A subtraction scheme at NLO for hybrid kT-factorization

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Hybrid kT-factorization at Born level

final-state momenta of particles and jets

kT-dependent PDF

collinear PDF

differential final-state parton-level phase space

#jets equal to #final-state partons

$$d\sigma^{\mathrm{HF,B}}(\{p\}_n) = \sum_{\bar\chi}\int_0^1 dx\int\frac{d^2k_\perp}{\pi}\,F(x,k_\perp)\int_0^1 d\bar x\,f_{\bar\chi}(\bar x)\,d\Phi\big(x,k_\perp,\bar x;\{p\}_n\big)\,\frac{\big|\overline{M}_{\star\bar\chi}\big|^2\big(x,k_\perp,\bar x;\{p\}_n\big)}{2x\bar x S}\,J_B(\{p\}_n)$$

Tree-level Matrix element with one initial-state space-like gluon, or "reggeon".
Uniquely defined for any number of final-state partons.
Calculable with Lipatov's effective action or the auxiliary parton method.

Hybrid kT-factorization

final-state momenta of particles and jets → $d\sigma^{HF,B}(\{p\}_n)$; kT-dependent PDF → $F(x,k_\perp)$; collinear PDF → $f_{\bar\chi}(\bar x)$; differential final-state parton-level phase space → $d\Phi$; #jets equal to #final-state partons → J_B

$$d\sigma^{\mathrm{HF,B}}(\{p\}_n) = \sum_{\bar\chi}\int_0^1 dx\int\frac{d^2k_\perp}{\pi}F(x,k_\perp)\int_0^1 d\bar x\, f_{\bar\chi}(\bar x)\, d\Phi\big(x,k_\perp,\bar x;\{p\}_n\big)\,\frac{\big|\overline{M}_{\star\bar\chi}\big|^2(x,k_\perp,\bar x;\{p\}_n)}{2x\bar xS}\,J_{\mathrm{B}}(\{p\}_n)$$

Tree-level Matrix element with one initial-state space-like gluon, or "reggeon".
Uniquely defined for any number of final-state partons.
Calculable with Lipatov's effective action or the auxiliary parton method.

In whatever way NLO hybrid kT-factorization is defined, there will be a real-radiation contribution of the type

$$d\sigma^{\mathrm{HF,R}}(\epsilon;\{p\}_{n+1}) = \sum_{\bar\chi}\int_0^1 dx\int\frac{d^2k_\perp}{\pi}F(x,k_\perp)\int_0^1 d\bar x\, f_{\bar\chi}(\bar x)\, d\Phi_\epsilon\big(x,k_\perp,\bar x;\{p\}_{n+1}\big)\,\frac{\big|\overline{M}_{\star\bar\chi}\big|^2(x,k_\perp,\bar x;\{p\}_{n+1})}{2x\bar xS}\,J_{\mathrm{R}}(\{p\}_{n+1})$$

dimensional regularization → $d\Phi_\epsilon$; one more final-state parton → $\{p\}_{n+1}$; #jets may be one fewer than #final-state partons → J_R

Hybrid kT-factorization: a real radiation contribution

The real radiation integral is plagued with final-state and initial-state singularities, leading to divergences. We want to decompose it as

$$d\sigma^{\text{HF,R}}(\epsilon;\{p\}_{n+1}) = d\sigma^{\text{HF,R,unresolved}}(\epsilon;\{p\}_n) + d\sigma^{\text{HF,R,resolved}}(\{p\}_{n+1}) + \mathcal{O}(\epsilon)$$

lives in Born phase space

is finite

The *subtraction method* (Frixione, Kunszt, Signer 1996, Catani, Seymour 1997) can achieve this.

In whatever way NLO hybrid kT-factorization is defined, there will be a real-radiation contribution of the type

$$d\sigma^{\text{HF,R}}(\epsilon;\{p\}_{n+1}) = \sum_{\bar{\chi}} \int_0^1 dx \int \frac{d^2k_\perp}{\pi} F(x,k_\perp) \int_0^1 d\bar{x}\, f_{\bar{\chi}}(\bar{x})\, d\Phi_\epsilon(x,k_\perp,\bar{x};\{p\}_{n+1}) \frac{\left|\overline{M}_{\star\bar{\chi}}\right|^2(x,k_\perp,\bar{x};\{p\}_{n+1})}{2x\bar{x}S} J_R(\{p\}_{n+1})$$

dimensional regularization

one more final-state parton

#jets may be one fewer than #final-state partons

The subtraction method

We want the Laurent expansion in ϵ of $\int_0^1 dx\, x^\epsilon \frac{f(x)}{x}$ but cannot perform the integral analytically.

$$\int_0^1 dx\, x^\epsilon \frac{f(x)}{x} = \int_0^1 dx\, x^\epsilon \frac{f(0)}{x} + \int_0^1 dx\, x^\epsilon \frac{f(x)-f(0)}{x} = \frac{f(0)}{\epsilon} + \int_0^1 dx \frac{f(x)-f(0)}{x} + \mathcal{O}(\epsilon)$$

- can be performed analytically (first term)
- can be expanded in eps (second term)
- can be performed numerically (integral on the right)

The subtraction method

We want the Laurent expansion in ϵ of $\int_0^1 dx\, x^\epsilon \frac{f(x)}{x}$ but cannot perform the integral analytically.

$$\int_0^1 dx\, x^\epsilon \frac{f(x)}{x} = \int_0^1 dx\, x^\epsilon \frac{f(0)}{x} + \int_0^1 dx\, x^\epsilon \frac{f(x)-f(0)}{x} = \frac{f(0)}{\epsilon} + \int_0^1 dx \frac{f(x)-f(0)}{x} + \mathcal{O}(\epsilon)$$

can be performed analytically — can be expanded in eps — can be performed numerically

In two dimensions

$$\int_0^1 dx\, x^\epsilon \int_0^1 dy\, y^\epsilon \frac{f(x,y)}{xy} = \frac{f(0,0)}{\epsilon^2} + \frac{1}{\epsilon}\int_0^1 dx \frac{f(x,0)-f(0,0)}{x} + \frac{1}{\epsilon}\int_0^1 dy \frac{f(0,y)-f(0,0)}{y}$$
$$+ \int_0^1 dx \ln(x) \frac{f(x,0)-f(0,0)}{x} + \int_0^1 dy \ln(y) \frac{f(0,y)-f(0,0)}{y}$$
$$+ \int_0^1 dx \int_0^1 dy \frac{f(x,y)-f(x,0)-f(0,y)+f(0,0)}{xy} + \mathcal{O}(\epsilon)$$

