Calculation of the soft anomalous dimensions with time-like and light-like Wilson lines

Zehao Zhu
University of Edinburgh

In collaboration with Einan Gardi

REF 18th Oct 2024

Introduction to Soft Anomalous Dimension

Full Amplitudes $M^I = S^{IJ} H^J$ Hard Function

Soft Function

$$\text{IR} + \text{UV} = 0 \longleftarrow \langle \Phi_{\beta_I} \Phi_{\beta_J} \cdots \rangle = \mathbf{1}$$

$$\Big\langle \Phi^{(m)}_{\beta_I} \Phi^{(m)}_{\beta_J} \cdots \Big\rangle$$

rescaling invariant variables

$$\gamma_{IJ} = \frac{2\beta_I \cdot \beta_J}{\sqrt{\beta_I^2}\sqrt{\beta_J^2}}$$

$$-\gamma_{IJ} = \alpha_{IJ} + \frac{1}{\alpha_{IJ}}$$

$$\Phi_\beta = \mathrm{P}\exp\left[\int_0^\infty ds \beta \cdot A(s\beta)\right]$$

IR Regulator

E.Gardi,arXiv:1310.5268

$$\int_0^\infty ds \to \int_0^\infty ds \exp\left(-ims\sqrt{\beta^2 - i0}\right)$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{-\beta \cdot k + i\varepsilon} \to \int \frac{d^D k}{(2\pi)^D} \frac{1}{-\beta \cdot k - m\sqrt{\beta^2} + i\varepsilon}$$

Introduction to Soft Anomalous Dimension

$$\frac{dZ}{d\log\mu^2} = -Z\Gamma$$

$$Z(\{\gamma_{IJ}\},\mu^2,\epsilon)\left\langle \Phi^{(m)}_{\beta_I}\Phi^{(m)}_{\beta_J}\cdots\right\rangle = \left\langle \Phi^{(m)}_{\beta_I}\Phi^{(m)}_{\beta_J}\cdots\right\rangle_{\text{ren}} = H(\{\gamma_{IJ}\},\mu^2,\epsilon)$$

$$Z(\{\gamma_{IJ}\},\mu^2,\epsilon) = \mathrm{P}\exp\left[\int_{\mu^2}^{\infty}\frac{d\tau^2}{\tau^2}\Gamma(\alpha_s(\tau^2,\epsilon),\{\gamma_{IJ}\})\right]$$

Finite! Multiplicative Renormalizability

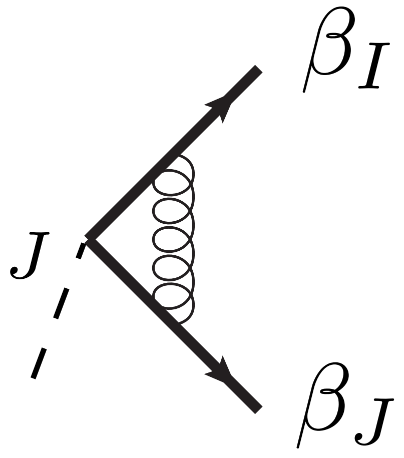
A. M. Polyakov, Nucl.Phys.B 164 (1980) 171-188

G. Korchemsky, A. Radyushkin, Nucl.Phys.B 283 (1987) 342-364

$$\alpha_s(\tau^2,\epsilon)\sim\alpha_s(\mu^2)\left(\frac{\tau^2}{\mu^2}\right)^{-\epsilon}$$

$$\left\langle \Phi^{(m)}_{\beta_I}\Phi^{(m)}_{\beta_J}\cdots\right\rangle = \mathrm{P}\exp\left[-\int_{\mu^2}^{\infty}\frac{d\tau^2}{\tau^2}\Gamma(\alpha_s(\tau^2,\epsilon),\{\gamma_{IJ}\})\right]H(\{\gamma_{IJ}\},\mu^2,\epsilon)$$

$$\left\langle \Phi^{(m)}_{\beta_I}\Phi^{(m)}_{\beta_J}\cdots\right\rangle = 1 + \alpha_s\sum_{I<J}\mathrm{T}_I\cdot\mathrm{T}_J$$



$$+\cdots$$

$$\propto \frac{1}{\epsilon}\frac{1+\alpha_{IJ}^2}{1-\alpha_{IJ}^2}\log(\alpha_{IJ}+i\varepsilon)$$

soft anomalous dimension at 1 loop

$$\Gamma^{(1)} = \frac{g_s^2}{4\pi^2}\sum_{I<J}\mathbf{T}_I\cdot\mathbf{T}_J\frac{1+\alpha_{IJ}^2}{1-\alpha_{IJ}^2}\log(\alpha_{IJ}+i\varepsilon)$$

Calculation with one lightlike line

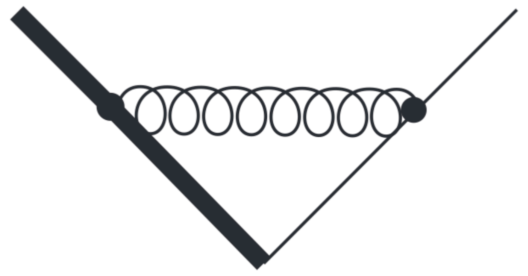
$$\left\langle \Phi^{(m)}_{\beta_I}\Phi_{\beta_J}\Phi_{\beta_K}\cdots\right\rangle \xrightarrow{\Phi_{\beta_K}\to\phi_{\beta_k}} \left\langle \Phi^{(m)}_{\beta_I}\Phi_{\beta_J}\phi_{\beta_k}\cdots\right\rangle$$

regularization scheme

J. M. Henn, C. Milloy, K. Yan, arXiv:2310.10145

lightlike Wilson line

at one loop



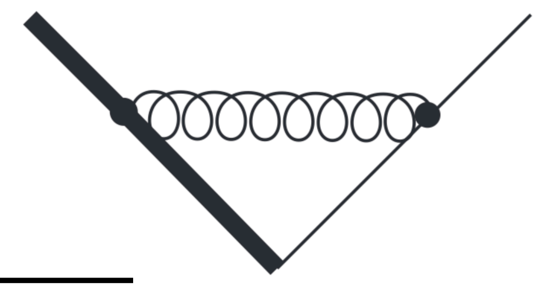
$$= e^{\epsilon\gamma_E}\beta_I\cdot\beta_k\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2}\frac{1}{-\beta_I\cdot k-\sqrt{\beta_I^2}}\frac{1}{\beta_k\cdot k}$$

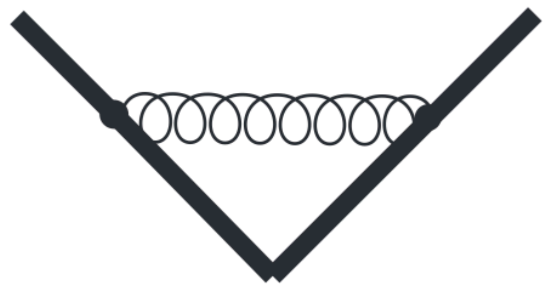
$$\longleftarrow\!\!\!\!\times\!\!\!\!\longrightarrow \quad Z(\{\gamma_{IJ}\},\mu^2,\epsilon)\Big|_{\beta_K^2\to 0}$$

$$= -\frac{\pi^{-2\epsilon}}{128}e^{\gamma_E\epsilon}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\Gamma(-\epsilon)\Gamma(2\epsilon) = \frac{\pi^{-2\epsilon}}{128}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\left[\boxed{\frac{1}{2\epsilon^2}}+\frac{5\pi^2}{24}+O\left(\epsilon^1\right)\right]$$

Calculation with one lightlike line

$$\left\langle \Phi^{(m)}_{\beta_I} \Phi_{\beta_J} \Phi_{\beta_K} \cdots \right\rangle \xrightarrow{\beta_K^2 \sim O(\lambda^2)} \left\langle \Phi^{(m)}_{\beta_I} \Phi_{\beta_J} \Phi_{\beta_K} \cdots \right\rangle \Big|_{\beta_K^2 \to 0}$$



