# Approximate Isospin symmetry in nuclear physics

## Applications to the study of exotic nuclei

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Master 1 - Internship
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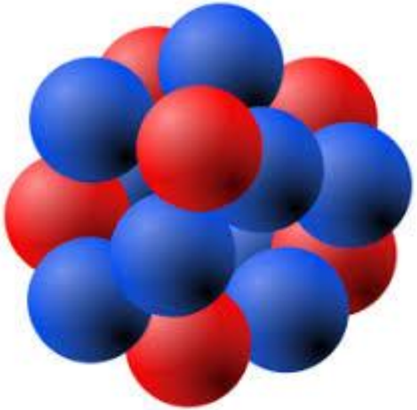
# Content of the Internship

- Study of the A-body problem characterized by its symmetries
- Use of a new approximate symmetry : the isospin
- Deepening of central potentials and collision theory
- Study of the nucleon-nucleon system and its non-central interaction with the isospin formalism
- Generalization of isospin in the nucleus and Phase-Shift Method
- Study of realistic interactions : Yukawa's works and QCD.
- Isospin in research : mirror nucleus and cluster methods.

# Symmetries at low energy

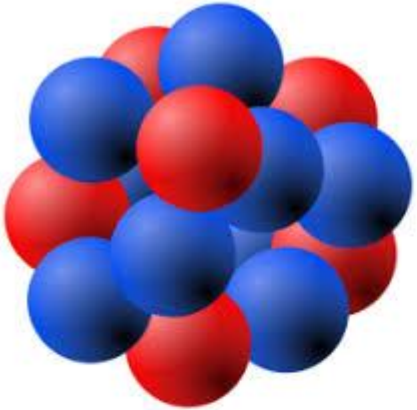


- Exact symmetries at low energy in the nucleus :
  - reflection
  - translation
  - rotation
  - time reversal
- These symmetries allow us to create complete sets of commuting observables, and to label the states with quantum numbers.
- Another exact symmetry from the nuclear interaction :

$$V_{nn}^{nuclear} = V_{pp}^{nuclear}$$

$\Longrightarrow$ Charge symmetry

# Approximate isospin symmetry



- Hamiltonian of a A-body system :

$$\hat{H} = \sum_{i=1}^{A} \hat{t}_i + \sum_{i<j}^{A} \hat{V}_{ij} + \sum_{i<j<k}^{A} \hat{V}_{ijk} + \cdots \approx \sum_{i=1}^{A} \hat{t}_i + \sum_{i<j}^{A} \hat{V}_{ij}$$

- Quasi-symmetry of the strong interaction between protons and neutrons :

$$|n\rangle = |\frac{1}{2} \ +\frac{1}{2}\rangle$$

$$|p\rangle = |\frac{1}{2} \ -\frac{1}{2}\rangle$$

2%

$$V_{nn}^{nuclear} = \underbrace{V_{pp}^{nuclear} \approx V_{np}^{nuclear}}_{\text{Charge invariance}}$$

- Both are projections of the same particle called “nucleon”, and characterized by their isospin.
- Even though this symmetry is an approximate one, it is widely used in nuclear physics.

# Bound and Scattering States

- Physical quantum states have to obey the postulates of quantum mechanics.
- Quantum states are divided in two groups : Bound and Scattering states.
- There are also resonance states which are "semi-bound" states with positive energy and can become a diffusion state by tunneling through the potential barrier at any time. These states only exist if the potential creates a dip in the positive part.
- Bound states are physical states described by a square-integrable stationary wave function :

$$\int_{-\infty}^{+\infty} \rho(x)\mathrm{d}x = 1.$$

- Scattering states are non-physical state described by a stationary wave function bounded at any point and not square-integrable. Consequently, they do not respect the postulates, but can be physically interpreted with the probability current :

$$\vec{J} \equiv \frac{i\hbar}{2m}\left(\Psi\vec{\nabla}\Psi^* - \Psi^*\vec{\nabla}\Psi\right)$$

# Nucleon-Nucleon Interaction

Experimentally, only one bound state possible :

- One proton and one neutron = deuteron
- Angular momentum : $J^\pi = 1^+$
- Describing the system with a central potential $\Longrightarrow$ the only bound state is necessarily a **S** state.
- For a pure **S** wave function : $\mu_d = \mu_n + \mu_p$ with $\mu_d$ the magnetic momentum of deuteron

  but we have : $\mu_d \approx 0.857406\mu_N \neq \mu_n + \mu_p \approx 0.879794\mu_N$

- These two results are not very far apart, so the ground state of the deuteron cannot be a **D** state or a **S** state, but rather a mixture of **D** and **S** states.
- Consequently, the nucleon-nucleon interaction **isn't governed by a purely central potential**.

