Assemblée Générale Groupe de Recherche Ondes Gravitationnelles

GRAVITATIONAL-RADIATION REACTION and FLUX-BALANCE EQUATIONS

Luc Blanchet

Institut d'Astrophysique de Paris

Based on a recent collaboration with

Guillaume Faye and David Trestini

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Einstein's quadrupole formula [Einstein 1918]

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{\mathfrak{J}}_{\mu\nu}^{2} - \frac{1}{3}\left(\sum_{\mu} \dddot{\mathfrak{J}}_{\mu\mu}\right)^2\right].$$

[Courtesy J. Mouette]

1. Quadrupole formula for the energy flux

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathrm{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 M_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 M_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}}$$

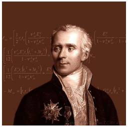
2. Quadrupole formula for the GW amplitude

$$\boxed{h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 r}\left\{\frac{\mathrm{d}^2 M_{ij}}{\mathrm{d}t^2}\left(t - \frac{r}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\mathrm{TT}} + \mathcal{O}\left(\frac{1}{r^2}\right)}$$

3. The quadrupole moment reduces to the usual Newtonian quadrupole

$$M_{ij}(t) = \int_{\mathrm{source}} \mathrm{d}^3\mathbf{x}\, \rho(\mathbf{x}, t)\left(x_i x_j - \frac{1}{3}\delta_{ij}\,\mathbf{x}^2\right) + \mathcal{O}\left(\frac{v}{c}\right)^2$$

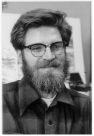
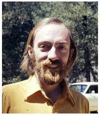
Gravitational radiation reaction




- Laplace [1776]: a finite speed of propagation of gravity would result in a damping of planetary orbits
- Poincaré [1907]: concept of "ondes gravifiques" and re-analysis of the Laplace effect

- Chandrasekhar & Esposito [1970]: radiation reaction is of order

$$\mathcal{O}\left(\frac{v}{c}\right)^5 \sim 2.5\text{PN}$$

- Burke & Thorne [1970]: simple expression of the radiation reaction

$$F^i_{\text{RR}} = \underbrace{-\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 M_{ij}}{\mathrm{d}t^5}}_{\text{small 2.5PN effect}} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

Flux-balance equations

Balance equations are associated with the ten symmetries of the Poincaré group

1. Energy [Einstein 1918]

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{G}{5c^5}\frac{\mathrm{d}^3 M_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 M_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{1}{c^7}\right)$$

2. Angular momentum [Papapetrou 1971; Thorne 1980]

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = -\frac{2G}{5c^5}\varepsilon_{ijk}\frac{\mathrm{d}^2 M_{jl}}{\mathrm{d}t^2}\frac{\mathrm{d}^3 M_{kl}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{1}{c^7}\right)$$

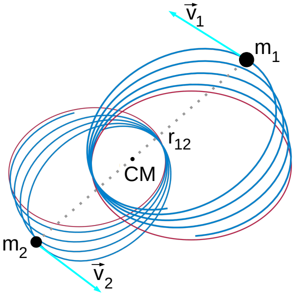
3. Linear momentum [Bonnor & Rotenberg 1961; Peres 1962; Bekenstein 1973; Thorne 1980]

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{G}{c^7}\left[\frac{2}{63}\frac{\mathrm{d}^4 M_{ijk}}{\mathrm{d}t^4}\frac{\mathrm{d}^3 M_{jk}}{\mathrm{d}t^3} + \frac{16}{45}\varepsilon_{ijk}\frac{\mathrm{d}^3 M_{jl}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 S_{kl}}{\mathrm{d}t^3}\right] + \mathcal{O}\left(\frac{1}{c^9}\right)$$

4. Center-of-mass position [Kozameh, Nieva & Quirega 2018; Blanchet & Faye 2019]

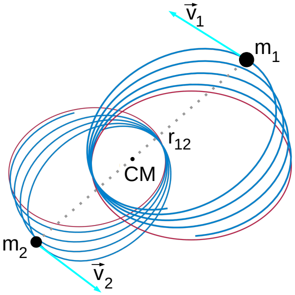
$$\frac{\mathrm{d}G_i}{\mathrm{d}t} = P_i - \frac{2G}{21c^7}\frac{\mathrm{d}^3 M_{ijk}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 M_{jk}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