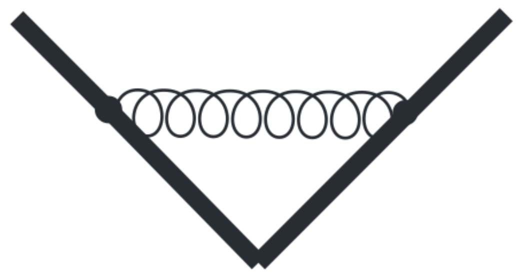


$$\Big|_{\beta_K^2 \to O(\lambda^2)} = -\left(\frac{m^2}{\mu^2}\right)^{-\epsilon} \frac{\pi^{\epsilon-2} e^{\gamma\epsilon}}{128} \frac{\pi}{\sin(2\pi\epsilon)} \left[\lambda^{-2\epsilon}\Gamma(\epsilon) + \frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\right] + O(\lambda)$$

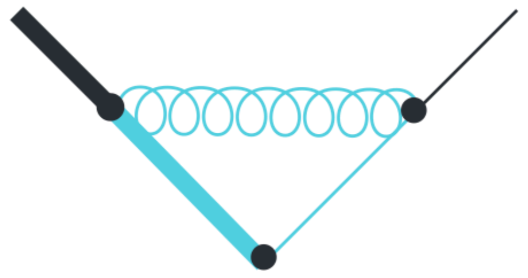
$$= \frac{\pi^{\epsilon-2} e^{\gamma\epsilon}}{128} \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} \left[\frac{\log(\lambda)}{\epsilon} + O\left(\epsilon^0\right)\right] + O(\lambda)$$

$$Z(\{\gamma_{IJ}\}, \mu^2, \epsilon)\Big|_{\beta_K^2 \to 0}$$

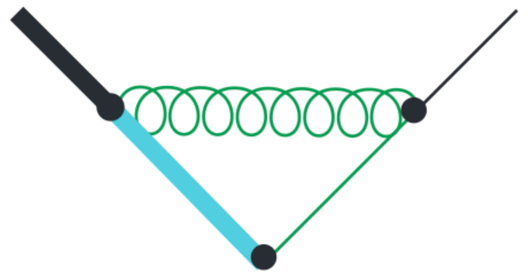
expansion by regions $\beta_K \to (\beta_K^+, \lambda^2 \beta_K^-, \lambda \beta_K^\perp)$ $\quad k \sim (1, \lambda^2, \lambda)$



$$\Big|_{\beta_K^2 \to O(\lambda^2)} =$$



$$+ \lambda^{-2\epsilon}$$



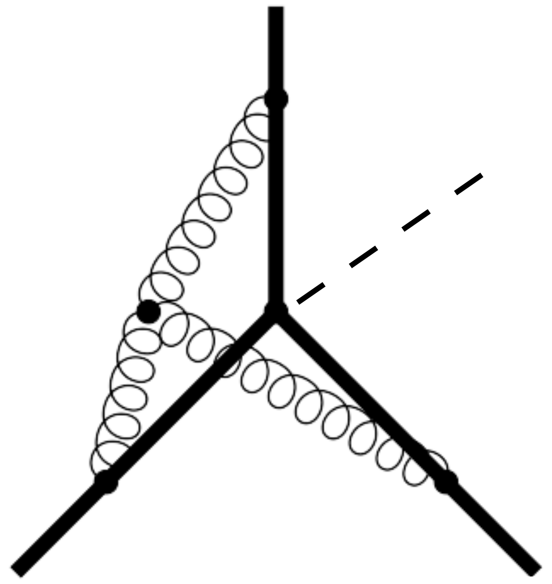
$$+ O(\lambda)$$

collinear mode

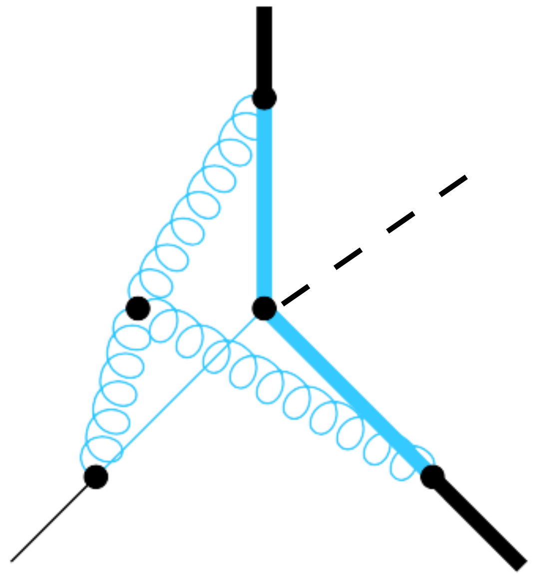
Calculation with one lightlike line

$$\left\langle \Phi^{(m)}_{\beta_I} \Phi_{\beta_J} \Phi_{\beta_K} \cdots \right\rangle \xrightarrow{\beta_K^2 \sim O(\lambda^2)} \left\langle \Phi^{(m)}_{\beta_I} \Phi_{\beta_J} \Phi_{\beta_K} \cdots \right\rangle \Big|_{\beta_K^2 \to 0}$$

$H \to t\bar{t}g$



$\Big|_{\beta_K^2 \to O(\lambda^2)}$ $=$

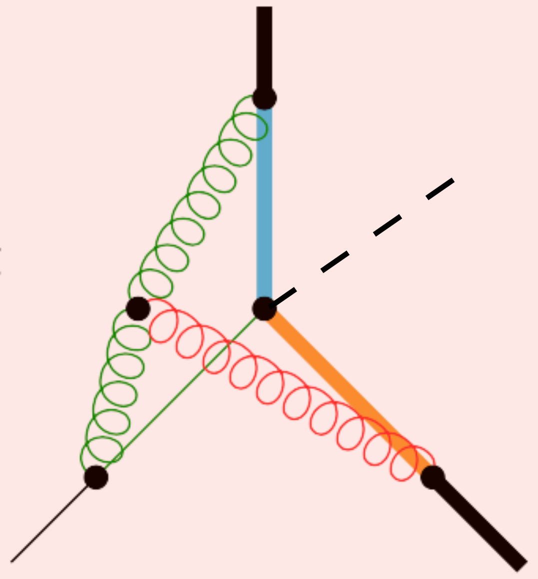


at two loops

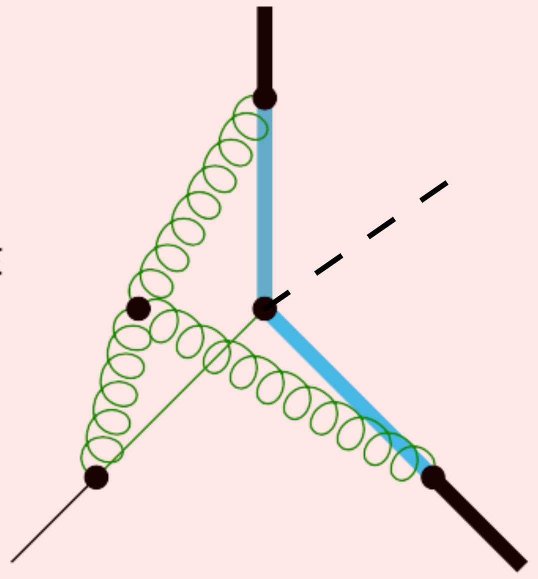
$$\left\langle \Phi^{(m)}_{\beta_I} \Phi_{\beta_J} \phi_{\beta_k} \cdots \right\rangle$$

