A new parton shower based on the small-x evolution equation

Yu Shi (石瑜)
CPHT, Ecole Polytechnique
yu.shi@polytechnique.edu

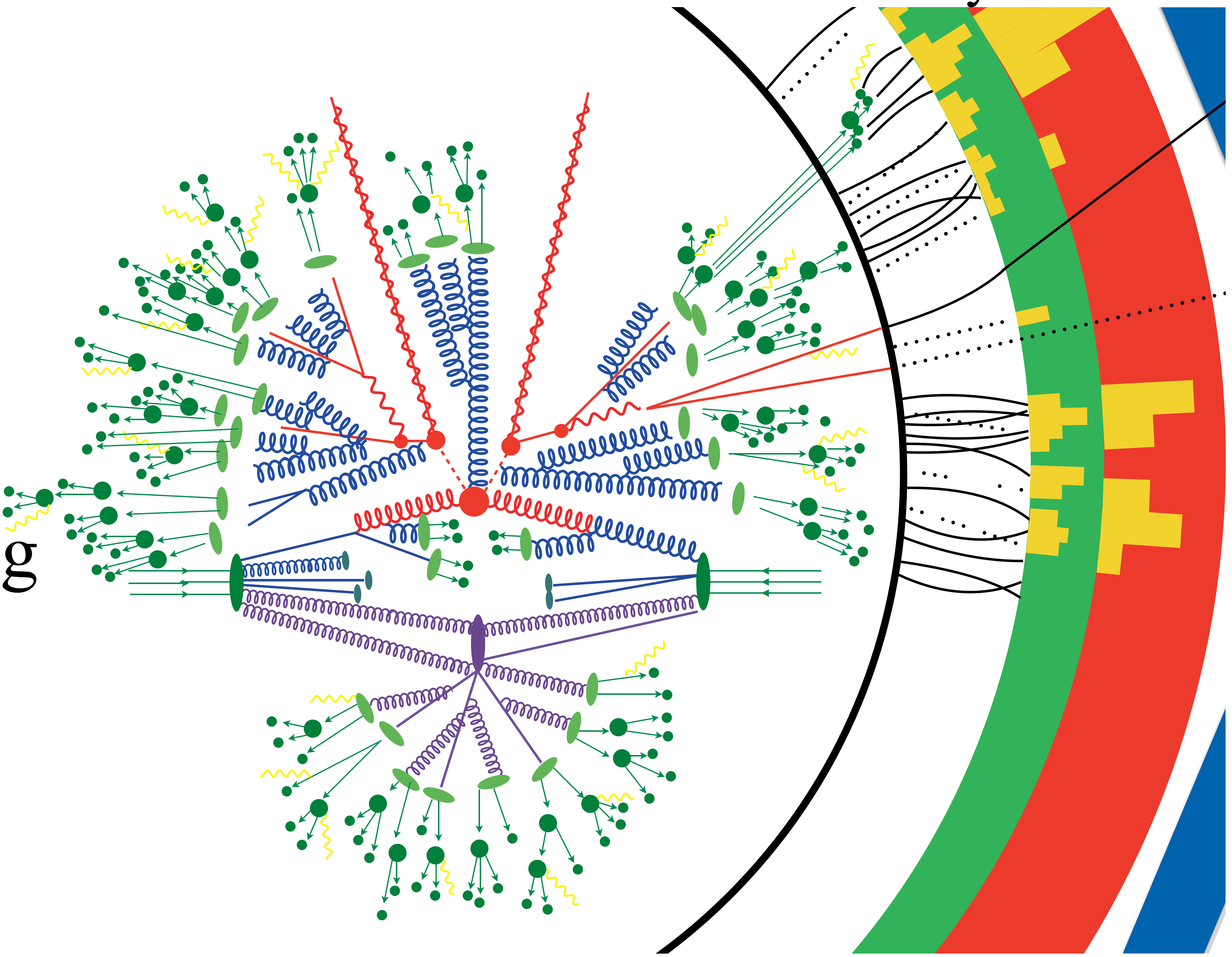
- YS, Shu-yi Wei and Jian Zhou, Phys.Rev.D 107, 016017 (2023).
- YS, Shu-yi Wei and Jian Zhou, Phys.Rev.D 108, 096025 (2023).
- Collaboration with Wei-yao Ke, Xin-nian Wang and Jian Zhou, working in progress.

REF 2024, Paris
October 17th, 2024

Why the parton shower and M.C. generator?

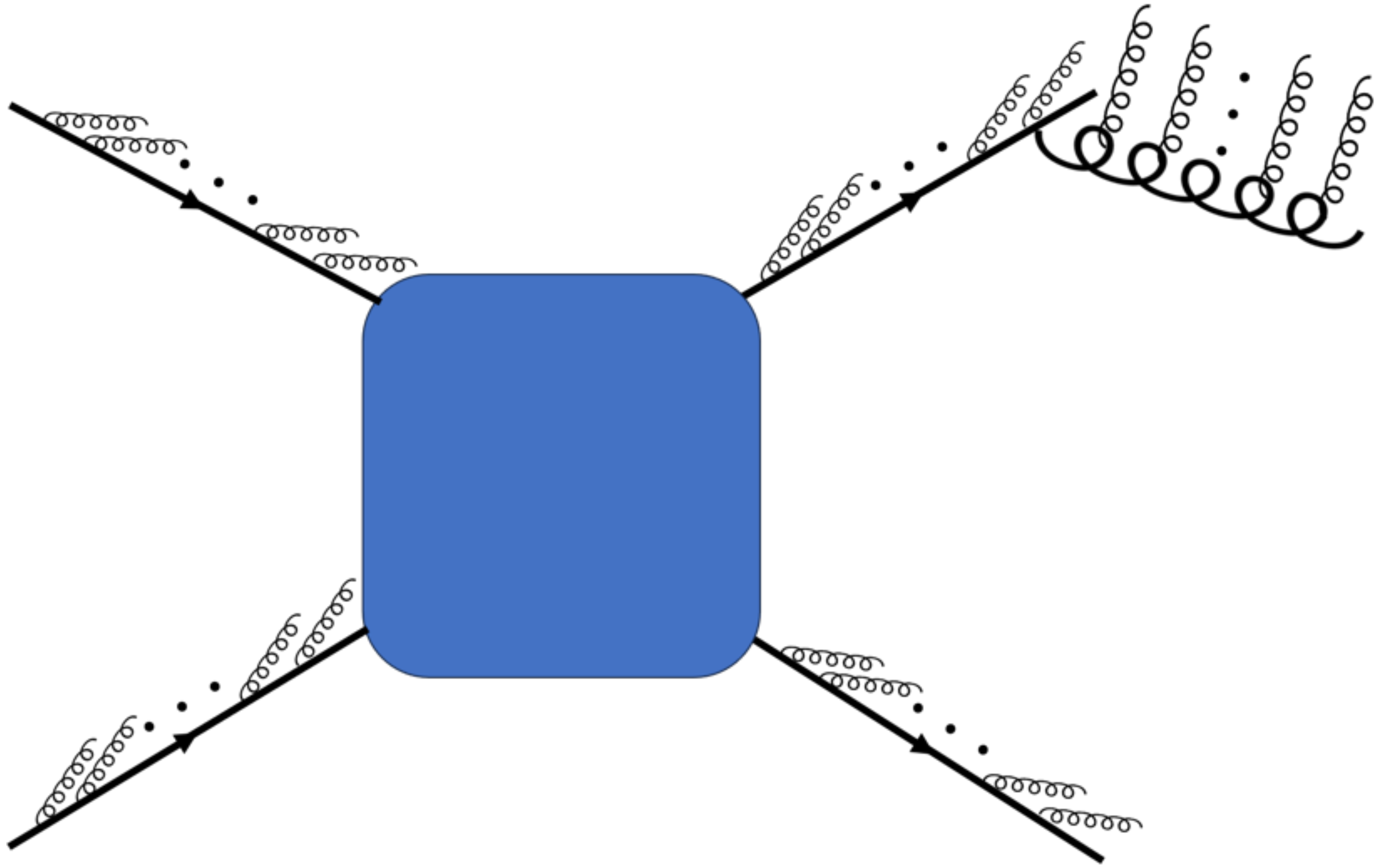
$$N_{\text{event}} = \mathscr{H}_{hard} \otimes x_1 f(x_1) \otimes x_2 f(x_2) \otimes D(z_1) \otimes D(z_2) \otimes S_{\text{ISR}} \otimes S_{\text{FSR}} \otimes P_{\text{MPI}} \otimes P_{\text{decay}} \ldots$$

- Describing fully exclusive hadronic state
- Including non-perturbative effects (MPI, SR…)
- Keeping momentum-conservation in each branching
- Impact studies for future experiments
- …



Credit: Benjamin Nachman

Parton shower algorithms in M.C. event generator



Parton shower algorithms are dedicated to simulating the radiation behavior of quarks and gluons.

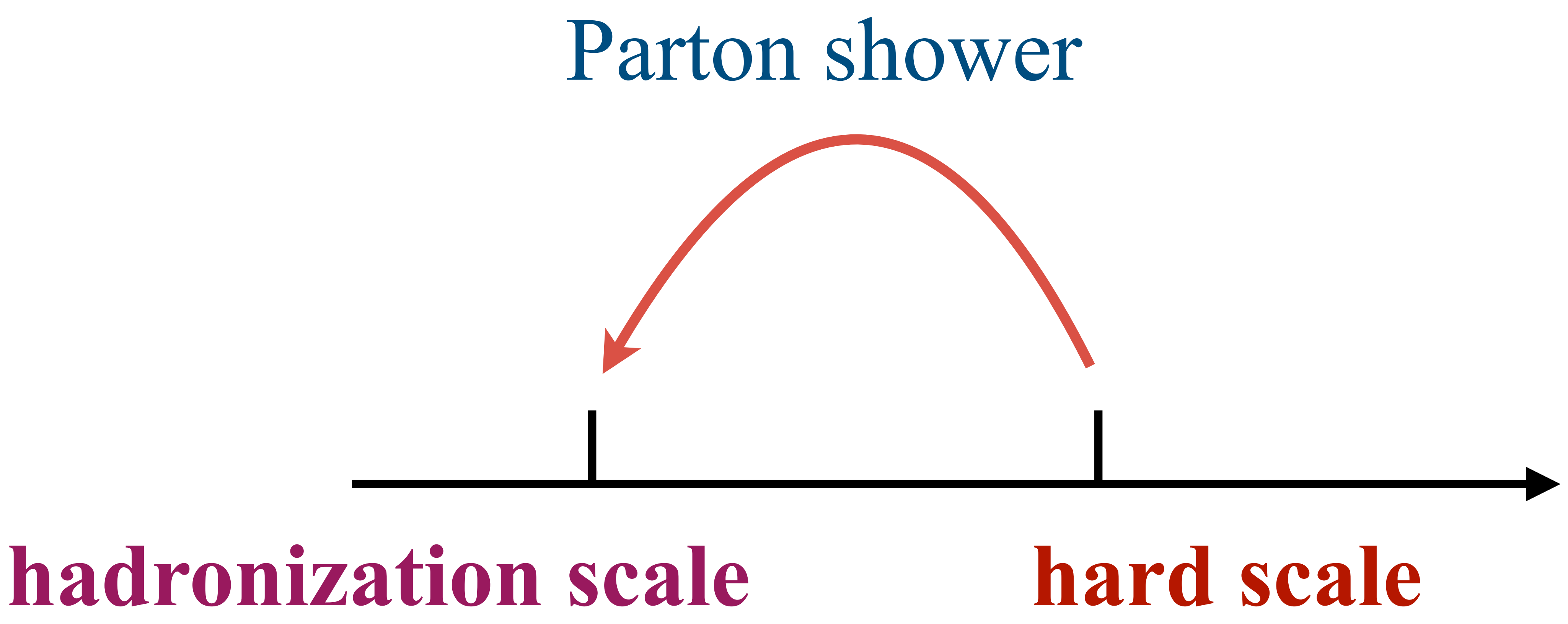
Parton shower: a model for the evolution from high scale to hadronization scale based on DGLAP/CCFM.

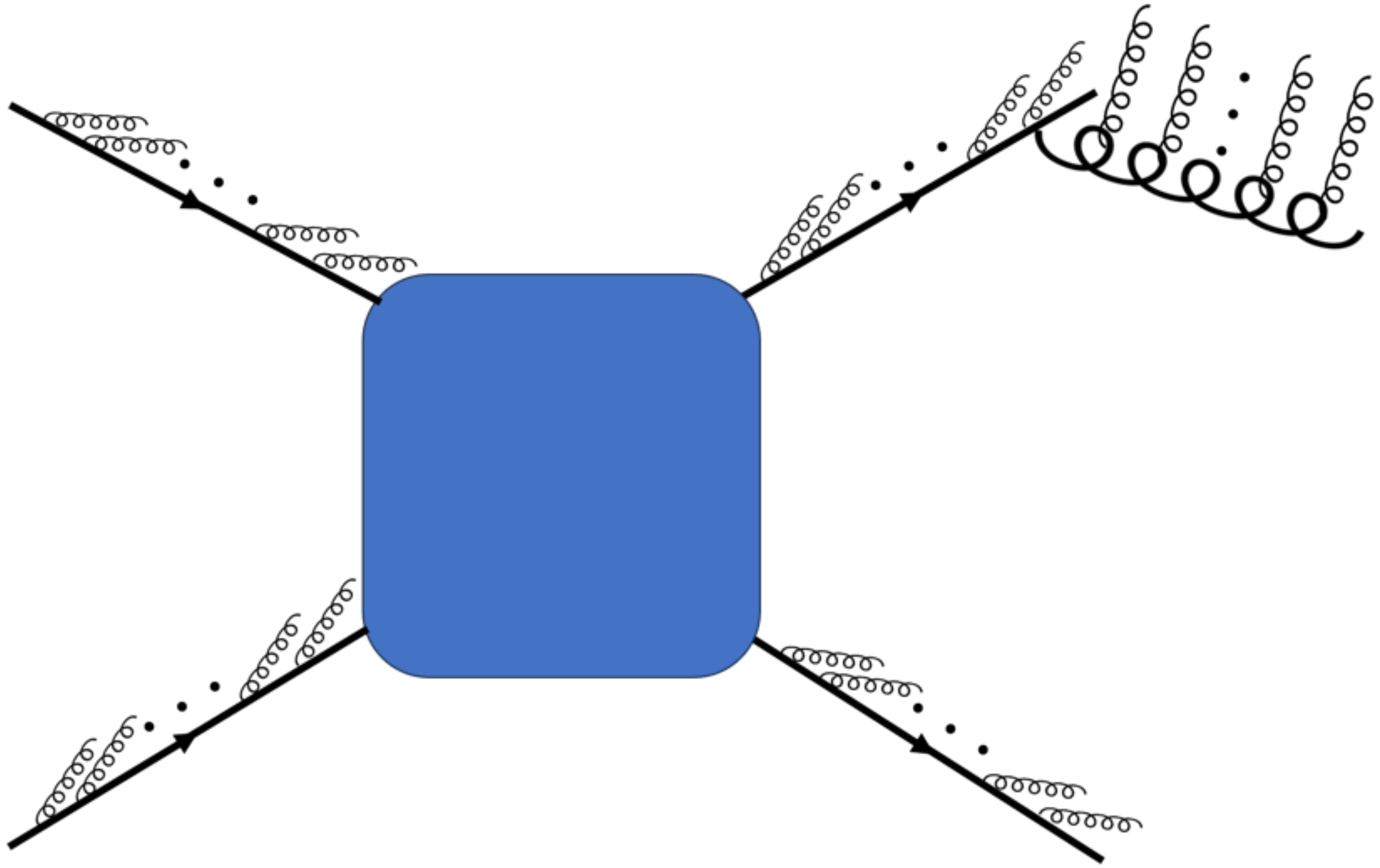
The same physics as resummation

Parton shower algorithms in M.C. event generator

Sudakov form factor

$$\Delta_a(t,t') = \exp\left\{ - \sum_{b\in\{q,g\}} \int_t^{t'} \frac{\mathrm{d}\bar{t}}{\bar{t}} \int_{z_{\min}}^{z_{\max}} \mathrm{d}z\, \frac{\alpha_s}{2\pi}\, \frac{1}{2}\, P_{ab}(z) \right\}$$





Parton shower algorithms are dedicated to simulating the radiation behavior of quarks and gluons.