Notation

The Born-level formula for the cross section in hybrid k_T-factorization:

$$\sigma_B = \frac{1}{\mathcal{S}_n}\int[dQ]\int d\Phi(Q;\{p\}_n)\,\mathcal{L}(Q;\{p\}_n)\,|\mathcal{M}|^2(Q;\{p\}_n)\,J_B(\{p\}_n)$$

Initial-state variables:

$$\int[dQ] = \int_0^1 dx\int_0^1 d\bar{x}\int d^2k_\perp \quad,\quad Q^\mu = k_\chi^\mu + k_{\bar{\chi}}^\mu \quad,\quad \begin{cases} k_\chi^\mu = xP^\mu + k_\perp^\mu & ,\quad P^\mu = (E,0,0,E) \\ k_{\bar{\chi}}^\mu = \bar{x}\bar{P}^\mu & ,\quad \bar{P}^\mu = (\bar{E},0,0,-\bar{E}) \end{cases}$$

Differential phase space for the final-state momenta $\{p\}_n$

$$d\Phi(Q;\{p\}_n) = \left(\prod_{l=1}^{n}\frac{d^4p_l}{(2\pi)^3}\delta_+(p_l^2 - m_l^2)\right)\frac{1}{(2\pi)^4}\,\delta\!\left(Q - \sum_{l=1}^{n}p_l\right)$$

The PDFs and flux factor:

$$\mathcal{L}(Q;\{p\}_n) = \frac{F_\chi(x,k_\perp,\mu_F(\{p\}_n))\,f_{\bar{\chi}}(\bar{x},\mu_F(\{p\}_n))}{8x\bar{x}E\bar{E}}$$

$|\mathcal{M}|^2(Q;\{p\}_n)$ tree-level matrix element without symmetry factors and averageing factors, they are captured by $\mathcal{S}_n$.
$J_B(\{p\}_n)$ denotes the jet function, demanding the number of jets to be equal to the number of final-state partons.

Singular limits at NLO: jets

For the real radiation, the jet function J_R does not avoid all singularities of the tree-level squared matrix element anymore, but allows one pair of partons to become collinear,

$$\text{one pair of partons to become collinear:} \quad p_r \| p_i \quad \Leftrightarrow \quad \vec{n}_r - \vec{n}_i \to \vec{0}$$
$$\text{one parton to become soft:} \quad p_r \to \text{soft} \quad \Leftrightarrow \quad E_r \to 0$$

The jet function behaves in those limits such that

$$J_R(\{p\}_{n+1}) \xrightarrow{p_r \to \text{soft}} J_B(\{p\}_n^{\not r}) \quad , \quad \{p\}_n^{\not r} \textit{ is obtained from } \{p\}_{n+1} \textit{ by removing momentum } p_r$$
$$J_R(\{p\}_{n+1}) \xrightarrow{p_r \| p_i} J_B(\{p\}_n^{\not r;i}) \quad , \quad \{p\}_n^{\not r;i} \textit{ is obtained by additionally replacing } p_i \textit{ with } (1+z_{ri})p_i \quad z_{ri} = E_r/E_i$$
$$J_R(\{p\}_{n+1}) \xrightarrow{p_r \| P, \bar{P}} J_B(\{p\}_n^{\not r}) \quad ,$$

(We assume p_r and also p_i to be light-like.)

The jet algorithm combines momenta that are “too collinear to be separate jets”, by adding them up. This sum of momenta is not on-shell (only at the limit) and cannot be the argument of a matrix element. This is one of the problems to be solved by a subtraction method.

Singular limits at NLO: matrix elements

Matrix elements are constructed from external momenta that must satisfy mometum conservation. When $(Q;\{p\}_{n+1})$ satisfies momentum conservation, then $(Q;\{p\}_n^{\not r})$ and $(Q;\{p\}_n^{\not r;i})$ do not. We must introduce deformed momenta to even write down the limits:

$$|\mathcal{M}|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\to\text{soft}} \hat{\mathcal{R}}^{\text{soft}}(p_r)\otimes\hat{\mathcal{A}}^{\text{soft}}(\tilde{Q};\{\tilde{p}\}_n^{\not r})$$

$$|\mathcal{M}|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\|p_i} \hat{\mathcal{R}}^{\text{F,col}}_{ir}(p_r)\otimes\hat{\mathcal{A}}^{\text{F,col}}_{ir}(\tilde{Q};\{\tilde{p}\}_n^{\not r;i})$$

$$|\mathcal{M}|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\|P} \hat{\mathcal{R}}^{\text{I,col}}_{\chi,r}(p_r)\otimes\hat{\mathcal{A}}^{\text{I,col}}_{\chi,r}(\tilde{Q}-x_rP;\{\tilde{p}\}_n^{\not r})$$

universal factor with singular behavior

spin-or color-correlated matrix element with n final-state particles

Singular limits at NLO: matrix elements

Matrix elements are constructed from external momenta that must satisfy mometum conservation. When $(Q;\{p\}_{n+1})$ satisfies momentum conservation, then $(Q;\{p\}_n^{\not r})$ and $(Q;\{p\}_n^{\not r;i})$ do not. We must introduce deformed momenta to even write down the limits:

$$\left|\mathcal{M}\right|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\to\text{soft}} \hat{\mathcal{R}}^{\text{soft}}(p_r)\otimes\hat{\mathcal{A}}^{\text{soft}}(\tilde{Q};\{\tilde{p}\}_n^{\not r})$$

$$\left|\mathcal{M}\right|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\|p_i} \hat{\mathcal{R}}^{\text{F,col}}_{ir}(p_r)\otimes\hat{\mathcal{A}}^{\text{F,col}}_{ir}(\tilde{Q};\{\tilde{p}\}_n^{\not r;i})$$

$$\left|\mathcal{M}\right|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\|P} \hat{\mathcal{R}}^{\text{I,col}}_{\chi,r}(p_r)\otimes\hat{\mathcal{A}}^{\text{I,col}}_{\chi,r}(\tilde{Q}-x_rP;\{\tilde{p}\}_n^{\not r})$$