The 4.5PN radiation-reaction for compact binaries



$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12} + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right]\boldsymbol{n}_{12} + \cdots\right\}}^{\text{1PN}} + \overbrace{\frac{1}{c^4}[\cdots]}^{\text{2PN}}$$
$$+ \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN} \\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN} \\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\text{4PN}} + \underbrace{\frac{1}{c^9}[\cdots]}_{\substack{\text{4.5PN} \\ \text{radiation reaction}}} + \mathcal{O}\left(\frac{1}{c^{10}}\right)$$

The 4.5PN radiation-reaction for compact binaries



$$
\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{G m_2}{r_{12}^2}\boldsymbol{n}_{12} + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \cdots\right]\boldsymbol{n}_{12} + \cdots\right\}}^{\text{1PN}} + \overbrace{\frac{1}{c^4}[\cdots]}^{\text{2PN}}
$$

$$
+ \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\text{4PN}} + \underbrace{\frac{1}{c^9}[\cdots]}_{\substack{\text{4.5PN}\\ \text{radiation reaction}}} + \mathcal{O}\left(\frac{1}{c^{10}}\right)
$$

The linear radiation-reaction potentials [Blanchet 1993, 1997]

- In an extension of the Burke-Thorne [1970] gauge the radiation reaction is described by the scalar, vector and tensor potentials

$$V_{\rm RR} = G\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\frac{(\ell+1)(\ell+2)}{\ell(\ell-1)}\partial_L\big\{M_L\big\}$$

$$V^i_{\rm RR} = -G\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\frac{\ell+2}{\ell-1}\left[\frac{2\ell+1}{\ell}\,\hat{\partial}_{iL}\big\{M_L^{(-1)}\big\} - \frac{\ell}{(\ell+1)c^2}\,\varepsilon_{iab}\partial_{aL-1}\big\{S_{bL-1}\big\}\right]$$

$$V^{ij}_{\rm RR} = G\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\frac{2\ell+1}{\ell-1}\left[\frac{2\ell+3}{\ell}\hat{\partial}_{ijL}\big\{M_L^{(-2)}\big\} - \frac{2\ell}{(\ell+1)c^2}\varepsilon_{ab(i}\hat{\partial}_{j)aL-1}\big\{S_{bL-1}^{(-1)}\big\}\right]$$

- We denote a monopolar anti-symmetric (retarded minus advanced) wave

$$\big\{M\big\}(t,r) \equiv \frac{M(t-r/c)-M(t+r/c)}{2r}$$

- Multipolar waves are obtained by apply (STF) multi-derivative operators $\hat{\partial}_L$

The 2PN radiation-reaction potentials [Blanchet, Faye & Trestini 2024]

- The main advantage of the extended Burke-Thorne gauge (in contrast to harmonic gauge) is that vector and tensor contributions are subdominant

$$V_{\mathrm{RR}} = \mathcal{O}\left(\frac{1}{c^{2\ell+1}}\right) = 2.5\mathrm{PN} + 3.5\mathrm{PN} + 4.5\mathrm{PN} + \cdots$$

$$V^i_{\mathrm{RR}} = \mathcal{O}\left(\frac{1}{c^{2\ell+3}}\right) = 3.5\mathrm{PN} + 4.5\mathrm{PN} + \cdots$$

$$V^{ij}_{\mathrm{RR}} = \mathcal{O}\left(\frac{1}{c^{2\ell+5}}\right) = 4.5\mathrm{PN} + \cdots$$

- Nevertheless a delicate PN non-linear iteration of the RR potentials has to be performed consistently to 2PN order

$$\begin{aligned}\mathcal{V}_{\mathrm{RR}} &= V_{\mathrm{RR}} + G^2\mathcal{V}_2\big[V_{\mathrm{sym}}, V_{\mathrm{RR}}, V^i_{\mathrm{RR}}, \cdots\big] + \cdots\\ \mathcal{V}^i_{\mathrm{RR}} &= V^i_{\mathrm{RR}} + \cdots\\ \mathcal{V}^{ij}_{\mathrm{RR}} &= V^{ij}_{\mathrm{RR}} + \cdots\end{aligned}$$

The 4.5PN RR part of the acceleration [Blanchet, Faye & Trestini 2024]

