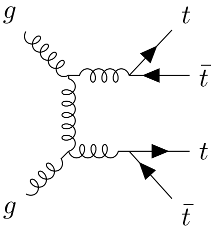
Soft gluon resummation for four-top production

Yehudi Simon



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Content

1. Resummation framework: how to improve fixed order
2. Resummation for top physics
3. Conclusion

Fixed order cross section calculation

$$\mathrm{d}\sigma_{ab\to f} = \mathrm{d}\sigma^{(0)} + \frac{\alpha_s}{2\pi}\mathrm{d}\sigma^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathrm{d}\sigma^{(2)} + ...$$

Calculations at fixed order (f. o.)

- $\mathcal{O}(\alpha_s)$, $\mathcal{O}(\alpha) \ll 1$: assuming convergence of perturbative series
- Truncation (finite set of diagrams): the more complex, the more computation ressources needed
- Matrix element computed pertubatively: Leading-Order (LO), Next-to-Leading-Order (NLO), ...
- Amplitudes not sufficient: $\mathrm{d}\Phi$ and PDFs

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(Collinear) Factorisation theorem → Mellin transform

$$\mathrm{d}\sigma_{pp\to f}(s, M^2, \mu_F) = \sum_{a,b} \int_0^1 \mathrm{d}x_A\, \mathrm{d}x_B\, f_{a/p}(x_A, \mu_F) f_{b/p}(x_B, \mu_F) \\ \mathrm{d}\hat{\sigma}_{ab\to f}(M^2/\hat{s}, M^2, \mu_F)\, \Theta(x_A x_B - M^2/s)$$

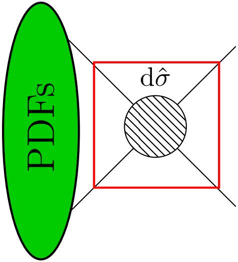
$$\Downarrow$$

$$\mathrm{d}\sigma_{pp\to f}(N-1, M^2) = \sum_{a,b} f_{a/p}(N, \mu_F)\, f_{b/p}(N, \mu_F)\, \mathrm{d}\hat{\sigma}_{ab}(N, M^2)$$

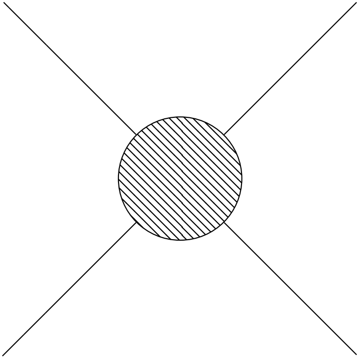
Partons & Mellin transform

$$F(N) = \int_0^1 dz\, z^{N-1}\, f(z)$$

- $f_{a(b)/p}$: PDFs of $a(b)$ in p
- μ_F: factorisation scale
- $\sqrt{\hat{s}} = \sqrt{s x_A x_B}$: partonic c.o.m. energy
- M: final invariant mass



About soft gluon threshold resummation



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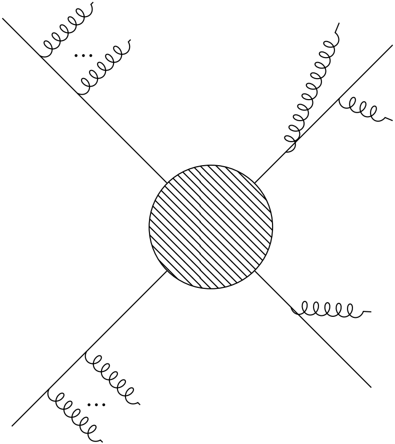

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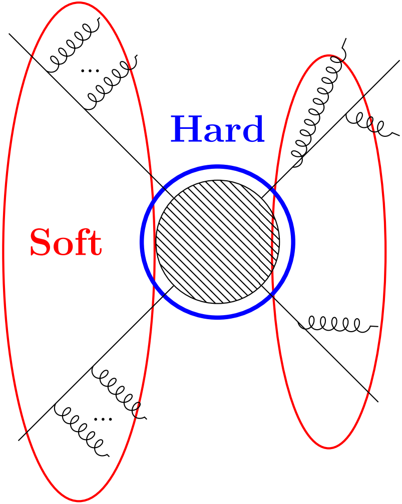