In k_T-factorization, we can choose to just deform the initial-state momenta:

$$\left|\mathcal{M}\right|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\to\text{soft}} \hat{\mathcal{R}}^{\text{soft}}(p_r)\otimes\hat{\mathcal{A}}^{\text{soft}}(Q-p_r;\{p\}_n^{\not r})$$

$$\left|\mathcal{M}\right|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\|p_i} \hat{\mathcal{R}}^{\text{F,col}}_{ir}(p_r)\otimes\hat{\mathcal{A}}^{\text{F,col}}_{ir}(Q-p_r+z_{ri}p_i;\{p\}_n^{\not r;i})$$

$$\left|\mathcal{M}\right|^2(Q;\{p\}_{n+1}) \xrightarrow{p_r\|P/\bar{P}} \hat{\mathcal{R}}^{\text{I,col}}_{\chi/\bar{\chi},r}(p_r)\otimes\hat{\mathcal{A}}^{\text{I,col}}_{\chi/\bar{\chi},r}(Q-p_r;\{p\}_n^{\not r})$$

This opens the possibility to construct subtraction terms with only deformed initial-state momenta.

Subtraction method

Real radiation contribution within dimensional regularization

$$\sigma_R(\epsilon) = \frac{1}{\mathcal{S}_{n+1}} \int [dQ] \int d\Phi(\epsilon; Q; \{p\}_{n+1}) \mathcal{L}(Q; \{p\}_{n+1})) \, |\mathcal{M}|^2 (Q; \{p\}_{n+1}) \, J_R(\{p\}_{n+1})$$

We want to split the real-radiation integral into a finite part and a divergent part that can be explicitly expressed as a Laurent expansion in ϵ within dimensional regularization

$$\sigma_R(\epsilon) = \sigma_R^{\text{div}}(\epsilon) + \sigma_R^{\text{resolved}} + \mathcal{O}(\epsilon) = \frac{1}{\epsilon^2}\,\sigma_R^{\text{div},(-2)} + \frac{1}{\epsilon}\,\sigma_R^{\text{div},(-1)} + \sigma_R^{\text{div},(0)} + \sigma_R^{\text{resolved}} + \mathcal{O}(\epsilon)$$

We define the finite "subtracted-real" integral as

$$\sigma_R^{\text{resolved}} = \frac{1}{\mathcal{S}_{n+1}} \int [dQ] \int d\Phi(Q; \{p\}_{n+1}) \Big\{ \mathcal{L}(Q; \{p\}_{n+1})) \, |\mathcal{M}|^2 (Q; \{p\}_{n+1}) \, J_R(\{p\}_{n+1}) - \sum_r \text{Subt}_r(Q; \{p\}_{n+1}) \Big\} \,,$$

that can be integrated numerically, and

$$\sigma_R^{\text{div}}(\epsilon) = \frac{1}{\mathcal{S}_{n+1}} \sum_r \int [dQ] \int d\Phi(\epsilon; Q; \{p\}_{n+1}) \text{Subt}_r(Q; \{p\}_{n+1}) = \frac{1}{\epsilon^2}\,\sigma_R^{\text{div},(-2)} + \frac{1}{\epsilon}\,\sigma_R^{\text{div},(-1)} + \sigma_R^{\text{div},(0)} \,,$$

for which the radiation should be integrable analytically.

Subtraction terms based on Somogyi, Trócsányi 2006

Final-state terms, with arguments $\left(Q - p_r + z_{ri}p_i; \{p\}_n^{\not r;i}\right)$ for amplitudes $\mathcal{M}$:

collinear terms $$\mathcal{R}_{ir}^{F,col} \otimes \mathcal{A}_{ir}^{F,col} = \frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\, \theta(n_r \cdot n_i < 2\zeta_0)\, \frac{\theta(E_r < E_i)}{p_i \cdot p_r}\, \mathcal{Q}_{ir}(z_{ri}) \otimes \left|\mathcal{M}_{ir}\right|^2$$

soft terms $$\mathcal{R}_{i}^{F,soft} \otimes \mathcal{A}_{i}^{F,soft} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\, \theta(E_r < E_0)\, \frac{2}{n_i \cdot p_r} \sum_b \frac{n_i \cdot n_b}{n_i \cdot p_r + n_b \cdot p_r}\, (\mathcal{M})^2_{\mathsf{color}(i,b)}$$

soft-collinear counter terms $$\mathcal{R}_{i}^{F,soco} \otimes \mathcal{A}_{i}^{F,soco} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\, \theta(E_r < E_0)\theta(n_r \cdot n_i < 2\zeta_0)\, \frac{2C_i}{p_i \cdot p_r}\, \frac{1}{z_{ri}}\, |\mathcal{M}|^2$$

Subtraction terms

Final-state terms, with arguments $\left(Q - p_r + z_{ri}p_i; \{p\}_n^{\not r;i}\right)$ for amplitudes $\mathcal{M}$:

collinear terms $\quad \mathcal{R}_{ir}^{F,col} \otimes \mathcal{A}_{ir}^{F,col} = \frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta(n_r\cdot n_i < 2\zeta_0)\,\frac{\theta(E_r < E_i)}{p_i\cdot p_r}\,\mathcal{Q}_{ir}(z_{ri}) \otimes \left|\mathcal{M}_{ir}\right|^2$

soft terms $\quad \mathcal{R}_i^{F,soft} \otimes \mathcal{A}_i^{F,soft} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta(E_r < E_0)\,\frac{2}{n_i\cdot p_r}\sum_b \frac{n_i\cdot n_b}{n_i\cdot p_r + n_b\cdot p_r}\,(\mathcal{M})^2_{\mathrm{color}(i,b)}$

soft-collinear counter terms $\quad \mathcal{R}_i^{F,soco} \otimes \mathcal{A}_i^{F,soco} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta(E_r < E_0)\theta(n_r\cdot n_i < 2\zeta_0)\,\frac{2C_i}{p_i\cdot p_r}\,\frac{1}{z_{ri}}\left|\mathcal{M}\right|^2$

split collinear and soft factors into two terms

$$\frac{1}{p_i\cdot p_j} = \frac{\theta(E_j < E_i)}{p_i\cdot p_j} + \frac{\theta(E_i < E_j)}{p_j\cdot p_i}$$

