



Small-Instanton induced Flavor Invariants and the Axion Potential

based on arXiv: [2402.09361](https://arxiv.org/abs/2402.09361)

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The Axion Quest

09 August 2024, Quy Nhon, Vietnam

Preliminary & Outline of this talk

- Axion potential:

How to make small(UV)-instanton becoming relevant?

QCD contribution & UV contributions via Instanton effects

- Small-instantons & Axion potential:

Topological susceptibilities and Flavour invariants

UV completions of small-instanton

Bounds from neutron EDM

Preliminary & Outline of this talk

Instanton #101:

QCD θ -vacuum = Superposition of n-vacua (energy degenerate but topologically distinct)

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta} |n\rangle = \cdots |0\rangle + e^{-i\theta} |1\rangle + \cdots$$


Instanton describes the tunnelling effect between degenerate n-vacua

Localized objects in Euclidean spacetime, satisfying the Euclidean equation of motion with non-trivial topologies and therefore minimize the Euclidean action.

Preliminary & Outline of this talk

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Explicit SU(2) BPST instanton solution with $Q = 1$: $\frac{g^2}{32\pi^2} \int d^4x G_{\mu\nu}^A \tilde{G}^{A,\mu\nu}(x) \Big|_{\text{inst.}} = Q$, where $Q \in \mathbb{Z}$.
(Background field configuration)

$$G_{\mu}^a(x) \Big|_{1\text{-inst.}} = 2 \eta_{a\mu\nu} \frac{(x - x_0)_{\nu}}{(x - x_0)^2 + \rho^2}$$

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Characterized by a set of collective coordinates \Rightarrow zero-modes (family of equivalent solutions)

x_0

Location of instanton

ρ

Size of instanton

Preliminary & Outline of this talk

Instanton #101: Path Integral with Instanton configurations

$$S_{\text{YM}}^{\text{inst.}} = \int d^4x \left(\frac{1}{4} G_{\mu\nu}^A G^{A,\mu\nu} + i\theta_{QCD} \frac{g^2}{32\pi^2} G_{\mu\nu}^A \tilde{G}^{A,\mu\nu} \right) \Big|_{(\text{a.-})\text{inst.}} = \frac{8\pi^2}{g^2} |Q| + i\theta_{QCD} Q$$

=> Within perturbative regime, $Q = \pm 1$ will dominate the Euclidean Path Integral

$$Z = \int \mathcal{D}A e^{-S_{\text{YM}}^{\text{inst.}}}$$

Preliminary & Outline of this talk

- Instanton #101: Path Integral with Instanton configurations

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Estimating instanton effects \Rightarrow vacuum-to-vacuum transition amplitude:

$$\langle 0|0 \rangle \Big|_{1\text{-inst.}} \sim \int \mathcal{D}\varphi_I^{(0)} \int \mathcal{D}\varphi_I^{(\neq 0)} e^{-\left[S_{\text{YM}}^{1\text{-inst.}} + \int d^4x \varphi_I^\dagger \left(\frac{\delta^2 \mathcal{L}}{\delta \varphi_I^2} \right) \varphi_I \right]}$$

Zero modes measure Non-zero modes measure

Preliminary & Outline of this talk

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Integrating out non-zero modes

$$\langle 0|0 \rangle \Big|_{1\text{-inst.}} = e^{-i\theta_{\text{QCD}}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \prod_{f=1}^{N_f} \left(\rho d\xi_f^{(0)} d\bar{\xi}_f^{(0)} \right) e^{\int d^4x (-\bar{\psi} J \psi + \text{h.c.})}$$

Instanton density

Preliminary & Outline of this talk

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$$d_N(\rho) \sim e^{-8\pi^2/g^2 (\Lambda=1/\rho)}$$

Strongly suppressed at high energy scale
(for asymptotic free theory)

Preliminary & Outline of this talk

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When small-instanton effects become relevant?

\Rightarrow Boost the QCD coupling at high energy scale

- Non-trivial embedding of QCD in UV theories: $SU(3)_1 \times SU(3)_2 \times \cdots \times SU(3)_k \rightarrow SU(3)_{\text{QCD}}$
- Extra-dimensions (5D instantons)

Agrawal and Howe (1710.04213)

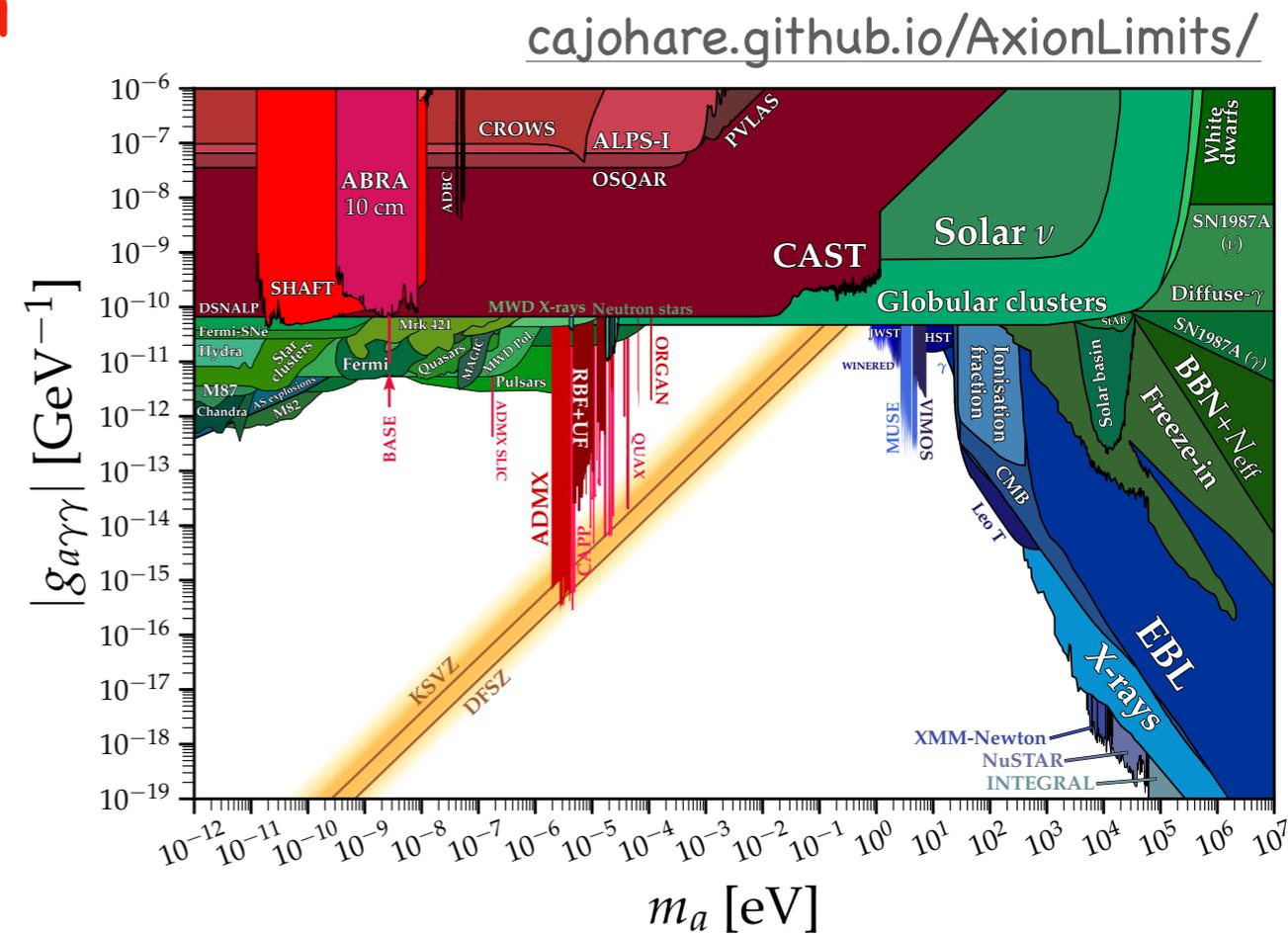
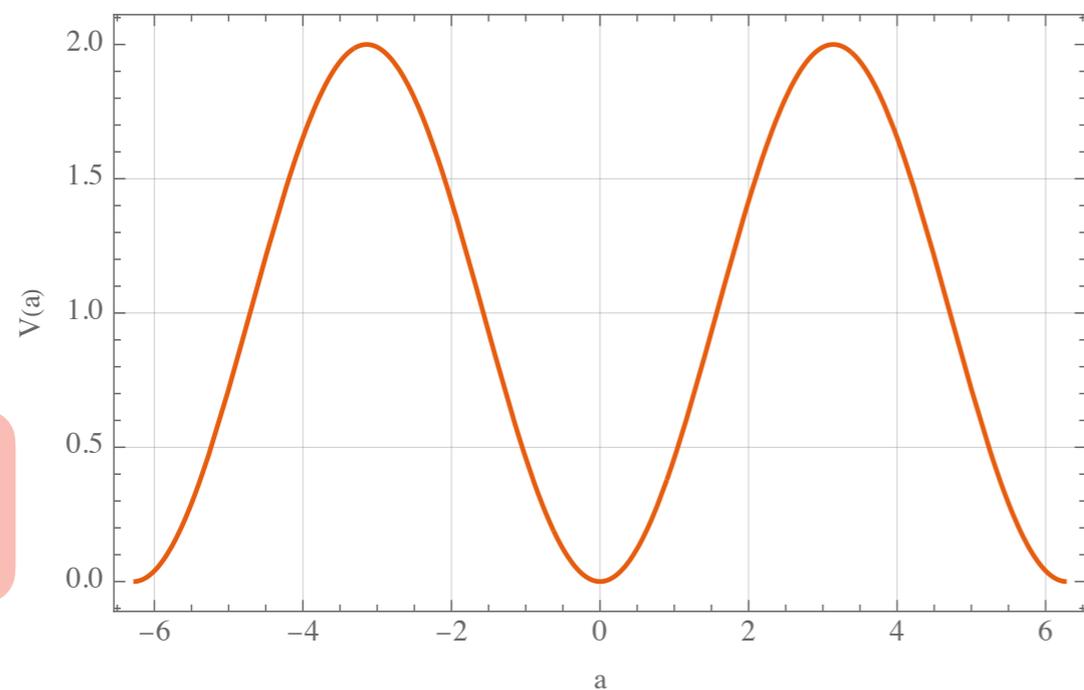
C. Csáki, M. Ruhdorfer, Y. Shirman (1912.02197)

