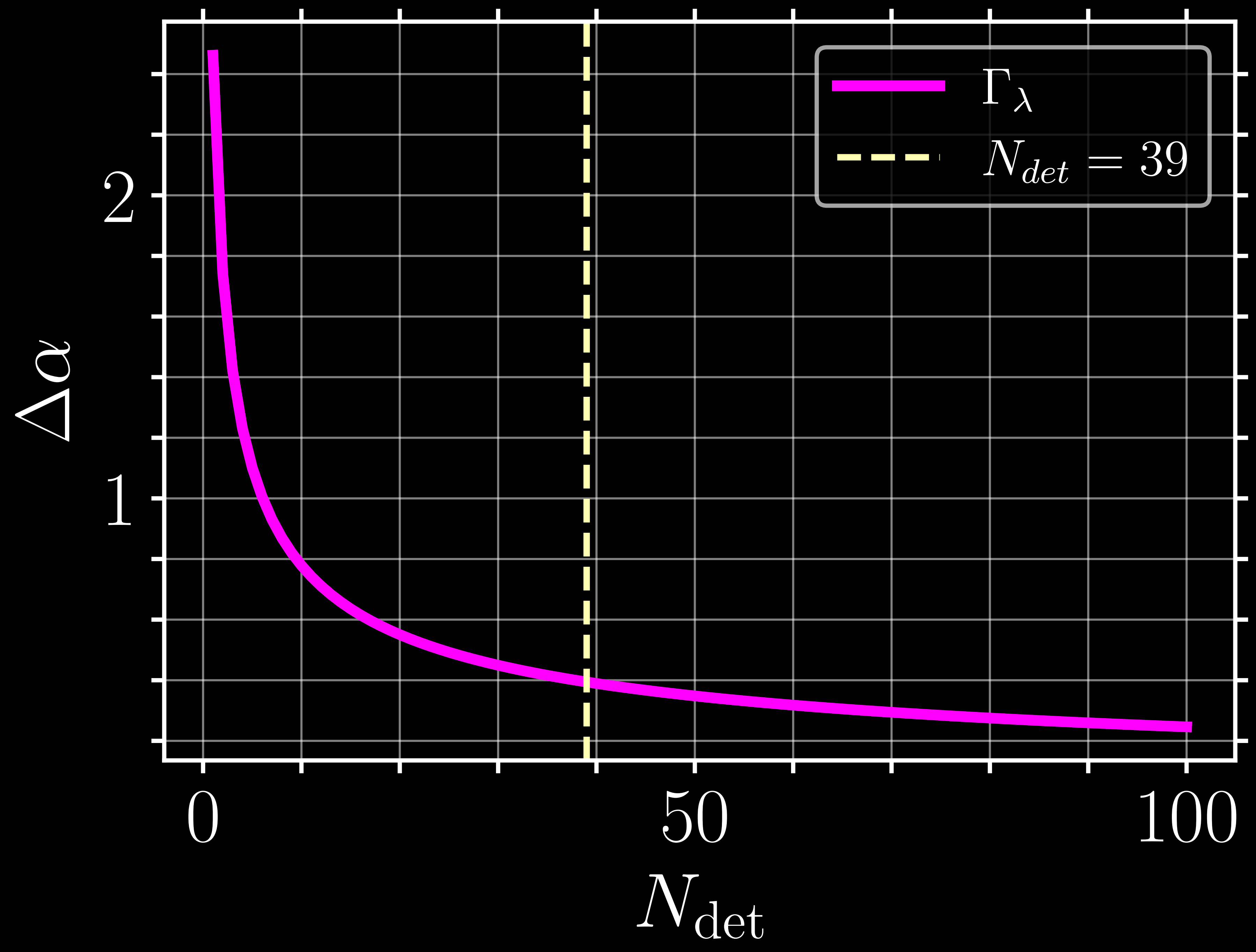
Population model with 2 parameters and 2 hyperparameters

$$p(M_1\,|\,\alpha) = \frac{\alpha}{M_{\max}^{\alpha} - M_{\min}^{\alpha}} M_1^{\alpha-1}$$