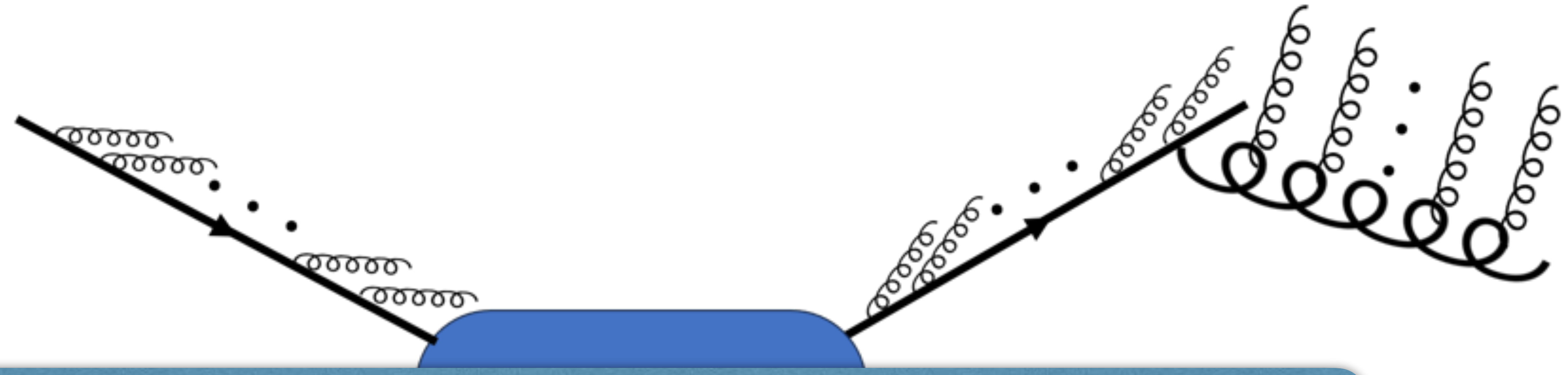
Parton shower: a model for the evolution from high scale to hadronization scale based on DGLAP/CCFM.

The same physics as resummation

Parton shower algorithms in M.C. event generator

Sudakov form factor

$$\Delta_a(t,t') = \exp\left\{-\sum_{b\in\{q,g\}}\int_t^{t'}\frac{\mathrm{d}\bar{t}}{\bar{t}}\int_{z_{\mathrm{min}}}^{z_{\mathrm{max}}}\mathrm{d}z\,\frac{\alpha_s}{2\pi}\,\frac{1}{2}\,P_{ab}(z)\right\}$$



Parton shower

hadronizatio

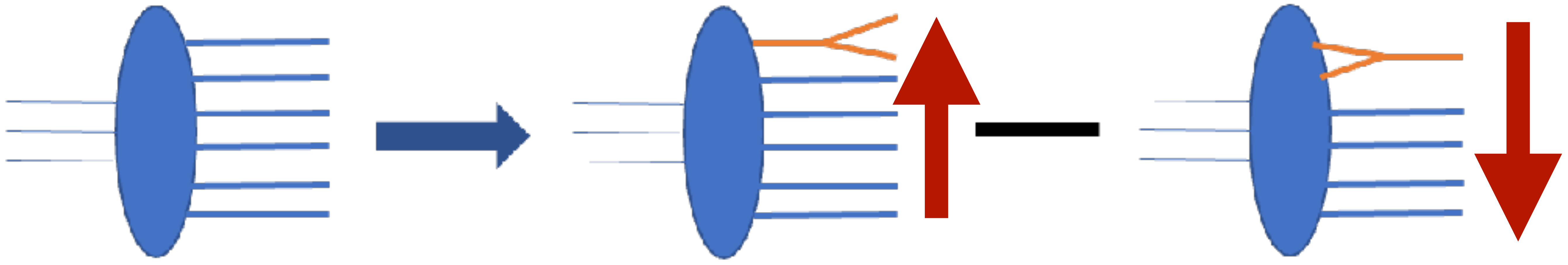
Can we use the parton shower to study the small-x physics?

Parton shower algorithms are dedicated to simulating the radiation behavior of quarks and gluons.

Parton shower: a model for the evolution from high scale to hadronization scale based on DGLAP/CCFM.

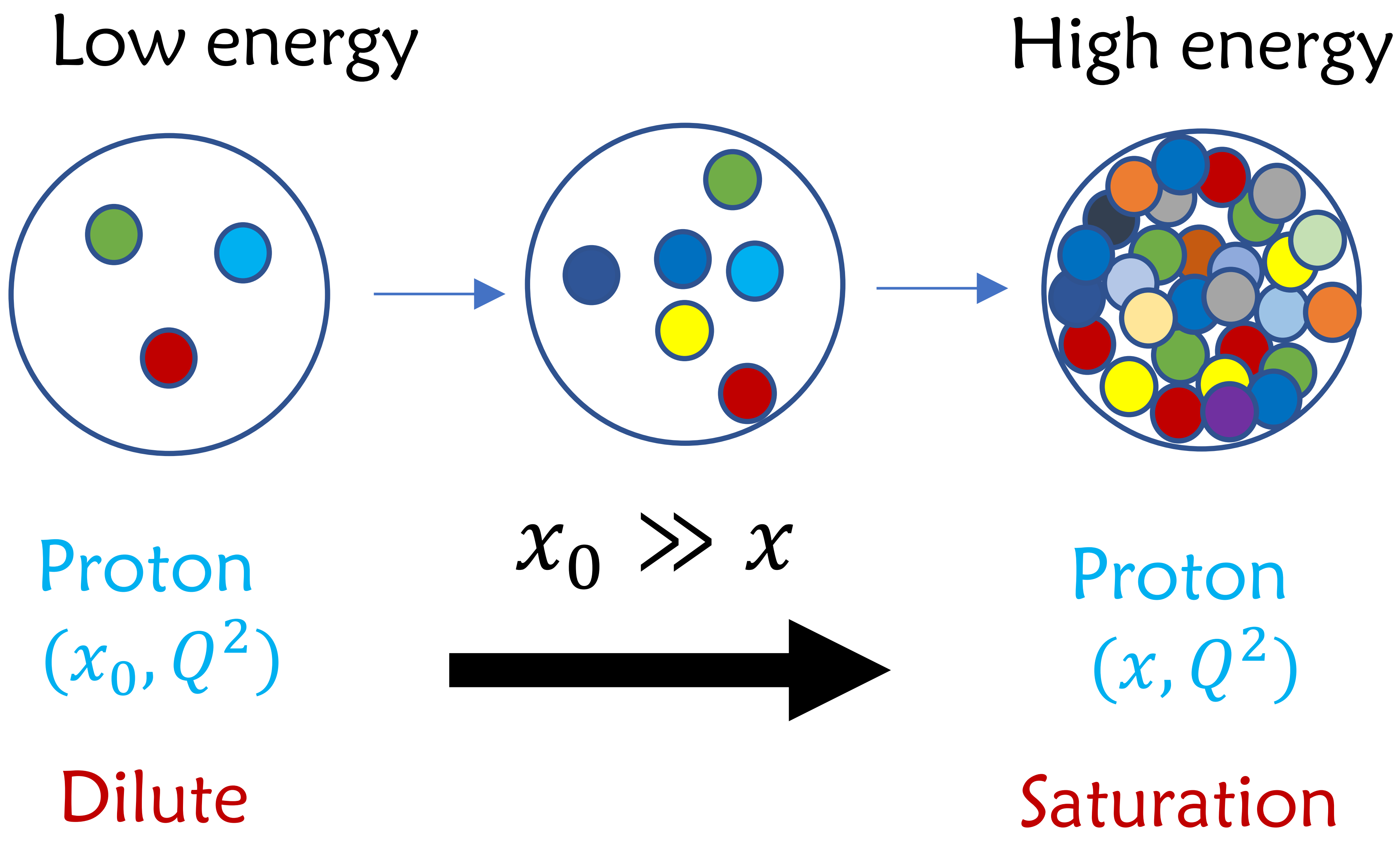
The same physics as resummation

Small-x non-linear evolution equations



Gluon splitting   Gluon fusion

- The small-x evolution equation $\ln(1/x)$



Low energy   High energy

Proton (x_0, Q^2)

Dilute

$x_0 \gg x$

Proton (x, Q^2)

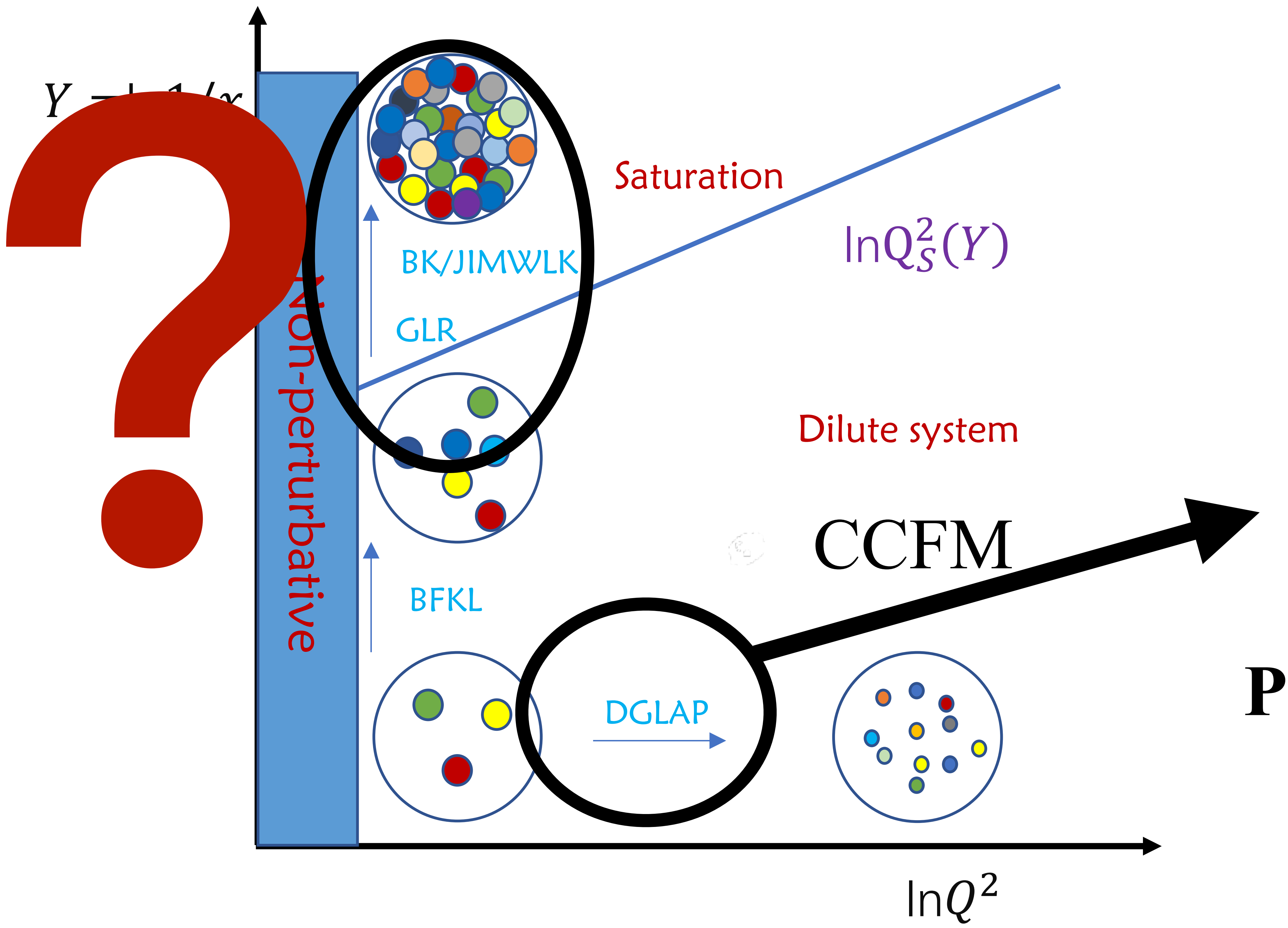
Saturation

- The GLR equation [Gribov, Levin, Ryskin, PR, 83]   Gluon fusion 2 → 1
- The BK equation [Balitsky, NPB, 96; Kovchegov, PRD, 98]   Gluon fusion 2 → 1, 3 → 1, 4 → 1…
- GLR/BK/JIMWLK equations are the non-linear evolution equations which describe gluons' non-linear evolution in the small-x region.