# Generalized Pauli Principle

$$\hat{P}^M \longrightarrow (-1)^L$$
$$\hat{P}^\sigma \longrightarrow (-1)^{S+1}$$
$$\hat{P}^\tau \longrightarrow (-1)^{T+1}$$

- Neutrons and protons being the same particle implies to update the Pauli Principle.
- Antisymmetry of the 2-nucleons wave function by switching the nucleons' position, spin and isospin give us eigenvalues. and leads to a selection rule

$$\psi(n_1, n_2) = -\psi(n_2, n_1)$$

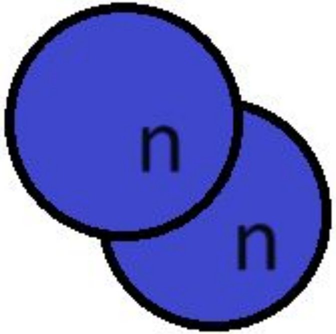
$$\hat{P}\,|\psi\rangle = -\,|\psi\rangle$$

$$\hat{P} = \hat{P}^M \hat{P}^\sigma \hat{P}^\tau$$

$$\hat{P}\,|\psi\rangle = (-1)^{L+S+1+T+1}\,|\psi\rangle \iff L+S+T \text{ is odd}$$

$\Longrightarrow$ Selection rule

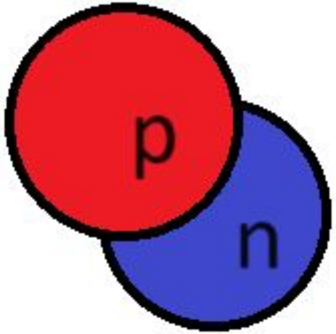
# Deuteron Ground State




$$|n\rangle = |\frac{1}{2} \ +\frac{1}{2}\rangle$$

$$|n\rangle = |\frac{1}{2} \ +\frac{1}{2}\rangle$$

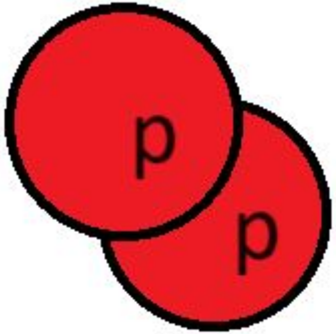
$$T = 1$$




$$|p\rangle = |\frac{1}{2} \ -\frac{1}{2}\rangle$$

$$|n\rangle = |\frac{1}{2} \ +\frac{1}{2}\rangle$$

$$T = 0, 1$$




$$|p\rangle = |\frac{1}{2} \ -\frac{1}{2}\rangle$$

$$|p\rangle = |\frac{1}{2} \ -\frac{1}{2}\rangle$$

$$T = 1$$

# Deuteron Ground State

$$L + S + T \text{ is odd}$$

$$|J - S| \leq L \leq J + S$$

$$J^{\pi} = 1^{+}$$

| Isospin | Spin | Angular | Parity | Allowed? |
|---|---|---|---|---|
| $T = 1$ | $S = 1$ | $L = 1$ | $-$ | Not Allowed |
| $T = 1$ | $S = 0$ | $L = 1$ | $-$ | Not Allowed |
| $T = 0$ | $S = 0$ | $L = 1$ | $-$ | Not Allowed |
| $T = 0$ | $S = 1$ | $L = 0$ | $+$ | Allowed |
| $T = 0$ | $S = 1$ | $L = 1$ | $-$ | Not Allowed |
| $T = 0$ | $S = 1$ | $L = 2$ | $+$ | Allowed |

$$|\psi^{gs}_{deuteron}\rangle = \alpha\,|3_{S1}\rangle + \beta\,|3_{D1}\rangle \quad |\alpha| = 96\%, |\beta| = 4\%$$