IR regions

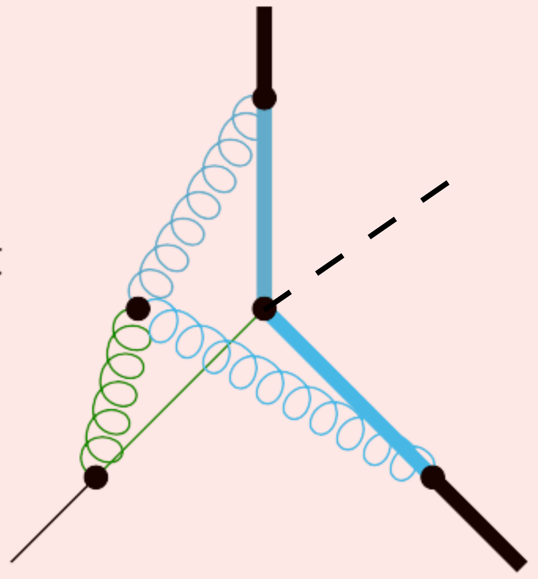
$+\lambda^{-6\epsilon}$



$+\lambda^{-4\epsilon}$

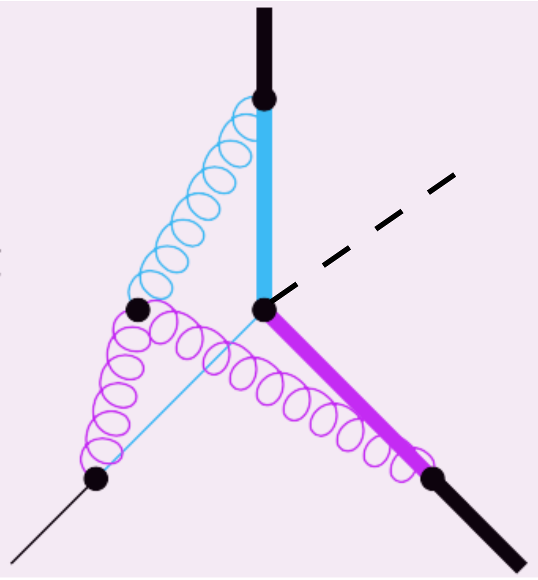


$+\lambda^{-2\epsilon}$



UV regions

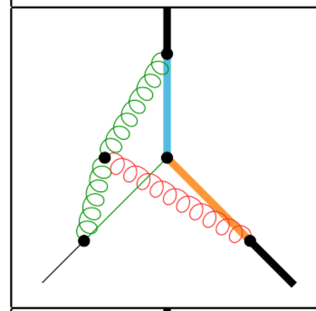
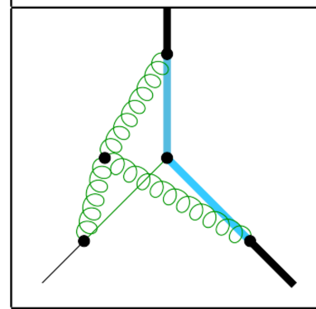
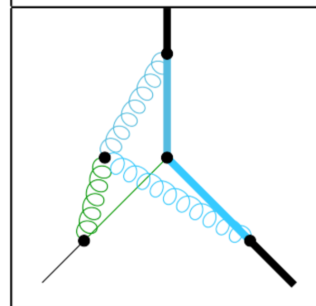
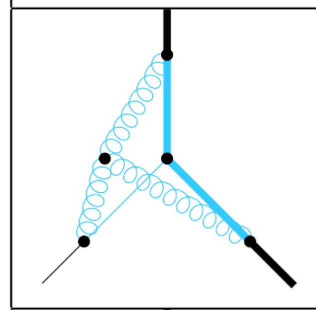
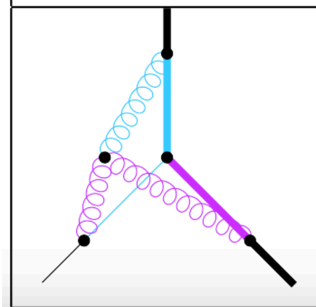
$+\lambda^{2\epsilon}$



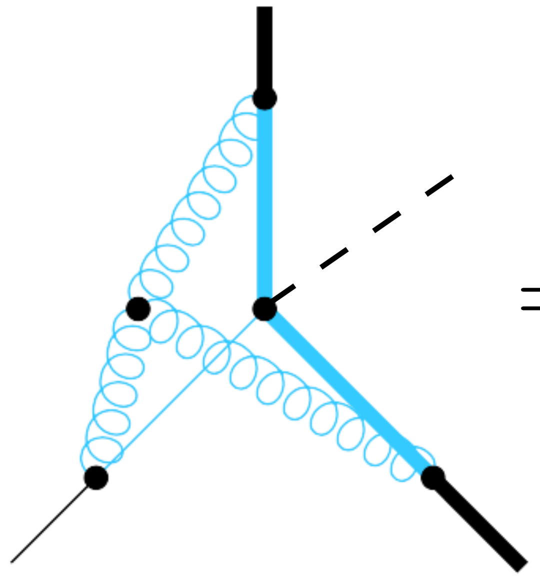
$+O(\lambda)$

$$\frac{1}{k^2 - 2\beta_k \cdot k + i\varepsilon}$$

$$O(k^2) > O(\beta_k \cdot k)$$

Scaling	k_1	k_2	k_3	D	N	Total	Region Type
	$(1,\lambda^2,\lambda)$	$(\lambda^2,\lambda^2,\lambda^2)$	$(1,\lambda^2,\lambda)$	λ^{-12}	1	$\lambda^{-6\epsilon}$	Soft, IR-collinear
	$(1,\lambda^2,\lambda)$	$(1,\lambda^2,\lambda)$	$(1,\lambda^2,\lambda)$	λ^{-8}	1	$\lambda^{-4\epsilon}$	IR-collinear, IR-collinear
	$(1,1,1)$	$(1,\lambda^2,\lambda)$	$(1,1,1)$	λ^{-4}	1	$\lambda^{-2\epsilon}$	Hard, IR-collinear
	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	1	1	1	Hard, Hhard
	$(1,1,1)$	$(\lambda^{-2},1,\lambda^{-1})$	$(\lambda^{-2},1,\lambda^{-1})$	λ^6	λ^{-2}	$\lambda^{2\epsilon}$	UV-collinear, Hard

Calculation with one lightlike line



Hard-Hard

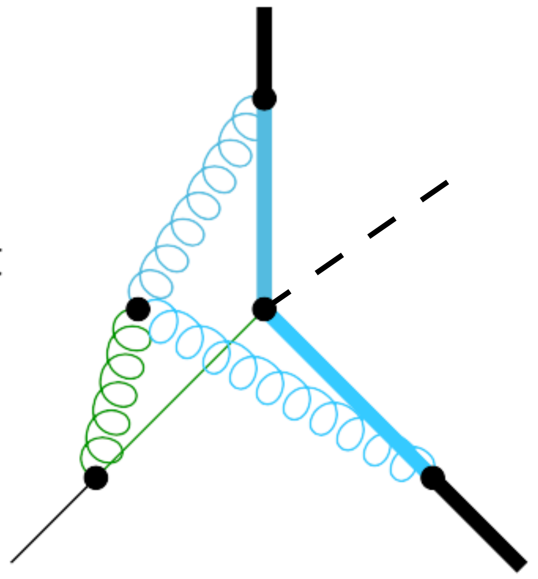
$$= \frac{\lambda^0 \pi^{2\epsilon-4}}{256}\left\{-\frac{1}{3\epsilon^4} - \frac{1}{3\epsilon^3}\log y_{IJk} - \frac{1}{\epsilon^2}\left[\frac{2\left(\alpha_{IJ}^2+1\right)\log\left(\alpha_{IJ}\right)\log\left(y_{IJk}\right)}{\alpha_{IJ}^2-1} + \frac{2}{3}\log^2\left(y_{IJk}\right) + \frac{17\pi^2}{18}\right]\right.$$

$$-\frac{1}{\epsilon}\left[F\left(\frac{y_{IJk}}{\alpha_{IJ}}\right) + F\left(\alpha_{IJ} y_{IJk}\right) - \frac{4\left(\alpha_{IJ}^2+1\right)\log^3\left(\alpha_{IJ}\right)}{3\left(\alpha_{IJ}^2-1\right)} - \frac{4\pi^2\left(\alpha_{IJ}^2+1\right)\log\left(\alpha_{IJ}\right)}{3\left(\alpha_{IJ}^2-1\right)}\right.$$

$$\left.\left.+\frac{\left(\alpha_{IJ}^2+1\right)M_{100}\left(\alpha_{IJ}\right)\log\left(y_{IJk}\right)}{\alpha_{IJ}^2-1} - \frac{1}{9}4\log^3\left(y_{IJk}\right) - \frac{7}{18}\pi^2\log\left(y_{IJk}\right) - \frac{74\zeta(3)}{9}\right] + O\left(\epsilon^0\right)\right\}$$