T. Gherghetta, V. V. Khoze, A. Pomarol, Y. Shirman (2001.05610)

Preliminary & Outline of this talk

● Axion potential: QCD contribution

Λ
 Λ_{CPV}
 Λ_{SI}
 Λ_{QCD}



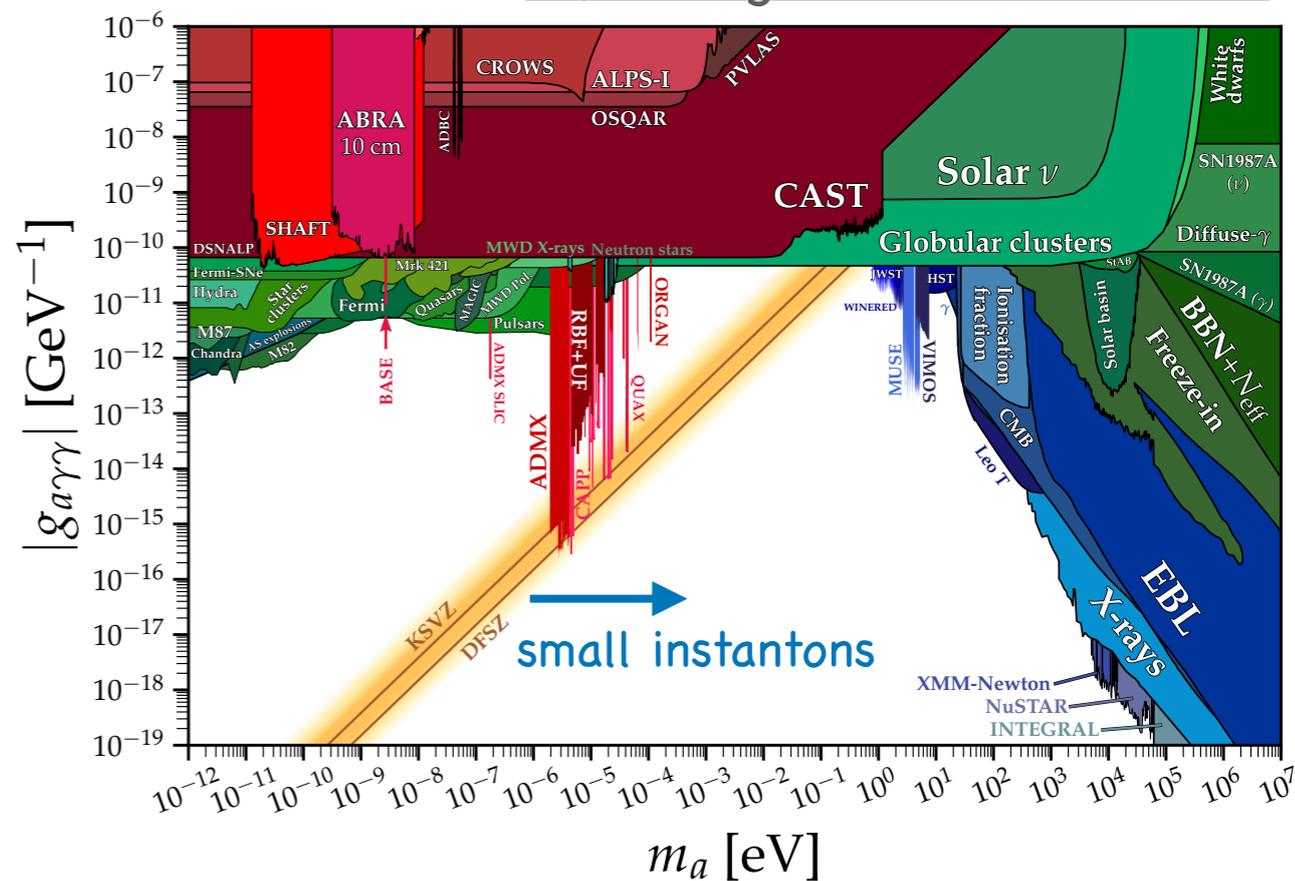
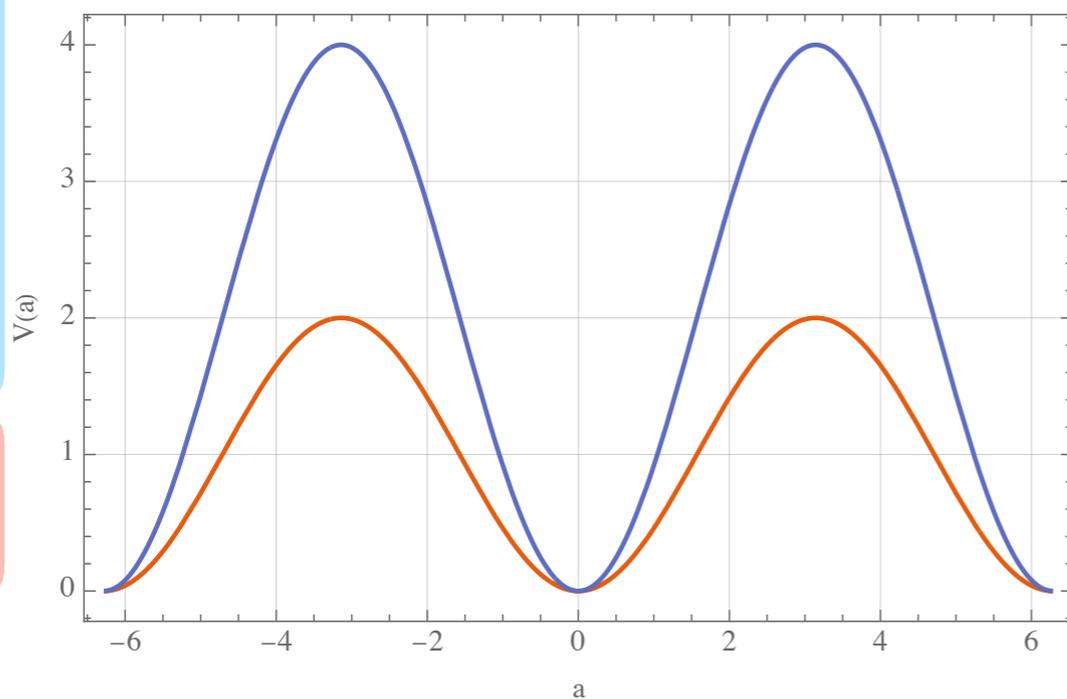
$$V(a) \sim m_\pi^2 f_\pi^2 \left[1 - \cos \frac{a}{f_a} \right] \rightarrow \left\langle \frac{a}{f_a} \right\rangle = 0$$

$$m_a^2 f_a^2 = \frac{m_u m_d}{(m_u + m_d)^2} m_\pi^2 f_\pi^2$$

Preliminary & Outline of this talk

- Axion potential: UV aligned contribution

cajohare.github.io/AxionLimits/

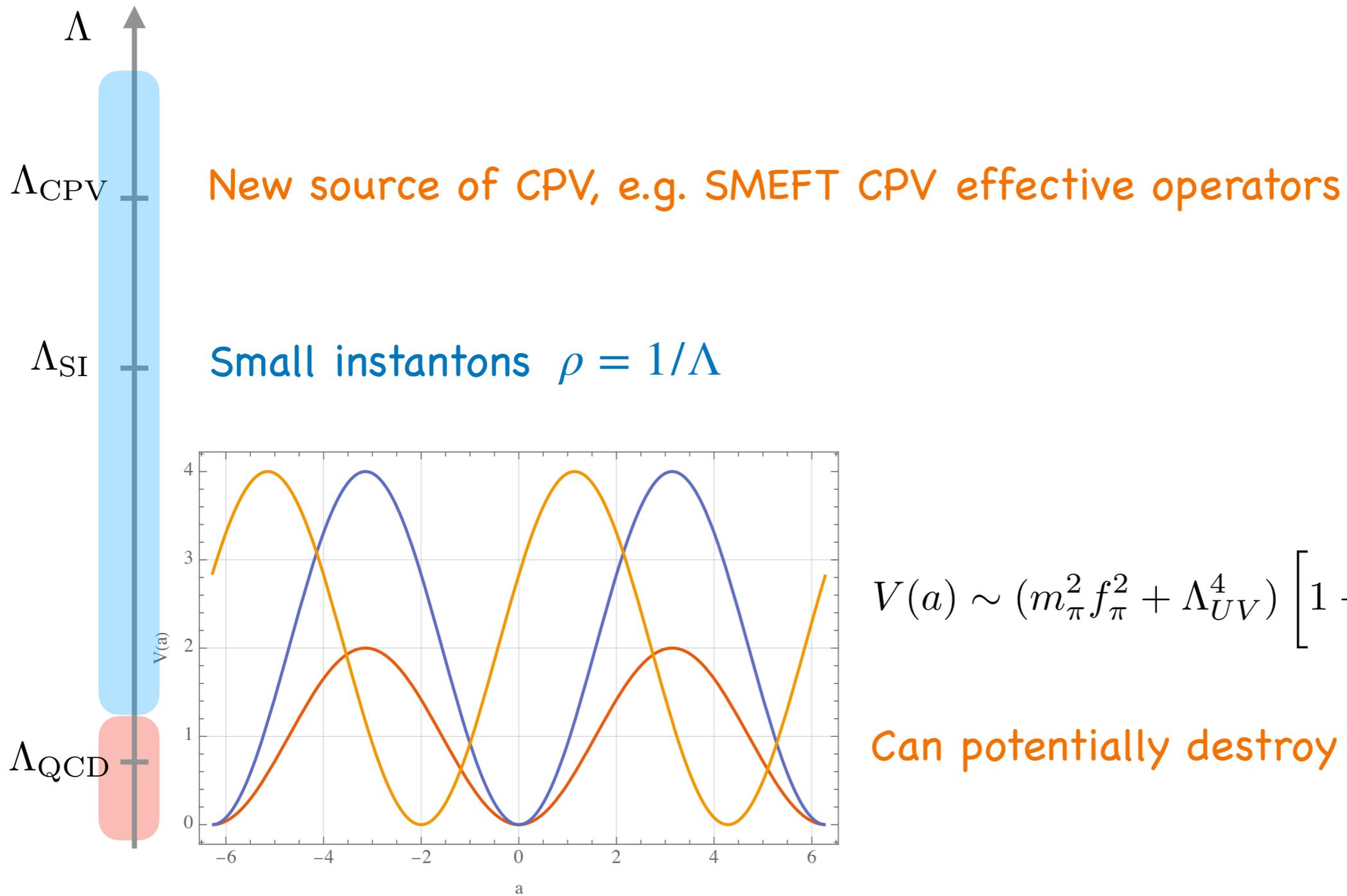


$$V(a) \sim (m_\pi^2 f_\pi^2 + \Lambda_{UV}^4) \left[1 - \cos \frac{a}{f_a} \right]$$

Enhance axion mass & solve strong CP problem

Preliminary & Outline of this talk

- Axion potential: UV misaligned contribution



$$V(a) \sim (m_\pi^2 f_\pi^2 + \Lambda_{UV}^4) \left[1 - \cos \left(\frac{a}{f_a} + \delta_{CPV} \right) \right]$$

Can potentially destroy the Axion solution

Preliminary & Outline of this talk

- CP-violation: The case of Standard Model(SM)

CPV is parametrised by Jarlskog invariant:

$$J_4 = \text{ImTr} \left([Y_u Y_u^\dagger, Y_d Y_d^\dagger]^3 \right)$$

=> CP is conserved iff $J_4 = 0$ (neglecting $\bar{\theta}$)

Jarlskog '85

Bernabeu, Branco, Gronau '86

Preliminary & Outline of this talk

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- CPV in the SM will not misalign the axion potential:
Appear at 4-loop (from threshold corrections) and
7-loop (from radiative corrections) level

$$\bar{\theta}_{\text{ind}}^{(\text{SM})} \sim 10^{-19}$$

Jarlskog '85

Bernabeu, Branco, Gronau '86

Ellis, Gaillard '79

Khriplovich '86

Georgi, Randall '86

Preliminary & Outline of this talk

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● BSM CP-violation: The case of SMEFT

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \sum_i \frac{C_i \mathcal{O}_i^{(6)}}{\Lambda^2}$$

Contain 1149 CP-odd couplings !!!