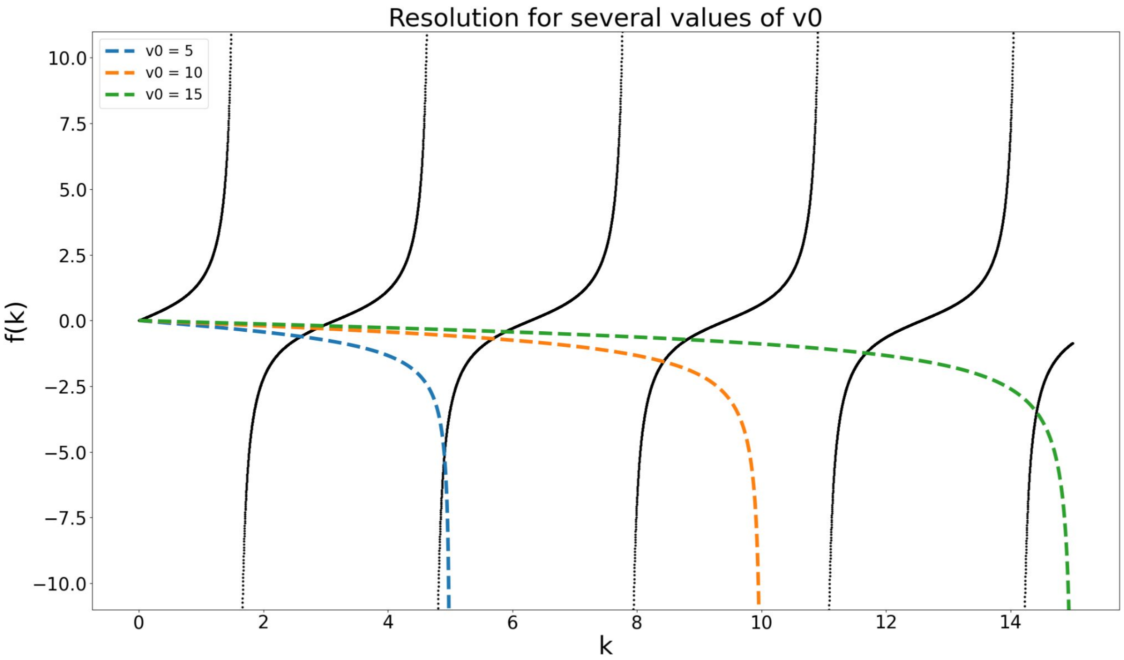
- We once again can deduce that the nucleon-nucleon potential is not central

# Square well in 3D

$$V(r) = \begin{cases} -V_0, & r \leq a, \\ 0, & r > a, \end{cases}$$

$$\boxed{\tan(k) = -\frac{k}{\sqrt{v_0^2 - k^2}}}$$

$$v_0 = \sqrt{\frac{2\mu a^2 V_0}{\hbar^2}} \qquad k = \sqrt{v_0^2 + \frac{2\mu a^2 E}{\hbar^2}}$$



# Phase-shift method

$$R_{kl}^{as}(r) = \frac{C_s}{kr}\left(e^{-i(kr-\frac{l\pi}{2})} - S_l e^{i(kr-\frac{l\pi}{2})}\right) \qquad S_l = e^{2i\delta_l}$$