Cancelled by each other

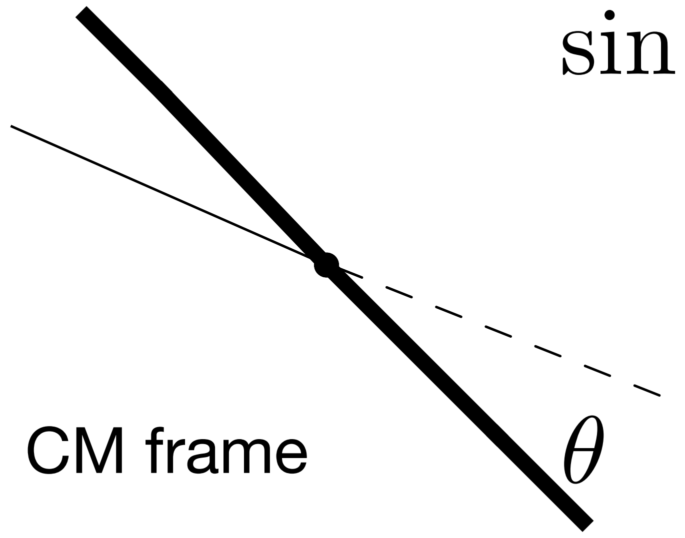
$\lambda^{-2\epsilon}$



Hard-IR Collinear

$$y_{IJk} + \alpha_{IJ}$$

$$1 + y_{IJk}\alpha_{IJ}$$



CM frame

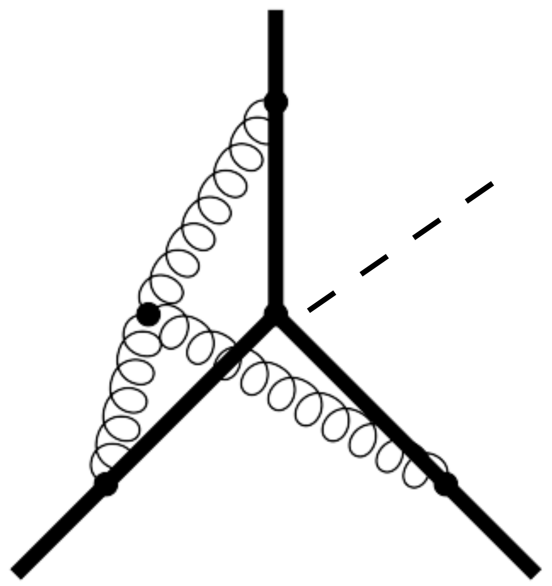
$$\sin\theta = 0$$

$$y_{IJk} = \frac{\beta_J \cdot \beta_k \sqrt{\beta_I^2}}{\beta_I \cdot \beta_k \sqrt{\beta_J^2}}$$

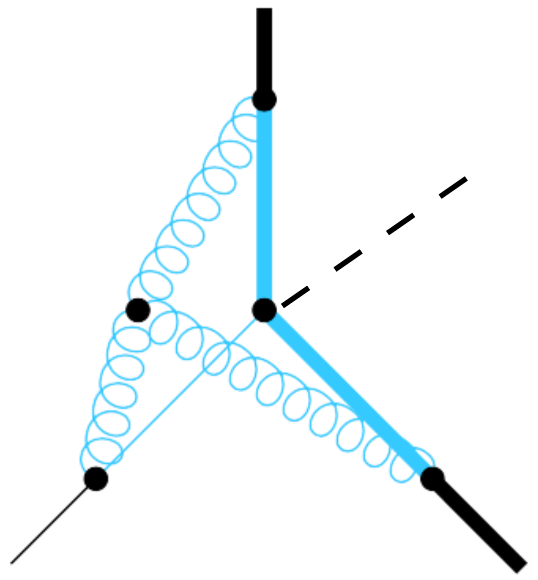
$$M_{100}(x) = 2\mathrm{Li}_2(x^2) - 2\log^2(x) - 2\zeta_2 + 4\log(x)\log(1-x^2)$$