- Threshold: $z = \frac{M^2}{\hat{s}} \to 1$
- Large logs arising from soft gluon emissions: $\log(1-z) \leftrightarrow \log(N)$
- Soft scale vs. Hard scale
- Factorisation properties: Soft "*universal*"

Resummation & exponentiation

$$I_{ij} = \int_0^1 \mathrm{d}z \frac{z^{N-1}-1}{1-z} \int_{\mu_F^2}^{(1-z)^2 M^2/4} \frac{\mathrm{d}k_t^2}{k_t^2} \frac{\alpha_s(k_t)}{2\pi} \left(A_i^{(1)} + A_j^{(1)}\right)$$

Thanks to factorisation, we can **resum** and exponentiate:

$$\mathscr{G}_{ij} = \sum_{n=0}^{+\infty} \frac{1}{n!} I_{ij}^n = e^{I_{ij}}$$

Generalises to Sudakov factor: depends only on the nature of coloured external massless particles, encodes (collinear-)soft radiations.

$$\mathscr{G}_{ij} = \exp\left(g_{1\,ij}(\lambda)\log N + g_{2\,ij}(\lambda) + \frac{\alpha_s}{2\pi} g_{3\,ij}(\lambda) + \ldots\right), \qquad \lambda = \frac{\alpha_s}{2\pi}\beta_0 \log N$$

Resummation & exponentiation

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Resummation & exponentiation

$$\boxed{I_{ij} = \int_0^1 \mathrm{d}z \frac{z^{N-1} - 1}{1 - z} \int_{\mu_F^2}^{(1-z)^2 M^2/4} \frac{\mathrm{d}k_t^2}{k_t^2} \frac{\alpha_s(k_t)}{2\pi} \left(A_i^{(1)} + A_j^{(1)}\right)}$$

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LL **NLL** **N^2LL**

General formalism: adding colour structure

$$\boxed{\mathrm{d}\hat{\sigma}^{N,\,res.}_{ij\to f}(\mu_R) \propto \mathrm{Tr}\Big(\mathbf{H}(\mu_R)\,\exp\Big[\int_{\mu_S}^{\mu_R}\Gamma^\dagger\Big]\,\mathbf{S}(\mu_S)\,\exp\Big[\int_{\mu_S}^{\mu_R}\Gamma\Big]\Big)\mathscr{G}_{ij}}$$

- $\mathbf{H}$: hard colour matrix, high energy process part
- $\mathbf{S}$: soft colour matrix, low energy emissions
- Γ: soft anomalous dimension colour matrix controlling the evolution over the RG of $\mathbf{S}$
- $\mathscr{G}_{ij}$: Sudakov factor encodes (collinear-)soft radiations

$$\mathbf{H} = \sum_{k=0}\left(\frac{\alpha_s}{2\pi}\right)^k \mathbf{H}^{(k)}, \quad \mathbf{S} = \sum_{k=0}\left(\frac{\alpha_s}{2\pi}\right)^k \mathbf{S}^{(k)}, \quad \Gamma = \sum_{k=0}\left(\frac{\alpha_s}{2\pi}\right)^k \Gamma^{(k)}$$

Matching resummation and fixed order

We can expand $\mathrm{d}\hat{\sigma}^{res.}$ to NLO in α_s in order to match to the NLO cross section (obtained with MG5aMC) → winning on all fronts !

Fixed order

$\mathrm{d}\sigma_{|_{\mathrm{NLO}}}$

$\alpha_s(L^2 + L + C)$

Valid away from threshold

Resummed

$\mathrm{d}\sigma^{res.}$

$e^{\alpha_s L^2 + \alpha_s L}$

Valid near threshold

Resummed @ f. o.