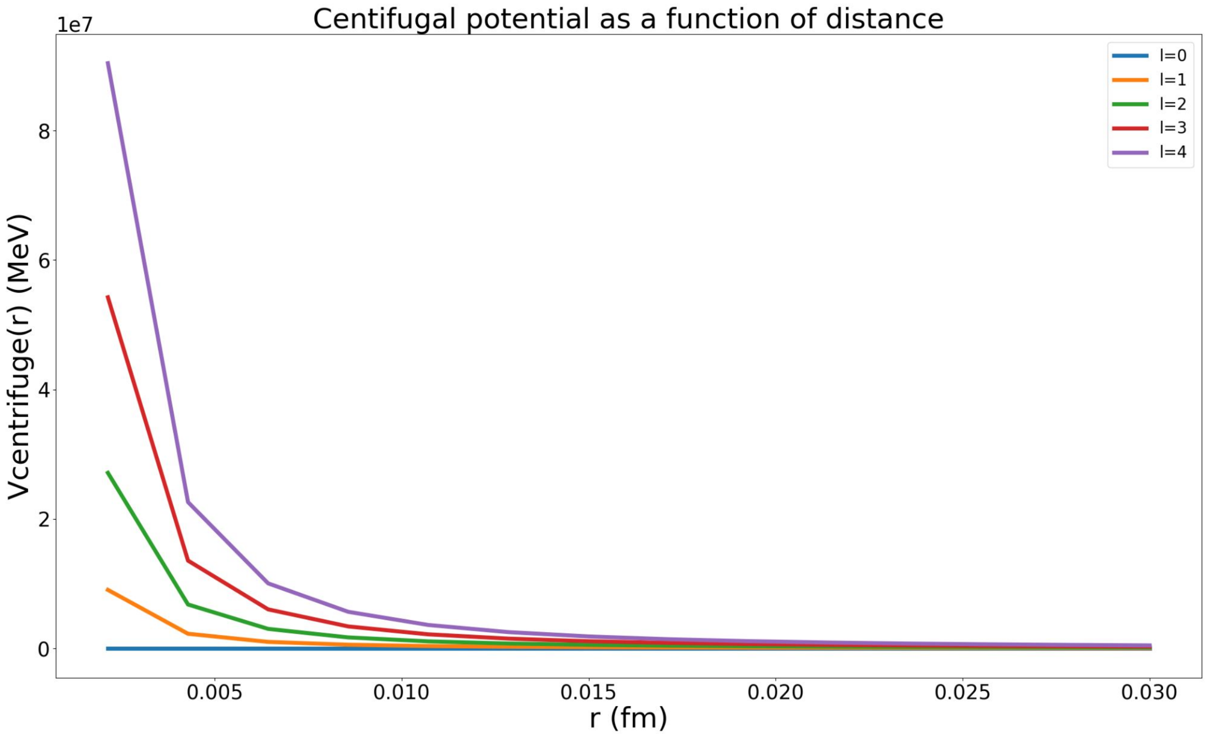
**Scattering amplitude :** $$f_k(\theta) = \frac{1}{2ik}\sum_{l=0}^{+\infty}(2l+1)(S_l-1)P_l(\cos\theta)$$

**Total elastic cross-section :** $$\sigma(E_k) = \frac{\pi}{k^2}\sum_l (2l+1)(S_l^*(E_k)-1)(S_l(E_k)-1)$$

# Phase-shift method

$$V_l^{eff}(r) = V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}$$

**Validity of this method :**



# Realistic interactions

- To remedy the inaccuracies provided by the central potential, empirical “realistic” interactions can be calculated and applied to model the nucleon-nucleon interaction. For instance, the Yukawa potential :

$$V_{\text{Yukawa}}(r) = \frac{C}{r}\exp(-\frac{r}{\lambda})$$

- These realistic interactions reproduce the bound and scattering states, which lead to the conservation of the nucleon-nucleon symmetry.

$$V_{\text{Bonn}} = (V_\pi^{ps} + V_\rho^v + V_\delta^s)\underbrace{\hat{t}_1.\hat{t}_2}_{\text{Charge invariant operator}} + V_\eta^{ps} + V_\omega^v + \frac{1}{2}(1+\hat{P}^\tau)V_\sigma^{s1} + \frac{1}{2}(1-\hat{P}^\tau)V_\sigma^{s0}$$

# Yukawa pion

**One-pion exchange potential :** $V_{OPEP} = V_\pi \boldsymbol{t_1} \cdot \boldsymbol{t_2}$

$$V_\pi = \frac{4}{3} f_\pi^2 \hbar c \left[ 4\boldsymbol{s_1} \cdot \boldsymbol{s_2} + \left(1 + \frac{3\lambda_\pi}{r} + \frac{3\lambda_\pi^2}{r^2}\right) S_{12} \right] \frac{e^{-r/\lambda_\pi}}{r}$$