Particles production in the DIS

Full exclusive process

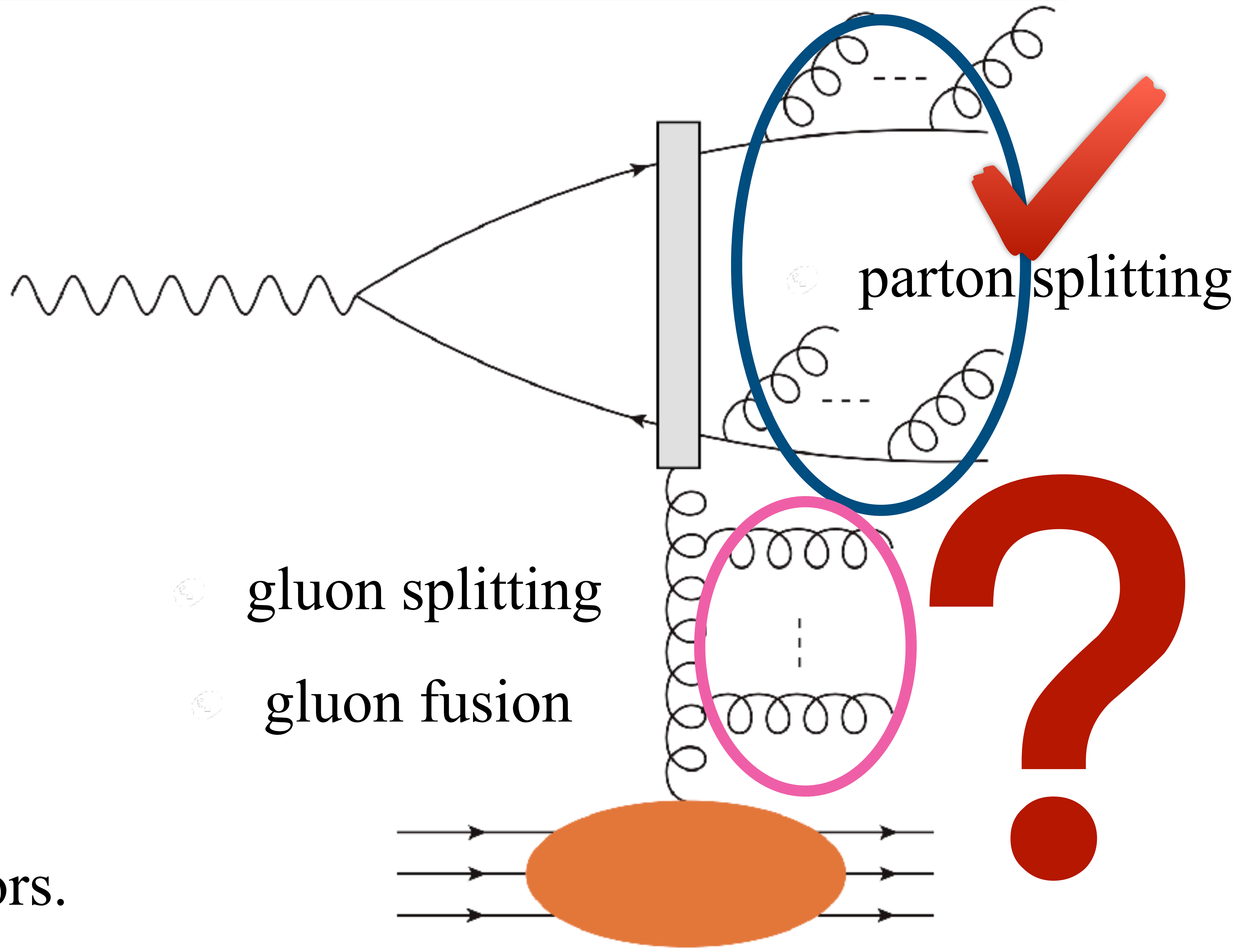
$$N_{\text{event}} = \mathscr{H}_{hard} \otimes \mathscr{N}(k_\perp) \otimes D(z) \otimes S_{\text{ISR}} \otimes S_{\text{FSR}} \otimes P_{\text{MPI}} \otimes P_{\text{decay}} \cdots$$



Cascade

CAT

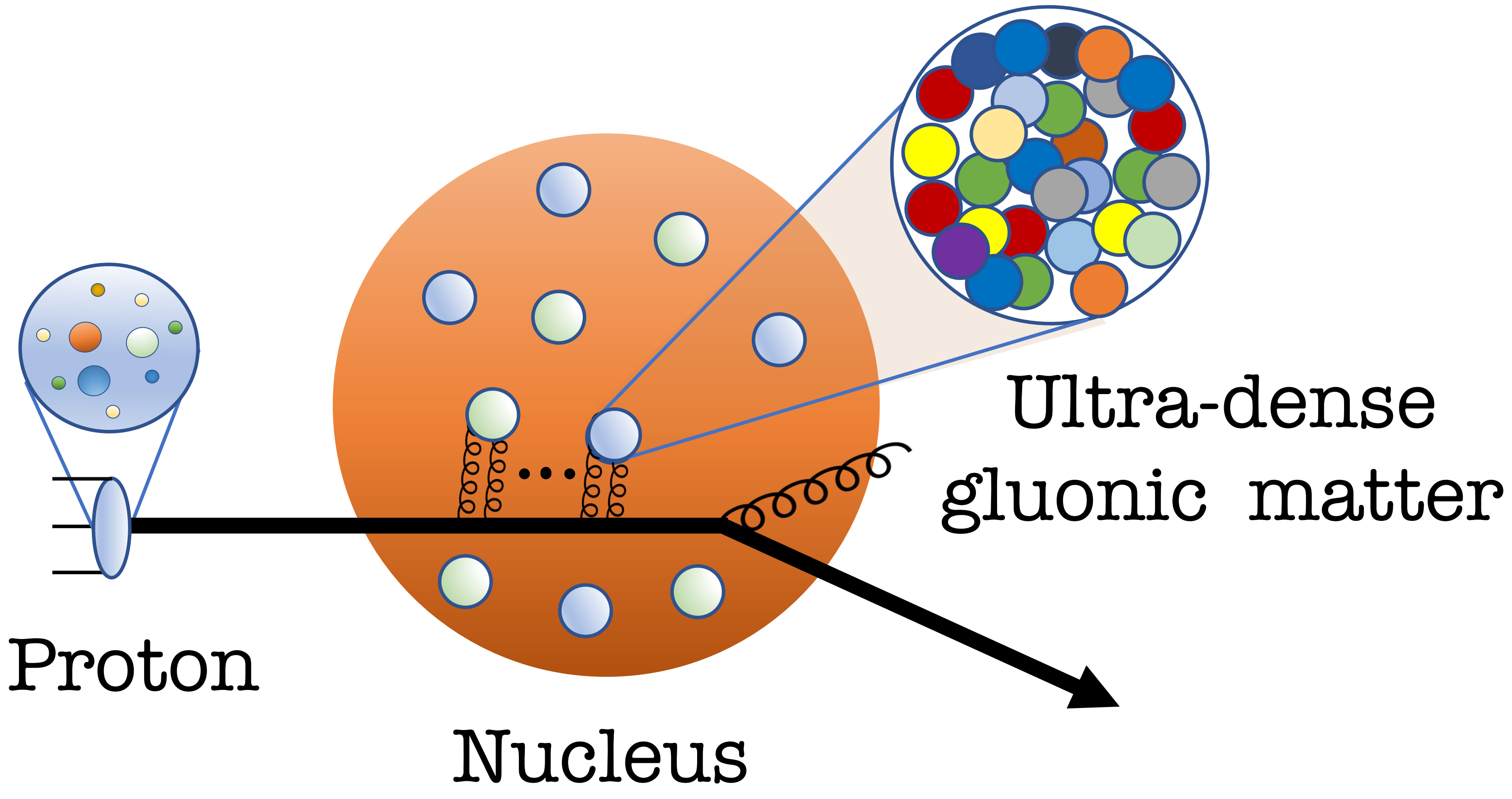
PYSHOW

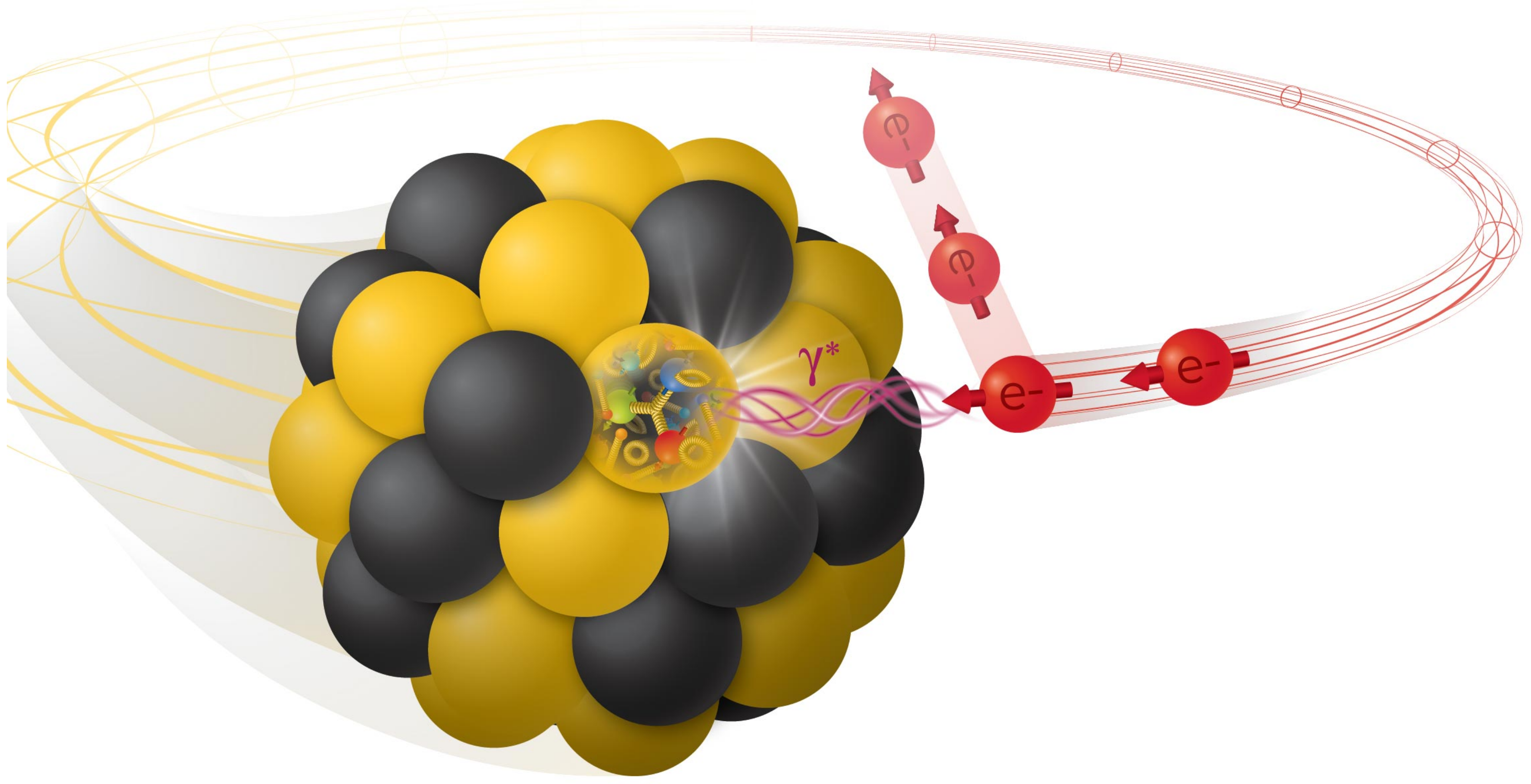


- gluon fusion effect is absent in all existing generators.
- Developing a P.S. algorithm based on the small-x nonlinear evolution equation is important.

Why do we need the small-x parton shower?

- gluon fusion effect is absent in all existing generators.
- Developing a P.S. algorithm including the gluon fusion effect is important.





- Studying the forward physics in pp&pA collisions at RHIC and LHC.
- Phenomenology in gamma-A collisions at UPC and future EIC.
- Cosmic ray event generator.

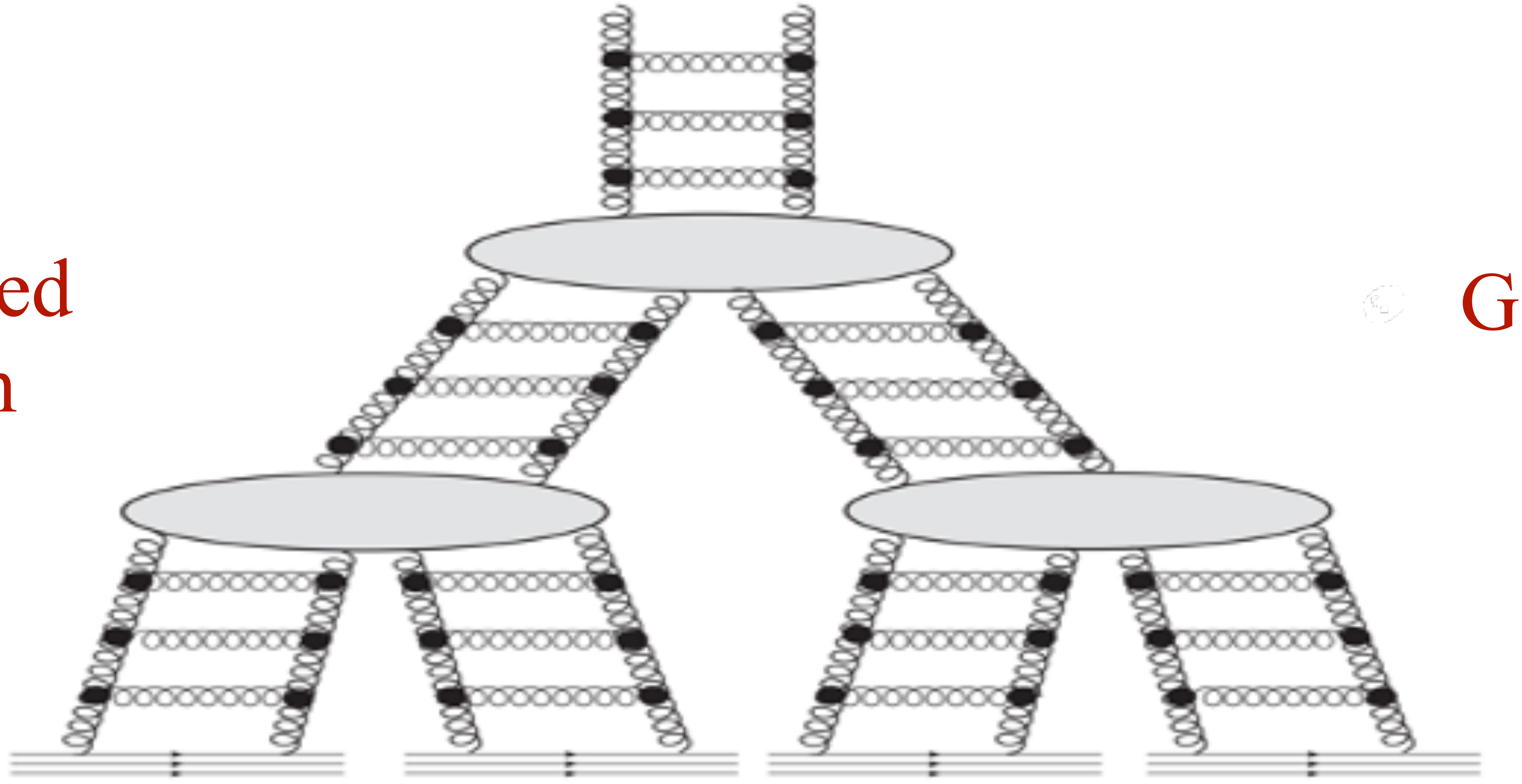
GLR evolution Equation

- The GLR equation [Gribov, Levin, Ryskin, PR, 83]

$$\frac{\partial N(\eta,k_\perp)}{\partial\eta}=\frac{\bar\alpha_s}{\pi}\left[\int\frac{\mathrm{d}^2l_\perp}{l_\perp^2}N(\eta,k_\perp+l_\perp)-\int_0^{k_\perp}\frac{\mathrm{d}^2l_\perp}{l_\perp^2}N(\eta,k_\perp)\right]-\bar\alpha_sN^2(\eta,k_\perp)$$

with the dipole gluon distribution $N(\eta,k_\perp)=\frac{2\alpha_s\pi^3}{N_cS_\perp}G(\eta,k_\perp)$

“fan” diagram resumed by the GLR equation



Gluon fusion 2 → 1

$$\eta=\ln\left(x_0/x\right)$$

- GLR equation is the non-linear evolution equation that describes the gluon diffusion process.