$\mathrm{d}\sigma^{res.}_{|_{\mathrm{NLO}}}$

$\alpha_s(L^2 + L)$

Double counting

Matching: $\mathrm{d}\sigma_{|_{\mathrm{N^kLO}}} + \mathrm{d}\sigma^{res.} - \mathrm{d}\sigma^{res.}_{|_{\mathrm{N^kLO}}}$ valid everywhere

$$(\mu_R, \mu_F) \in \mu_0 \times \Big\{(1,1);\ (1,2);\ (2,1);\ (2,2);\ (1/2,1);\ (1,1/2);\ (1/2,1/2)\Big\}$$

Content

1. Resummation framework: how to improve fixed order

2. Resummation for top physics

3. Conclusion

Top sector: SM & beyond

Up-to-date physics: 4 t

- First mesure by ATLAS and CMS with 2018 data, $\sqrt{s} = 13$ TeV
- Observation $\approx 2 \times$ SM: 1.8 σ deviation
- $\mathcal{O}(25\%)$ uncertainty both on exp. & theory
- CMS four top candidate
- Higgs laboratory: *e.g.* Constraint on Γ_H, [ATLAS 2407.10631]
- Connection to NP using SMEFT at LHC and future colliders [Vryonidou *et al.* 2022]

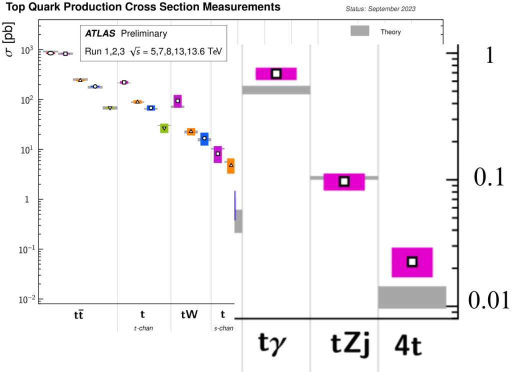


Figure: [ATL-PHYS-PUB-2023-028]
Preliminary summary for top physics

- High multiplicity final state
  → NNLO out of reach for now...

Resummation ingredients for $t\bar{t}t\bar{t}$ at NLL

$$\boxed{\mathrm{d}\hat{\sigma}^{N,\,res.}_{ij\to f}(\mu_R) \propto \mathrm{Tr}\Big(\mathbf{H}(\mu_R)\,\exp\Big[\int_{\mu_S}^{\mu_R}\Gamma^{\dagger}\Big]\,\mathbf{S}(\mu_S)\,\exp\Big[\int_{\mu_S}^{\mu_R}\Gamma\Big]\Big)\mathscr{G}_{ij}}$$

- Color basis choice (6×6 for $q\bar{q}$ and 13×13 for gg)
- $\mathbf{H}$: extracted from `MadLoop` up to $\mathbf{H}^{(1)}$
- Soft matrices from Mathematica code (Eikonal integrals)
- Sudakov well known

⚠ $\longrightarrow$ some freedom of choosing what is expanded or not

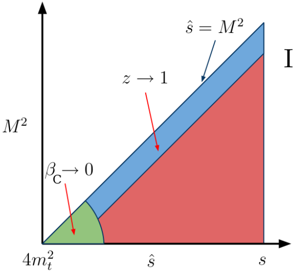
⚠ $\longrightarrow$ $\neq$ frameworks (abs. thresh./inv. mass, SCET/pQCD...)

Which logs are resummed / What is exponentiated?

Soft gluon resummation: threshold defintions

Absolute threshold limit

- Definition : $\sqrt{\hat{s}} \to 4m_t \Longrightarrow M \to 4m_t$
- Γ diagonal in the natural color basis 🥳
- Simplified dependence to the kinematics 🤩
- $\dfrac{\mathrm{d}\sigma}{\mathrm{d}M}$ not accessible 😢
- → [PRL Kulesza et al.]