$$\lambda_\pi = \frac{\lambda_{\pi^\pm} + \lambda_{\pi^0}}{2}$$

$$\boldsymbol{s_1} \cdot \boldsymbol{s_2} = \frac{1}{4}(2P_\sigma - 1)$$

$$\boldsymbol{t_1} \cdot \boldsymbol{t_2} = \frac{1}{4}(2P_\tau - 1)$$

$P_\sigma$ : spin exchange operator
$P_\tau$ : isospin exchange operator

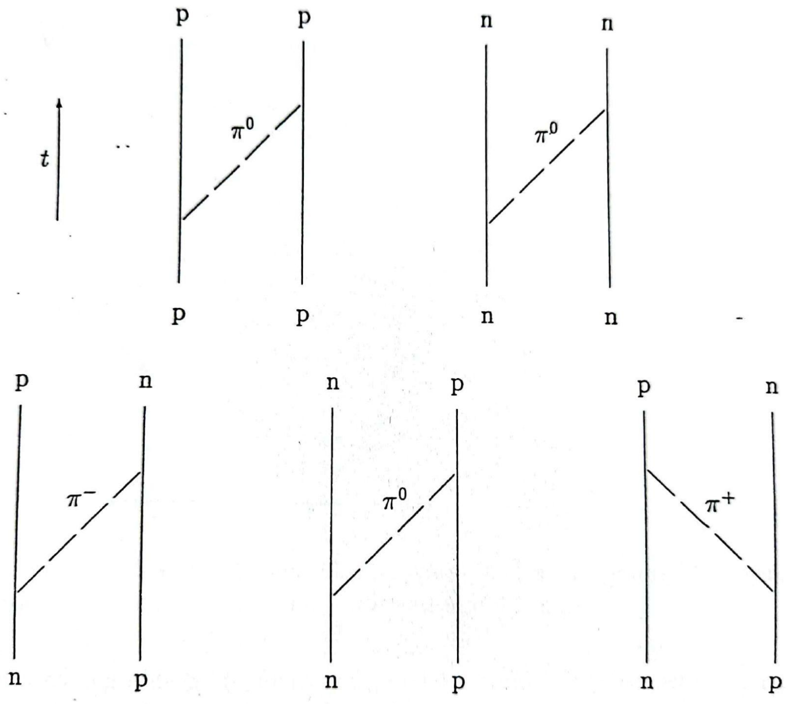



FIG. 2.6: Echanges de pions entre nucléons.

# Isospin and nucleus

- The notion of isospin can be extended to systems which have more than two particles. We can define the total isospin number as :

$$\vec{T} = \sum t_i$$

- The states space of an A-body system is then the tensorial product of each singular states space :

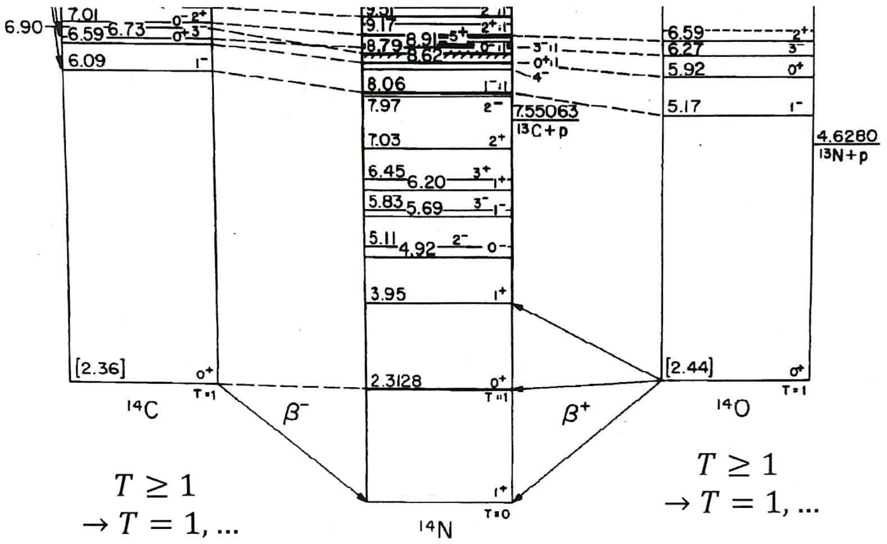
$$\epsilon = \epsilon_1 \otimes \epsilon_2 \otimes ...\epsilon_A$$

- For a nucleus with N nucleons and Z protons, we have the relationship :

$$\left|\frac{N-Z}{2}\right| \leq T \leq \frac{N+Z}{2}$$

# Towards research, study on mirror nuclei

exemple: $^{14}$C ($T_z = 1$), $^{14}$N ($T_z = 0$), $^{14}$O ($T_z = -1$): multiplet $T = 1$