GLR evolution Equation

- Resolved and unresolved branching

[YS, Wei, Zhou, PRD, 2023]

$$\int \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta, k_\perp + l_\perp) \approx \int_\mu \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta, k_\perp + l_\perp) + \int_0^\mu \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta, k_\perp)$$

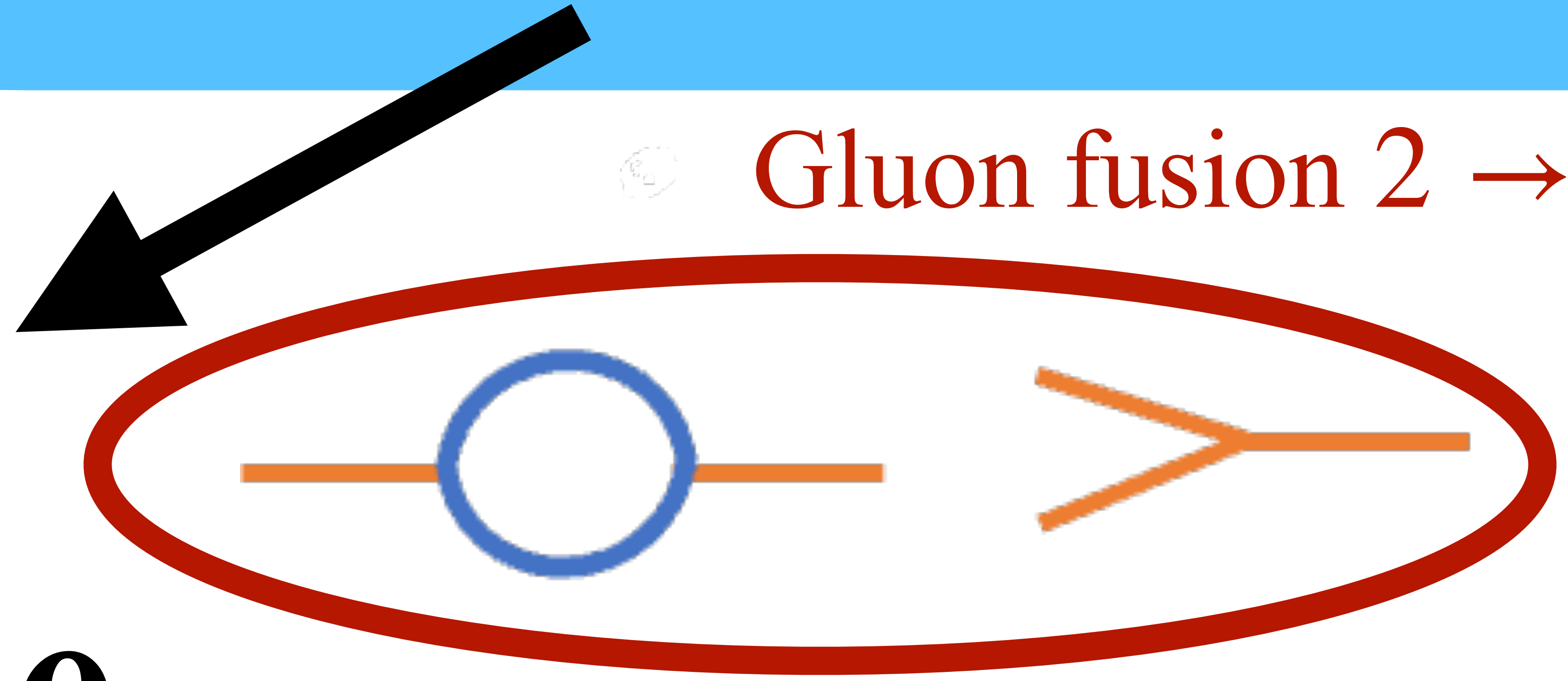
- Non-Sudakov form factor resums the virtual and non-linear term

$$\Delta(\eta, k_\perp) = \exp\left\{ -\bar{\alpha}_s \int_{\eta_0}^{\eta} d\eta' \left[\ln \frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp) \right] \right\}$$

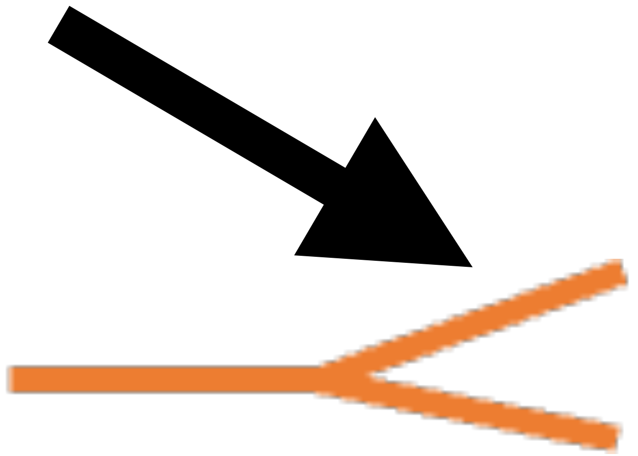
- Non-Sudakov form factor is the probability of gluon evolution without gluon splitting.
- The integral GLR equation (folded one)

Independent on the choice of μ

$$N(\eta, k_\perp) = N(\eta_0, k_\perp)\Delta(\eta, k_\perp) + \frac{\bar{\alpha}_s}{\pi} \int_{\eta_0}^{\eta} \mathrm{d}\eta' \frac{\Delta(\eta, k_\perp)}{\Delta(\eta', k_\perp)} \int_\mu \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta', l_\perp + k_\perp)$$



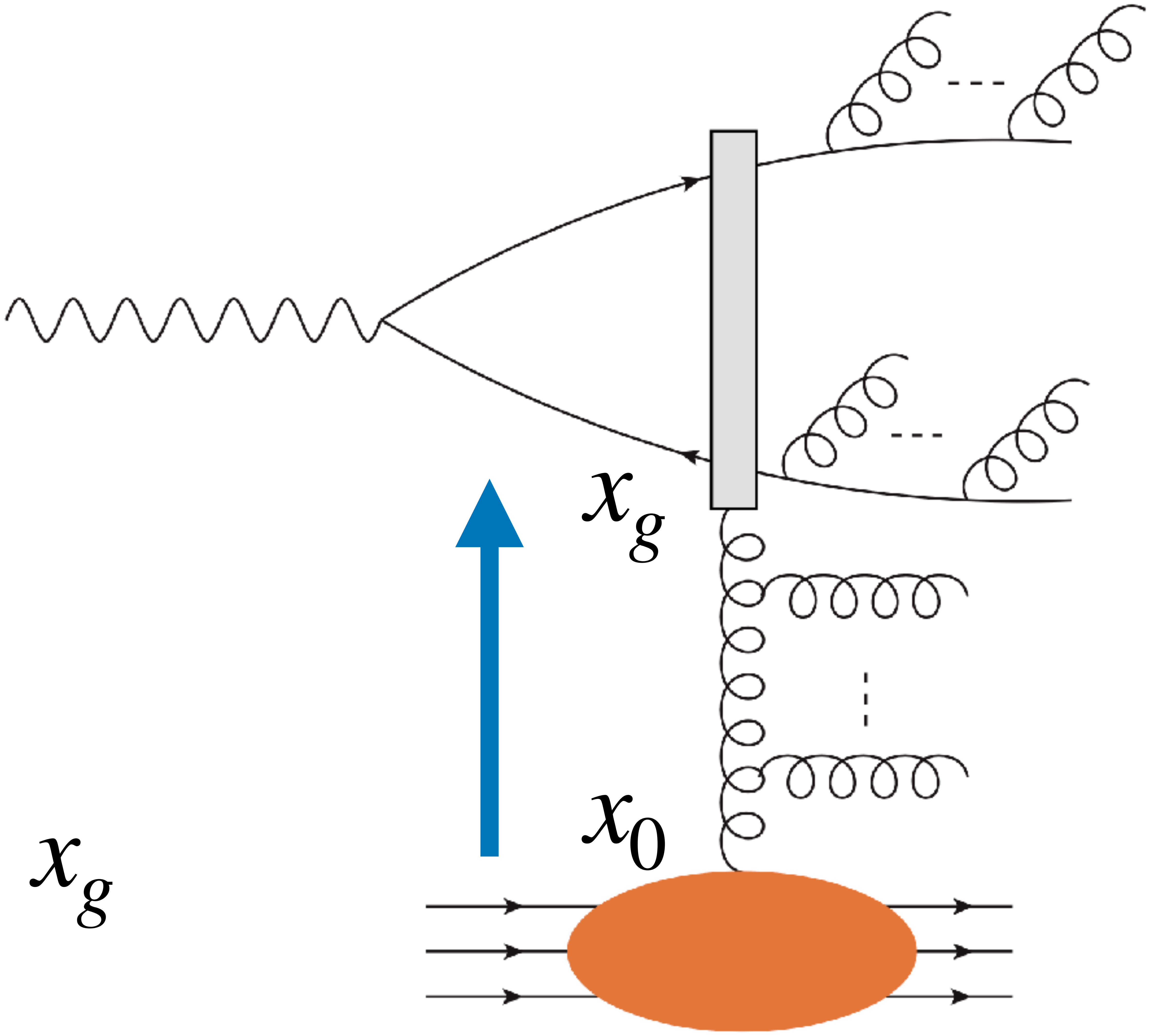
Gluon fusion 2 → 1



Gluon splitting

e

The forward evolution algorithm



$$x_0 \gg x_g$$

The forward evolution algorithm

First step: non-Sudakov form factor

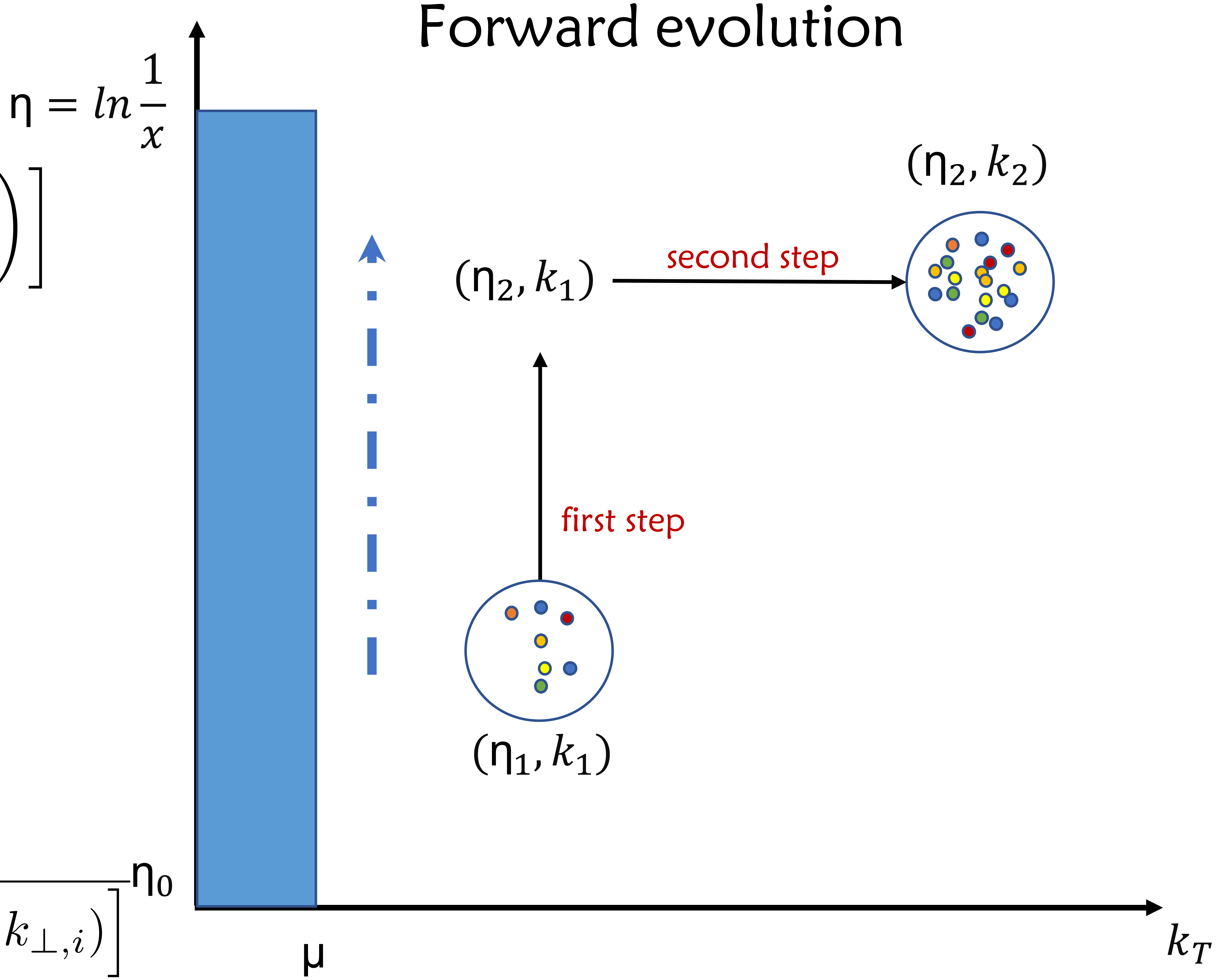
$$\mathcal{R} = \exp\left[-\bar{\alpha}_s \int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta' \left(\ln\frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp)\right)\right]$$

Second step: Real splitting kernel

$$\mathcal{R}_2 \int_{\mu}^{P_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} = \int_{\mu}^{|l_\perp|} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp}$$

The generated event has to be re-weighted

$$\mathcal{W}(\eta_i, \eta_{i+1}; k_{\perp,i}) = \frac{\int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta \ln(P_\perp^2/\mu^2)}{\int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta \left[\ln(k_{\perp,i}^2/\mu^2) + N(\eta, k_{\perp,i})\right]}$$



The forward evolution algorithm

First step: non-Sudakov form factor

$$\mathcal{R} = \exp\left[-\bar{\alpha}_s \int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta' \left(\ln \frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp)\right)\right]$$

Second step: Real splitting kernel

$$\mathcal{R}_2 \int_{\mu}^{P_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} = \int_{\mu}^{|l_\perp|} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp}$$