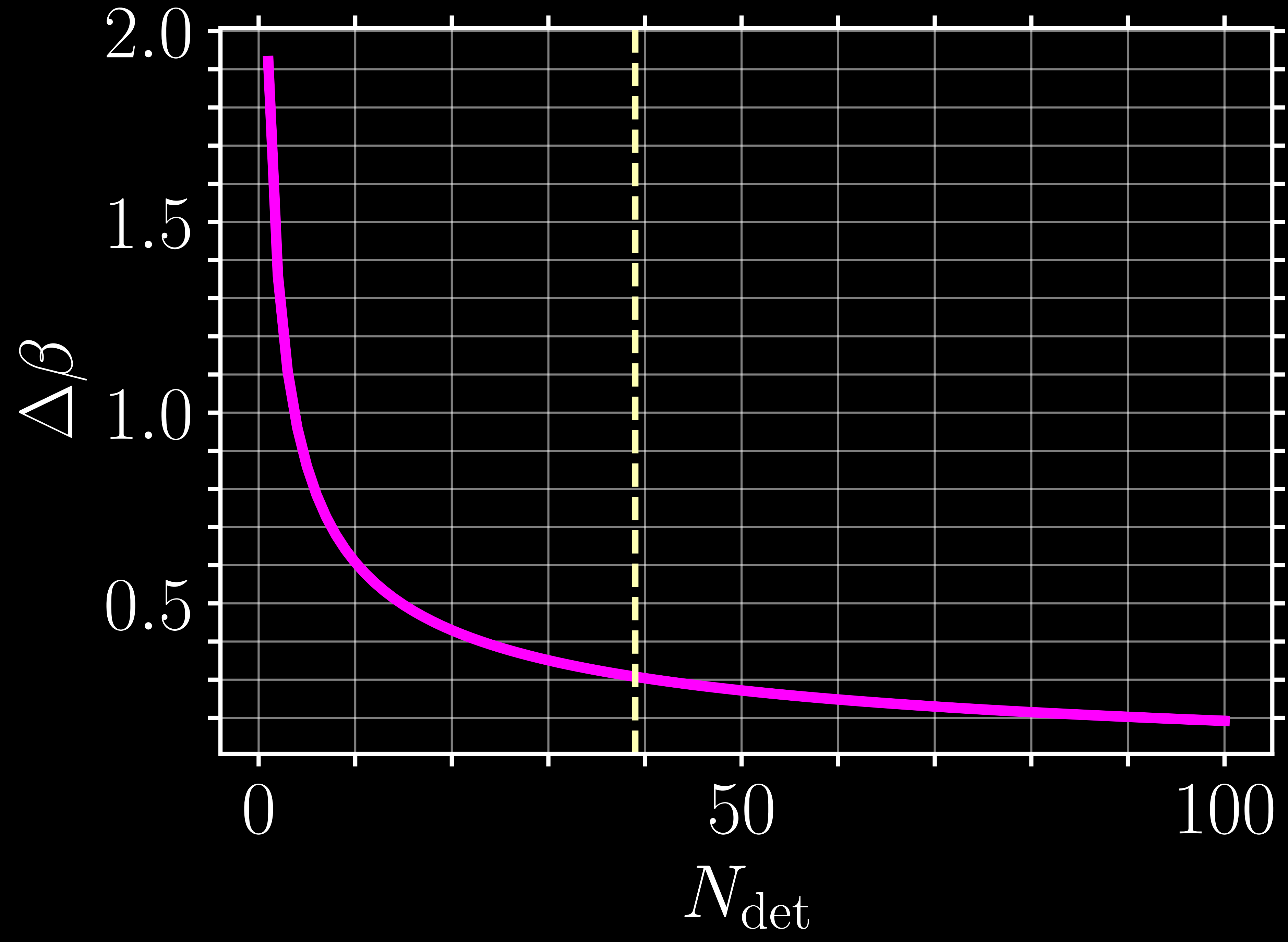
$$p(M_2\,|\,\beta, M_1) \sim \left(\frac{M_2}{M_1}\right)^{\beta}$$