$$F(x) = H_{-1,0,0}(x) - \frac{\pi^2}{4}H_{-1}(x)$$

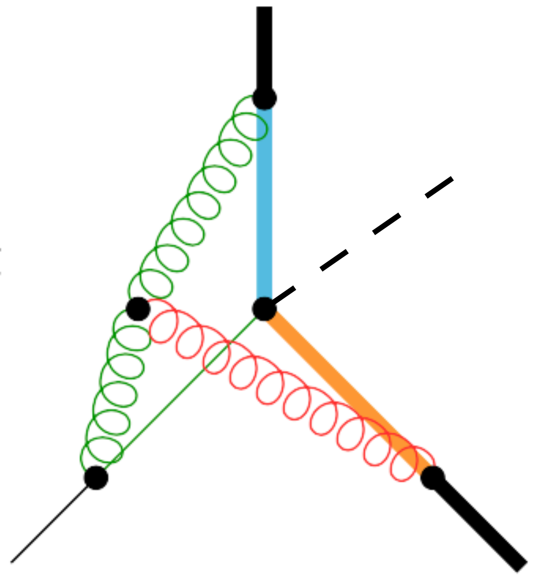
Calculation with one lightlike line



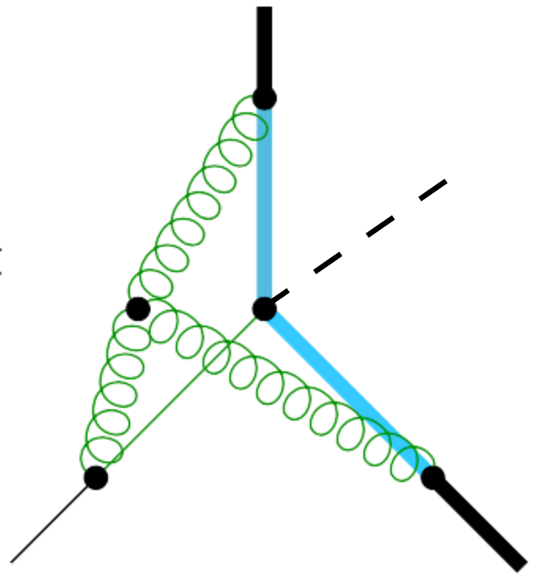
$\Big|_{\beta_K^2 \to O(\lambda^2)} =$



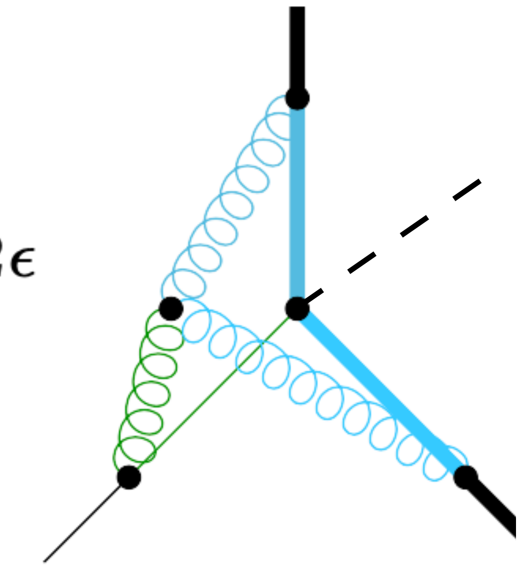
$+\lambda^{-6\epsilon}$



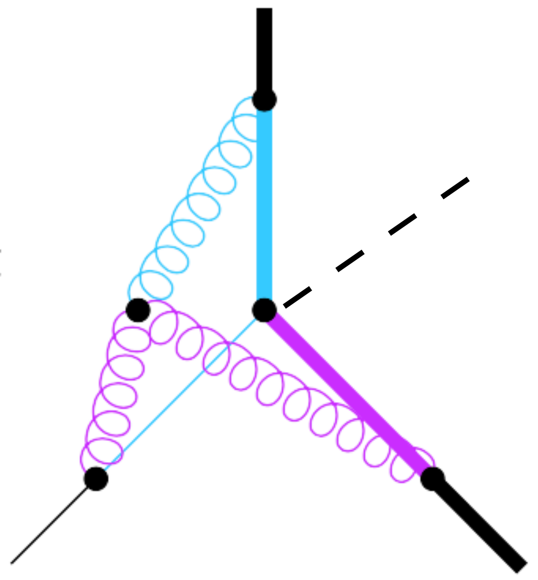
$+\lambda^{-4\epsilon}$



$+\lambda^{-2\epsilon}$



$+\lambda^{2\epsilon}$



$+O(\lambda)$

$$= \frac{\pi^{2\epsilon-4}}{128\epsilon}\left\{\frac{\alpha_{IJ}^2+1}{\alpha_{IJ}^2-1}\log(\alpha_{IJ})\log\left(y_{IJk}\right)\log\left(\lambda^2 y_{IJk}\right)+\log\left(y_{IJk}\right)\left(\log^2(\alpha_{IJ})+\log(\lambda)\log\left(\lambda y_{IJk}\right)\right)\right\}+O\left(\lambda^1,\epsilon^0\right)$$

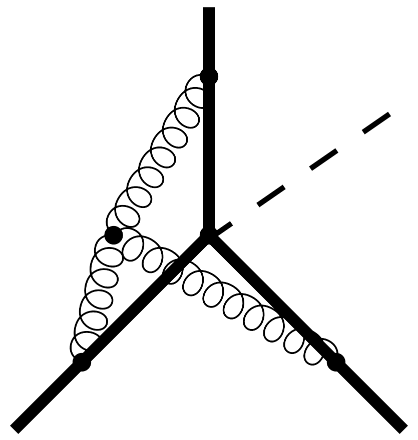
$$\left\langle \Phi_{\beta_I}^{(m)}\Phi_{\beta_J}\Phi_{\beta_K}\cdots\right\rangle \longleftarrow Z(\{\gamma_{IJ}\},\mu^2,\epsilon)$$

$y_{IJk}+\alpha_{IJ}$

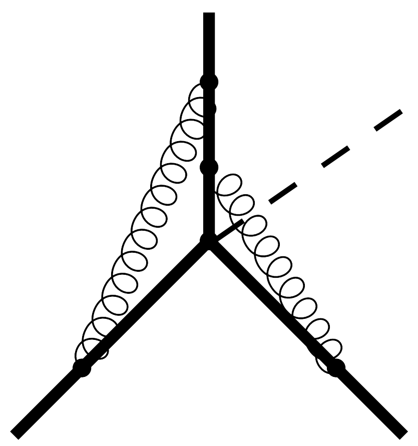
$1+y_{IJk}\alpha_{IJ}$

$$\left\langle \Phi_{\beta_I}^{(m)}\Phi_{\beta_J}\phi_{\beta_k}\cdots\right\rangle \longleftarrow\!\!\!\!\times\;\; Z(\{\gamma_{IJ}\},\mu^2,\epsilon)\Big|_{\beta_K^2\to 0}$$

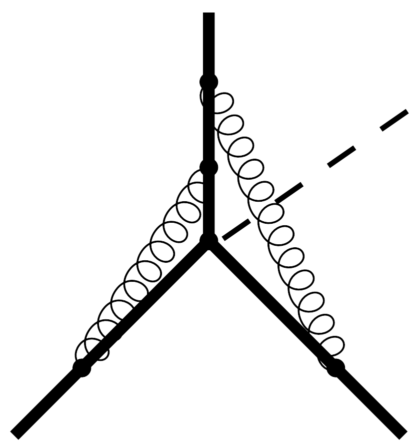
$\alpha_s^2 \sum_{(I,J,K)} f^{abc}\mathrm{T}_I^a\mathrm{T}_J^b\mathrm{T}_K^c\Big[$



$+\frac{1}{2}\Big($



$-$



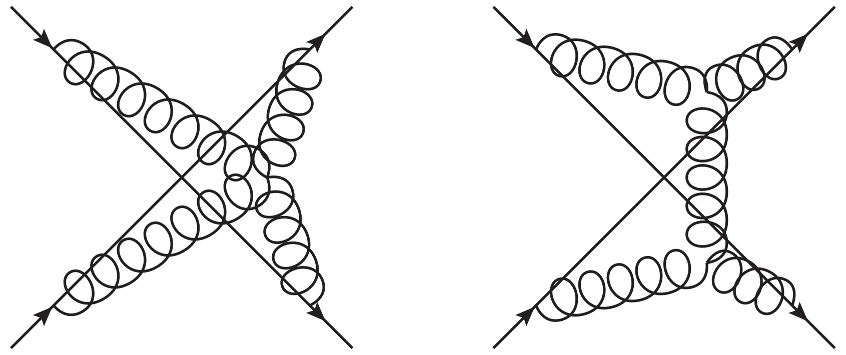
$\Big)+\cdots\Big]$

Calculation with more lightlike lines

$$\Gamma_{\text{one mass}}(\alpha_s, \lambda^2, \beta_I, \{\beta_i\}) = \text{dipole} + f(\alpha_s) \sum_{i,j,k,l} \mathcal{T}_{ijkl} + \sum_{i,j,k,l} \mathcal{T}_{ijkl} F_4(\rho_{ijkl}, \rho_{ilkj}, \alpha_s(\lambda^2, \epsilon)) + \sum_{i,j} \mathcal{T}_{ijII} F_{h2}(z_{ij}, \alpha_s(\lambda^2, \epsilon))$$

$$+ \sum_{i,j,k} \mathcal{T}_{ijkI} F_{h3}(z_{ij}, z_{ik}, z_{jk}, \alpha_s(\lambda^2, \epsilon)) + O(\alpha_s^4)$$

Ø.Almelid,C.Duhr,E.Gardi,arXiv:1507.00047

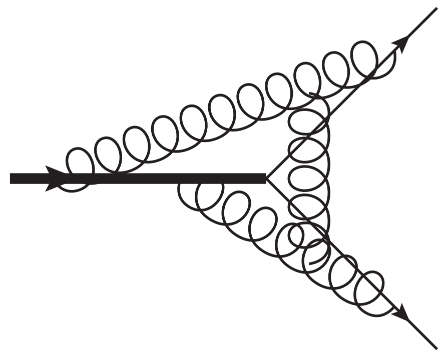


$\cdots$

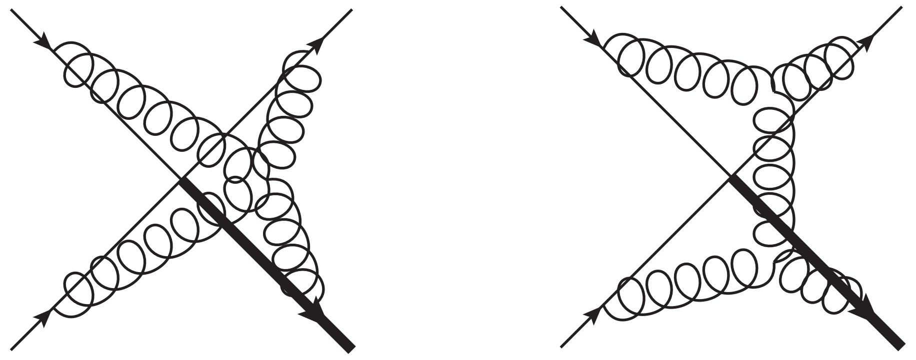
$$\rho_{ijkl} = \frac{\beta_i \cdot \beta_j \beta_k \cdot \beta_l}{\beta_i \cdot \beta_k \beta_j \cdot \beta_l} = z_{ijkl} \bar{z}_{ijkl}$$

$$\rho_{ilkj} = \frac{\beta_i \cdot \beta_l \beta_k \cdot \beta_j}{\beta_i \cdot \beta_k \beta_j \cdot \beta_l} = (1 - z_{ijkl})(1 - \bar{z}_{ijkl})$$