A long calculation yields the RR acceleration of particle 1

$$a^i_{\mathrm{RR}\,1} = a^i_{2.5\mathrm{PN}\,1} + a^i_{3.5\mathrm{PN}\,1} + a^i_{4.5\mathrm{PN}\,1} + \mathcal{O}\left(\frac{1}{c^{11}}\right)$$

where the PN pieces are given by

$$\begin{aligned}
a^i_{2.5\mathrm{PN}\,1} &= -\frac{2G}{5c^5} y_1^a M^{(5)}_{ia} \\
a^i_{3.5\mathrm{PN}\,1} &= \frac{G}{c^7}\Bigg\{ -\frac{11}{105} y_1^b M^{(7)}_{ib} y_1^2 + \frac{17}{105} y_1^{iab} M^{(7)}_{ab} - \frac{8}{15} y_1^b M^{(6)}_{ib} (v_1 y_1) \\
&\qquad + \frac{G M^{(5)}_{ia}}{r_{12}} \left(\frac{7}{5} m_2 n_{12}^a r_{12} + \frac{1}{5} m_2 y_1^a \right) \\
&\qquad - \frac{16}{45} \varepsilon_{ibj} S^{(6)}_{aj} y_1^{ab} - \frac{16}{45} \varepsilon_{ibj} v_1^a y_1^b S^{(5)}_{aj} - \frac{32}{45} \varepsilon_{iaj} v_1^a y_1^b S^{(5)}_{bj} + \cdots \Bigg\} \\
a^i_{4.5\mathrm{PN}\,1} &= \frac{G}{c^9}\Big\{ \text{a very long expression} \Big\}
\end{aligned}$$

The 4.5PN flux-balance equations [Blanchet, Faye & Trestini 2024]

- From the RR acceleration a long calculation (in the source's near zone) yields

$$\boxed{\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= -\mathcal{F}_E & \qquad \frac{\mathrm{d}J^i}{\mathrm{d}t} &= -\mathcal{F}_J^i \\ \frac{\mathrm{d}P^i}{\mathrm{d}t} &= -\mathcal{F}_P^i & \qquad \frac{\mathrm{d}G^i}{\mathrm{d}t} &= P^i - \mathcal{F}_G^i \end{aligned}}$$

- The fluxes to 2PN relative order are given by exactly the same expressions as computed at future null infinity, *e.g.*

$$\mathcal{F}_E = \frac{G}{c^5}\left(\frac{1}{5}M_{ij}^{(3)}M_{ij}^{(3)} + \frac{1}{c^2}\left[\frac{1}{189}M_{ijk}^{(4)}M_{ijk}^{(4)} + \frac{16}{45}S_{ij}^{(3)}S_{ij}^{(3)}\right] + \frac{1}{c^4}\left[\cdots\right]\right) + \mathcal{O}\left(\frac{1}{c^{11}}\right)$$

- The left sides are composed by the conservative pieces (obtained with 2PN precision) plus RR contributions in the form of Schott [1915] terms, *e.g.*

$$\begin{aligned} E &= E_{\text{cons}} + E_{\text{RR}} \\ \text{with} \quad E_{\text{RR}} &= E_{2.5\text{PN}} + E_{3.5\text{PN}} + E_{4.5\text{PN}} + \mathcal{O}\left(\frac{1}{c^{11}}\right) \end{aligned}$$

Restriction to the center-of-mass frame [Blanchet, Faye & Trestini 2024]

- The balance equations for linear momentum and CM position are

$$\frac{\mathrm{d}P^i}{\mathrm{d}t} = -\mathcal{F}_P^i \qquad\qquad \frac{\mathrm{d}G^i}{\mathrm{d}t} = P^i - \mathcal{F}_G^i$$

  where P^i and G^i correspond to the matter system (the compact binary) while the right-hand sides are attribuable to the emitted radiation

- Integrating we obtain

$$P^i(t) = P_0^i - \int_{t_0}^{t} \mathrm{d}t'\, \mathcal{F}_P^i(t')$$

$$G^i(t) = P_0^i\big(t - t_0\big) + G_0^i - \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' \mathcal{F}_P^i(t'') - \int_{t_0}^{t} \mathrm{d}t' \mathcal{F}_G^i(t')$$

  where P_0^i and G_0^i are two integration constants

Restriction to the center-of-mass frame [Blanchet, Faye & Trestini 2024]