- ★ 2 parameters $\overrightarrow{\theta} = \{M_1, M_2\}$
- ★ 2 hyperparameters $\overrightarrow{\lambda} = \{\alpha, \beta\}$ with $\alpha_{\rm true} \approx 0$ and $\beta_{\rm true} \approx 1.1$
  (Spectral indexes for the PL of the primary (mass ratio) distributions)
- ★ Ndet=39

$$\Delta\alpha_{\rm tot} = \sqrt{\left(\Gamma_\lambda\, N_{\rm det}\right)^{-1}} \approx 0.39$$



$$\Delta\beta_{\rm tot} = \sqrt{\left(\Gamma_\lambda\, N_{\rm det}\right)^{-1}} \approx 0.31$$



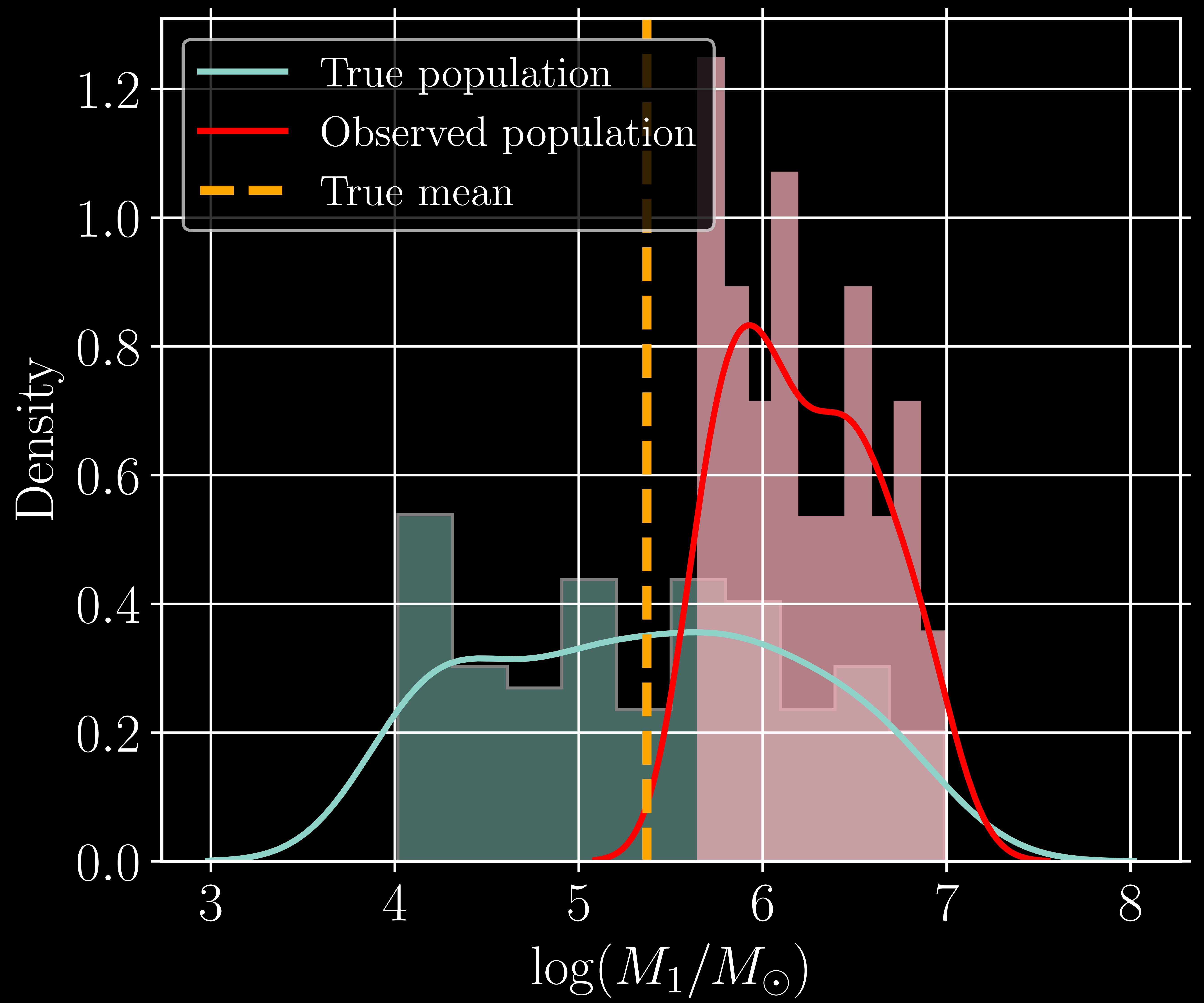
Population model with 1 parameter and 1 hyperparameter