$$\frac{(n_i\cdot n_j)}{(n_i\cdot p_r)(p_r\cdot n_j)} = \frac{1}{n_i\cdot p_r}\,\frac{n_i\cdot n_j}{n_i\cdot p_r + p_r\cdot n_j} + \frac{1}{n_j\cdot p_r}\,\frac{n_i\cdot n_j}{n_j\cdot p_r + p_r\cdot n_i}$$

to avoid double counting when summing over all radiators and all radiation

Subtraction terms

Final-state terms, with arguments $\left(Q - p_r + z_{ri}p_i; \{p\}_n^{\not r;i}\right)$ for amplitudes $\mathcal{M}$:

collinear terms $$\mathcal{R}_{ir}^{F,col} \otimes \mathcal{A}_{ir}^{F,col} = \frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\ \theta(n_r\cdot n_i < 2\zeta_0)\ \frac{\theta(E_r < E_i)}{p_i\cdot p_r}\boxed{\mathcal{Q}_{ir}(z_{ri})}\otimes\left|\mathcal{M}_{ir}\right|^2$$

soft terms $$\mathcal{R}_{i}^{F,soft} \otimes \mathcal{A}_{i}^{F,soft} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\ \theta(E_r < E_0)\ \frac{2}{n_i\cdot p_r}\sum_b \frac{n_i\cdot n_b}{n_i\cdot p_r + n_b\cdot p_r}\left(\mathcal{M}\right)^2_{\text{color}(i,b)}$$

soft-collinear counter terms $$\mathcal{R}_{i}^{F,soco} \otimes \mathcal{A}_{i}^{F,soco} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\ \theta(E_r < E_0)\theta(n_r\cdot n_i < 2\zeta_0)\ \frac{2C_i}{p_i\cdot p_r}\frac{1}{z_{ri}}\left|\mathcal{M}\right|^2$$

splitting function in terms of energy ratios

$$\mathcal{Q}(z) = \mathcal{P}\left(\frac{1-z}{z}\right)$$

Subtraction terms

Final-state terms, with arguments $\left(Q - p_r + z_{ri}p_i; \{p\}_n^{\not r;i}\right)$ for amplitudes $\mathcal{M}$:

collinear terms $\mathcal{R}_{ir}^{F,col} \otimes \mathcal{A}_{ir}^{F,col} = \frac{4\pi\alpha_s}{\mu^{-2\epsilon}} \boxed{\theta(n_r{\cdot}n_i < 2\zeta_0)} \frac{\theta(E_r < E_i)}{p_i{\cdot}p_r} \mathcal{Q}_{ir}(z_{ri}) \otimes |\mathcal{M}_{ir}|^2$

soft terms $\mathcal{R}_i^{F,soft} \otimes \mathcal{A}_i^{F,soft} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}} \boxed{\theta(E_r < E_0)} \frac{2}{n_i{\cdot}p_r} \sum_b \frac{n_i{\cdot}n_b}{n_i{\cdot}p_r + n_b{\cdot}p_r} (\mathcal{M})^2_{color(i,b)}$

soft-collinear counter terms $\mathcal{R}_i^{F,soco} \otimes \mathcal{A}_i^{F,soco} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}} \boxed{\theta(E_r < E_0)\theta(n_r{\cdot}n_i < 2\zeta_0)} \frac{2C_i}{p_i{\cdot}p_r} \frac{1}{z_{ri}} |\mathcal{M}|^2$

restrict terms to phase space regions where they matter

Both $\sigma_R^{div,(0)}$ and $\sigma_R^{resolved}$ depend on the exact values of E_0, ζ_0, ξ_0,
but $\sigma_R^{div,(-2)}$, $\sigma_R^{div,(-1)}$, and $\sigma_R^{div,(0)} + \sigma_R^{resolved}$ should not.

Subtraction terms

Final-state terms, with arguments $\big(Q - p_r + z_{ri}p_i; \{p\}_n^{\not r;i}\big)$ for amplitudes $\mathcal{M}$:

collinear terms
$$\mathcal{R}_{ir}^{F,\text{col}} \otimes \mathcal{A}_{ir}^{F,\text{col}} = \frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta(n_r\!\cdot\! n_i < 2\zeta_0)\,\frac{\theta(E_r < E_i)}{p_i\!\cdot\! p_r}\,\mathcal{Q}_{ir}(z_{ri}) \otimes \big|\mathcal{M}_{ir}\big|^2$$

soft terms
$$\mathcal{R}_{i}^{F,\text{soft}} \otimes \mathcal{A}_{i}^{F,\text{soft}} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta(E_r < E_0)\,\frac{2}{n_i\!\cdot\! p_r}\sum_b \frac{n_i\!\cdot\! n_b}{n_i\!\cdot\! p_r + n_b\!\cdot\! p_r}\big(\mathcal{M}\big)^2_{\text{color}(i,b)}$$

soft-collinear counter terms
$$\mathcal{R}_{i}^{F,\text{soco}} \otimes \mathcal{A}_{i}^{F,\text{soco}} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta(E_r < E_0)\theta(n_r\!\cdot\! n_i < 2\zeta_0)\,\frac{2C_i}{p_i\!\cdot\! p_r}\frac{1}{z_{ri}}\big|\mathcal{M}\big|^2$$

Initial-state terms, with arguments $\big(Q - p_r; \{p\}_n^{\not r}\big)$ for amplitudes $\mathcal{M}$:

$$\mathcal{R}_{\chi r}^{I,\text{col}} \otimes \mathcal{A}_{\chi r}^{I,\text{col}} = \frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta\big(\bar{x}_r < \xi_0 x_r\big)\,\frac{-2}{S\bar{x}_r x}\,\mathcal{Q}_{\chi r}(-x_r/x) \otimes \big|\mathcal{M}_{\chi r}\big|^2$$

$$\mathcal{R}_{\chi}^{I,\text{soft}} \otimes \mathcal{A}_{\chi}^{I,\text{soft}} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta(E_r < E_0)\,\frac{2}{n_\chi\!\cdot\! p_r}\sum_b \frac{n_\chi\!\cdot\! n_b}{n_\chi\!\cdot\! p_r + n_b\!\cdot\! p_r}\big(\mathcal{M}\big)^2_{\text{color}(\chi,b)}$$

$$\mathcal{R}_{\chi}^{I,\text{soco}} \otimes \mathcal{A}_{\chi}^{I,\text{soco}} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\,\theta(E_r < E_0)\theta\big(\bar{x}_r < \xi_0 x_r\big)\,\frac{4C_\chi}{Sx_r\bar{x}_r}\big|\mathcal{M}\big|^2$$