=> Generalise Jarlskog invariant to study CPV in the SMEFT systematically

Jarlskog '85

Bernabeu, Branco, Gronau '86

Ellis, Gaillard '79

Khriplovich '86

Georgi, Randall '86

Bonnefoy, Gendy, Grojean, Ruderman

2112.03889, 2302.07288

Preliminary & Outline of this talk

● CP-violation: The case of SMEFT

Considering non-perturbative effects => Use θ_{QCD} as a spurion:

| | $U(3)_Q$ | $U(3)_u$ | $U(3)_d$ | $U(3)_L$ | $U(3)_e$ |
|----------------------------|-------------------|-------------------------|-------------------------|-------------------|-------------------------|
| $e^{i\theta_{\text{QCD}}}$ | $\mathbf{1}_{+6}$ | $\mathbf{1}_{-3}$ | $\mathbf{1}_{-3}$ | $\mathbf{1}_0$ | $\mathbf{1}_0$ |
| Y_u | $\mathbf{3}_{+1}$ | $\bar{\mathbf{3}}_{-1}$ | $\mathbf{1}_0$ | $\mathbf{1}_0$ | $\mathbf{1}_0$ |
| Y_d | $\mathbf{3}_{+1}$ | $\mathbf{1}_0$ | $\bar{\mathbf{3}}_{-1}$ | $\mathbf{1}_0$ | $\mathbf{1}_0$ |
| Y_e | $\mathbf{1}_0$ | $\mathbf{1}_0$ | $\mathbf{1}_0$ | $\mathbf{3}_{+1}$ | $\bar{\mathbf{3}}_{-1}$ |

SM has one more CP-odd flavour invariant:

$$J_\theta = \text{Im}[e^{-i\theta_{\text{QCD}}} \det(Y_u Y_d)]$$

Built flavour invariants featuring θ_{QCD} for CP-violating SMEFT operators:

$$\mathcal{O}_{quqd}^{(1)} = \bar{Q}u\bar{Q}d$$

$$\mathcal{I}(C_{quqd}^{(1,8)}) = \text{Im} \left[e^{-i\theta_{\text{QCD}}} \epsilon^{ABC} \epsilon^{abc} \epsilon^{DEF} \epsilon^{def} Y_{u,Aa} Y_{u,Bb} C_{quqd,CcDd}^{(1,8)} Y_{d,Ee} Y_{d,Ff} \right]$$

Note: $\bar{Q}u\bar{Q}d$ has 81 CP-odd phases

Bonnefoy, Gendy, Grojean, Ruderman

Preliminary & Outline of this talk

- Small instanton & Axion potential: UV misaligned contributions

Small instantons generate axion potential of the form:

$$V(a) = \chi_{\mathcal{O}}(0) \frac{a}{f_a} + \frac{1}{2} \chi(0) \left(\frac{a}{f_a} \right)^2 \longrightarrow \left\langle \frac{a}{f_a} \right\rangle \equiv \theta_{\text{ind}} = -\frac{\chi_{\mathcal{O}}(0)}{\chi(0)}$$

Induced by CP-violating operator This talk

Preliminary & Outline of this talk

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Induced by CP-violating operator
This talk

Coefficients in the potential can be computed from following correlators: Witten '79

$$\chi(0) = -i \lim_{k \rightarrow 0} \int d^4x e^{ikx} \left\langle 0 \left| T \left\{ \frac{g^2}{32\pi^2} G\tilde{G}(x), \frac{g^2}{32\pi^2} G\tilde{G}(0) \right\} \right| 0 \right\rangle \Big|_{1-(a.-)inst.}$$

$$\chi_{\mathcal{O}}(0) = -i \lim_{k \rightarrow 0} \int d^4x e^{ikx} \left\langle 0 \left| T \left\{ \frac{g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}_a^{\mu\nu}(x), \frac{C_{\mathcal{O}}^{ij\dots}}{\Lambda_{\text{CP}}^{D-4}} \mathcal{O}^{D,ij\dots}(0) \right\} \right| 0 \right\rangle \Big|_{1-(a.-)inst.}$$

Evaluating these correlation functions within perturbative regime and one-(anti)instanton approximation. Making connection with SMEFT flavour invariants => Simplify the calculations

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

● Core technique 1: Path Integral & Instanton background

$$\begin{aligned}\chi_{\mathcal{O}}(0) &= -i \lim_{k \rightarrow 0} \int d^4x e^{ikx} \left\langle 0 \left| T \left\{ \frac{g^2}{32\pi^2} G\tilde{G}(x), \frac{C_{\mathcal{O}}}{\Lambda_{\mathcal{CP}}^2} \mathcal{O}[\varphi_I, \varphi](0) \right\} \right| 0 \right\rangle, \\ &= e^{-i\theta_{\text{QCD}}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \prod_{f=1}^{N_f} (\rho d\xi_f^{(0)} d\bar{\xi}_f^{(0)}) \\ &\quad \times \int \mathcal{D}\varphi e^{-S_0[\varphi] - S_{\text{int}}[\varphi_I, \varphi]} \int d^4x \frac{g^2}{32\pi^2} G\tilde{G}(x) \times \frac{C_{\mathcal{O}}}{\Lambda_{\mathcal{CP}}^2} \mathcal{O}[\varphi_I, \varphi](0) \Big|_{1-(\text{a.-})\text{inst.}}\end{aligned}$$

Fields with instanton solutions (e.g. gluon, quark): φ_I

=> Expand the fields in their eigenmodes, replace zero mode wave function by instanton solutions, **and integrate out non-zero modes:**

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

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Fields without instanton solutions: φ

=> Integrate over without performing the eigenmode expansion

Small instanton & Axion potential: Evaluating the correlator $\chi_{\theta}(0)$

● Core technique 2: Fermion zero mode & Grassmann integral

Fermion eigenmode expansion & fermion zero-mode solutions:

$$\psi_f(x) = \sum_k \xi_f^{(k)} \psi^{(k)}(x); \quad \bar{\psi}_f(x) = \sum_k \bar{\xi}_f^{(k)} \bar{\psi}^{(k)}(x)$$

$$-i\not{D}\Big|_{1\text{-inst.}} \psi^{(0)}(x) = 0. \quad \longrightarrow \quad \psi^{(0)}(x)\Big|_{1\text{-inst.}} = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \frac{1}{\pi} \frac{\rho}{[(x-x_0)^2 + \rho^2]^{3/2}} \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \quad \varphi_{\alpha m} = \epsilon^{\alpha m}$$

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

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Fermion zero modes & Grassmann integral give rise to determinant-like structures:

● The well-known Grassmann integration identity:

$$\int d^3\xi_1 d^3\xi_2 e^{\xi_1 A \xi_2} = \det A$$

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

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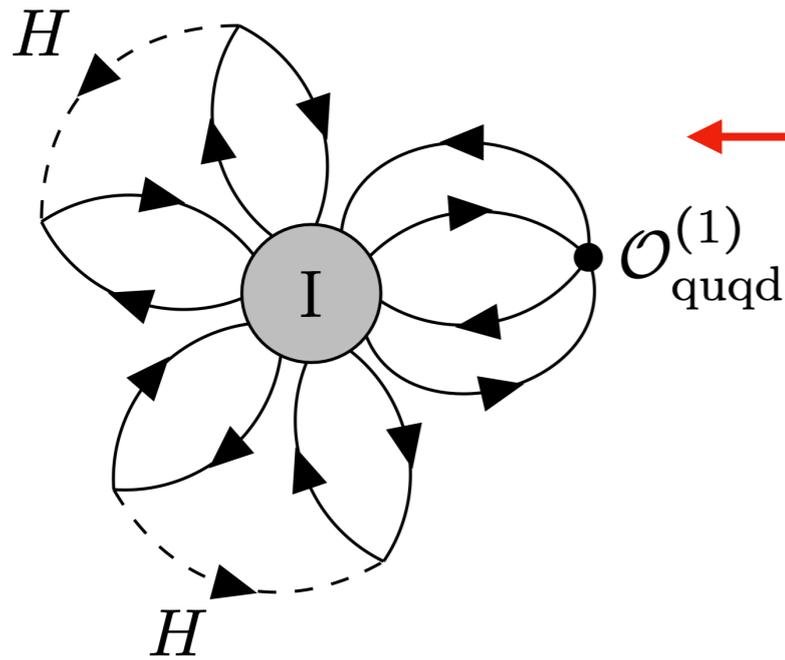
- Generalise the Grassmann integration identity for operator insertion:

Example:
$$\int d^3\xi_1 d^3\xi_2 e^{\xi_1 A \xi_2} \xi_1 B \xi_2 = \frac{1}{2} \epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} A_{i_1 j_1} A_{i_2 j_2} B_{i_3 j_3}$$

$$\int d^3\xi_1 d^3\xi_2 d^3\xi_3 d^3\xi_4 e^{\xi_1 A \xi_2 + \xi_3 B \xi_4} \xi_1 C \xi_2 = \frac{1}{2} \epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} A_{i_1 j_1} A_{i_2 j_2} C_{i_3 j_3} \det B$$

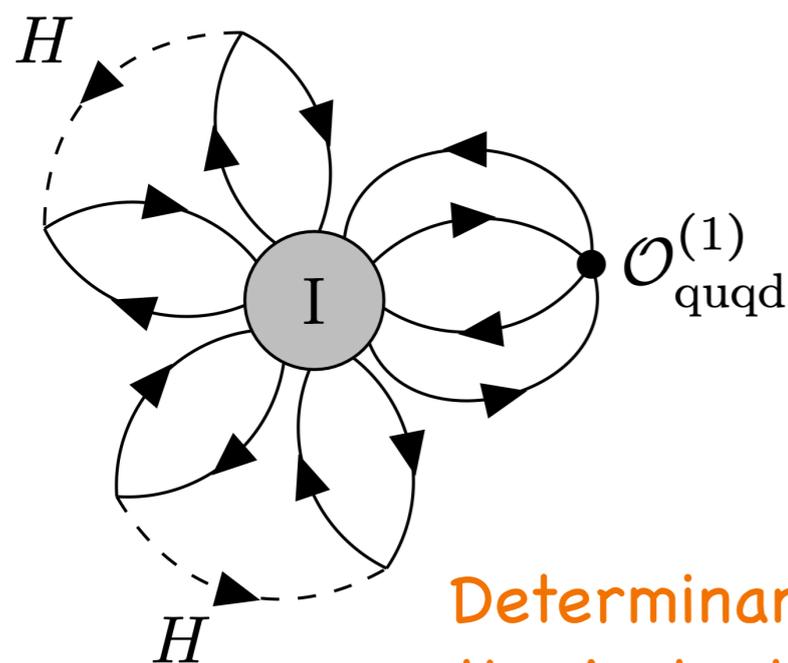
Determinant-like contraction

Topological Susceptibilities & Flavor invariants: Four-quark operator



$$\leftarrow \chi_{\text{quqd}}^{(1)}(0)^{1\text{-inst.}} = -i \lim_{k \rightarrow 0} \int d^4x e^{ikx} \left\langle 0 \left| T \left\{ \frac{1}{32\pi^2} G\tilde{G}(x), \frac{C_{\text{quqd}}^{(1)}}{\Lambda_{\text{CP}}^2} \mathcal{O}_{\text{quqd}}^{(1)}(0) \right\} \right| 0 \right\rangle$$