[Gair et al. (2023) MNRAS 519 2736]

$$p(M_1 \mid \alpha) = \frac{\alpha}{M_{\max}^{\alpha} - M_{\min}^{\alpha}} M_1^{\alpha-1}$$

- ★ 1 parameter $\theta = M_1$
- ★ 1 hyperparameter $\lambda = \alpha$ with $\alpha_{\text{true}} \approx 0$
- ★ $M_{\min} = 10^4 M_\odot$
- ★ $M_{\max} = 10^7 M_\odot$

- ★ $d_{th} = 5 \times 10^5 M_\odot$
- ★ $\Gamma = \frac{1}{\sigma^2}$ with $\sigma = 0.1$
- ★ $N_{\text{obs}} = 100$
- ★ $N_{\text{det}} = 39$



Population model with 1 parameter and 1 hyperparameter

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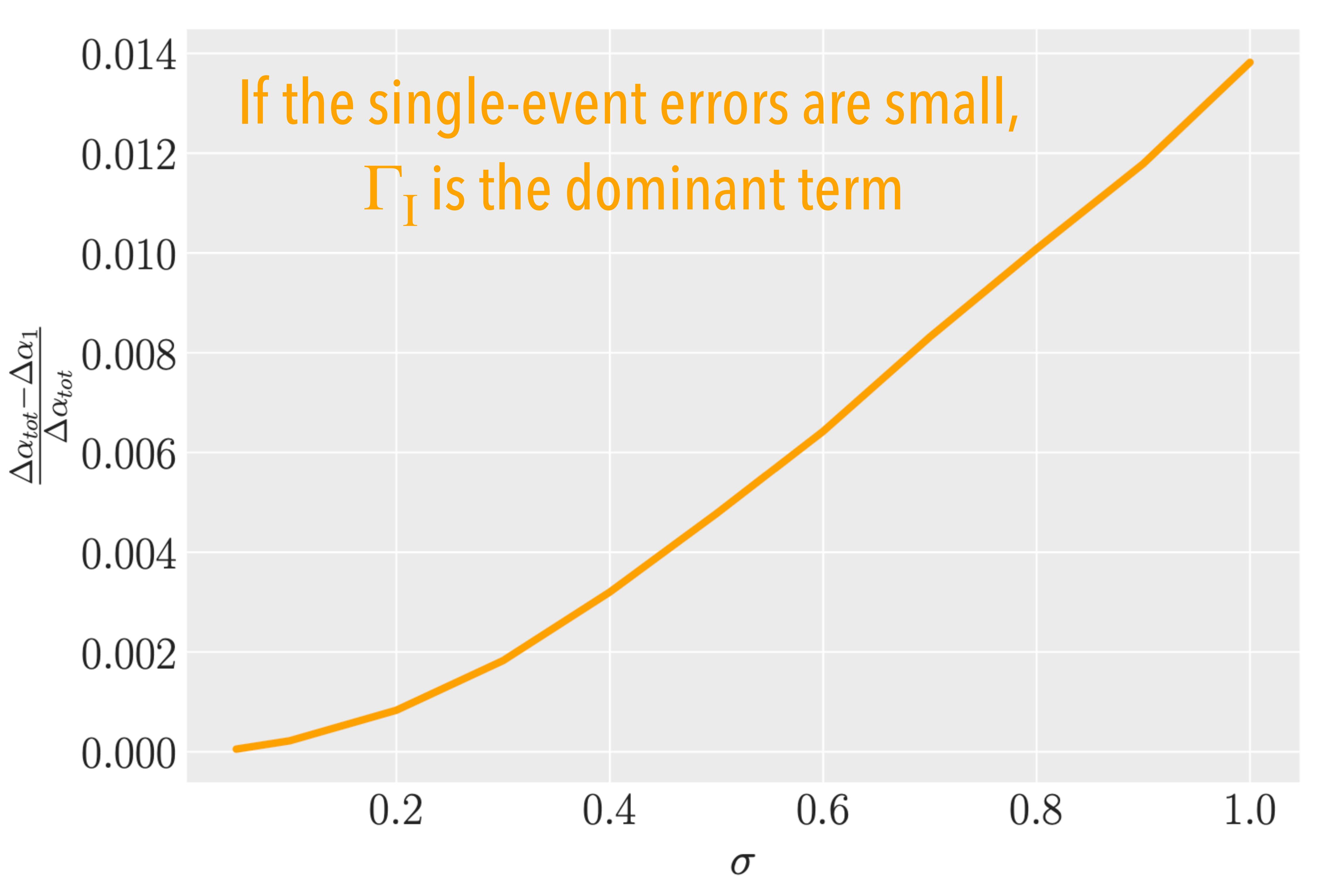
- ★ 1 hyperparameter $\lambda = \alpha$ with $\alpha_{\rm true} \approx 0$
- ★ $M_{\rm min} = 10^4 M_\odot$
- ★ $M_{\rm max} = 10^7 M_\odot$