Subtraction terms

While $k_\chi^\mu = xP^\mu + k_\perp^\mu$ is space-like, there is an initial-state singularity related to the space-like gluon if the radiative momentum becomes collinear to P, with splitting function (AvH, Motyka, Ziarko 2022)

$$\mathcal{Q}_{\chi r}(\zeta) = \frac{2N_c}{\zeta(1+\zeta)^2} \quad \Leftrightarrow \quad \mathcal{P}_{\chi r}(z) \equiv -z\mathcal{Q}_\chi(z-1) = \frac{2N_c}{z(1-z)}$$

Initial-state terms, with arguments $\left(Q - p_r; \{p\}_n^{\not r}\right)$ for amplitudes $\mathcal{M}$:

$$\mathcal{R}_{\chi r}^{\mathsf{I,col}} \otimes \mathcal{A}_{\chi r}^{\mathsf{I,col}} = \frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\, \theta\left(\bar{x}_r < \xi_0 x_r\right) \frac{-2}{S\bar{x}_r x} \mathcal{Q}_{\chi r}(-x_r/x) \otimes \left|\mathcal{M}_{\chi r}\right|^2$$

$$\mathcal{R}_{\chi}^{\mathsf{I,soft}} \otimes \mathcal{A}_{\chi}^{\mathsf{I,soft}} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\, \theta(E_r < E_0) \frac{2}{n_\chi \cdot p_r} \sum_b \frac{n_\chi \cdot n_b}{n_\chi \cdot p_r + n_b \cdot p_r} \left(\mathcal{M}\right)^2_{\mathsf{color}(\chi,b)}$$

$$\mathcal{R}_{\chi}^{\mathsf{I,soco}} \otimes \mathcal{A}_{\chi}^{\mathsf{I,soco}} = -\frac{4\pi\alpha_s}{\mu^{-2\epsilon}}\, \theta(E_r < E_0)\theta\left(\bar{x}_r < \xi_0 x_r\right) \frac{4C_\chi}{S x_r \bar{x}_r} \left|\mathcal{M}\right|^2$$

Subtracted-real integral

$$\sigma_R(\epsilon) = \sigma_R^{\text{div}}(\epsilon) + \sigma_R^{\text{resolved}} + \mathcal{O}(\epsilon)$$

We define the finite "subtracted-real" (resolved) integral as

$$\sigma_R^{\text{resolved}} = \frac{1}{\mathcal{S}_{n+1}} \int [dQ] \int d\Phi(Q;\{p\}_{n+1}) \Big\{ \mathcal{L}(Q;\{p\}_{n+1}) \, |\mathcal{M}|^2(Q;\{p\}_{n+1}) \, J_R(\{p\}_{n+1}) - \sum_r \text{Subt}_r(Q;\{p\}_{n+1}) \Big\} ,$$

where the r-sum is over all final-state partons, and where $\text{Subt}_r(Q;\{p\}_{n+1})$ is given by

$$\begin{aligned}
\text{Subt}_r(Q;\{p\}_{n+1}) = & \sum_i \mathcal{L}(Q - p_r + z_{ri}p_i;\{p\}_n^{\not r;i}) \quad \mathcal{R}_{ir}^{F}(p_r) \otimes \mathcal{A}_{ir}^{F}(Q - p_r + z_{ri}p_i;\{p\}_n^{\not r;i}) \, J_B(\{p\}_n^{\not r;i}) \\
& + \sum_{a\in\{\chi,\bar\chi\}} \mathcal{L}(Q - p_r \quad ;\{p\}_n^{\not r}) \, \mathcal{R}_a^{\text{I,soft}}(p_r) \otimes \mathcal{A}_a^{\text{I,soft}}(Q - p_r \quad ;\{p\}_n^{\not r}) \, J_B(\{p\}_n^{\not r}) \\
& + \sum_{a\in\{\chi,\bar\chi\}} \mathcal{L}(Q - p_r \quad ;\{p\}_n^{\not r}) \, \mathcal{R}_a^{\text{I,soco}}(p_r) \otimes \mathcal{A}_a^{\text{I,soco}}(Q - p_r \quad ;\{p\}_n^{\not r}) \, J_B(\{p\}_n^{\not r}) \\
& + \quad \mathcal{L}(Q - \bar x_r \bar P - p_{\perp r};\{p\}_n^{\not r}) \, \mathcal{R}_{\chi,r}^{\text{I,col}}(p_r) \otimes \mathcal{A}_{\chi,r}^{\text{I,col}}(Q - p_r \quad ;\{p\}_n^{\not r}) \, J_B(\{p\}_n^{\not r}) \\
& + \quad \mathcal{L}(Q - x_r P - p_{\perp r};\{p\}_n^{\not r}) \, \mathcal{R}_{\bar\chi,r}^{\text{I,col}}(p_r) \otimes \mathcal{A}_{\bar\chi,r}^{\text{I,col}}(Q - p_r \quad ;\{p\}_n^{\not r}) \, J_B(\{p\}_n^{\not r})
\end{aligned}$$

Subtracted-real integral

$$\sigma_R(\epsilon) = \sigma_R^{\text{div}}(\epsilon) + \sigma_R^{\text{resolved}} + \mathcal{O}(\epsilon)$$

We define the finite "subtracted-real" (resolved) integral as

$$\sigma_R^{\text{resolved}} = \frac{1}{\mathcal{S}_{n+1}} \int [dQ] \int d\Phi(Q;\{p\}_{n+1}) \Big\{ \mathcal{L}(Q;\{p\}_{n+1})) \, |\mathcal{M}|^2(Q;\{p\}_{n+1}) \, J_R(\{p\}_{n+1}) - \sum_r \text{Subt}_r(Q;\{p\}_{n+1}) \Big\} \,,$$

where the r-sum is over all final-state partons, and where $\text{Subt}_r(Q;\{p\}_{n+1})$ is given by