Topological Susceptibilities & Flavor invariants: Four-quark operator

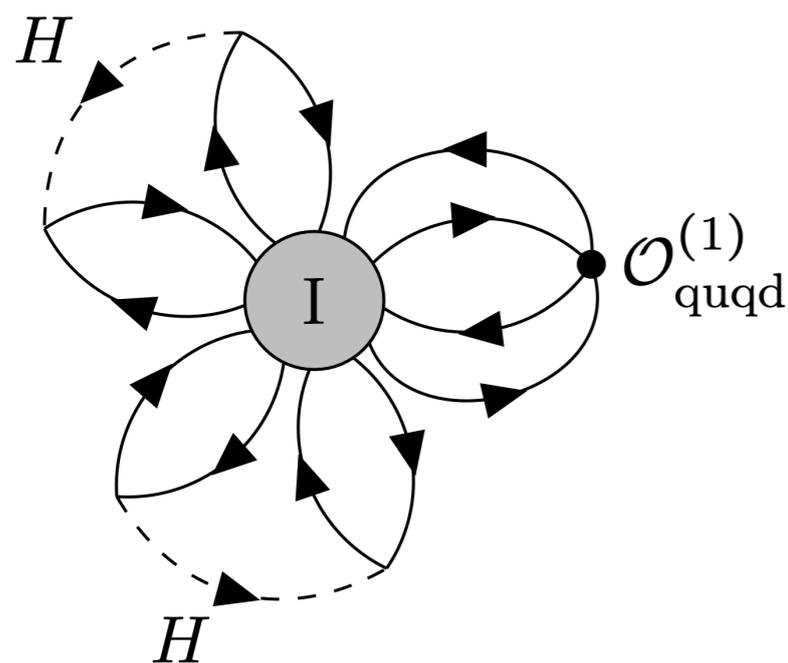


Determinant-like flavour invariants naturally arise in the instanton calculations

$$\begin{aligned} \chi_{\text{quqd}}^{(1)}(0)^{1-\text{inst.}} &= \frac{1}{4\Lambda_{\text{CP}}^2} \left[e^{-i\theta_{\text{QCD}}} \epsilon^{i_1 i_2 m} \epsilon^{j_1 j_2 n} Y_{u, i_1 j_1} Y_{u, i_2 j_2} C_{\text{quqd}, mnop}^{(1)} \epsilon^{k_1 k_2 o} \epsilon^{l_1 l_2 p} Y_{d, k_1 l_1} Y_{d, k_2 l_2} \right. \\ &\quad \left. + e^{-i\theta_{\text{QCD}}} \epsilon^{i_1 i_2 m} \epsilon^{j_1 j_2 n} Y_{u, i_1 j_1} Y_{u, i_2 j_2} C_{\text{quqd}, onmp}^{(1)} \epsilon^{k_1 k_2 o} \epsilon^{l_1 l_2 p} Y_{d, k_1 l_1} Y_{d, k_2 l_2} \right] \int d^4 x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \rho^6 \\ &\times \underbrace{\int \mathcal{D}H \mathcal{D}H^\dagger e^{-S_0[H, H^\dagger]} \left[\int d^4 x_1 d^4 x_2 (\bar{\psi}^{(0)} H_I^\dagger \epsilon^{IJ} P_R \psi^{(0)})(x_1) (\bar{\psi}^{(0)} \epsilon_{JK} H^K P_R \psi^{(0)})(x_2) \right]^2}_{= 2! \left[\int d^4 x_1 d^4 x_2 (\bar{\psi}^{(0)} P_R \psi^{(0)})(x_1) \Delta_H(x_1 - x_2) \epsilon_{IJ} \epsilon^{JI} (\bar{\psi}^{(0)} P_R \psi^{(0)})(x_2) \right]^2 \equiv 2! \mathcal{I}^2} \\ &\times \left(\epsilon_{MNP} \epsilon^{MNP} \bar{\psi}^{(0)} P_R \psi^{(0)} \bar{\psi}^{(0)} P_R \psi^{(0)} \right) (0) \int d^4 x \frac{G \tilde{G}(x)}{32\pi^2}. \end{aligned}$$

Fermion zero modes

Topological Susceptibilities & Flavor invariants: Four-quark operator

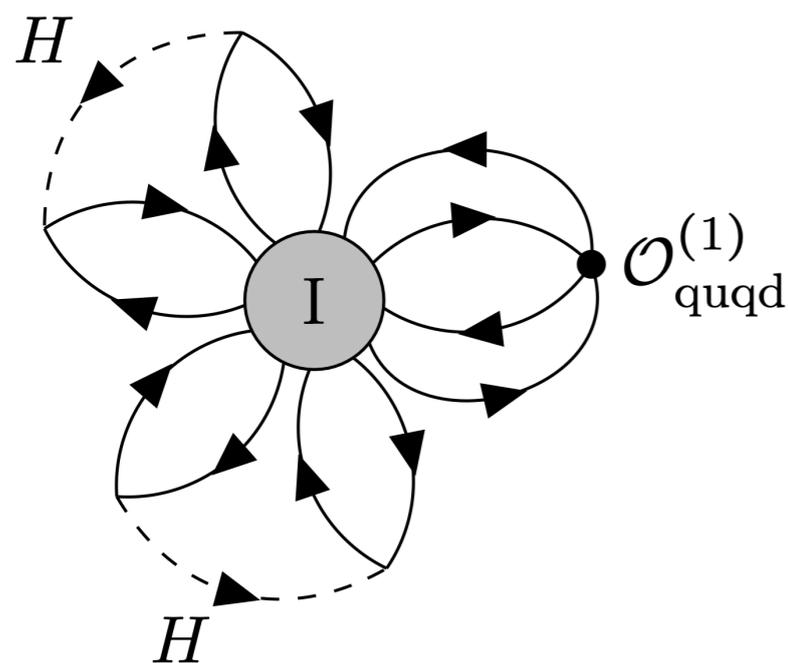


Plugging explicit form of fermion zero modes
Integrate over loop momenta,
collective coordinates

$$\chi_{\text{quqd}}^{(1)(\text{UV})}(0) = \frac{i}{\Lambda_{\text{CP}}^2} \left(\mathcal{A}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) + \mathcal{B}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) \right) \int \frac{d\rho}{\rho^5} d_N(\rho) \frac{2!}{(6\pi^2)^2} \frac{2}{5\pi^2 \rho^2}$$

Contraction of Yukawa matrices
encapsulated in the Flavour invariants

Topological Susceptibilities & Flavor invariants: Four-quark operator



Can also use Instanton Naive Dimensional Analysis (NDA), result up to $\mathcal{O}(1)$

Csáki, D'Agnolo, Kuflik, Ruhdorfer (2311.09285)

$$\chi_{\text{quqd}}^{(1)(UV)}(0) = \frac{i}{\Lambda_{\mathcal{CP}}^2} \left(\mathcal{A}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) + \mathcal{B}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) \right) \int \frac{d\rho}{\rho^5} d_N(\rho) \frac{1}{(256\pi^6)\rho^2}$$

Contraction of Yukawa matrices
encapsulated in the Flavour invariants

Combining Flavour invariants & Instanton NDA
=> quickly estimate 't Hooft flower diagrams

Topological Susceptibilities & Flavor invariants: Semi-leptonic operator

$$\mathcal{O}_{\text{lequ}}^{(1)} = \bar{L}e\bar{Q}u + \text{h.c.}$$

● Now start from the Flavour invariants:

$$\text{Im} \left(I_{\text{lequ}}^{(1)} \right) = \text{Im} \left[e^{-i\theta_{\text{QCD}}} \epsilon^{i_1 i_2 m} \epsilon^{j_1 j_2 n} Y_{u, i_1 j_1} Y_{u, i_2 j_2} C_{\text{lequ}, opmn}^{(1)} Y_{e, po}^\dagger \det Y_d \right] = \mathcal{I}_{0000}^0 \left(C_{\text{lequ}}^{(1)} \right)$$

Trace-like contraction

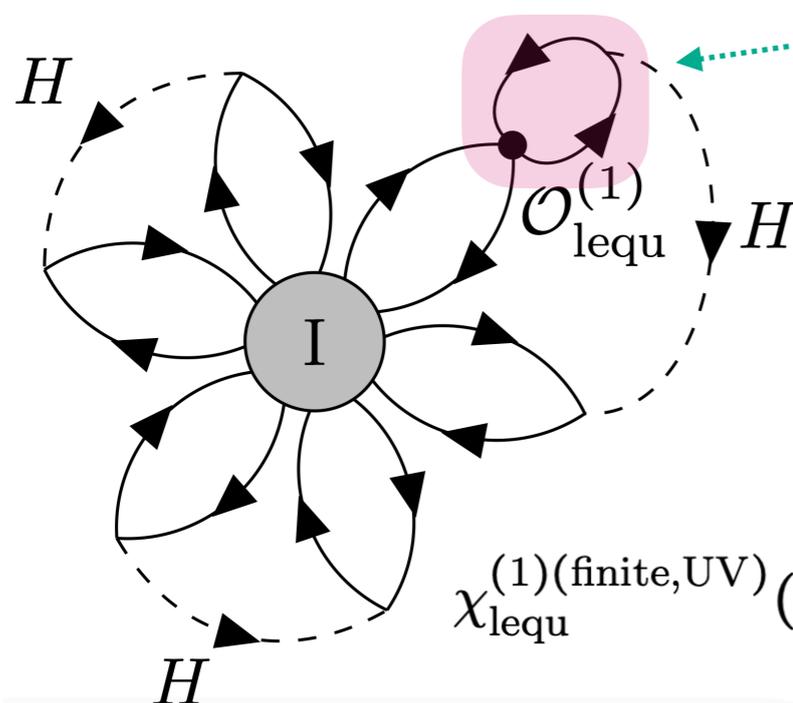
Topological Susceptibilities & Flavor invariants: Semi-leptonic operator

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Trace-like contraction

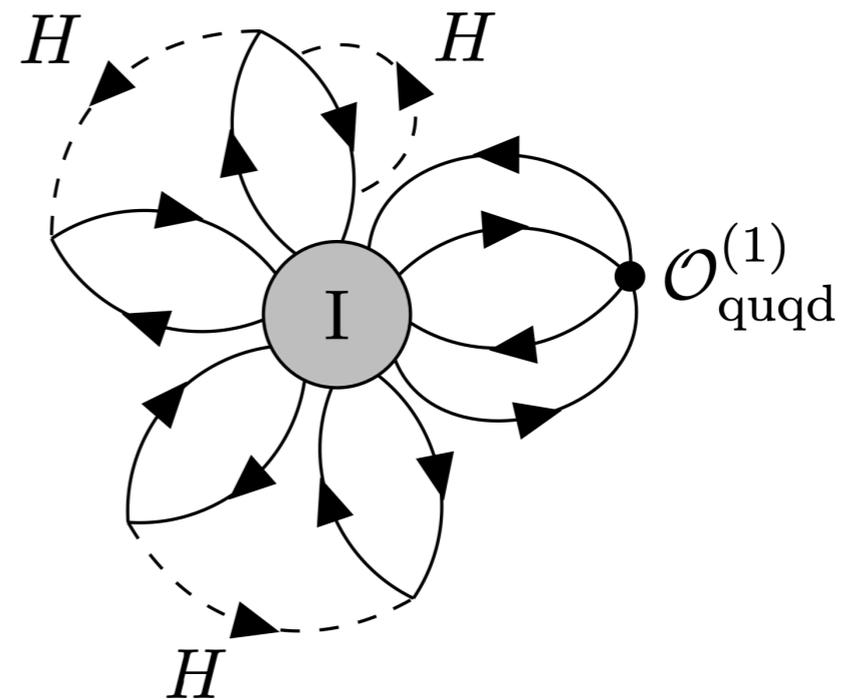
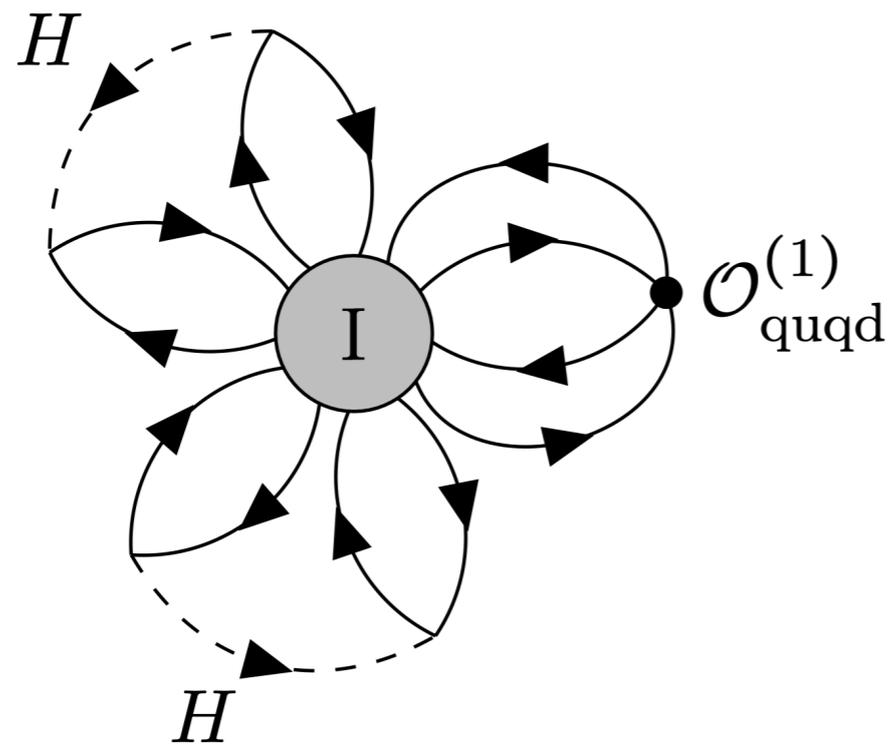