Z.Liu,N.Schalch,arXiv:2207.02864



$$z_{ij} = \frac{\beta_I^2 (\beta_i \cdot \beta_j)}{2(\beta_I \cdot \beta_i)(\beta_I \cdot \beta_j)}$$



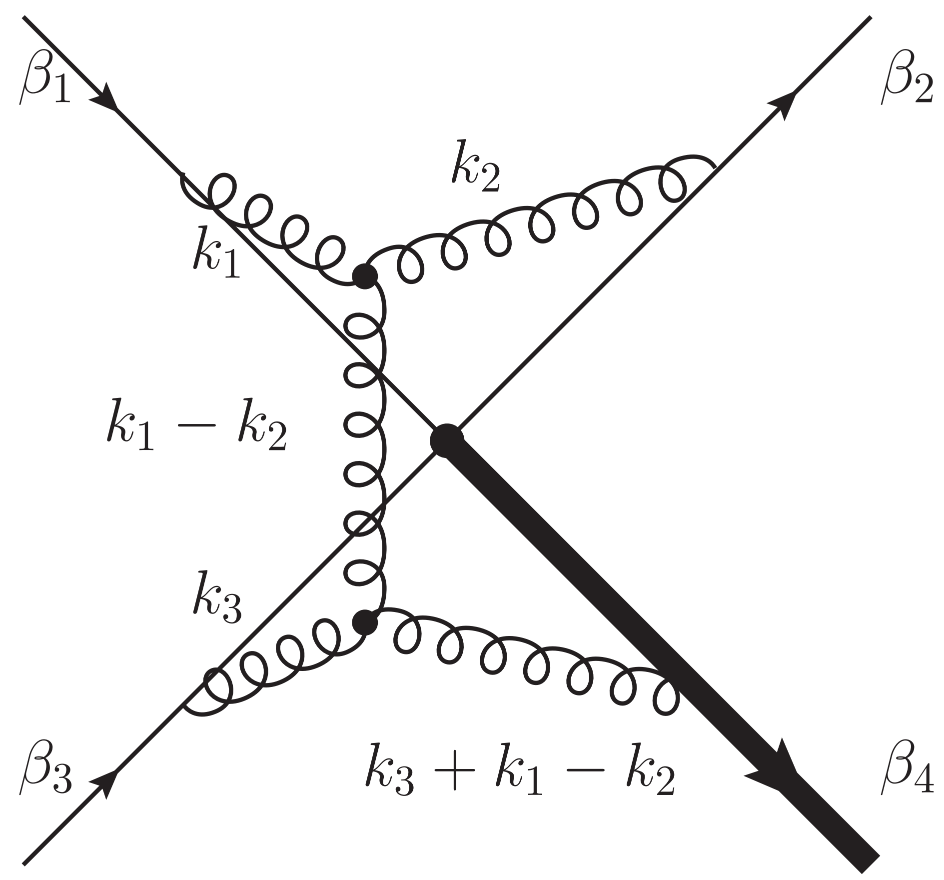
$\cdots$

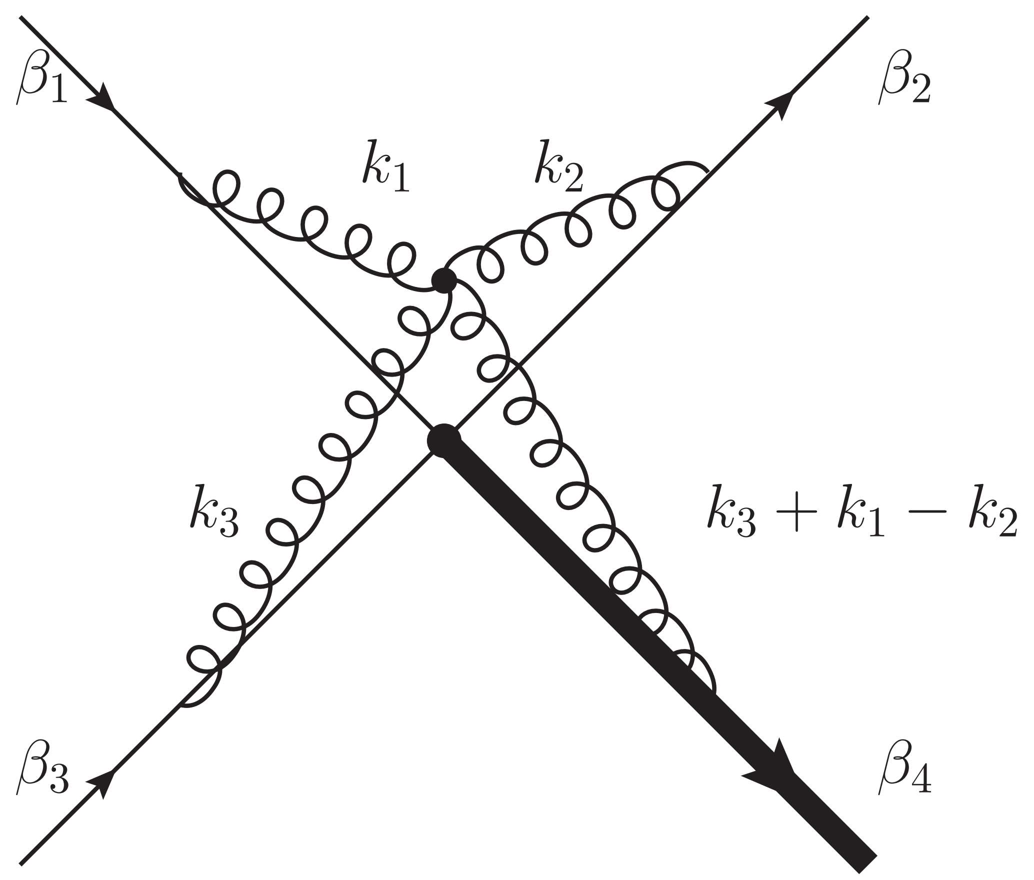
$$z_{ij} = \frac{\beta_I^2 (\beta_i \cdot \beta_j)}{2(\beta_I \cdot \beta_i)(\beta_I \cdot \beta_j)}$$

$$z_{ik} = \frac{\beta_I^2 (\beta_i \cdot \beta_k)}{2(\beta_I \cdot \beta_i)(\beta_I \cdot \beta_k)}$$

$$z_{jk} = \frac{\beta_I^2 (\beta_j \cdot \beta_k)}{2(\beta_I \cdot \beta_j)(\beta_I \cdot \beta_k)}$$

Set-up for the differential equations





$P_1 = k_1^2 \quad P_2 = k_2^2 \quad P_3 = k_3^2 \quad P_4 = (k_1 - k_2)^2$

$P_5 = (k_3 + k_1 - k_2)^2 \quad P_6 = -\beta_1 \cdot k_1 \quad P_7 = -\beta_2 \cdot k_2$

$P_8 = -\beta_3 \cdot k_3 \quad P_9 = -\beta_4 \cdot (k_3 + k_1 - k_2) - m\sqrt{\beta_4^2}$