$$\begin{aligned}
\text{Subt}_r(Q;\{p\}_{n+1}) = \; & \sum_i \mathcal{L}\big(Q - p_r + z_{ri}p_i;\{p\}_n^{\not r;i}\big) \quad \mathcal{R}_{ir}^{F}(p_r) \otimes \mathcal{A}_{ir}^{F}\big(Q - p_r + z_{ri}p_i;\{p\}_n^{\not r;i}\big) \, J_B\big(\{p\}_n^{\not r;i}\big) \\
& + \sum_{a\in\{\chi,\bar\chi\}} \mathcal{L}\big(Q - p_r \;;\{p\}_n^{\not r}\big) \, \mathcal{R}_a^{\text{l,soft}}(p_r) \otimes \mathcal{A}_a^{\text{l,soft}}\big(Q - p_r \;;\{p\}_n^{\not r}\big) \, J_B\big(\{p\}_n^{\not r}\big) \\
& + \sum_{a\in\{\chi,\bar\chi\}} \mathcal{L}\big(Q - p_r \;;\{p\}_n^{\not r}\big) \, \mathcal{R}_a^{\text{l,soco}}(p_r) \otimes \mathcal{A}_a^{\text{l,soco}}\big(Q - p_r \;;\{p\}_n^{\not r}\big) \, J_B\big(\{p\}_n^{\not r}\big) \\
& + \quad \mathcal{L}\big(Q - \bar x_r \bar P - p_{\perp r};\{p\}_n^{\not r}\big) \, \mathcal{R}_{\chi,r}^{\text{l,col}}(p_r) \otimes \mathcal{A}_{\chi,r}^{\text{l,col}}\big(Q - p_r \;;\{p\}_n^{\not r}\big) \, J_B\big(\{p\}_n^{\not r}\big) \\
& + \quad \mathcal{L}\big(Q - x_r P - p_{\perp r};\{p\}_n^{\not r}\big) \, \mathcal{R}_{\bar\chi,r}^{\text{l,col}}(p_r) \otimes \mathcal{A}_{\bar\chi,r}^{\text{l,col}}\big(Q - p_r \;;\{p\}_n^{\not r}\big) \, J_B\big(\{p\}_n^{\not r}\big)
\end{aligned}$$

subtract recoil also from arguments of the PDFs and the flux factor

This is allowed if the recoil vanishes at the singular limit.

For the initial-state collinear terms, this cannot be the whole momentum p_r, only the part that vanishes

Integrated subtraction terms

$$\sigma_R(\epsilon) = \sigma_R^{\text{div}}(\epsilon) + \sigma_R^{\text{resolved}} + \mathcal{O}(\epsilon)$$

$$\sigma_R^{\text{div}}(\epsilon) = \frac{1}{\mathcal{S}_{n+1}} \sum_r \int [dQ] \int d\Phi\big(Q;\{p\}_n^{\not r}\big)\, \mathcal{L}\big(Q;\{p\}_n^{\not r}\big)\big)\, J_B\big(\{p\}_n^{\not r}\big)$$

$$\times \left\{ \sum_i \mathcal{J}_{ir}^{F}\big(\epsilon, Q, \{p\}_n^{\not r}\big) \otimes \mathcal{A}_{ir}^{F}\big(Q;\{p\}_n^{\not r}\big) + \sum_{a\in\{\chi,\bar\chi\}} \mathcal{J}_{ar}^{I}\big(\epsilon, Q, \{p\}_n^{\not r}\big) \otimes \mathcal{A}_{ar}^{I}\big(Q;\{p\}_n^{\not r}\big) \right\},$$

$$\mathcal{J}_{ir}^{F}\big(\epsilon, Q, \{p\}_n^{\not r}\big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\,(1 - z_{ri})\, \boxed{\mathcal{R}_{ir}^{F}(p_r)}\, \Theta(p_r - z_{ri}p_i)$$

$$\mathcal{J}_{a}^{I,\text{soft/soco}}\big(\epsilon, Q, \{p\}_n^{\not r}\big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\, \boxed{\mathcal{R}_{a}^{I,\text{soft/soco}}(p_r)}\, \Theta(p_r)$$

$$\mathcal{J}_{\chi r}^{I,\text{col}}\big(\epsilon, Q, \{p\}_n^{\not r}\big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\, \boxed{\mathcal{R}_{\chi r}^{I,\text{col}}(p_r)}\, \Theta(p_r)\, \frac{\mathcal{L}\big(Q + x_r P;\{p\}_n^{\not r}\big)}{\mathcal{L}\big(Q;\{p\}_n^{\not r}\big)}$$

where $\quad \Theta(q) = \theta(-x < x_q < 1 - x)\,\theta(-\bar{x} < \bar{x}_q < 1 - \bar{x})$

Singular factors form simple integrands suitable for analyic integration

Integrated subtraction terms

$$\sigma_R(\epsilon) = \sigma_R^{\text{div}}(\epsilon) + \sigma_R^{\text{resolved}} + \mathcal{O}(\epsilon)$$

$$\sigma_R^{\text{div}}(\epsilon) = \frac{1}{\mathcal{S}_{n+1}} \sum_r \int [dQ] \int d\Phi\big(Q;\{p\}_n^{\not r}\big)\, \mathcal{L}\big(Q;\{p\}_n^{\not r}\big)\, J_B\big(\{p\}_n^{\not r}\big)$$

$$\times \left\{ \sum_i \mathcal{J}_{ir}^F\big(\epsilon, Q, \{p\}_n^{\not r}\big) \otimes \mathcal{A}_{ir}^F\big(Q;\{p\}_n^{\not r}\big) + \sum_{a\in\{\chi,\bar\chi\}} \mathcal{J}_{ar}^I\big(\epsilon, Q, \{p\}_n^{\not r}\big) \otimes \mathcal{A}_{ar}^I\big(Q;\{p\}_n^{\not r}\big) \right\},$$

$$\mathcal{J}_{ir}^F\big(\epsilon, Q, \{p\}_n^{\not r}\big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\, (1-z_{ri})\, \mathcal{R}_{ir}^F(p_r)\, \Theta(p_r - z_{ri}p_i)$$

$$\mathcal{J}_a^{\text{I,soft/soco}}\big(\epsilon, Q, \{p\}_n^{\not r}\big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\, \mathcal{R}_a^{\text{I,soft/soco}}(p_r)\, \Theta(p_r)$$

$$\mathcal{J}_{\chi r}^{\text{I,col}}\big(\epsilon, Q, \{p\}_n^{\not r}\big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\, \mathcal{R}_{\chi r}^{\text{I,col}}(p_r)\, \Theta(p_r)\, \frac{\mathcal{L}\big(Q + x_r P;\{p\}_n^{\not r}\big)}{\mathcal{L}\big(Q;\{p\}_n^{\not r}\big)}$$

where $\Theta(q) = \theta(-x < x_q < 1-x)\,\theta(-\bar{x} < \bar{x}_q < 1-\bar{x})$

Initial-state collinear integrated subtraction term involves the PDF

Cannot be (completely) integrated analytically.