1-loop suppression induced by leptonic fields

$$\chi_{\text{lequ}}^{(1)(\text{finite,UV})}(0) = \frac{i}{\Lambda_{\text{CP}}^2} \mathcal{I}_{0000}^0 \left(C_{\text{lequ}}^{(1)} \right) \int \frac{d\rho}{\rho^5} d_N(\rho) \frac{3!}{(6\pi^2)^2} \frac{11 + 30 (\log(\rho\Lambda_{\text{CP}}) + \gamma_E - \log 2)}{600\pi^4 \rho^2}$$

- Anticipating how CPV SMEFT operators participate in the instanton computations
- Classifying the leading effects from the Wilson coefficients

Topological Susceptibilities & Flavor invariants: Higher-order Invariants



$$\mathcal{A}_{a_1, b_1, c_1, d_1}^{a_2, b_2, c_2, d_2} (C_{\text{quqd}}^{(1,8)}) = \text{Im} \left[e^{-i\theta_{\text{QCD}}} \epsilon^{ABC} \epsilon^{abc} \epsilon^{DEF} \epsilon^{def} Y_{u, Aa} Y_{u, Bb} \left(X_u^{a_1} X_d^{b_1} X_u^{c_1} X_d^{d_1} \right)_C^{C'} \right. \\ \left. \times C_{\text{quqd}, C'cD'd}^{(1,8)} \left(X_u^{a_2} X_d^{b_2} X_u^{c_2} X_d^{d_2} \right)_D^{D'} Y_{d, Ee} Y_{d, Ff} \right],$$

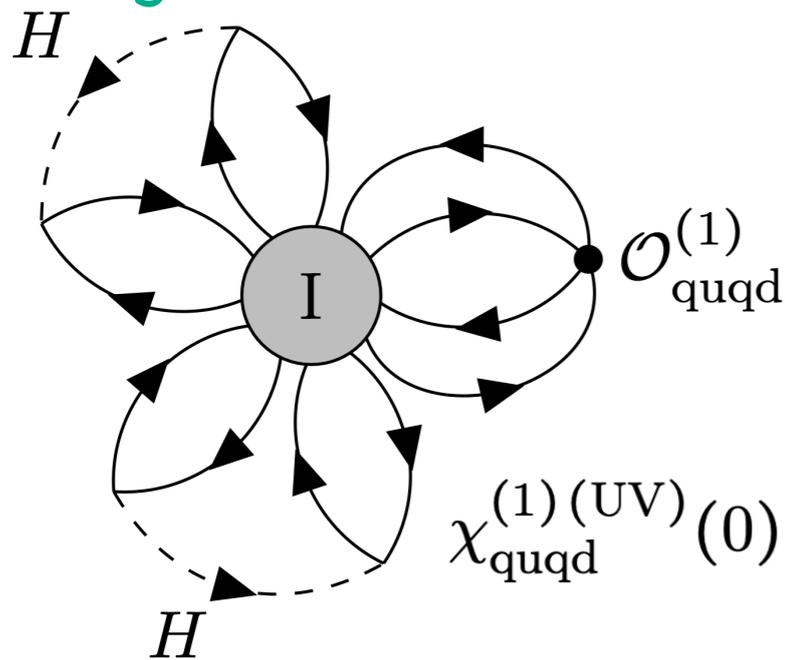
$$X_{u,d} = Y_{u,d} Y_{u,d}^\dagger$$

=> Set $X=1$ for the lowest order flavour invariants

Topological Susceptibilities & Flavor invariants: Four-quark operator

#Integration over the size of instanton

$$\mathcal{O}_{\text{quqd}}^{(1)} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



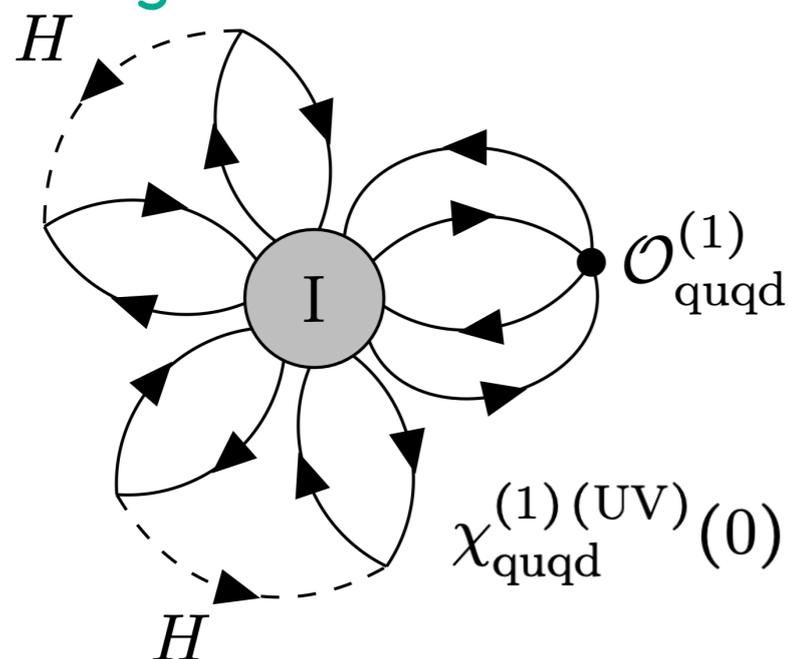
$$\chi_{\text{quqd}}^{(1)(\text{UV})}(0) = \frac{i}{\Lambda_{\text{QCD}}^2} \left(\mathcal{A}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) + \mathcal{B}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) \right) \int \frac{d\rho}{\rho^5} d_N(\rho) \frac{2!}{(6\pi^2)^2} \frac{2}{5\pi^2 \rho^2}$$

ρ -integral is IR divergent
 \Rightarrow Need a physical IR cut-off

Topological Susceptibilities & Flavor invariants: Four-quark operator

#Integration over the size of instanton

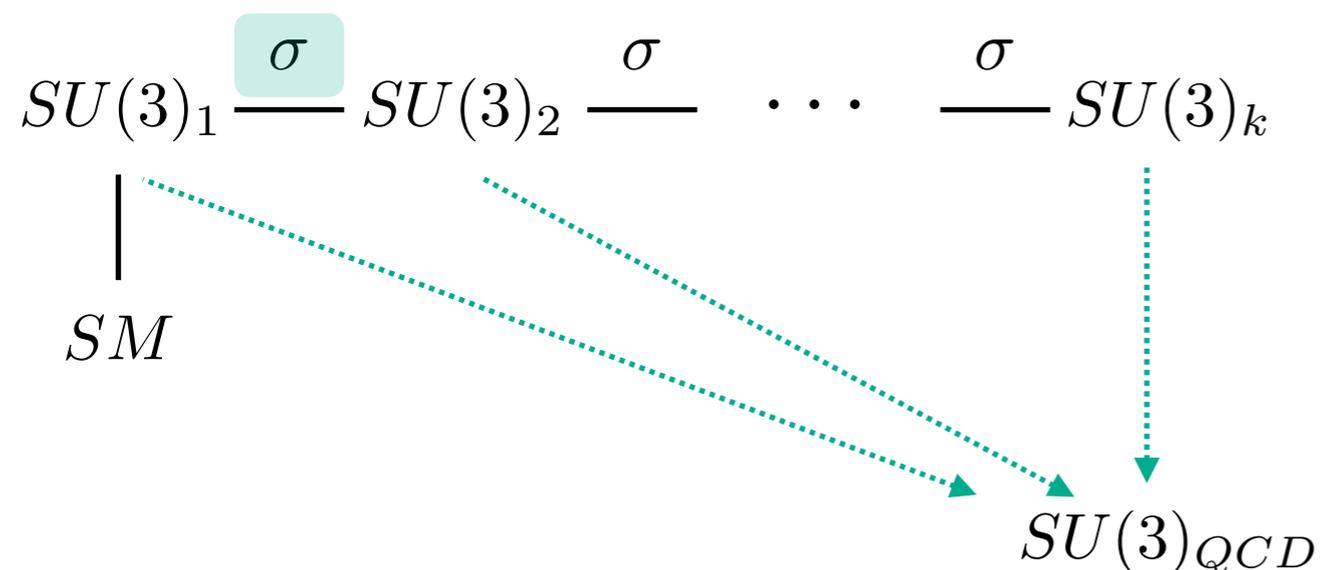
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- Possible UV completion of small-instantons:

Product of Gauge groups



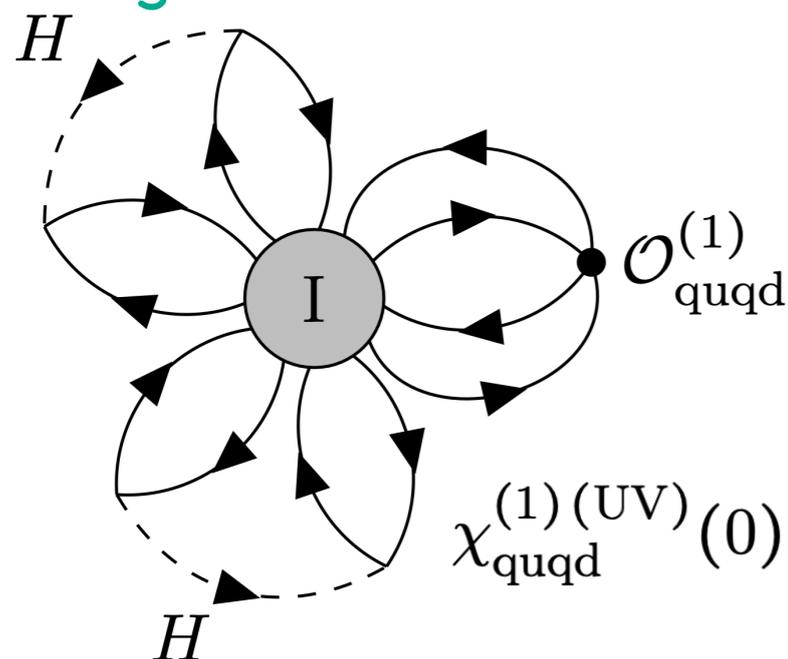
Agrawal and Howe (1710.04213)