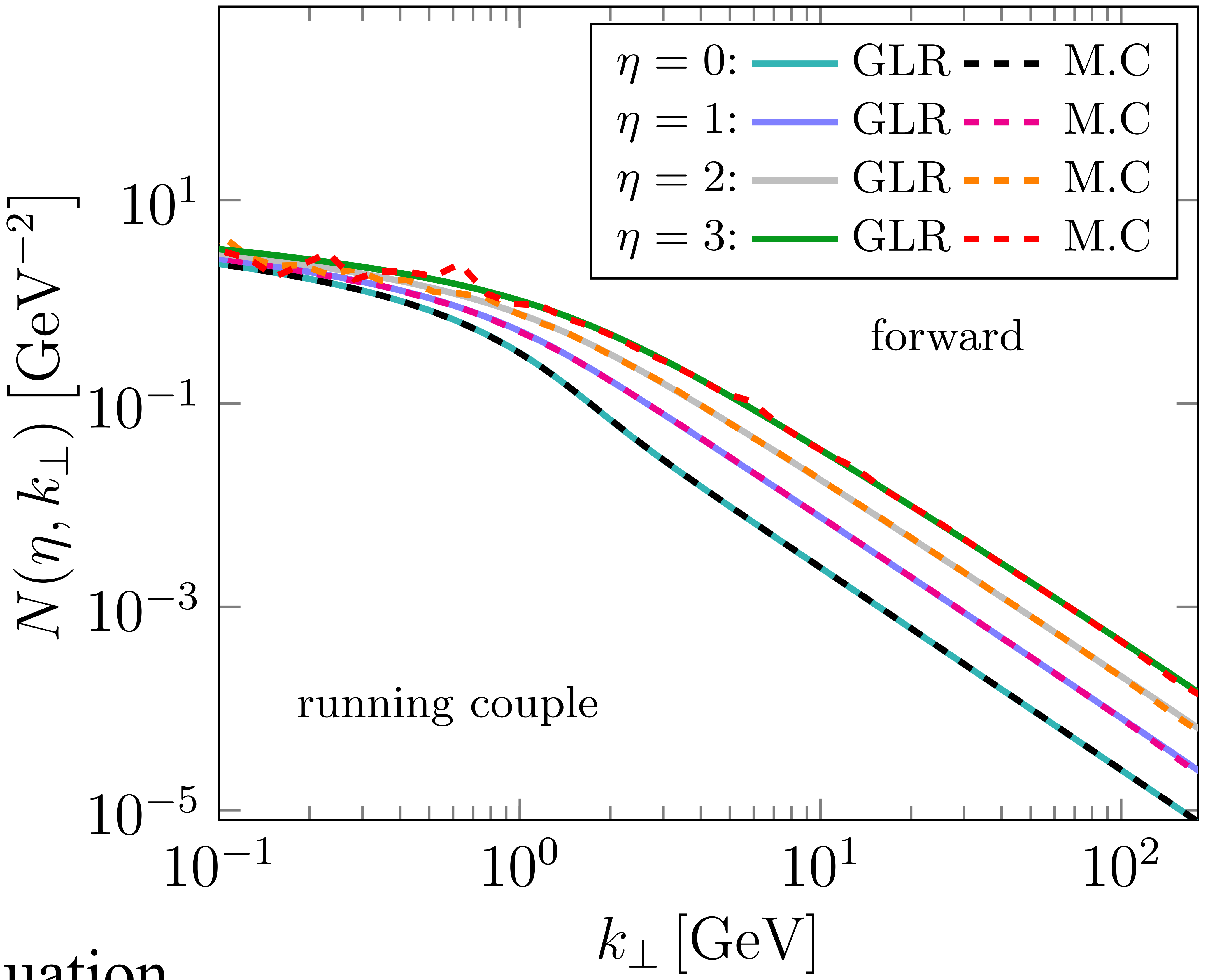
The generated event has to be re-weighted

$$\mathcal{W}(\eta_i, \eta_{i+1}; k_{\perp,i}) = \frac{\int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta \ln(P_\perp^2/\mu^2)}{\int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta \left[\ln(k_{\perp,i}^2/\mu^2) + N(\eta, k_{\perp,i})\right]}$$

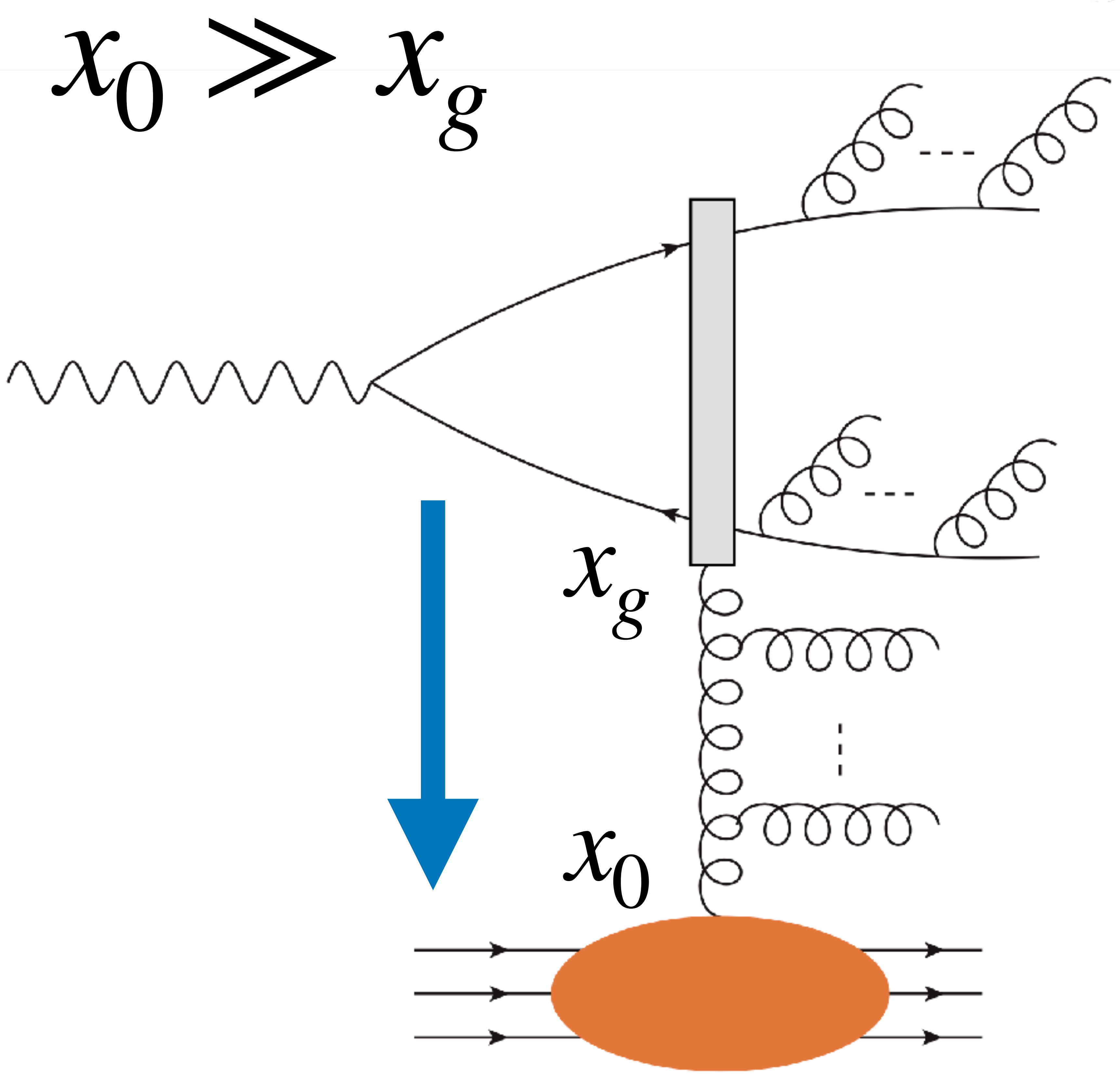
- Agree with the numerical solutions of the GLR equation.

The initial condition likes

$$N(\eta = 0, k_\perp) = \int \frac{d^2 r_\perp}{2\pi} e^{-ik_\perp \cdot r_\perp} \frac{1}{r_\perp^2} \left(1 - \exp\left[-\frac{1}{4} Q_{s0}^2 r_\perp^2 \ln(e + \frac{1}{\Lambda r_\perp})\right]\right)$$



The backward evolution algorithm



First step: backward non-Sudakov form factor

$$\mathcal{R} = \exp\left[-\frac{\bar{\alpha}_s}{\pi}\int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta \int_{\mu} \frac{\mathrm{d}^2 l_\perp}{l_\perp^2}\frac{N(\eta, k_{\perp,i+1}+l_\perp)}{N(\eta, k_{\perp,i+1})}\right]$$

Second step: Real splitting

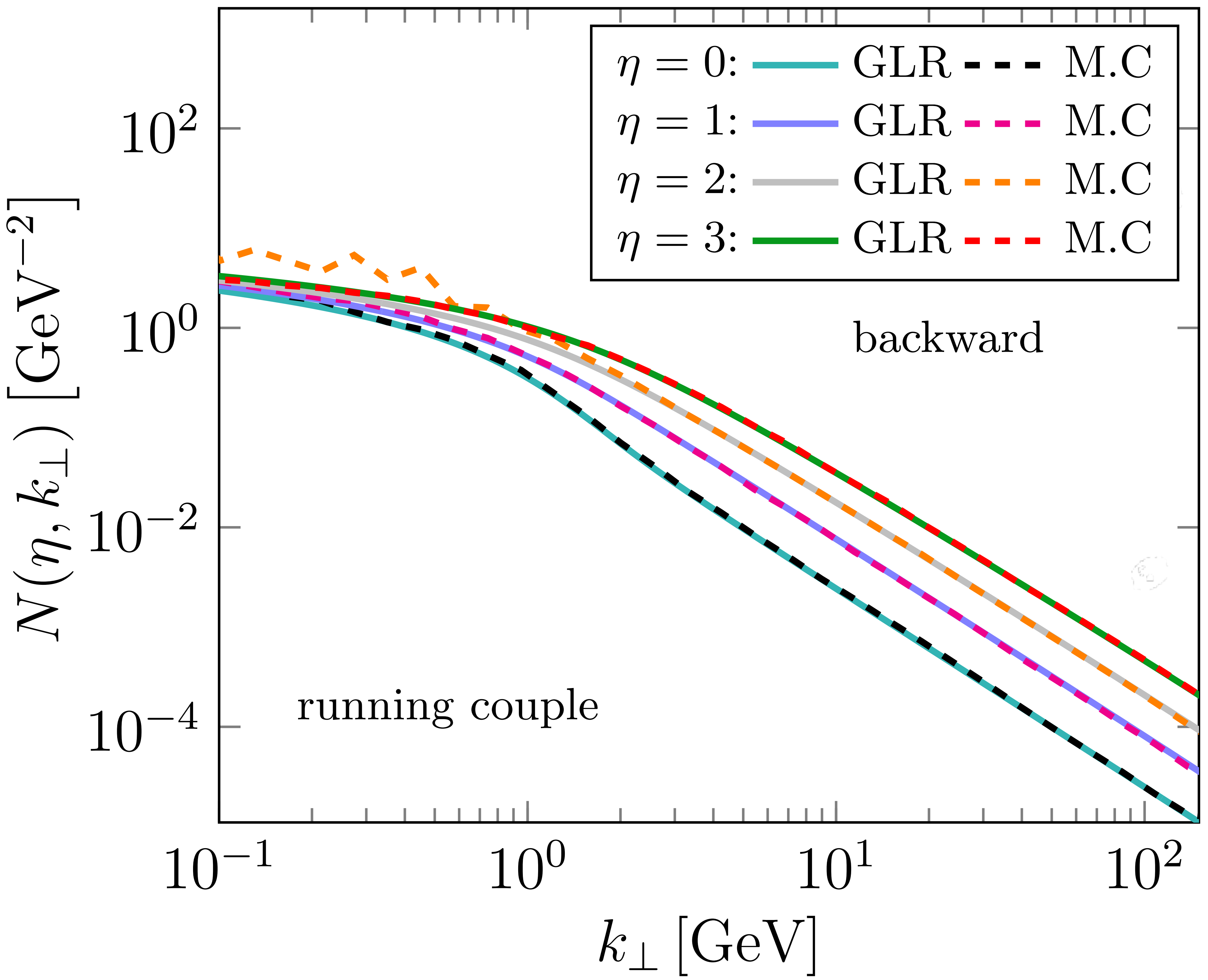
$$\frac{\bar{\alpha}_s}{\pi}\int_{\mu}^{l_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} N(\eta_i, k_{\perp,i+1}+l'_\perp) = \mathcal{R}_2 \frac{\bar{\alpha}_s}{\pi}\int_{\mu}^{P_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} N(\eta_i, k_{\perp,i+1}+l'_\perp)$$

The generated event has to be re-weighted

$$\mathcal{W}_{\mathrm{backward}} = \frac{1}{\mathcal{W}_{forward}}$$

- As a more efficient procedure, the backward evolution approach is also presented.
- Using the numerical solution of the GLR equation $N(\eta, k_\perp)$ to guide the backward evolution.

The backward evolution algorithm



First step: backward non-Sudakov form factor

$$\mathcal{R} = \exp\left[-\frac{\bar{\alpha}_s}{\pi}\int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta \int_{\mu} \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} \frac{N(\eta, k_{\perp,i+1} + l_\perp)}{N(\eta, k_{\perp,i+1})}\right]$$

Second step: Real splitting

$$\frac{\bar{\alpha}_s}{\pi}\int_{\mu}^{l_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} N(\eta_i, k_{\perp,i+1} + l'_\perp) = \mathcal{R}_2 \frac{\bar{\alpha}_s}{\pi}\int_{\mu}^{P_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} N(\eta_i, k_{\perp,i+1} + l'_\perp)$$

The generated event has to be re-weighted

$$\mathcal{W}_{\text{backward}} = \frac{1}{\mathcal{W}_{forward}}$$

- As a more efficient procedure, the backward evolution approach is also presented.
- Using the numerical solution of the GLR equation $N(\eta, k_\perp)$ to guide the backward evolution.