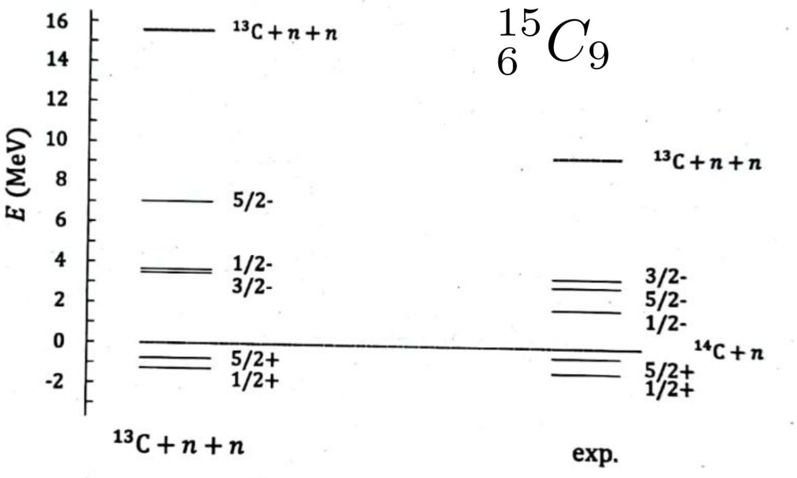



$T \geq 1$
$\rightarrow T = 1, \ldots$

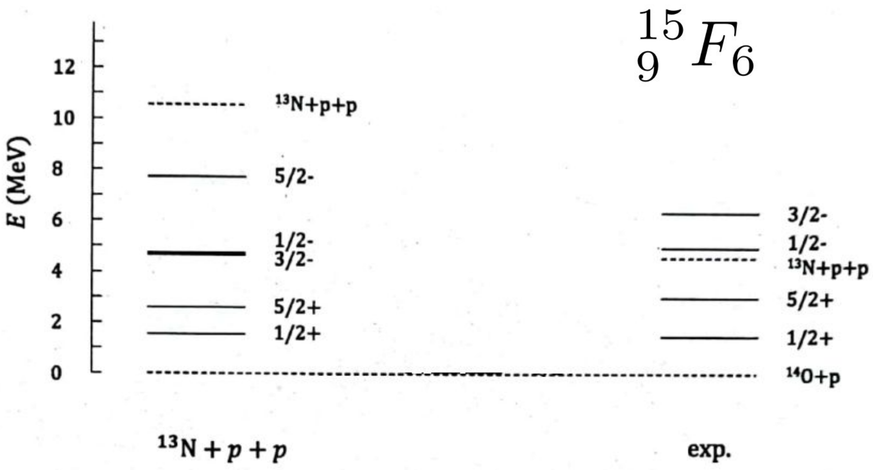
$T \geq 1$
$\rightarrow T = 1, \ldots$

$T \geq 0$
$\rightarrow T = 0{,}1, \ldots$

# Towards research, study on mirror nuclei




C(15) spectrum in the three-clusters models




F(15) spectrum in the three-clusters models

*Equivalence between two- and three- cluster models: Application to the* $^{15}_{6}C_9$ *and* $^{15}_{9}F_6$ *nuclei*
*P.Descouvemont and M.Dufour (2024)*

# Conclusion

- A-body problem and its symmetries at low energy
- Approximate Isospin symmetry : application to the case of the nucleon-nucleon interaction and the deuteron
- Use of the phase shift method as an alternative to calculate unbound states
- Realistic interactions to model the nucleon-nucleon non-central potential
- Study of how the Approximate Isospin symmetry is still currently used in research

# Bibliography


- *Equivalence between two- and three-cluster models: Application to the $^{15}_{6}C_9$ and the $^{15}_{9}F_6$ nuclei.* P.Descouvemont and M.Dufour, Physical Review C 109, 034303 (2024)
- D.Baye lessons: Nuclear interactions
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- *Mécanique quantique, une introduction générale illustrée*, D.Baye M.Dufour and B.Fuks, Ellipses 2017

# Annexe
# Square Well

Radial Schrodinger equation of a particle of mass $\mu$ :

$$H\psi(r) = E\psi(r)$$

But : $H = -\frac{\hbar^2}{2\mu}\Delta + V(r)$ ,with $\Delta = \frac{1}{r}\frac{\partial^2 r}{\partial r^2} - \frac{L^2}{\hbar^2 r^2}$ in spherical coordinates with central potential.