This results in the so-called P-operator (Catani, Seymour 1997).

Integrated subtraction terms

$$\sigma_R(\epsilon) = \sigma_R^{div}(\epsilon) + \sigma_R^{resolved} + \mathcal{O}(\epsilon)$$

$$\sigma_R^{div}(\epsilon) = \frac{1}{\mathcal{S}_{n+1}} \sum_r \int [dQ] \int d\Phi(Q;\{p\}_n^{\not r})\, \mathcal{L}(Q;\{p\}_n^{\not r})\, J_B(\{p\}_n^{\not r})$$
$$\times \left\{ \sum_i \mathcal{J}_{ir}^F(\epsilon, Q, \{p\}_n^{\not r}) \otimes \mathcal{A}_{ir}^F(Q;\{p\}_n^{\not r}) + \sum_{a\in\{\chi,\bar\chi\}} \mathcal{J}_{ar}^I(\epsilon, Q, \{p\}_n^{\not r}) \otimes \mathcal{A}_{ar}^I(Q;\{p\}_n^{\not r}) \right\},$$

$$\mathcal{J}_{ir}^F(\epsilon, Q, \{p\}_n^{\not r}) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\,(1-z_{ri})\,\mathcal{R}_{ir}^F(p_r)\,\Theta(p_r - z_{ri}p_i)$$

$$\mathcal{J}_a^{I,soft/soco}(\epsilon, Q, \{p\}_n^{\not r}) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\,\mathcal{R}_a^{I,soft/soco}(p_r)\,\Theta(p_r)$$

$$\mathcal{J}_{\chi r}^{I,col}(\epsilon, Q, \{p\}_n^{\not r}) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\,\mathcal{R}_{\chi r}^{I,col}(p_r)\,\Theta(p_r)\,\frac{\mathcal{L}(Q + x_r P;\{p\}_n^{\not r})}{\mathcal{L}(Q;\{p\}_n^{\not r})}$$

where $\Theta(q) = \theta(-x < x_q < 1-x)\,\theta(-\bar{x} < \bar{x}_q < 1-\bar{x})$

Integration limits not in terms of natural integration variables

This makes the integrals unnecessarily cumbersome.

Integrated subtraction terms

$$\sigma_R(\epsilon) = \sigma_R^{\text{div}}(\epsilon) + \sigma_R^{\text{resolved}} + \mathcal{O}(\epsilon)$$

$$\sigma_R^{\text{div}}(\epsilon) = \frac{1}{\mathcal{S}_{n+1}} \sum_r \int [dQ] \int d\Phi\big(Q;\{p\}_n^{\not r}\big)\, \mathcal{L}\big(Q;\{p\}_n^{\not r}\big)\, J_B\big(\{p\}_n^{\not r}\big)$$

$$\times \left\{ \sum_i \mathcal{J}_{ir}^{F}\big(\epsilon, Q, \{p\}_n^{\not r}\big) \otimes \mathcal{A}_{ir}^{F}\big(Q;\{p\}_n^{\not r}\big) + \sum_{a\in\{\chi,\bar\chi\}} \mathcal{J}_{ar}^{I}\big(\epsilon, Q, \{p\}_n^{\not r}\big) \otimes \mathcal{A}_{ar}^{I}\big(Q;\{p\}_n^{\not r}\big) \right\},$$

$$\mathcal{J}_{ir}^{F}\big(\epsilon, Q, \{p\}_n^{\not r}\big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\,(1 - z_{ri})\, \mathcal{R}_{ir}^{F}(p_r)\, \boxed{\Theta(p_r - z_{ri}p_i)}$$

$$\mathcal{J}_a^{I,\text{soft/soco}}\big(\epsilon, Q, \{p\}_n^{\not r}\big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\, \mathcal{R}_a^{I,\text{soft/soco}}(p_r)\, \boxed{\Theta(p_r)}$$

$$\mathcal{J}_{\chi r}^{I,\text{col}}\big(\epsilon, Q, \{p\}_n^{\not r}\big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\, \delta_+(p_r^2)\, \mathcal{R}_{\chi r}^{I,\text{col}}(p_r)\, \boxed{\Theta(p_r)}\, \frac{\mathcal{L}\big(Q + x_r P;\{p\}_n^{\not r}\big)}{\mathcal{L}\big(Q;\{p\}_n^{\not r}\big)}$$

where $\boxed{\Theta(q) = \theta(-x < x_q < 1 - x)\,\theta(-\bar{x} < \bar{x}_q < 1 - \bar{x})}$

Notice that $\Theta = 1$ whenever $\mathcal{R}$ is divergent $\quad\Longrightarrow\quad$ use $\Theta = 1 + [\Theta - 1]$

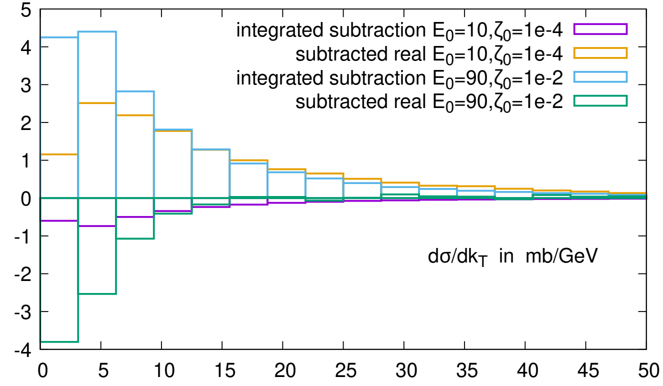
The integral with 1 can be performed analytically, the one with $[\Theta - 1]$ is finite and can be performed numerically.