Parton shower algorithms

GLR v.s. DGLAP/CCFM

$$\Delta(\eta, k_\perp) = \exp\left\{-\bar{\alpha}_s \int_{\eta_0}^{\eta} d\eta' \left[\ln\frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp)\right]\right\}$$

$$\Delta_a(t,t') = \exp\left\{-\sum_{b\in\{q,g\}} \int_t^{t'} \frac{\mathrm{d}\bar{t}}{\bar{t}} \int_{z_{\mathrm{min}}}^{z_{\mathrm{max}}} \mathrm{d}z\, \frac{\alpha_s}{2\pi} \frac{1}{2} P_{ab}(z)\right\}$$

gluon splitting
gluon fusion

parton splitting

The evolution variable:

$\eta = \ln(1/x)$ Q

The generated event:

reweight Unitary

Joint the kt resummation in the small-x region

- multiple well-separated hard scales in some processes: (dijet and dihadron)
- the invariant mass of di-jet Q^2, the total transverse momentum of di-jet $k_\perp$ and the momentum fraction x_g

$$P_{gg}(\xi) = 2C_A\frac{(\xi^2 - \xi + 1)^2}{\xi(1-\xi)}$$

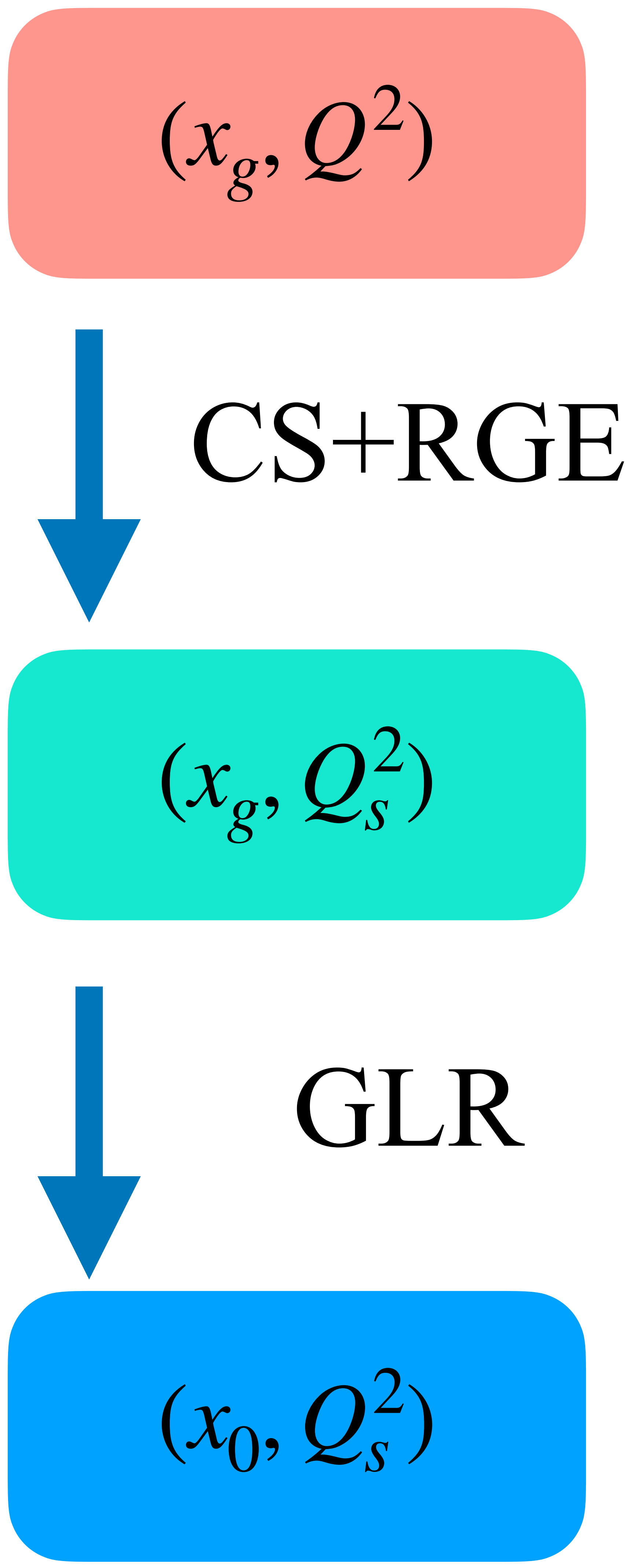
$\xi \to 0$ $\ln(1/x)$ ✓ Small-x evolution equation

$\xi \to 1$ $\ln^2\left(Q^2/k_\perp^2\right)$ $\ln\left(Q^2/k_\perp^2\right)$ ✓ CS +RGE evolution equation

[Mueller, Xiao, Yuan, PRL, 12; Zheng, Aschenauer, Lee, Xiao, PRD, 14;]

$$N(Q^2,\eta,k_\perp) = \int\frac{d^2b_\perp}{(2\pi)^2}e^{ik_\perp\cdot b_\perp}e^{-S(\mu_b^2,Q^2)}\int d^2l_\perp e^{-il_\perp\cdot b_\perp}N(\eta,l_\perp)$$

- In our parton shower, we need resum both small-x and soft-collinear logarithms.



(x_g, Q^2)

CS+RGE

(x_g, Q_s^2)

GLR

(x_0, Q_s^2)

Joint the kt resummation in the small-x region

- Combine CS + RGE [Collins, Soper, 81; Collins, Soper, Sterman, 85; Xiao, Yuan, Zhou, NPB, 17; YS, Wei, Zhou, PRD, 2023]

$$\frac{\partial N(Q^2,\eta,k_\perp)}{\partial \ln Q^2} = \frac{\bar{\alpha}_s}{2\pi}\int_0^Q \frac{d^2 l_\perp}{l_\perp^2}\left[N(Q^2,\eta,k_\perp+l_\perp) - N(Q^2,\eta,k_\perp)\right] + \bar{\alpha}_s\beta_0 N(Q^2,\eta,k_\perp)$$

where $N(Q^2,\eta,k_\perp) \equiv N(\mu^2=Q^2,\zeta^2=Q^2,\eta,k_\perp)$

- The integral equation (folded one)

$$N(Q^2,\eta,k_\perp) = N(Q_0^2,\eta,k_\perp)\Delta_s(Q^2) + \int_{Q_0^2}^{Q^2}\frac{dt}{t}\frac{\Delta_s(Q^2)}{\Delta_s(t)}\frac{\bar{\alpha}_s(t)}{2\pi}\int_{\Lambda_{\rm cut}}^{Q}\frac{d^2 l_\perp}{l_\perp^2}N(t,\eta,k_\perp+l_\perp)$$

With Sudakov form factor

$$\Delta_s(Q^2) = \exp\left[-\int_{Q_0^2}^{Q^2}\frac{dt}{t}\frac{\bar{\alpha}_s(t)}{2}\left(\ln\frac{t}{\Lambda_{\rm cut}^2} - 2\beta_0\right)\right]$$

The forward and backward evolution of CS+RGE

First step: Sudakov form factor

$$\mathcal{R} = \exp\left[-\int_{Q_i^2}^{Q_{i+1}^2} \frac{dt}{t}\bar{\alpha}_s(t)\left(\frac{1}{2}\ln\frac{t}{\Lambda_{\rm cut}^2} - \beta_0\right)\right]$$

$$\mathcal{R} = \exp\left[-\int_{Q_i^2}^{Q_{i+1}^2} \frac{dt}{t}\frac{\bar{\alpha}_s(t)}{2\pi}\int_{\Lambda_{\rm cut}}^{\sqrt{t}} \frac{d^2 l_\perp}{l_\perp^2}\frac{N(t,\eta,k_{\perp,i+1}+l_\perp)}{N(t,\eta,k_{\perp,i+1})}\right.$$

Second step: Real splitting kernel

$$\int_{\Lambda_{\rm cut}}^{Q_{i+1}} \frac{d^2 l'_\perp}{l'^2_\perp} = \mathcal{R}_2\int_{\Lambda_{\rm cut}}^{|l_\perp|} \frac{d^2 l'_\perp}{l'^2_\perp}$$

$$\mathcal{R}\int_{\Lambda_{\rm cut}}^{Q_i} \frac{d^2 l'_\perp}{l'^2_\perp} N(Q_i^2,\eta,k_{\perp,i+1}+l'_\perp) = \int_{\Lambda_{\rm cut}}^{l_{\perp,i}} \frac{d^2 l'_\perp}{l'^2_\perp} N(Q_i^2,\eta,k_{\perp,i+1}+l'_\perp)$$

The generated event has to be re-weighted

$$\mathcal{W}_{\rm CS}(Q_{i+1}^2, Q_i^2) = \frac{\int_{Q_i^2}^{Q_{i+1}^2}\frac{dt}{t}\alpha_s(t)\ln\frac{t}{\Lambda_{\rm cut}^2}}{\int_{Q_i^2}^{Q_{i+1}^2}\frac{dt}{t}\alpha_s(t)\left[\ln\frac{t}{\Lambda_{\rm cut}^2} - 2\beta_0\right]}$$

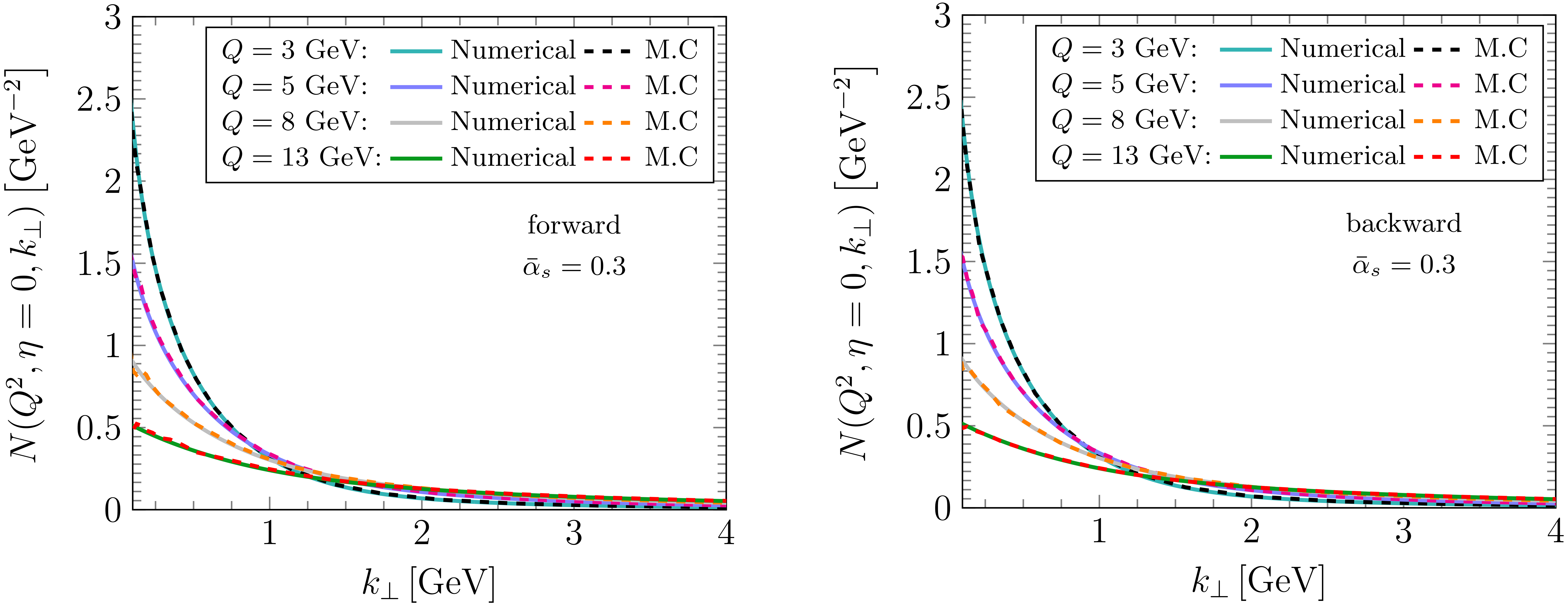
$$\mathcal{W}_{\rm backward} = \frac{1}{\mathcal{W}_{forward}}$$

Ignoring the single log, the event is unitary.

The forward & backward evolution of CS+RGE

- The initial condition is given as

$$N(Q_0 = 3\ \mathrm{GeV}, \eta = 0, k_\perp) = \int \frac{d^2 r_\perp}{2\pi} e^{ik_\perp \cdot r_\perp} \frac{1}{r_\perp^2} \left[1 - e^{-\frac{Q_s^2 r_\perp^2}{4} \log(\frac{1}{r_\perp \Lambda} + e)}\right]$$



- Agree with the numerical solutions.

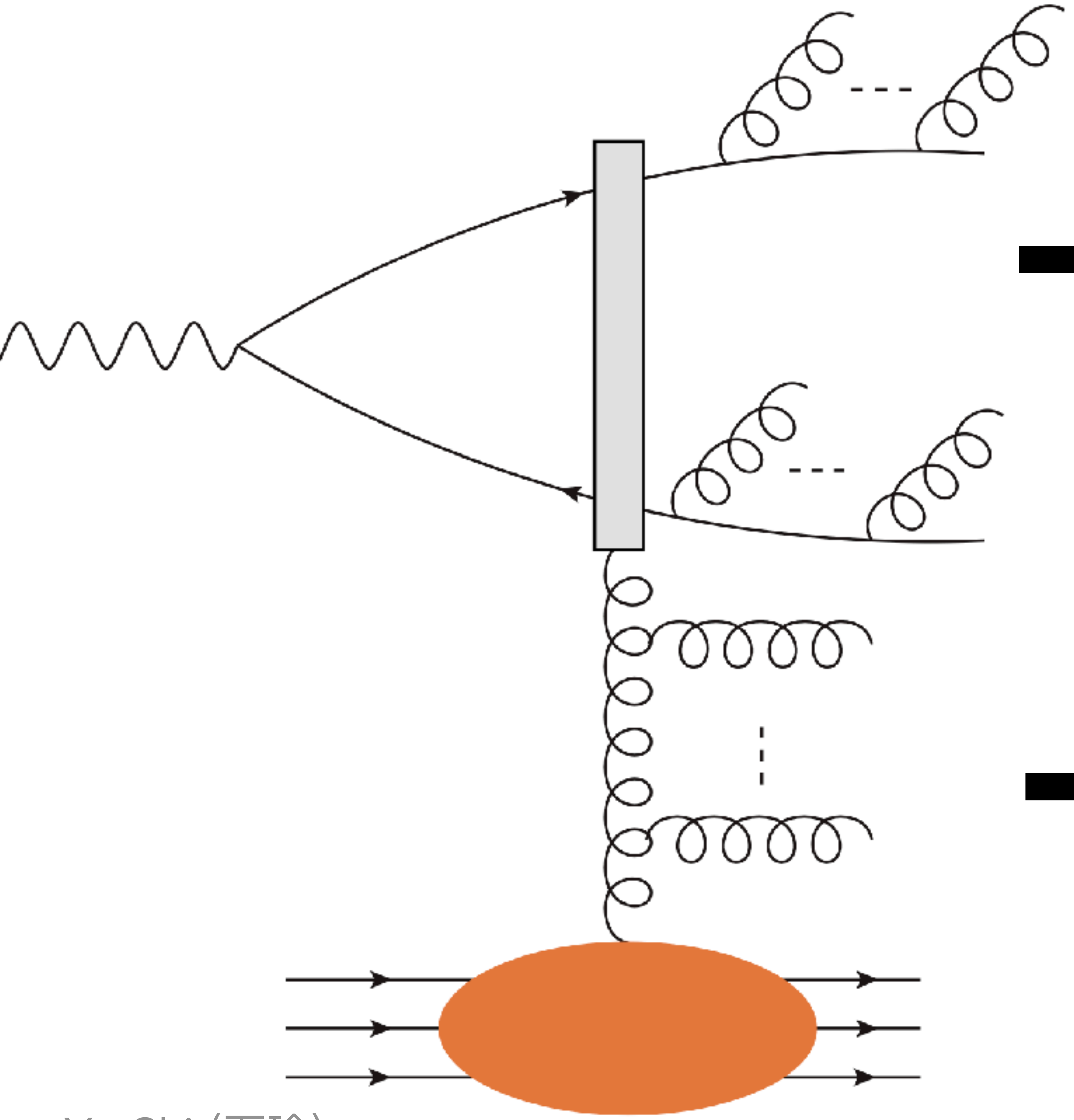
Particles production in the DIS

$$N_{\text{event}} = \mathscr{H}_{hard} \otimes \mathscr{N}(k_\perp) \otimes D(z) \otimes S_{\text{ISR}} \otimes S_{\text{FSR}} \otimes P_{\text{MPI}} \otimes P_{\text{decay}} \cdots$$

$$\frac{d\sigma^{\gamma^* A \to q\bar{q}X}}{dy_1 dy_2 d^2P_\perp d^2q_\perp} = \frac{S_\perp N_c \alpha_{\text{em}} e_q^2}{3\pi^2} x_\gamma f_\gamma(x_\gamma, \mu) \frac{z(1-z)}{P_\perp^4} \left(z^2 + (1-z)^2\right) N(x_g, q_\perp)$$

Working in progress

[Dominguez, Marquet, Xiao, Yuan, PRD, 11]



Pythia: collinear radiations

non-linear radiations
from small-x region

Pythia
hadronization….