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{\partial^2 r}{\partial r^2} - \frac{L^2}{\hbar^2 r^2}\right) + V(r)\right]\psi(r) = E\psi(r)$$

As $L^2$ commute with $H$, we introduce a spherical harmonic : $\psi(r) = Y_{lm}(\theta,\phi)R_l(r)$

$$Y_{lm}(\theta,\phi)\left(-\frac{\hbar^2}{2\mu r}\frac{\partial^2 r}{\partial r^2} + V(r)\right)R_l(r) + \frac{R_l(r)L^2Y_{lm}(\theta,\phi)}{2\mu r^2} = EY_{lm}(\theta,\phi)R_l(r)$$

But : $L^2Y_{lm}(\theta,\phi) = \hbar^2 l(l+1)Y_{lm}(\theta,\phi)$ , so we have :

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{\partial^2 r}{\partial r^2} - \frac{l(l+1)}{r^2}\right) + V(r)\right]R_l(r) = ER_l(r)$$

$$\Longleftrightarrow \begin{cases} \left(-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}\right) - V_0\right)u_l(r) = Eu_l(r), & 0 \le r \le a, \\ -\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}\right)u_l(r) = Eu_l(r), & r > a, \end{cases}$$

with $u_l(r) = rR_l(r)$

# Annexe
# Square Well

We put $l = 0$ to solve the radial equation and we introduce $\rho = \frac{r}{a}$

$$\begin{cases} \left(\frac{d^2}{d\rho^2} + \frac{2\mu a^2 V_0}{\hbar^2} + \frac{2\mu a^2 E}{\hbar^2}\right) u_0(\rho) = 0, & \rho \leq 1, \\ \left(\frac{d^2}{d\rho^2} + \frac{2\mu a^2 E}{\hbar^2}\right) u_0(\rho) = 0, & \rho > 1, \end{cases}$$

$$\Longleftrightarrow \begin{cases} \left(\frac{d^2}{d\rho^2} + v_0^2 - \epsilon^2\right) u_0(\rho) = 0, & \rho \leq 1, \\ \left(\frac{d^2}{d\rho^2} - \epsilon^2\right) u_0(\rho) = 0, & \rho > 1, \end{cases}$$

with $v_0 = \sqrt{\frac{2\mu a^2 V_0}{\hbar^2}}$ and $\epsilon = \sqrt{-\frac{2\mu a^2 E}{\hbar^2}}$

So, we have:

$$\begin{cases} u_0(\rho) = A_1 e^{ik\rho} + B_1 e^{-ik\rho}, & \rho \leq 1, \\ u_0(\rho) = A_2 e^{\epsilon\rho} + B_2 e^{-\epsilon\rho}, & \rho > 1, \end{cases}$$

with $k = \sqrt{v_0^2 - \epsilon^2}$

But, when $r \to 0$ ($\rho \to 0$), $u_0(\rho)$ must cancel, so as $u_0(\rho) = A_1 \sin(k\rho) + B_1 \cos(k\rho)$ and $\cos(k\rho) \to 1$ when $r \to 0$, $B_1 = 0$. Moreover, $u_0(\rho) \to 0$ when $r \to \infty$ so $A_2 = 0$ (because $e^{\epsilon\rho} \to \infty$ when $r \to \infty$). So we have:

$$\begin{cases} u_{0,1}(\rho) = A_1 sin(k\rho), & \rho \leq 1, \\ u_{0,2}(\rho) = B_2 e^{-\epsilon\rho}, & \rho > 1, \end{cases}$$