[JHEP Yang et al.]
Threshold regions

Invariant mass threshold

- Definition : $M \to \sqrt{\hat{s}}$
- Γ more complex 🤔
- $\dfrac{\mathrm{d}\sigma}{\mathrm{d}M}$ accessible 😎
- Absolut threshold limit should coincide
- → this work

$t\bar{t}t\bar{t}$ production: preliminary results

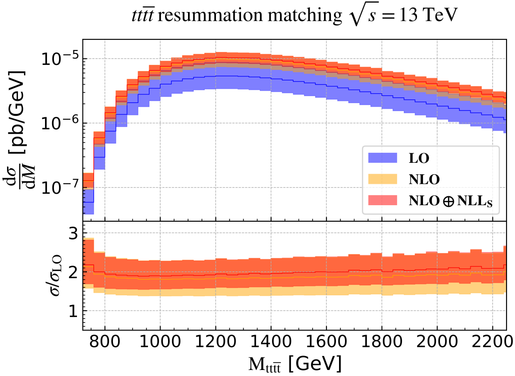


Figure: Matching of NLO to NLL accuracy, no virtual loops included, $\mu_0 = M_{tt\bar{t}}/2$

Four top production

- Scale uncertainty reduction: +28.6% − 25.2% to +20.6% − 18.2% → **precision**
- Central value increase: +7% → *accuracy*
- Multiplicity: challenging phase space integration (`Vegas` MC integrator)
- Need to be optimized to allow dynamical scale integration

$$\mathbf{H}^{(0)}, \quad \mathbf{S}^{(1)}, \quad \Gamma^{(1)} \quad \in \mathbb{M}_{6/13\times 6/13}(\mathscr{F}); \qquad g_1, \quad g_2 \quad \in \mathscr{F}(\lambda, \mu_F, \mu_R, M_{t\bar{t}t\bar{t}})$$

$t\bar{t}t\bar{t}$ production: other scale choice

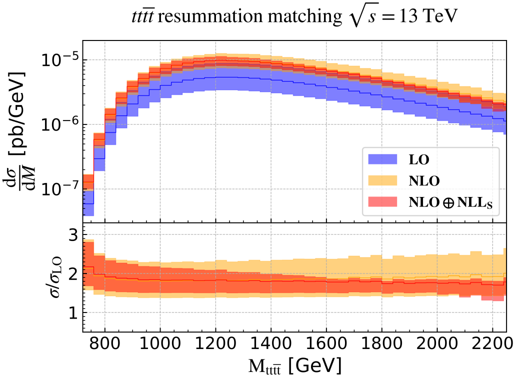

$t\bar{t}t\bar{t}$ resummation matching $\sqrt{s} = 13$ TeV

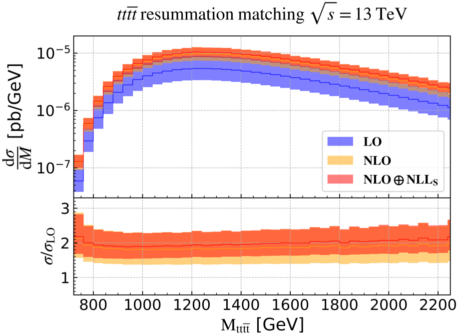
Figure: Matching of NLO to NLL accuracy, no virtual loop, $\mu_0 = 2m_t$

Four top production

- Even more **precision**
- Reduction of central value → call for $\mathbf{H}^{(1)}$ for *accuracy*
- High CPU cost for inclusion of loops (gluon channel)
- Dynamical scale choice seems more robust → compare once full $\mathbf{H}^{(1)}$ included

$$\mathbf{H}^{(0)}, \quad \mathbf{S}^{(1)}, \quad \Gamma^{(1)} \quad \in \mathbb{M}_{6/13\times 6/13}(\mathscr{F}); \qquad g_1, \quad g_2 \quad \in \mathscr{F}(\lambda, \mu_F, \mu_R, M_{t\bar{t}t\bar{t}})$$

$t\bar{t}t\bar{t}$ production: (naïve) comparison



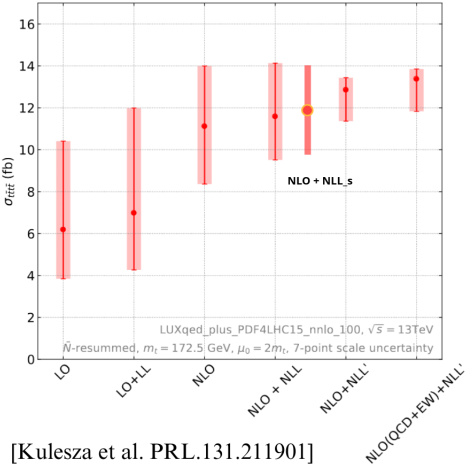


Figure: Matching of NLO to NLL accuracy, no virtual loops included, $\mu_0 = M_{t\bar{t}t\bar{t}}/2$