Particles production in the DIS

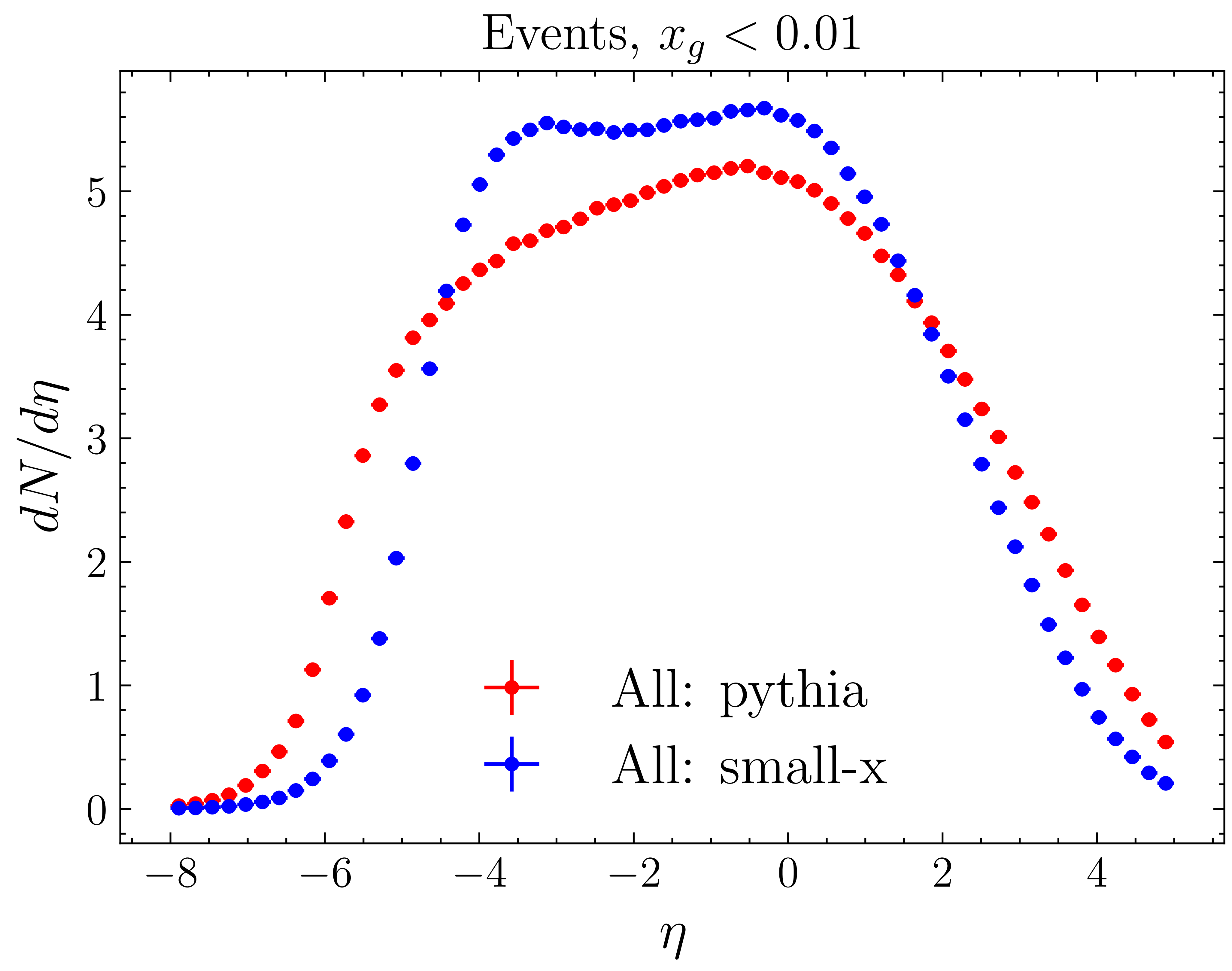
Lepton-proton collider at HERA (Photon is quasi-real photon.)

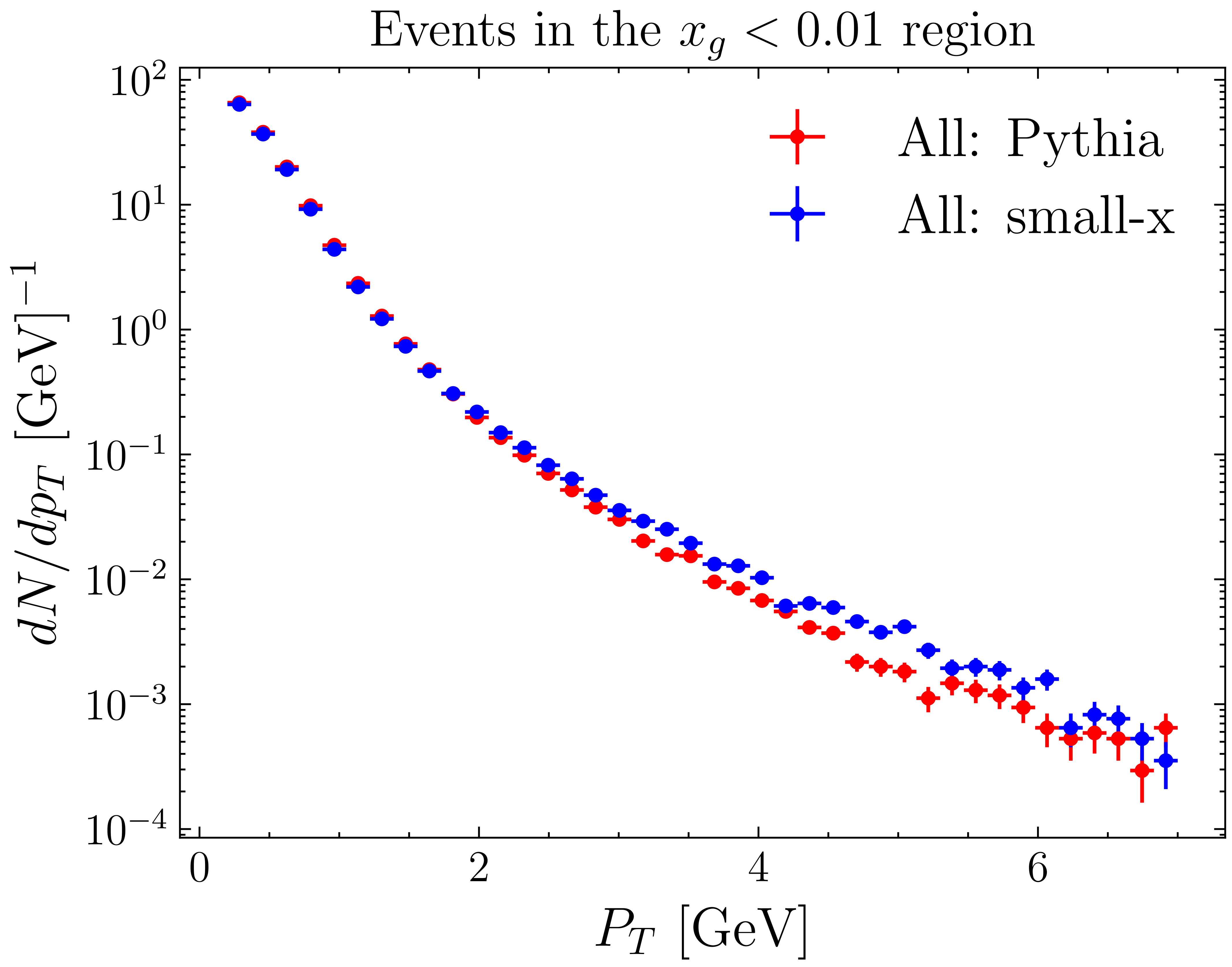
Preliminary results

Small-x Cascade

Pythia + hadronization

Working in progress





Summary and outlook

- The first parton shower algorithm incorporating gluon fusion is based on the GLR evolution equation.
- This work enables the Monte Carlo generator to simultaneously resum large-kt and small-x logarithms in the small-x regime for the first time.
- Our work paves the way for developing an event generator that incorporates the saturation effect.
- Particles production in ep&eA collisions is working in progress.
- We also plan to integrate our algorithms into eHIJING.

Thank you !

Backups

GLR evolution Equation

- The GLR equation [Gribov, Levin, Ryskin, PR, 83]

Gluon fusion 2 → 1

$$\frac{\partial G(\eta,k_\perp)}{\partial\eta}=\frac{\bar{\alpha}_s}{\pi}\left[\int\frac{\mathrm{d}^2l_\perp}{l_\perp^2}G(\eta,k_\perp+l_\perp)-\int_0^{k_\perp}\frac{\mathrm{d}^2l_\perp}{l_\perp^2}G(\eta,k_\perp)\right]-g_{\mathrm{TPV}}\frac{\alpha_s^2}{S_\perp(8\pi)^2}G^2(\eta,k_\perp)$$

$g_{\mathrm{TPV}}=8(2\pi)^4$

the dipole gluon distribution $G(\eta,k_\perp)$ $N(\eta,k_\perp)=\frac{2\alpha_s\pi^3}{N_cS_\perp}G(\eta,k_\perp)$

$$\frac{\partial N(\eta,k_\perp)}{\partial\eta}=\frac{\bar{\alpha}_s}{\pi}\left[\int\frac{\mathrm{d}^2l_\perp}{l_\perp^2}N(\eta,k_\perp+l_\perp)-\int_0^{k_\perp}\frac{\mathrm{d}^2l_\perp}{l_\perp^2}N(\eta,k_\perp)\right]-\bar{\alpha}_sN^2(\eta,k_\perp)$$

- this form is the same as the BK equation in the momentum space [Balitsky, NPB 96; Kovchegov, PRD 99]

Gluon fusion 2 → 1, 3 → 1, 4 → 1…

$$\frac{\partial\mathcal{N}(\eta,k_\perp)}{\partial\eta}=\frac{\bar{\alpha}_s}{\pi}\left[\int\frac{\mathrm{d}^2l_\perp}{l_\perp^2}\mathcal{N}(\eta,l_\perp+k_\perp)-\int_0^{k_\perp}\frac{\mathrm{d}^2l_\perp}{l_\perp^2}\mathcal{N}(\eta,k_\perp)\right]-\bar{\alpha}_s\mathcal{N}^2(\eta,k_\perp)$$

WW gluon distribution $\mathcal{N}(\eta,k_\perp)=\int\frac{\mathrm{d}^2r_\perp}{2\pi}\frac{e^{-ik_\perp\cdot r_\perp}}{r_\perp^2}\left[1-\frac{1}{N_c}\langle U^\dagger(0)U(r_\perp)\rangle\right]$ [Kovchegov, PRD, 00; Marquet, Soyez, NPA, 05]

- It is hard to develop a parton shower based on BK equation!!!!

Kinematical constraint in the GLR evolution equation

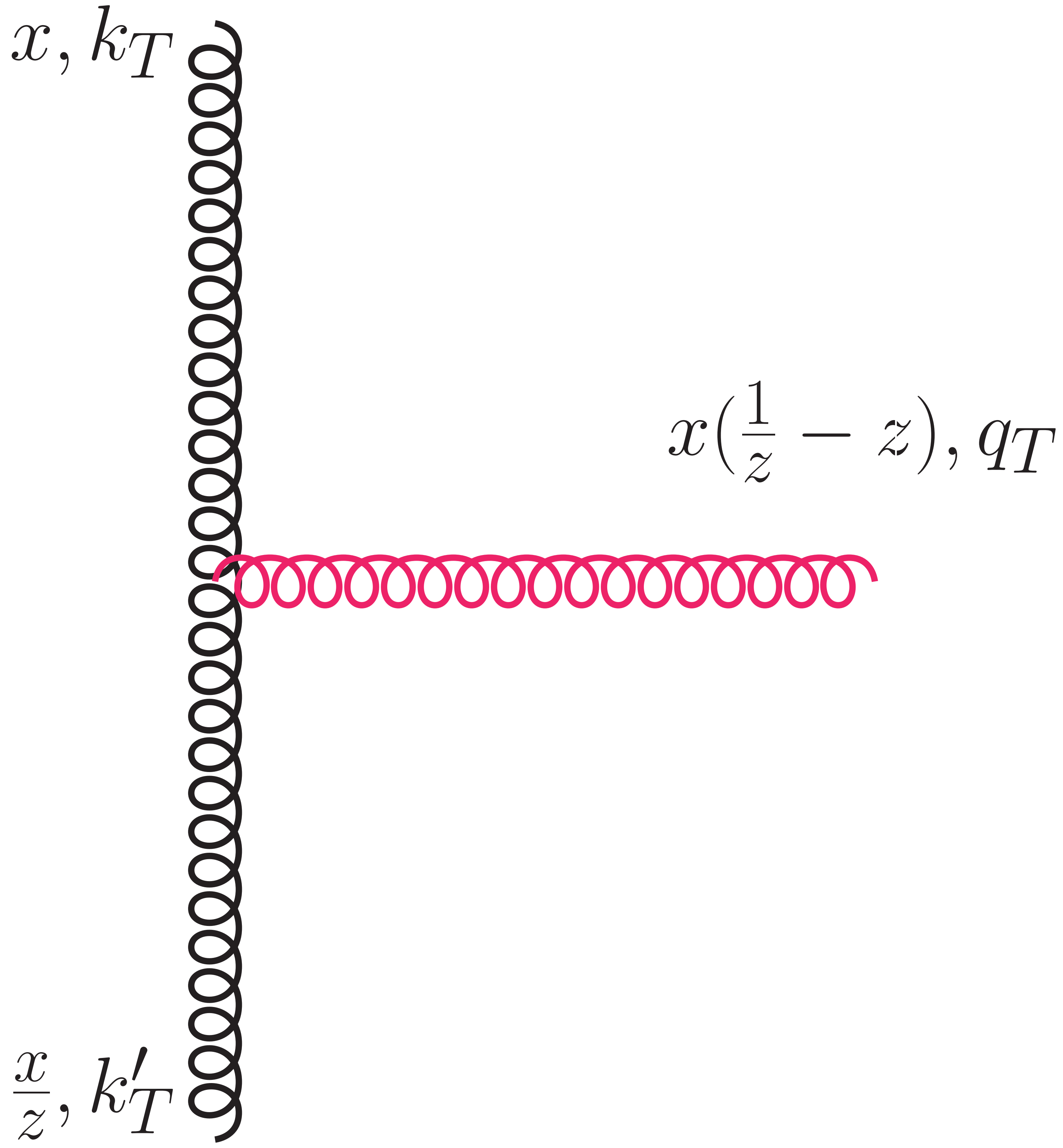
[Kwiecinski, Martin, Sutton, Z. Phys. C, 96;