C. Csáki, M. Ruhdorfer, Y. Shirman (1912.02197)

Topological Susceptibilities & Flavor invariants: Four-quark operator

#Integration over the size of instanton

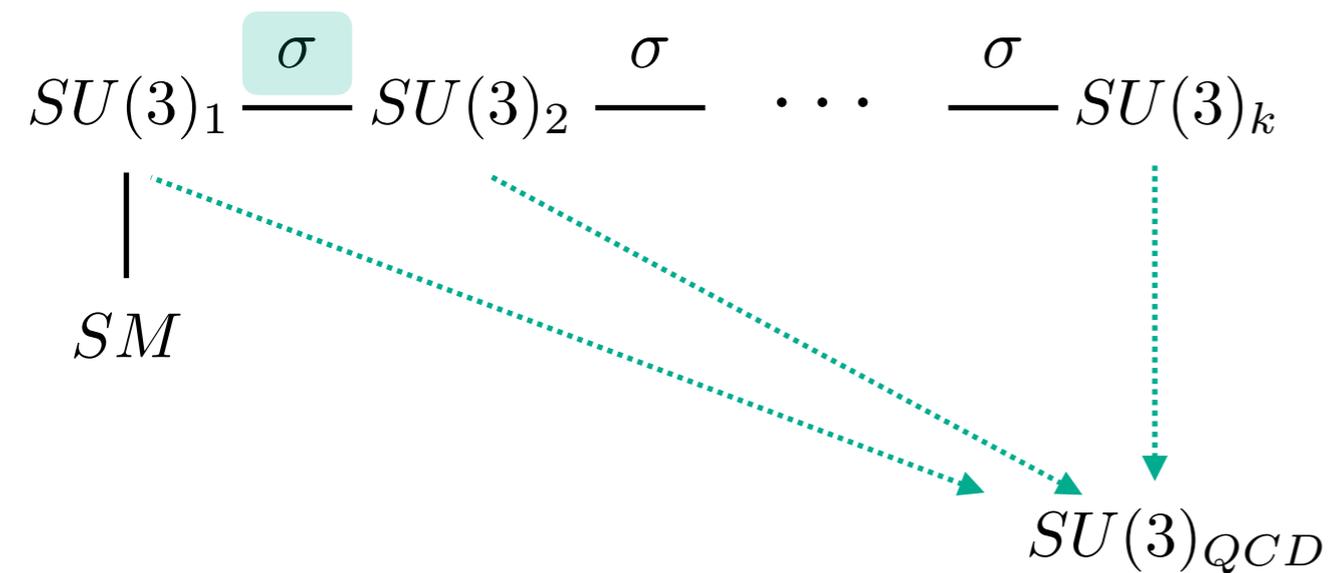
$$\mathcal{O}_{\text{quqd}}^{(1)} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



$$\chi_{\text{quqd}}^{(1)(\text{UV})}(0) = \frac{i}{\Lambda_{\text{QCD}}^2} \left(\mathcal{A}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) + \mathcal{B}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) \right) \int \frac{d\rho}{\rho^5} d_N(\rho) \frac{2!}{(6\pi^2)^2} \frac{2}{5\pi^2 \rho^2}$$

- Possible UV completion of small-instantons:

Product of Gauge groups



Boost the coupling of each QCD subgroup:

$$\frac{1}{g_{\text{QCD}}^2(\mu)} = \frac{1}{g_1^2(\mu)} + \frac{1}{g_2^2(\mu)} + \dots + \frac{1}{g_k^2(\mu)}$$

Provide a physical cut-off:

$$d_N(\rho) \rightarrow d_N(\rho) e^{-2\pi^2 \rho^2 \sum |\langle \sigma \rangle|^2}$$

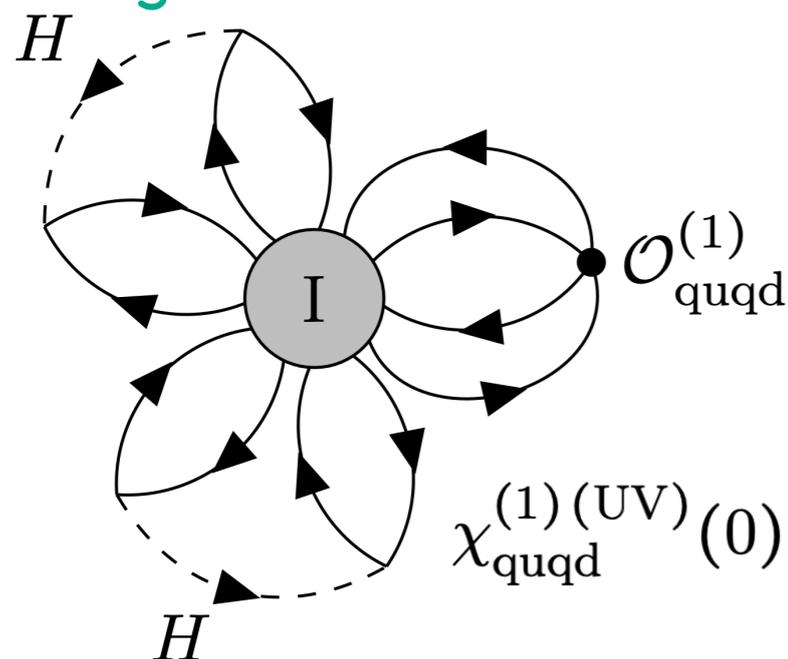
Agrawal and Howe (1710.04213)

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Topological Susceptibilities & Flavor invariants: Four-quark operator

#Integration over the size of instanton

$$\mathcal{O}_{\text{quqd}}^{(1)} = \bar{Q}u\bar{Q}d + \text{h.c.}$$



$$\chi_{\text{quqd}}^{(1)(\text{UV})}(0) = \frac{i}{\Lambda_{\mathcal{CP}}^2} \left(\mathcal{A}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) + \mathcal{B}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) \right) \int \frac{d\rho}{\rho^5} d_N(\rho) \frac{2!}{(6\pi^2)^2} \frac{2}{5\pi^2 \rho^2}$$

- Possible UV completion of small-instantons:

5D instantons

Uplift BPST instanton to a compact extra dimension of size R

$$d_N(\rho) \rightarrow d_N(\rho) e^{R/\rho}$$

Bounds from non-measurement of theta-induced: Four-quark operator

- Example: Product of gauge groups

$$\mathcal{O}_{\text{quqd}}^{(1)} = \bar{Q}u\bar{Q}d + \text{h.c.}$$

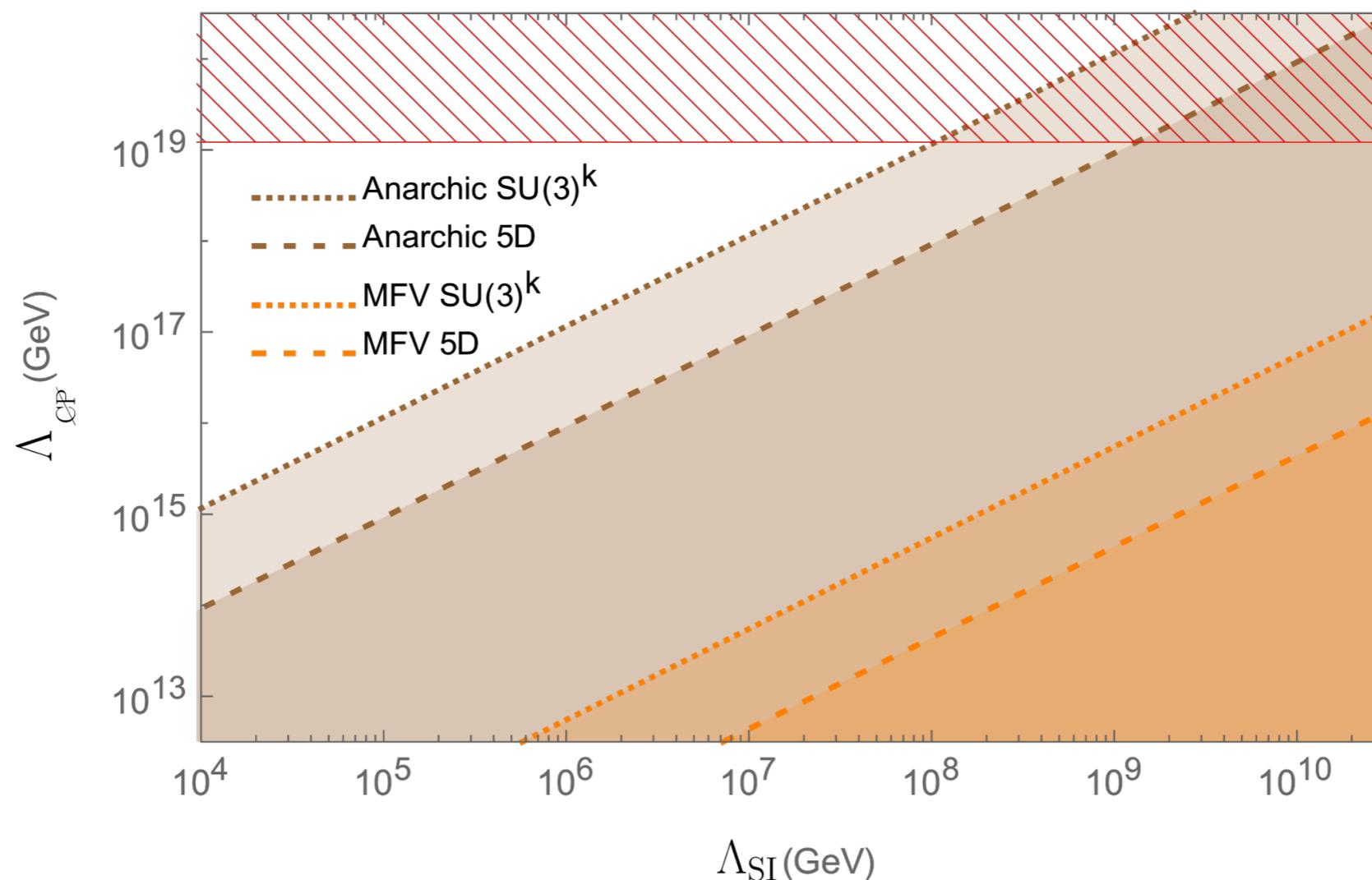
$$\theta_{\text{ind}} = \frac{16\pi^2}{5(b_0 - 6)K_\theta} \left(\mathcal{A}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) + \mathcal{B}_{0000}^{0000} \left(C_{\text{quqd}}^{(1)} \right) \right) \frac{\Lambda_{\text{SI}}^2}{\Lambda_{\text{CP}}^2}$$

Test different flavour scenarios

Finite ratio in the decoupling limit

$$K_\theta = \text{Re} \left[e^{-i\theta_{\text{QCD}}} \det(Y_u Y_d) \right]$$

$$\mathcal{O}_{\text{quqd}}^{(1)}$$



Conclusions

- Enhancing the axion mass via small-instanton also (accidentally) enhances CPV effects that misalign potential
 - => Dangerous effects which spoil Axion solution
 - => The quality of Axion solution depends on UV scenarios
- The estimation of these effects can be made easier with the help of Determinant-like Flavour Invariants and Instanton Naive Dimensional Analysis
 - => Allow to study the contributions of any SMEFT operators to the axion potential

Bonus slides

Preliminary & Outline of this talk

- Small instanton & Axion potential: UV misaligned contributions

Small instantons generate axion potential of the form:

$$V(a) = \chi_{\mathcal{O}}(0) \frac{a}{f_a} + \frac{1}{2} \chi(0) \left(\frac{a}{f_a} \right)^2 \longrightarrow \left\langle \frac{a}{f_a} \right\rangle \equiv \theta_{\text{ind}} = -\frac{\chi_{\mathcal{O}}(0)}{\chi(0)}$$

Induced by CP-violating operator
This talk

Coefficients in the potential can be computed from following correlators: Witten '79