When $r = a$ ($\rho = 1$), there is continuity of the wave function and its derivative:

$$\frac{u'_{0,1}(\rho = 1)}{u_{0,1}(\rho = 1)} = \frac{u'_{0,2}(\rho = 1)}{u_{0,2}(\rho = 1)}$$

$$\Longleftrightarrow \frac{kA_1 cos(k)}{A_1 sin(k)} = -\epsilon \frac{e^{-\epsilon} B_2}{e^{-\epsilon} B_2}$$

$$\Longleftrightarrow tan(k) = -\frac{k}{\sqrt{v_0^2 - k^2}}$$

# Annexe Isospin (Formalism)

- Isospin operator and similarities to spin formalism :

$$\hat{S}_x \longleftrightarrow \hat{t}_1 \qquad \hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \longleftrightarrow \hat{t}^2 = \hat{t}_1^2 + \hat{t}_2^2 + \hat{t}_3^2$$

$$\hat{S}_y \longleftrightarrow \hat{t}_2 \qquad [\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z \longleftrightarrow [\hat{t}_1, \hat{t}_2] = i\hat{t}_3$$

$$\hat{S}_z \longleftrightarrow \hat{t}_3$$

- Eigenvalues :

$$\hat{t}^2 \left|tm_t\right\rangle = t(t+1) \left|tm_t\right\rangle \qquad \hat{t}_3 \left|tm_t\right\rangle = m_t \left|tm_t\right\rangle$$

# Annexe Isospin (Ladder operators)

- Ladder operators and its eigenvalues

$$\hat{t}_{\pm} = \hat{t}_1 \pm i\hat{t}_2$$

$$\hat{t}_+ |n\rangle = 0$$
$$\hat{t}_+ |p\rangle = |n\rangle$$
$$\hat{t}_- |n\rangle = |p\rangle$$
$$\hat{t}_- |p\rangle = 0$$

$$\begin{aligned}\hat{t}_+\hat{t}_- &= (\hat{t}_1 + i\hat{t}_2)(\hat{t}_1 - i\hat{t}_2) \\ &= \hat{t}_1^2 + \hat{t}_2^2 - i[\hat{t}_1, \hat{t}_2] \\ &= \hat{t}^2 - \hat{t}_3^2 + \hat{t}_3\end{aligned}$$

$$\begin{aligned}\langle tm_t| \hat{t}_+\hat{t}_- |tm_t\rangle &= \langle tm_t| \hat{t}^2 - \hat{t}_3^2 + \hat{t}_3 |tm_t\rangle \\ &= t(t+1) - m_t(m_t - 1)\end{aligned}$$

$$\hat{t}_+ |tm_t\rangle = \sqrt{t(t+1) - m_t(m_t+1)}\,|tm_t\rangle$$
$$\hat{t}_- |tm_t\rangle = \sqrt{t(t+1) - m_t(m_t-1)}\,|tm_t\rangle$$

## Annexe Isospin (Projection operators)

$$P^n = \frac{1}{2} + \hat{t}_3$$

$$P^p = \frac{1}{2} - \hat{t}_3$$

$$\hat{P}^n + \hat{P}^p = \hat{1}$$

$$|\psi\rangle = \alpha |n\rangle + \beta |p\rangle$$

$$\begin{aligned} \hat{P}^n |\psi\rangle &= (\frac{1}{2} + \hat{t}_3) |\psi\rangle \\ &= \frac{\alpha}{2} |n\rangle + \frac{\beta}{2} |p\rangle + \frac{\alpha}{2} |n\rangle - \frac{\alpha}{2} |p\rangle \\ &= \alpha |n\rangle \end{aligned}$$

## Annexe Isospin (Charge operators and total isospin)

$$\hat{q} = e(\frac{1}{2} + \hat{t}_3)$$

$$\begin{aligned} \hat{q}\,|n\rangle &= \frac{e}{2}\,|n\rangle - e\hat{t}_3\,|n\rangle \\ &= \frac{e}{2}\,|n\rangle - e\frac{1}{2}\,|n\rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} \hat{q}\,|p\rangle &= \frac{e}{2}\,|p\rangle - e\hat{t}_3\,|p\rangle \\ &= \frac{e}{2}\,|n\rangle + e\frac{1}{2}\,|n\rangle \\ &= e\,|p\rangle \end{aligned}$$

$$\hat{T}^2\,|TM_T\rangle = T(T+1)\,|TM_T\rangle$$

$$\hat{T}_3\,|TM_T\rangle = M_T\,|TM_T\rangle$$

$$\hat{Q}\,|TM_T\rangle = e(\frac{1}{2}A + \hat{t}_3)$$

$$\hat{Q}\,|TM_T\rangle = Ze\,|TM_T\rangle$$