[Kulesza et al. PRL.131.211901]

$$\mathbf{H}^{(0)},\ \mathbf{S}^{(1)},\ \Gamma^{(1)} \in \mathbb{M}_{6/13\times 6/13}(\mathscr{F})$$

$$g_1,\ g_2 \in \mathscr{F}(\lambda, \mu_R, \mu_F, M_{t\bar{t}t\bar{t}})$$

See L. Moreno's talk [REF 2023]

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Summary

Takeaway message

- Higher orders: *accuracy*, **precision** and correct kinematics
- Resummation: cure pathologically large logarimths, better convergence of the series, reduction of the scale uncertainty
- High multiplicity final states computationally demanding but achievable, requires optimization
- Need the full one loop pieces in order to have sensible results
- Many prospects to continue the study (NNLL, joint resummation, scheme consistency...)

→ Stay tunned for the complete NLL soft gluon threshold resummation at differential level for $t\bar{t}t\bar{t}$!

Thank you for your attention!

Backup

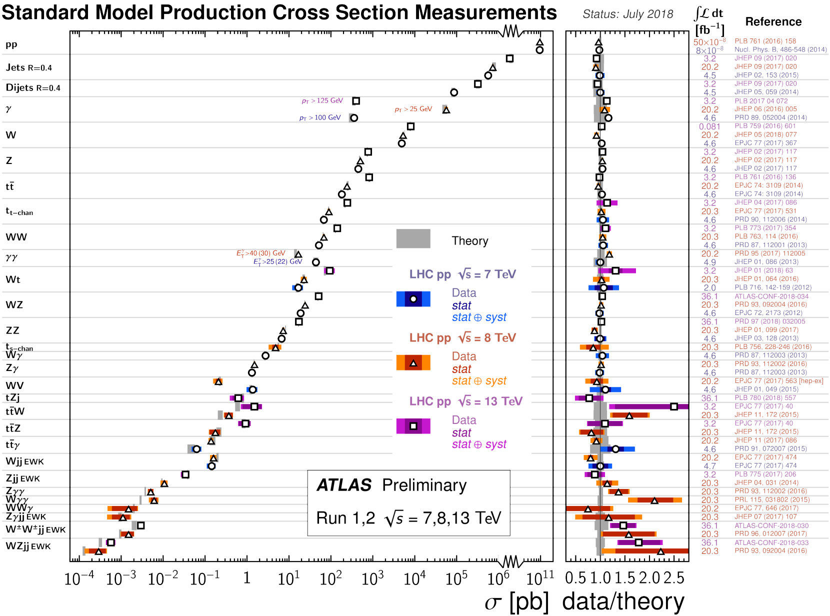


Figure: [ATLAS]: SM summary plots

Parton Distribution Functions

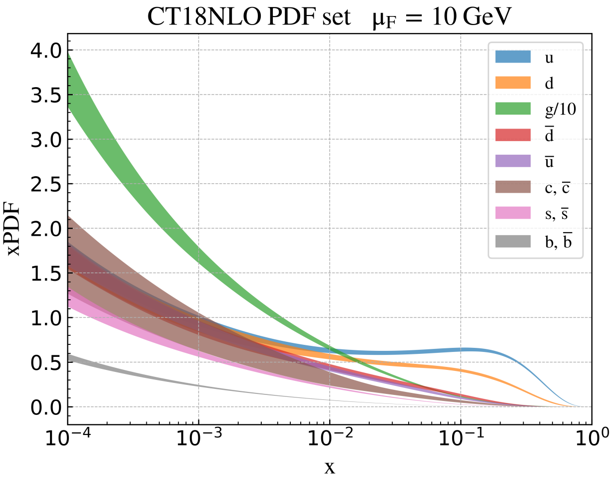


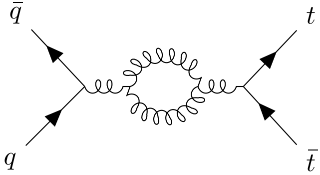
Figure: LHAPDF CT18NLO set [PRD.103.014013] displayed for $\mu_F = 10$ GeV

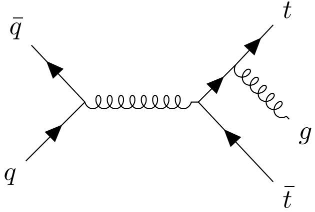
Fixed order calculation: NLO and beyond

Examples of calculations at f. o.