- The definition of the CM frame is then

$$\boxed{P_0^i = G_0^i = 0}$$

where by CM we mean the one of the total matter plus radiation system

- We pose for the non-local-in-time integrated fluxes

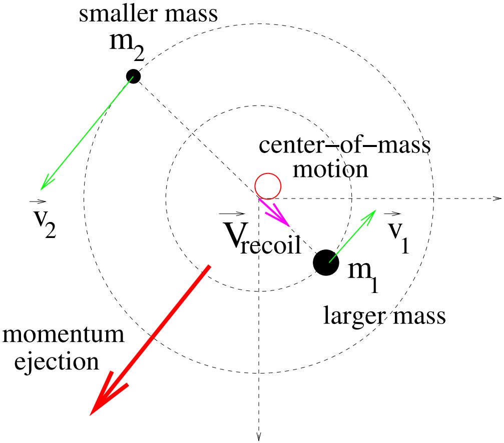
$$\Pi^i(t) \equiv \int_{-\infty}^{t} \mathrm{d}t'\, \mathcal{F}_P^i(t') \quad \text{and} \quad \Gamma^i(t) \equiv \int_{-\infty}^{t} \mathrm{d}t'\, \Pi^i(t') + \int_{-\infty}^{t} \mathrm{d}t' \mathcal{F}_G^i(t')$$

- The CM frame condition reads

$$\boxed{G^i + \Gamma^i = 0 \quad \text{which implies} \quad P^i + \Pi^i = 0}$$

- This involves the non-local ejected momentum or GW recoil of the source

Restriction to the center-of-mass frame [Blanchet, Faye & Trestini 2024]



$$M\, V^i_{\text{recoil}}(t) = -\Pi^i(t) = -\int_{-\infty}^{t} \mathrm{d}t'\, \mathcal{F}^i_P(t')$$

The flux-balance approach to RR

[Iyer & Will 1993, 1995; Gopakumar, Iyer & Iyer 1997]

- The RR acceleration is determined in the center-of-mass frame by imposing the flux-balance equations for energy and angular momentum

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}_E \qquad\qquad \frac{\mathrm{d}J^i}{\mathrm{d}t} = -\mathcal{F}_J^i$$

- The result for the CM relative acceleration up to order 4.5PN is of the form

$$\boxed{\begin{aligned} a^i_{\mathrm{RR}} = -\frac{8}{5}\frac{G^2m^2\nu}{c^3r^3}\Big[&-\big(A_{2.5\mathrm{PN}} + A_{3.5\mathrm{PN}} + A_{4.5\mathrm{PN}}\big)\dot{r}n^i \\ &+ \big(B_{2.5\mathrm{PN}} + B_{3.5\mathrm{PN}} + B_{4.5\mathrm{PN}}\big)\,v^i\Big] + \mathcal{O}\left(\frac{1}{c^{11}}\right)\end{aligned}}$$

- Remembering that RR is intrinsically gauge-dependent the coefficients depend on 20 arbitrary gauge parameters up to 4.5PN order

$$\begin{cases} \alpha_3, \beta_2 \qquad \Longleftarrow \ 2.5\mathrm{PN} \\ \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \rho_5 \qquad \Longleftarrow \ 3.5\mathrm{PN} \\ \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8, \psi_9, \chi_6, \chi_8, \chi_9 \qquad \Longleftarrow \ 4.5\mathrm{PN} \end{cases}$$

The flux-balance approach to RR

[Iyer & Will 1993, 1995; Gopakumar, Iyer & Iyer 1997]

- The flux-balance method is based on a fundamental ansatz that the 4.5PN CM acceleration is local-in-time
- Yet, we have proved that this is incorrect at 4.5PN order, because the definition of the CM frame must take into account the radiation contribution
- We find that the 4.5PN CM acceleration and energy and angular momentum in [Gopakumar, Iyer & Iyer 1997] must be corrected by

$$\delta a^i_{\text{RR}} = \frac{G\Delta}{r^2c^2}\left(2n^iv^j + n^jv^i\right)\left[\Pi^j + \mathcal{F}^j_G\right] - \frac{\Delta}{mc^2}v^iv^j\left[\mathcal{F}^j_P + \dot{\mathcal{F}}^j_G\right]$$
$$\delta E_{\text{RR}} = \frac{\nu\Delta}{c^2}v^2v^i\left[\Pi^i + \mathcal{F}^i_G\right]$$
$$\delta J^i_{\text{RR}} = \frac{\nu\Delta}{c^2}\varepsilon_{ijk}x^jv^kv^l\left[\Pi^l + \mathcal{F}^l_G\right]$$

- After correction we can uniquely determine the GII parameters which constitues a non-trivial check of the RR potentials and correctness