- ★ $d_{th} = 5 \times 10^5 M_\odot$
- ★ $\Gamma = \dfrac{1}{\sigma^2}$ with $\sigma = 0.1$
- ★ $N_{\rm obs} = 100$
- ★ $N_{\rm det} = 39$

$$\Gamma_\lambda = 29.35 \rightarrow \Delta\alpha_{\rm tot} = \sqrt{\left(\Gamma_\lambda\, N_{\rm det}\right)^{-1}} \approx 0.18$$

$$\Gamma_{\rm I} = 29.36 \rightarrow \Delta\alpha_{\rm I} = \sqrt{\left(\Gamma_{\rm I}\, N_{\rm det}\right)^{-1}} \approx 0.18$$

$$\Gamma_{\rm II} + \Gamma_{\rm III} + \Gamma_{\rm IV} + \Gamma_{\rm V} \approx 10^{-13}$$



Population model with 2 parameter and 2 hyperparameter

- ★ 2 parameter $\overrightarrow{\theta} = \{M_1, M_2\}$
- ★ 2 hyperparameter $\overrightarrow{\lambda} = \{\alpha, \beta\}$ with $\alpha_{\text{true}} \approx 0$ and $\beta_{\text{true}} \approx 1.1$
- ★ Ndet=39

$$\Delta\alpha_{\text{tot}} = \sqrt{\left(\Gamma_\lambda\, N_{\text{det}}\right)^{-1}} \approx 0.39$$

$$\Delta\alpha_{\text{I}} = \sqrt{\left(\Gamma_{\text{I}}\, N_{\text{det}}\right)^{-1}} \approx 0.23$$

$$\Delta\beta_{\text{tot}} = \sqrt{\left(\Gamma_\lambda\, N_{\text{det}}\right)^{-1}} \approx 0.31$$

$$\Delta\beta_{\text{I}} = \sqrt{\left(\Gamma_{\text{I}}\, N_{\text{det}}\right)^{-1}} \approx 0.18$$