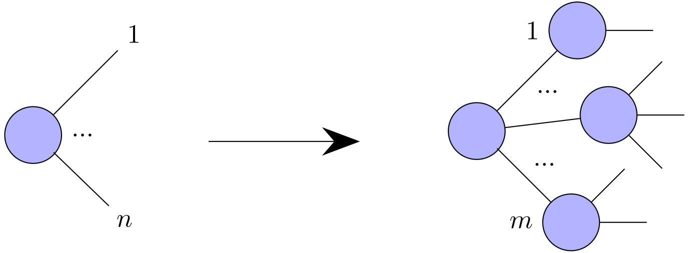
- Well known processes (N^{3}LO): (di)-Higgs production, Drell-Yan
- More complex (N^{2}LO): $t\bar{t}$, vector boson pair (γ, Z, $W^{\pm}$), $\gamma\gamma\gamma$
- Known only at NLO: $t\bar{t}t\bar{t}$

Multiplicity of the final state increases complexity ($\mathscr{M}$ and dΦ integration). NLO computations automated in MadGraph5_aMC@NLO (MG5aMC), but not beyond.





Phase space factorisation



$$\mathrm{d}\Phi_n(P;\,\{p_i\}_{i\leq n}) = \prod_{i=1}^{m}\left(\frac{\mathrm{d}Q_i^2}{2\pi}\mathrm{d}\Phi_{n_i}(Q_i;\,\{p_k^{(i)}\}_{k\leq n_i})\right)\mathrm{d}\Phi_m(P;\,\{Q_j\}_{j\leq m})$$

For a single soft gluon emission it gives:

$$\boxed{\mathrm{d}\Phi_{n+1}(P;\,\{p_i\},k) \simeq \mathrm{d}\Phi_n(M;\,\{p_i\})\frac{\mathrm{d}^3k}{(2\pi)^3 2E_k}\frac{\mathrm{d}M^2}{\hat{s}}\delta\left(z-1+\frac{2E_k}{\sqrt{\hat{s}}}\right)}$$

Phase space factorisation: Mellin space

For mutliple soft gluon emissions:

$$\mathrm{d}\Phi_{n+n_k}(P;\,\{p_i\},\{k_j\}) \propto \mathrm{d}\Phi_n(M;\,\{p_i\})\delta\Big(z - \prod_j z_j\Big)$$

with $z_j = 1 - \dfrac{2E_{k_j}}{\sqrt{\hat{s}}}$. The phase space factorises in Mellin space:

$$\boxed{\mathrm{d}\Phi_{n+n_k}(N) \propto \int_0^1 dz\; z^{N-1}\;\delta\Big(z - \prod_j z_j\Big) = \prod_j z_j^{N-1}}$$

Expected behaviours

- We expect the ratio $\frac{1}{d\sigma^0/dM^2}\left(\frac{d\sigma^{res.}}{dM^2} - \frac{d\sigma^{res.}}{dM^2}\Big|_{NLO}\right) \xrightarrow[M^2 \ll S_h]{} 0$

  Away from threshold the logarithmic terms are not important and the behaviour is captured by the first orders of the expansion.

- We expect also $\frac{1}{d\sigma^0/dM^2}\left(\frac{d\sigma^{NLO}}{dM^2} - \frac{d\sigma^{res.}}{dM^2}\Big|_{NLO}\right) \xrightarrow[M^2 \to S_h]{} 0$

  In the threshold regime, the resummed expanded reproduces the behaviour of original cross section.