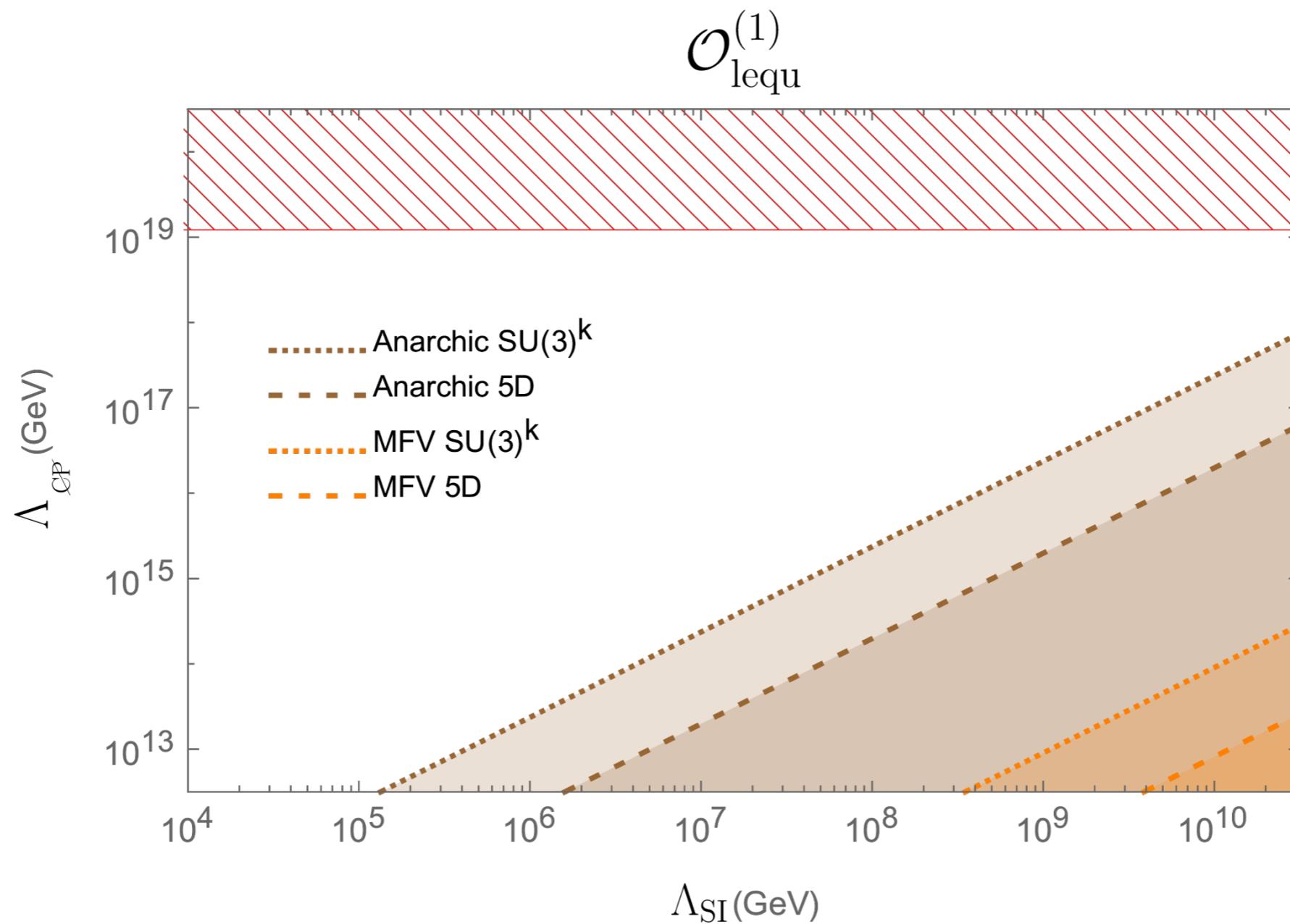
$$\chi(0) = -i \lim_{k \rightarrow 0} \int d^4x e^{ikx} \left\langle 0 \left| T \left\{ \frac{g^2}{32\pi^2} G\tilde{G}(x), \frac{g^2}{32\pi^2} G\tilde{G}(0) \right\} \right| 0 \right\rangle$$

$$\chi_{\mathcal{O}}(0) = -i \lim_{k \rightarrow 0} \int d^4x e^{ikx} \left\langle 0 \left| T \left\{ \frac{g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}_a^{\mu\nu}(x), \frac{C_{\mathcal{O}}^{ij\dots}}{\Lambda_{\text{CP}}^{D-4}} \mathcal{O}^{D,ij\dots}(0) \right\} \right| 0 \right\rangle$$

Note: In strongly coupled regime, one should use non-perturbative methods
 => Current algebra, QCD light cone sum rules

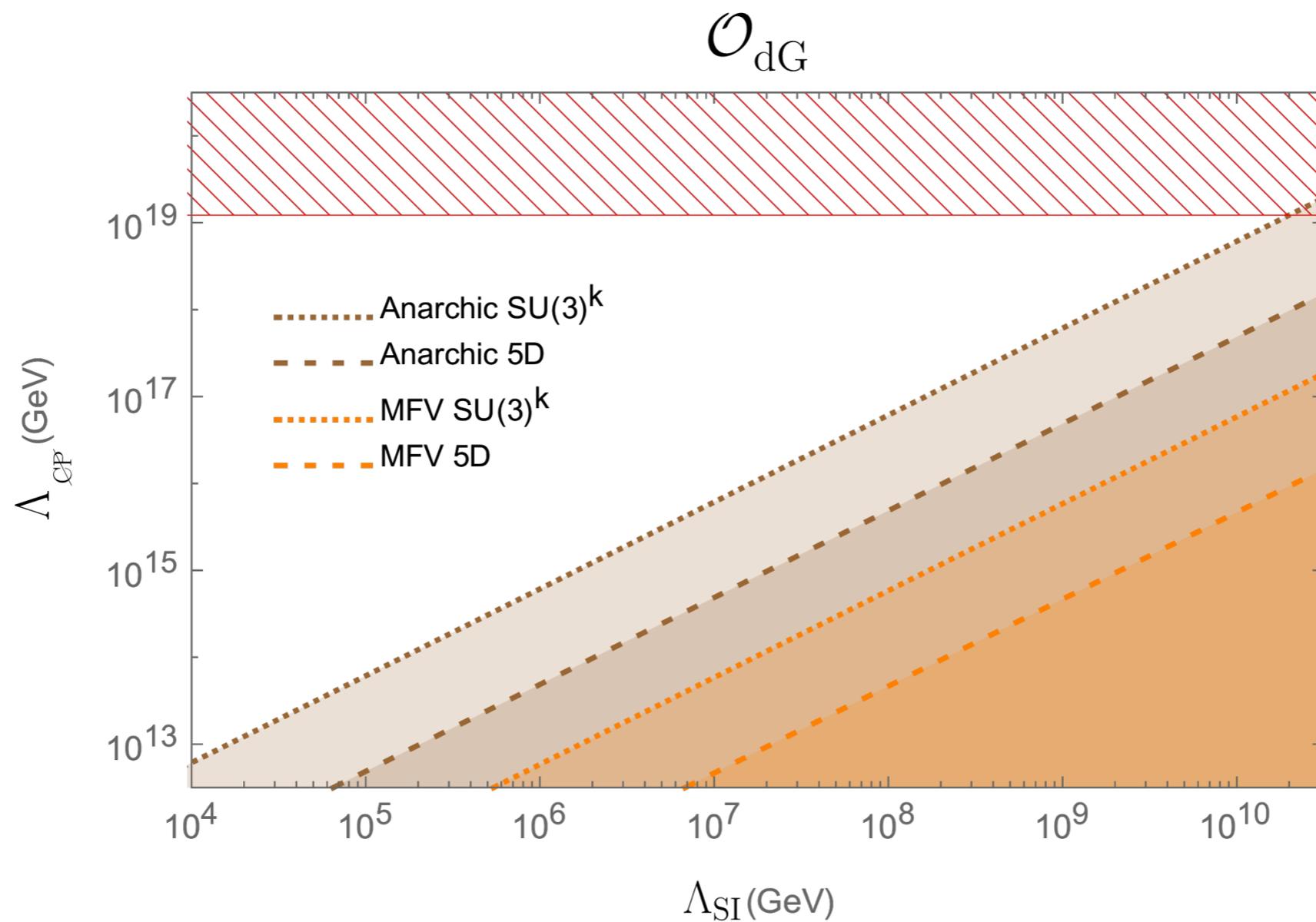
Bounds from non-measurement of theta-induced: Semi-leptonic operator

$$\mathcal{O}_{\text{lequ}}^{(1)} = \bar{L}e\bar{Q}u + \text{h.c.}$$

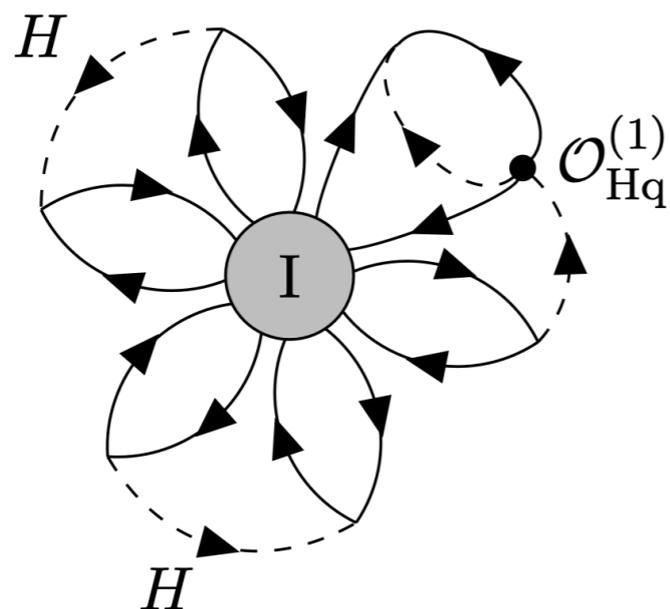


Bounds from non-measurement of theta-induced: Gluon dipole operator

$$\mathcal{O}_{uG} = (\bar{Q}\sigma^{\mu\nu}T^A u)\tilde{H}G_{\mu\nu}^A + \text{h.c.}$$



Topological Susceptibilities & Flavor invariants: Higher-order Invariants



(c) Instanton diagram with an insertion of a non-chirality-flipping effective operator $\mathcal{O}_{\text{Hq}}^{(1)}$.

$$\mathcal{I}_{abcd}(C_{\text{Hq}}^{(1,3)}) \equiv \text{Im} \left[e^{-i\theta_{\text{QCD}}} \epsilon^{IJK} \epsilon^{ijk} Y_{u,Ii} Y_{u,Jj} \left(X_u^a X_d^b X_u^c X_d^d C_{\text{Hq}}^{(1,3)} Y_u \right)_{Kk} \det Y_d \right]$$

$$X_{u,d} = Y_{u,d} Y_{u,d}^\dagger$$

=> Set $X=1$ for the lowest order flavour invariants

Preliminary

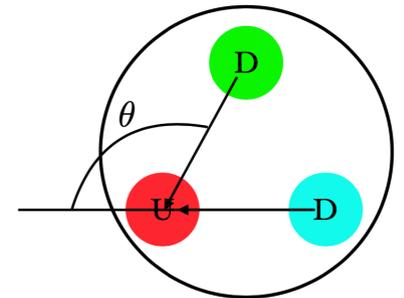
● Strong CP problem & Axion solution

1.) QCD vacuum allows an effective (CP violating) term in the Lagrangian:

$$\mathcal{L} \supset \bar{\theta} \frac{g_s^2}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu}$$

#Key feature: $\bar{\theta} = \theta_{\text{QCD}} - \arg(\det M_q)$ received contributions from both Strong & Electroweak sectors \Rightarrow theta-bar expected to be $O(1)$

2.) Bound from Neutron EDM: $\bar{\theta} < 10^{-10}$



Strong CP problem: Why is theta-bar so small?

Alternative questions: why no CP-violation in QCD? What make theta-bar so small?
(any mechanism behind?)