Deak, Kutak, Li, Stasto, EPJC, 19]

- The key observation is that the virtuality of a gluon should arise mainly from the transverse momentum

$$k_T^2 > |k^+k^-| \qquad k^- = k'^- - q^- \simeq -q^- = -q_T^2/q^+$$

$$k^+k^- \simeq -\frac{k^+}{q^+}q_T^2 = -\frac{k^+}{k'^+ - k^+}q_T^2 = -\frac{z}{1-z}q_T^2$$



- The on-shell condition give the kinematical constraint

$$q_T^2 < \frac{1-z}{z}k_T^2 \qquad \eta \longrightarrow \eta + \ln\frac{k_\perp^2}{k_\perp^2 + l_\perp^2}$$

- The kinematic constrainted GLR equation can be modified as

$$\frac{\partial N(\eta, k_\perp)}{\partial \eta} = \frac{\bar{\alpha}_s}{\pi}\int \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N\left(\eta + \ln\frac{k_\perp^2}{k_\perp^2 + l_\perp^2}, l_\perp + k_\perp\right) - \frac{\bar{\alpha}_s}{\pi}\int_0^{k_\perp}\frac{\mathrm{d}^2 l_\perp}{l_\perp^2}N(\eta, k_\perp) - \bar{\alpha}_s N^2(\eta, k_\perp)$$

Fixed boundary condition: forward

First step: non-Sudakov form factor

$$\mathcal{R} = \exp\left[-\bar{\alpha}_s \int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta' \left(\ln \frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp)\right)\right]$$

Second step: Real splitting kernel

$$\mathcal{R} = \frac{1}{\mathcal{C}} \frac{\bar{\alpha}_s}{\pi} \int_{\Lambda_{\text{cut}}}^{l_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} \exp\left\{-\bar{\alpha}_s \int_{\eta_i}^{\eta_{i+1} + \ln \frac{(k_{\perp,i} - l'_\perp)^2}{(k_{\perp,i} - l'_\perp)^2 + l'^2_\perp}} \mathrm{d}\eta \left[\ln \frac{k^2_{\perp,i}}{\Lambda^2_{\text{cut}}} + N(\eta, k_{\perp,i})\right]\right\},$$

$$\mathcal{C} = \frac{\bar{\alpha}_s}{\pi} \int_{\Lambda_{\text{cut}}}^{\min[P_\perp, \sqrt{(k_{\perp,i} - l'_\perp)^2 \frac{1-z}{z}}]} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} \exp\left\{-\bar{\alpha}_s \int_{\eta_i}^{\eta_{i+1} + \ln \frac{(k_{\perp,i} - l'_\perp)^2}{(k_{\perp,i} - l'_\perp)^2 + l'^2_\perp}} \mathrm{d}\eta \left[\ln \frac{k^2_{\perp,i}}{\Lambda^2_{\text{cut}}} + N(\eta, k_{\perp,i})\right]\right\},$$

The generated event has to be re-weighted

$$\mathcal{W}_{kc,1}(\eta_i, \eta_{i+1}; k_{\perp,i}) = \frac{(\eta_{i+1} - \eta_i) \int_{\Lambda_{\text{cut}}}^{\min\left[P_\perp, \sqrt{\frac{1-z}{z}(k_{\perp,i} - l_\perp)^2}\right]} \frac{\mathrm{d}^2 l_\perp}{l^2_\perp} e^{-\bar{\alpha}_s \int_{\eta_{i+1}}^{\eta_{i+1} + \ln \frac{(k_{\perp,i} - l_\perp)^2}{(k_{\perp,i} - l_\perp)^2 + l^2_\perp}} \mathrm{d}\eta \left[\ln \frac{k^2_{\perp,i}}{\Lambda^2_{\text{cut}}} + N(\eta, k_{\perp,i})\right]}}{(\eta_{i+1} - \eta_i) \ln \frac{k^2_{\perp,i}}{\Lambda^2_{\text{cut}}} + \int_{\eta_i}^{\eta_{i+1}} d\eta N(\eta, k_{\perp,i})}$$

Frozen boundary condition: backward

- First step: backward non-Sudakov form factor

$$\Pi_{ns}(\eta_{i+1}, \eta_i; k_{\perp,i+1}) = \exp\left[-\frac{\bar{\alpha}_s}{\pi}\int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta \int_{\Lambda_{\mathrm{cut}}}^{P_\perp} \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} \frac{N\left(\eta + \ln\left[\frac{k_{\perp,i+1}^2}{k_{\perp,i+1}^2 + l_\perp^2}\right], k_{\perp,i+1} + l_\perp\right)}{N(\eta, k_{\perp,i+1})}\right]$$

- Second step: Real splitting kernel

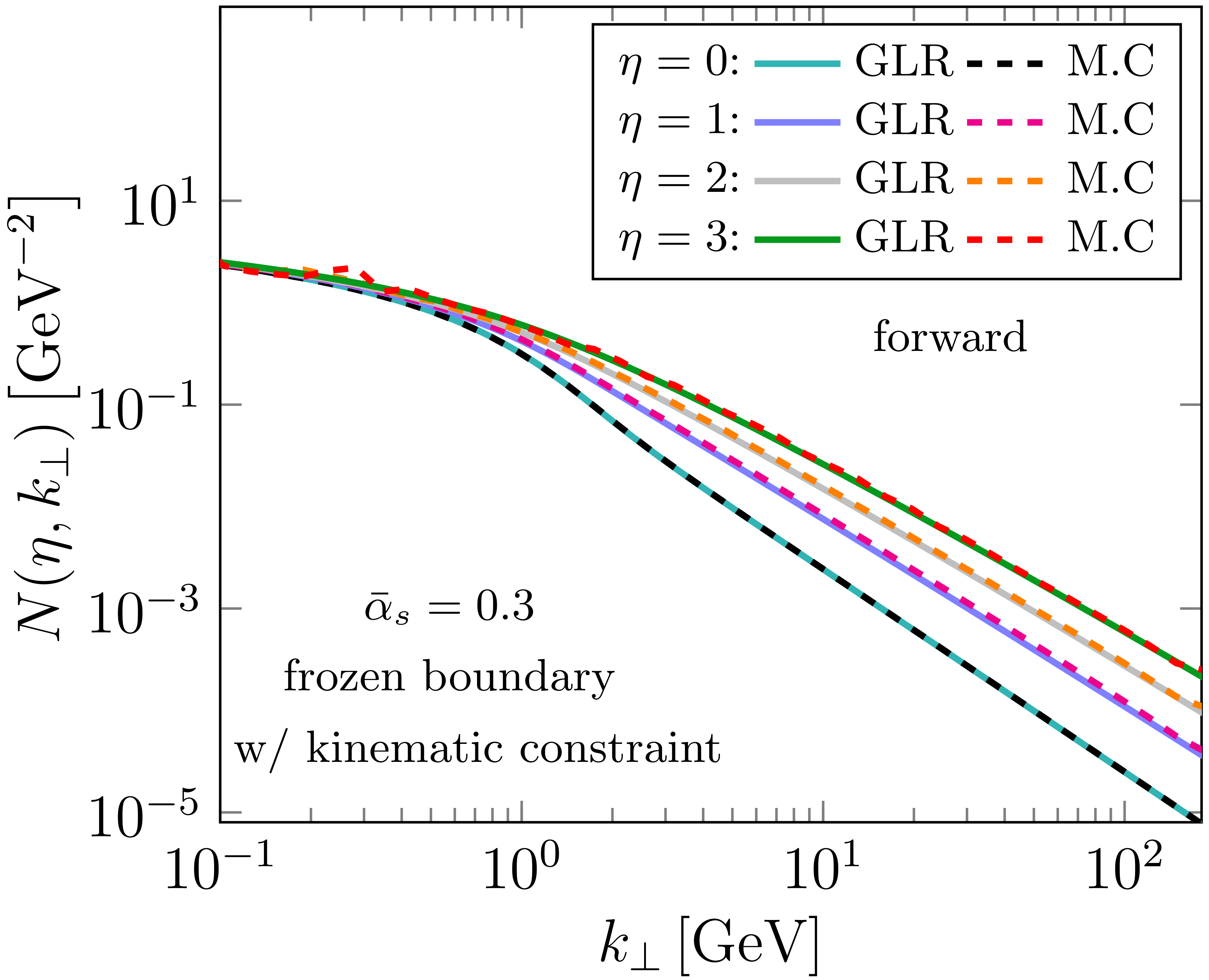
$$\mathcal{R} = \frac{1}{\mathcal{C}}\frac{\bar{\alpha}_s}{\pi}\int_{\Lambda_{\mathrm{cut}}}^{l_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} N\left(\eta_{i+1} + \ln\left[\frac{k_{\perp,i+1}^2}{k_{\perp,i+1}^2 + l'^2_\perp}\right], k_{\perp,i+1} + l'_\perp\right)$$

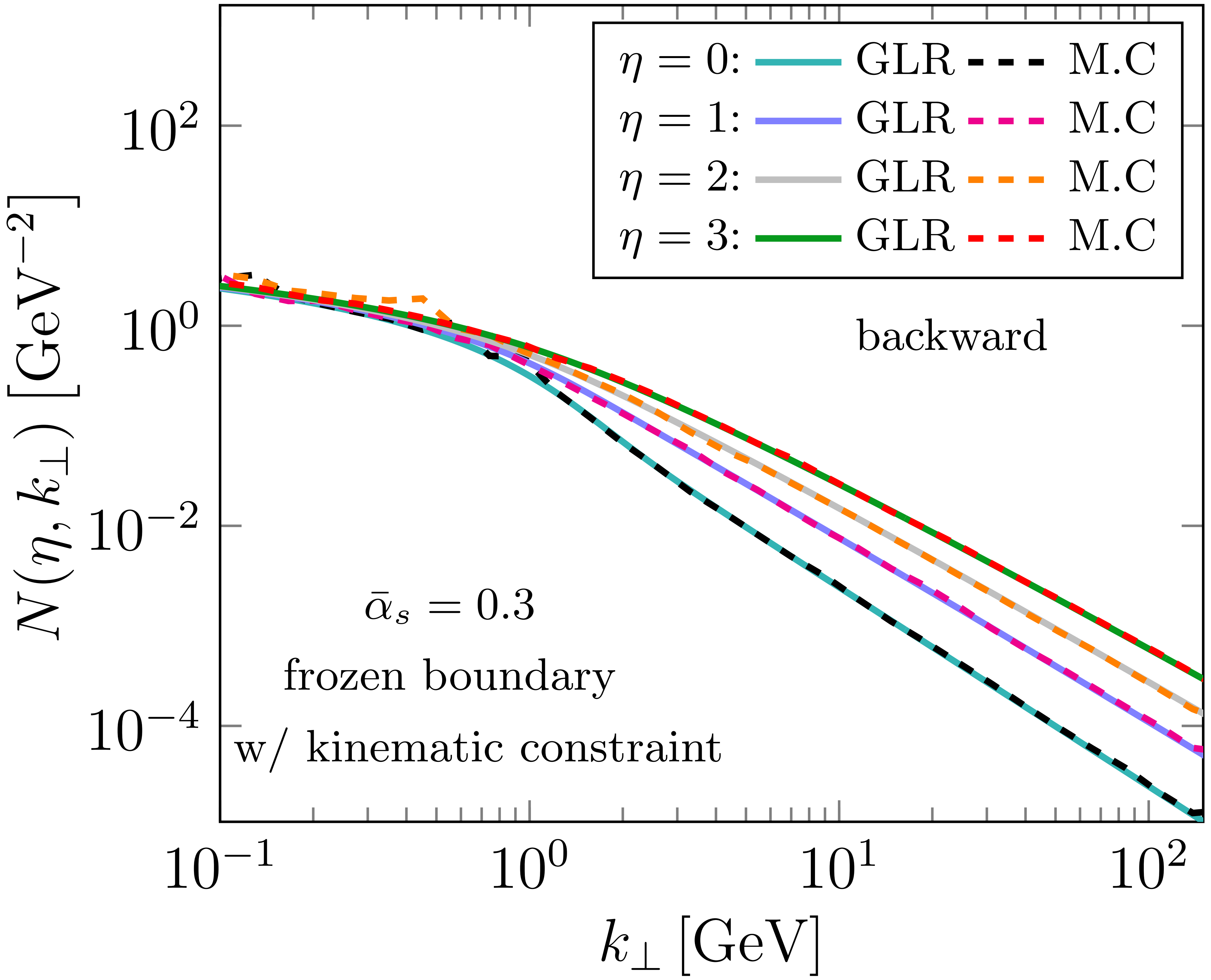
$$\mathcal{C} = \frac{\bar{\alpha}_s}{\pi}\int_{\Lambda_{\mathrm{cut}}}^{P_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} N\left(\eta_{i+1} + \ln\left[\frac{k_{\perp,i+1}^2}{k_{\perp,i+1}^2 + l'^2_\perp}\right], k_{\perp,i+1} + l'_\perp\right).$$

- The generated event has to be re-weighted

$$\mathcal{W}_{\mathrm{backward}} = \frac{1}{\mathcal{W}_{forward}}$$

The forward & backward evolution





- The kinematic constrainted GLR equation can be modified as

$$\frac{\partial N(\eta, k_\perp)}{\partial \eta} = \frac{\bar{\alpha}_s}{\pi} \int \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N\left(\eta + \ln \frac{k_\perp^2}{k_\perp^2 + l_\perp^2}, l_\perp + k_\perp\right) - \frac{\bar{\alpha}_s}{\pi} \int_0^{k_\perp} \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta, k_\perp) - \bar{\alpha}_s N^2(\eta, k_\perp)$$