$P_{10} = k_1 \cdot k_3 \quad P_{11} = \beta_2 \cdot k_1 \quad P_{12} = \beta_3 \cdot k_1$

$P_{13} = \beta_4 \cdot k_1 \quad P_{14} = \beta_1 \cdot k_2 \quad P_{15} = \beta_3 \cdot k_2$

$P_{16} = \beta_4 \cdot k_2 \quad P_{17} = \beta_1 \cdot k_3 \quad P_{18} = \beta_2 \cdot k_3$

$$I_{\nu_1\nu_2\cdots\nu_{18}}(z_{12}, z_{13}, z_{23}) = \mathrm{e}^{3\epsilon\gamma_E} \int \frac{d^D k_1}{i\pi^{D/2}} \int \frac{d^D k_2}{i\pi^{D/2}} \int \frac{d^D k_3}{i\pi^{D/2}} \prod_{i=1}^{18} \frac{1}{P_i^{\nu_i}}$$

$$z_{12} = \frac{\beta_4^2(\beta_1 \cdot \beta_2)}{2(\beta_1 \cdot \beta_4)(\beta_2 \cdot \beta_4)} \qquad z_{13} = \frac{\beta_4^2(\beta_1 \cdot \beta_3)}{2(\beta_1 \cdot \beta_4)(\beta_3 \cdot \beta_4)} \qquad z_{23} = \frac{\beta_4^2(\beta_2 \cdot \beta_3)}{2(\beta_2 \cdot \beta_4)(\beta_3 \cdot \beta_4)}$$

$$\frac{\partial}{\partial z_{ij}} \vec{I} = \mathbf{A}_{ij} \vec{I} \qquad 19 \times 19$$

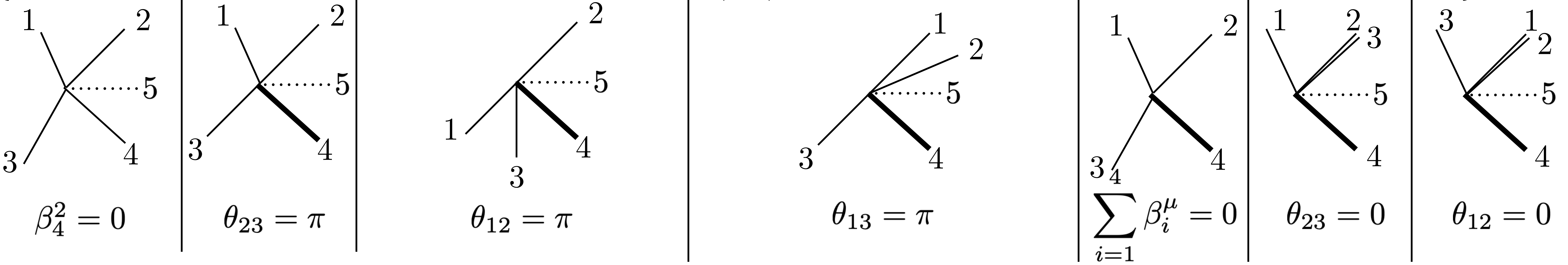
Solving the DEs

$$d\overrightarrow{I}' = \epsilon \sum_i d\log w_i \mathbf{C}_i \overrightarrow{I}' \qquad z_{13} = \frac{\lambda(-\bar{z}+\lambda+z)}{(z-1)z(\bar{z}-1)\bar{z}} \quad z_{12} = \frac{\lambda(-\bar{z}+\lambda+z)}{(z-1)(\bar{z}-1)} \quad z_{23} = \frac{\lambda(-\bar{z}+\lambda+z)}{z\bar{z}}$$

$$\{w_i\} = \{z_{12}, 1-z_{12}, z_{13}, 1-z_{13}, z_{23}, 1-z_{23}, \Lambda(z_{12},z_{13},z_{23}), \Lambda(z_{12},z_{13},z_{23}) + 4z_{12}z_{13}z_{23}\}$$

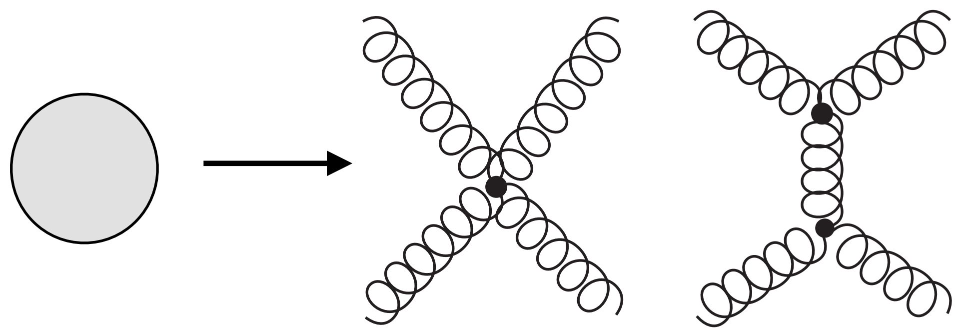
$$\Lambda(z_{12},z_{13},z_{23}) = z_{12}^2 + z_{13}^2 + z_{23}^2 - 2z_{12}z_{13} - 2z_{12}z_{23} - 2z_{13}z_{23}$$

$$\{w_i\} = \{\lambda, \bar{z}-\lambda-z, \lambda+z, \bar{z}-\lambda, -\lambda-z+1, -\bar{z}+\lambda+1, z(-\bar{z})+\lambda+z, -z\bar{z}+\bar{z}-\lambda, \bar{z}-2\lambda-z, 1-z, 1-\bar{z}, z, \bar{z}\}$$



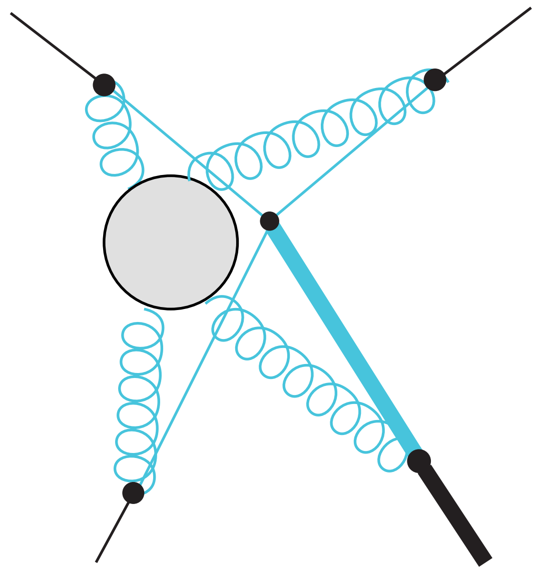
Next steps

$$\left\langle \Phi^{(m)}_{\beta_4}\Phi_{\beta_3}\Phi_{\beta_2}\Phi_{\beta_1}\cdots\right\rangle\Big|_{\beta_3^3=\beta_2^3=\beta_1^3=O(\lambda^2)} = \left\langle \Phi^{(m)}_{\beta_4}\phi_{\beta_3}\phi_{\beta_2}\phi_{\beta_1}\cdots\right\rangle + \cdots$$