To obtain a sensible cross section in all ranges we may consider the combination: $\sigma_{|_{NLO}} + \sigma^{res.} - \sigma^{res.}_{|_{NLO}}$

Threshold resummation for Drell-Yan-like processes

- Only massless initial quarks emit gluons
  → apply to all similar processes: Drell-Yan-like
- Universal Sudakov *soft part* known up to N^3LL

$$\Delta^{\mathrm{res}}_{q\bar{q}}(N, M^2, \mu_F^2)\Big|_{\mathrm{N^kLL}} = \tilde{g}_{0,q\bar{q}}(M^2, \mu_F^2, \mu_R^2)\Big|_{\mathrm{N^kLO}} \times \exp\left(g_{1,q\bar{q}}(\lambda)\ln\overline{N} + \sum_{j=2}^{k+1} a_s^{j-2} g_{j,q\bar{q}}(\lambda)\right)$$

- Can be used for BSM studies: exotic lepton pair productions
  → require some fixed order pieces: adapt `UFO` and use MG5aMC!

Application to $t\bar{t}$ production: differential cross section

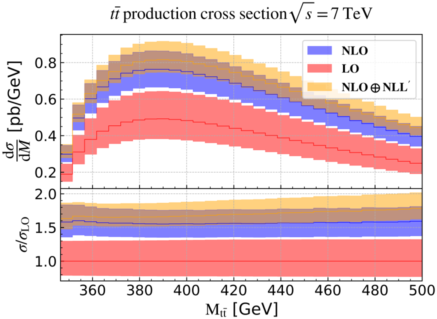


Figure: Effect of resummation for differential cross section

Top pair production

- Central scale: $\mu_0 = M_{t\bar{t}}/2$
- Enhancement between 3% (production thresh.) & 12% at $M_{t\bar{t}} = 500$ GeV
- Scale uncertainty reduction: 26.6% → 21.4%

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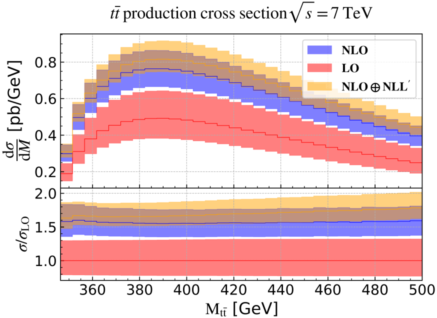

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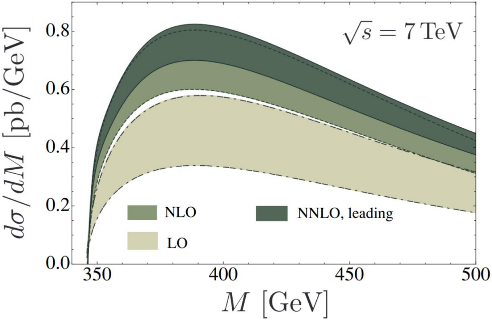

Figure: [JHEP09(2010)097] SCET $t\bar{t}$ resummation

g functions

$$\begin{aligned}
\tilde{g}^{(1)}_{0,q\bar{q}} &= \frac{-64}{3} + \frac{64}{3}\zeta_2 - 8L_{fr} + 8L_{qr}\,, \\
\tilde{g}^{(2)}_{0,q\bar{q}} &= \frac{-1291}{9} + \frac{64\zeta_2}{9} + \frac{368\zeta_2^2}{3} + \frac{4528\zeta_3}{27} \\
&\quad + \frac{188L_{fr}^2}{3} + \frac{4L_{qr}^2}{3} \\
&\quad + L_{fr}\left(\frac{1324}{9} - \frac{1888\zeta_2}{9} + \frac{32\zeta_3}{3}\right) \\
&\quad + L_{qr}\left(\frac{148}{9} - 64L_{fr} + \frac{416\zeta_2}{9} - \frac{32\zeta_3}{3}\right),
\end{aligned} \tag{1}$$

with $L_{qr} = \ln \frac{M^2}{\mu_R^2}$, $L_{fr} = \ln \frac{\mu_F^2}{\mu_R^2}$ and ζ_n being the Riemann zeta function.