3.) **Axion solution:** dynamically relaxes theta-bar to zero

$$\mathcal{L} \supset \left(\bar{\theta} + \frac{a}{f_a} \right) \frac{g_s^2}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu}$$

BPST instanton

$$G_{\mu}^a(x)|_{1\text{-inst.}} = 2\eta_{a\mu\nu} \frac{(x-x_0)_{\nu}}{(x-x_0)^2 + \rho^2}, \quad \eta_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu}, & \mu, \nu \in \{1, 2, 3\} \\ -\delta_{a\nu}, & \mu = 0 \\ +\delta_{a\mu}, & \nu = 0 \\ 0, & \mu = \nu = 0 \end{cases}$$

An important property of the one-(anti-)instanton solution is that it satisfies the (*anti*-) *self-dual* equation

$$G_{\mu\nu}^a = \pm \tilde{G}_{\mu\nu}^a, \quad (\text{C.6})$$

and thus, due to the Bianchi identity, automatically solves the gluon equation of motion $D^{\mu}G_{\mu\nu}^a = D^{\mu}\tilde{G}_{\mu\nu}^a = 0$. With all of these properties, the one-(anti-)instanton solution then yields the finite QCD classical action

$$S_{\text{YM}}^{1\text{-inst.}} = \int d^4x \left(\frac{1}{4} G_{\mu\nu}^A G^{A,\mu\nu} + i\theta_{\text{QCD}} \frac{g^2}{32\pi^2} G_{\mu\nu}^A \tilde{G}^{A,\mu\nu} \right) \Big|_{1\text{-(a.-)inst.}} = \frac{8\pi^2}{g^2} \pm i\theta_{\text{QCD}}. \quad (\text{C.7})$$

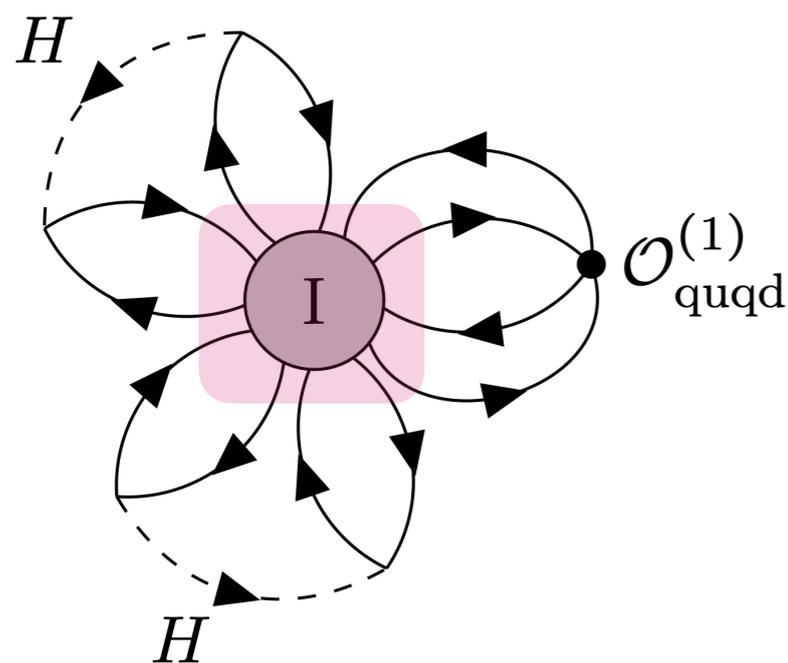
Instanton density

$$d_N(\rho) = C[N] \left(\frac{8\pi^2}{g^2} \right)^{2N} e^{-8\pi^2/g^2(1/\rho)}$$

$$C[N] = \frac{C_1 e^{-C_2 N}}{(N-1)!(N-2)!} e^{0.292 N_f}$$

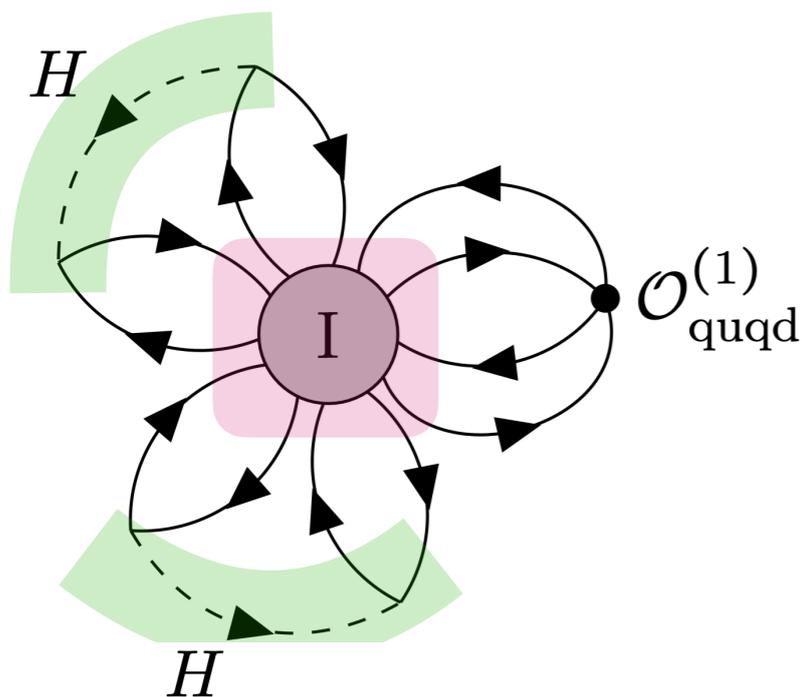
$$\frac{8\pi^2}{g^2(1/\rho)} = \frac{8\pi^2}{g_0^2(\Lambda_{UV})} - b_0 \log \rho \Lambda_{UV}, \quad b_0 = \frac{11}{3} N - \frac{2}{3} N_f$$

Topological Susceptibilities & Flavor invariants: Four-quark operator



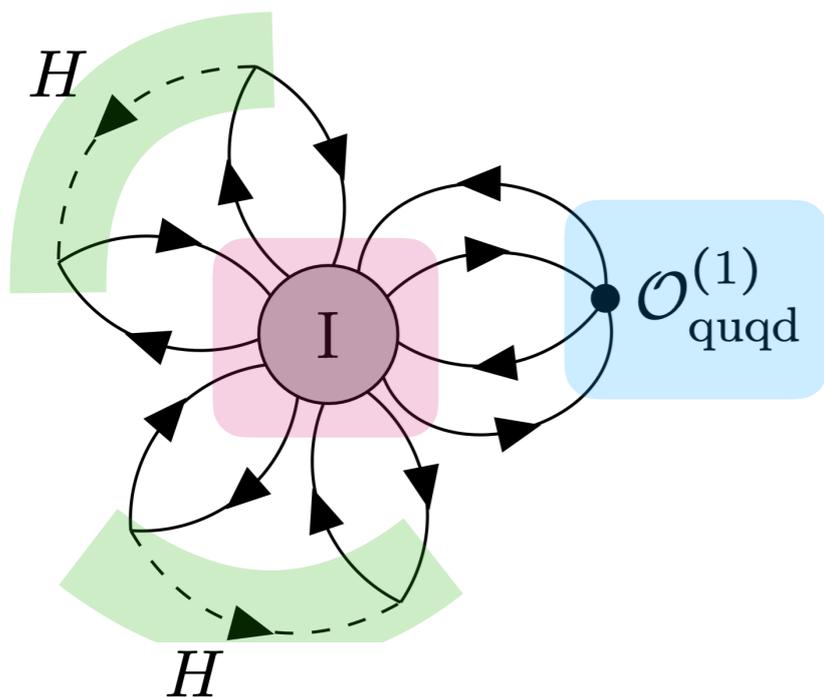
$$\begin{aligned}
 \chi_{\text{quqd}}^{(1)}(0)^{1\text{-inst.}} &= -i \lim_{k \rightarrow 0} \int d^4x e^{ikx} \left\langle 0 \left| T \left\{ \frac{1}{32\pi^2} G\tilde{G}(x), \frac{C_{\text{quqd}}^{(1)}}{\Lambda_{\text{CP}}^2} \mathcal{O}_{\text{quqd}}^{(1)}(0) \right\} \right| 0 \right\rangle, \\
 &= e^{-i\theta_{\text{QCD}}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H \mathcal{D}H^\dagger e^{-S_0[H, H^\dagger]} \int \prod_{f=1}^3 \left(\rho^2 d\xi_{u_f}^{(0)} d\xi_{d_f}^{(0)} d^2 \bar{\xi}_{Q_f}^{(0)} \right) \\
 &\times e^{\int d^4x (\bar{Q} Y_u \tilde{H} u + \bar{Q} Y_d H d + \text{h.c.})(x)} \frac{1}{32\pi^2} \int d^4x G\tilde{G}(x) \left(\frac{C_{\text{quqd}}^{(1)}}{\Lambda_{\text{CP}}^2} \bar{Q} u \bar{Q} d(0) + \text{h.c.} \right), \\
 &\quad Q = 1
 \end{aligned}$$

Topological Susceptibilities & Flavor invariants: Four-quark operator



$$\begin{aligned}
 \chi_{\text{quqd}}^{(1)}(0)^{1\text{-inst.}} &= -i \lim_{k \rightarrow 0} \int d^4x e^{ikx} \left\langle 0 \left| T \left\{ \frac{1}{32\pi^2} G\tilde{G}(x), \frac{C_{\text{quqd}}^{(1)}}{\Lambda_{\text{CP}}^2} \mathcal{O}_{\text{quqd}}^{(1)}(0) \right\} \right| 0 \right\rangle, \\
 &= e^{-i\theta_{\text{QCD}}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H \mathcal{D}H^\dagger e^{-S_0[H, H^\dagger]} \int \prod_{f=1}^3 \left(\rho^2 d\xi_{u_f}^{(0)} d\xi_{d_f}^{(0)} d^2 \bar{\xi}_{Q_f}^{(0)} \right) \\
 &\times e^{\int d^4x (\bar{Q} Y_u \tilde{H} u + \bar{Q} Y_d H d + \text{h.c.})(x)} \frac{1}{32\pi^2} \int d^4x G\tilde{G}(x) \left(\frac{C_{\text{quqd}}^{(1)}}{\Lambda_{\text{CP}}^2} \bar{Q} u \bar{Q} d(0) + \text{h.c.} \right), \\
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 \end{aligned}$$

Topological Susceptibilities & Flavor invariants: Four-quark operator



$$\begin{aligned}
 \chi_{\text{quqd}}^{(1)}(0)^{1\text{-inst.}} &= -i \lim_{k \rightarrow 0} \int d^4x e^{ikx} \left\langle 0 \left| T \left\{ \frac{1}{32\pi^2} G\tilde{G}(x), \frac{C_{\text{quqd}}^{(1)}}{\Lambda_{\text{QCD}}^2} \mathcal{O}_{\text{quqd}}^{(1)}(0) \right\} \right| 0 \right\rangle, \\
 &= e^{-i\theta_{\text{QCD}}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H \mathcal{D}H^\dagger e^{-S_0[H, H^\dagger]} \int \prod_{f=1}^3 \left(\rho^2 d\xi_{u_f}^{(0)} d\xi_{d_f}^{(0)} d^2 \bar{\xi}_{Q_f}^{(0)} \right) \\
 &\times e^{\int d^4x (\bar{Q} Y_u \tilde{H} u + \bar{Q} Y_d H d + \text{h.c.})(x)} \frac{1}{32\pi^2} \int d^4x G\tilde{G}(x) \left(\frac{C_{\text{quqd}}^{(1)}}{\Lambda_{\text{QCD}}^2} \bar{Q} u \bar{Q} d(0) + \text{h.c.} \right), \\
 &\quad Q = 1
 \end{aligned}$$