- Region by region
- Bootstrapping method

Hard

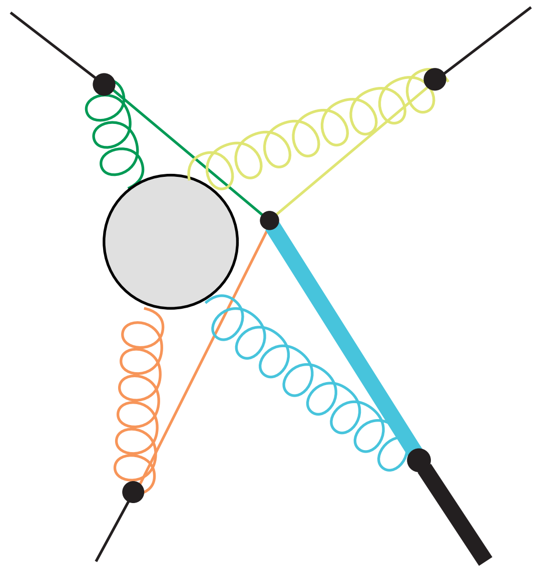


Variables

UT basis element

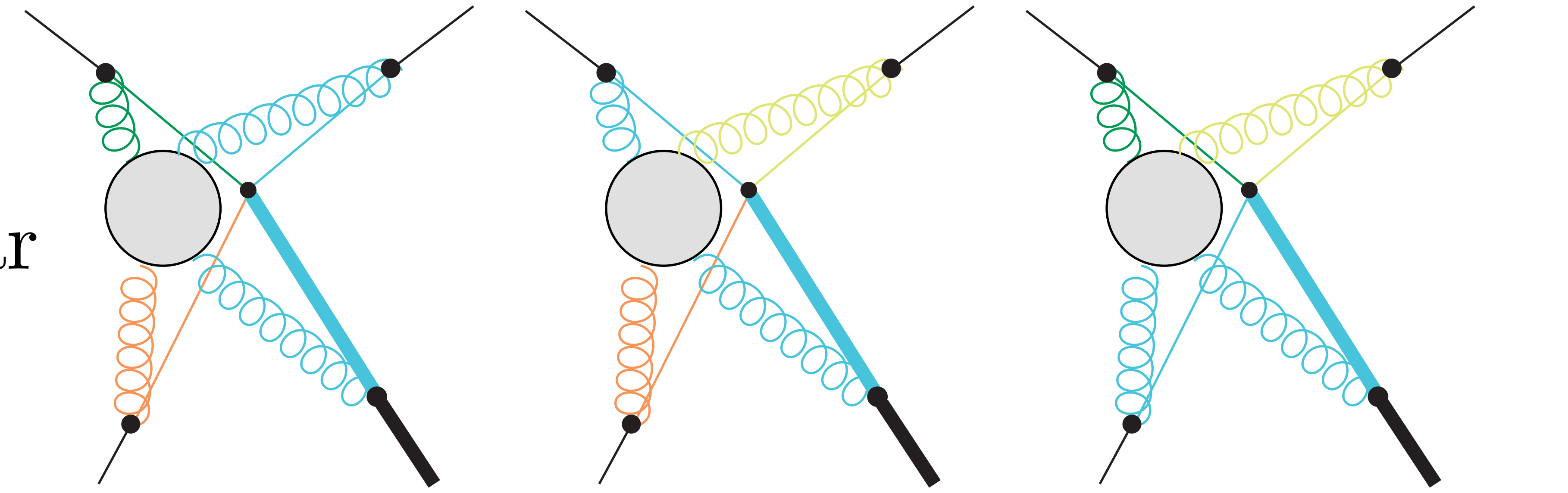
Symbol letters

Triple collinear

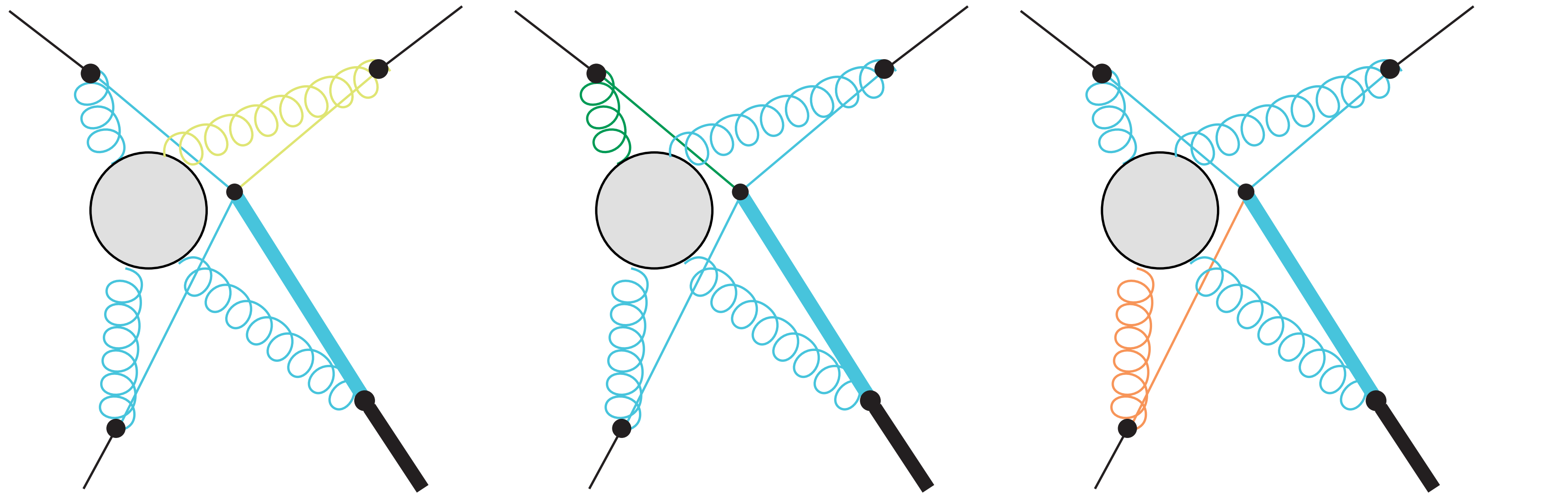


$\cdots$ 56 regions

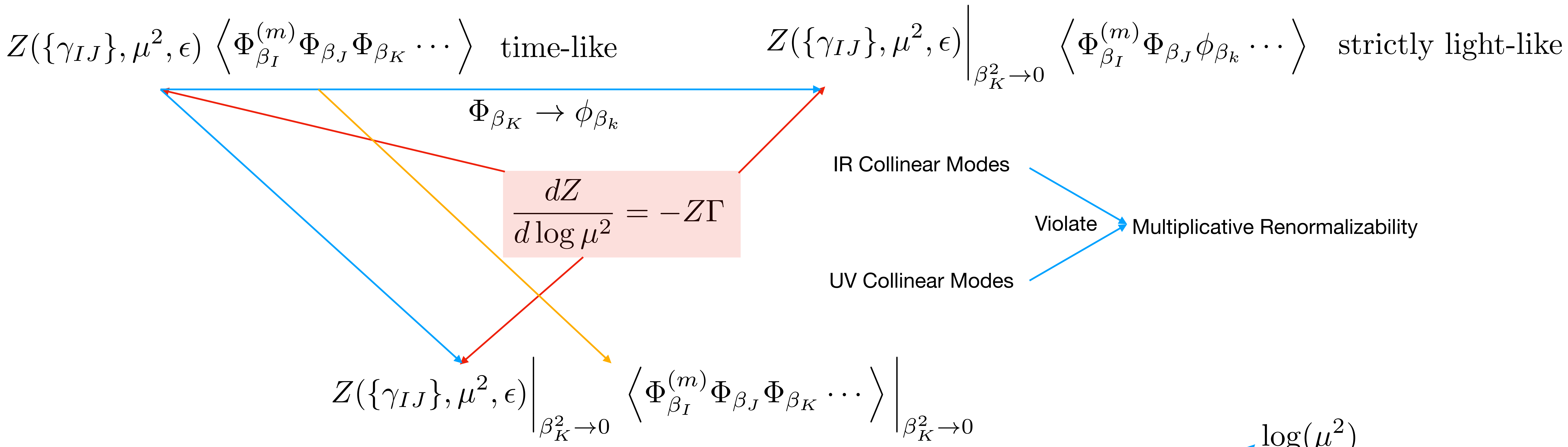
Double collinear



Single collinear



Summary



future plan

$\left\langle \Phi^{(m)}_{\beta_4} \Phi_{\beta_3} \Phi_{\beta_2} \Phi_{\beta_1} \cdots \right\rangle$ at three loops

asymptotic expansion

- γ_{cusp} → $\log(\mu^2)$, → rescaling invariance
- fully connected diagrams

Thank you!