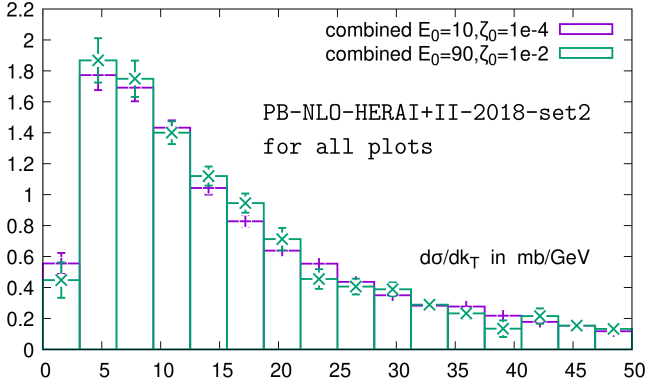
Integrating numerically means simply increasing Monte Carlo phase space, not an integral for each phase space point.

Numerical test results

$\sigma_R^{\text{div},(0)} + \sigma_R^{\text{resolved}}$ for dijets

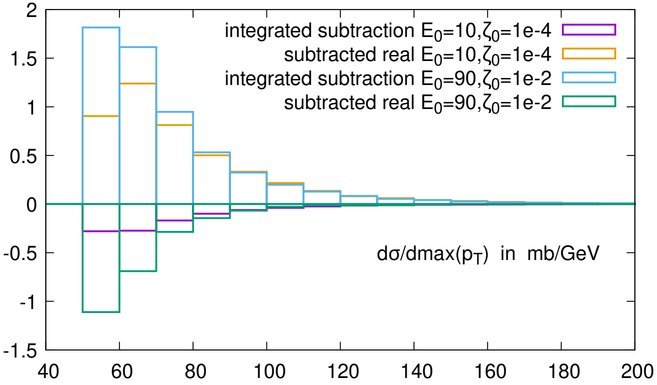
$gg^\star \to ggg,\ gg^\star \to u\bar{u}g,\ ug^\star \to ugg,$
$ug^\star \to u\bar{u}d,\ ug^\star \to u\bar{u}u,\ (u \leftrightarrow d)$

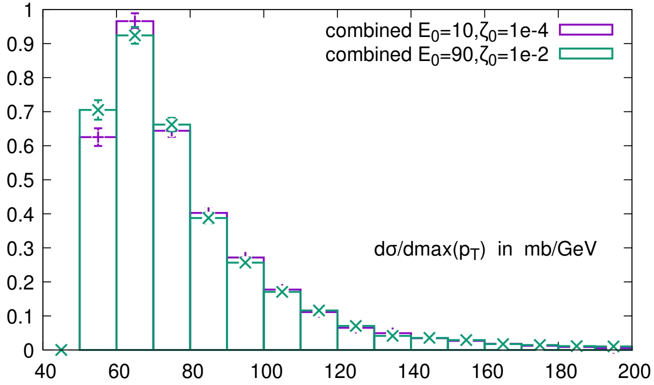




finite errorbars:

$\Rightarrow$ subtraction terms correct





independence of E_0, η_0, ξ_0:

$\Rightarrow$ integrated subtraction terms correct

Divergences

All poles in ϵ of the integrated subtraction terms are *the same as in the on-shell case*, except the initial-state collinear divergence

$$\sigma^{\mathrm{I,col,div}}_{\chi r}(\epsilon) = \frac{1}{\mathcal{S}_n}\int [dQ]\int d\Phi\big(Q;\{p\}_n\big)\,\mathcal{L}\big(Q;\{p\}_n\big)\,\big|\mathcal{M}\big|^2\big(Q;\{p\}_n\big)\,J_B\big(\{p\}_n\big)$$
$$\times\frac{\alpha_s}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left\{\frac{N_c}{\epsilon^2}-\frac{1}{\epsilon}\int_0^1 dz\,\mathcal{P}^{\mathrm{reg}}_{\chi r}(z)\,\frac{\ell_\chi(x/z)}{z}\,\theta(z>x)\right\}$$

with

$$\ell_\chi(x/z) = \frac{F_\chi(x/z,k_\perp,\mu_F)}{F_\chi(x,k_\perp,\mu_F)} \quad\text{and}\quad \mathcal{P}^{\mathrm{reg}}_{\chi g}(z) = 2N_c\left[\frac{1}{(1-z)_+}+\frac{1}{z}\right]$$

Compare with the "usual" on-shell collinear side

$$\sigma^{\mathrm{I,col,div}}_{\bar\chi r}(\epsilon) \quad\text{with}\quad \ell_{\bar\chi}(\bar x/z) = \frac{f_{\bar\chi}(\bar x/z,\mu_F)}{f_{\bar\chi}(x,\mu_F)} \quad\text{and}\quad \mathcal{P}^{\mathrm{reg}}_{\bar\chi g}(z) = 2N_c\left[\frac{1}{(1-z)_+}+\frac{1}{z}+z(1-z)-2\right]$$

$\sigma^{\mathrm{I,col,div}}_{\chi r}$ and $\sigma^{\mathrm{I,col,div}}_{\bar\chi r}$ do not cancel against virtual divergences.

$\sigma^{\mathrm{I,col,div}}_{\bar\chi r}$ together with the virtual left-over divergence is subtracted within the usual factorization prescription.

Something similar must happen with $\sigma^{\mathrm{I,col,div}}_{\chi r}$.

Conclusions

- A subtraction scheme was presented to separate the real radiation integral in hybrid k_T-factorization for arbitrary processes into a finite resolved part, and a divergent unresolved part with all $1/\epsilon$ poles expliced.
- It was tested and found to
  - lead to convergent phase space integrals for the resolved part $\Rightarrow$ the subraction terms are correct
  - lead to final results that are independent of arbitrary phase space restrictions $\Rightarrow$ the integrated subtraction terms are correct
- There is an initial-state collinear singularity when the radiation becomes collinear to the hadron "producing the reggeon", with splitting function

$$\mathcal{P}_{\chi}(z) = \frac{2N_c}{z(1-z)}$$

Backup slides

On-shell limit

Space-like (LO) matrix elements have desired on-shell limit only after azimuthal integration:

$$\left|\mathcal{M}(k_\perp)\right|^2 \xrightarrow{|k_\perp|\to 0} \mathcal{M}_\mu^*(0)\,\frac{k_\perp^\mu k_\perp^\nu}{|k_\perp|^2}\,\mathcal{M}_\nu(0) \xrightarrow{\int d\varphi_\perp} \left|\mathcal{M}(0)\right|^2$$

As a consequence, point-wise cancellation of singularities fails at $|k_\perp| = 0$:

$$\left|\mathcal{M}(k_\perp, r_\perp)\right|^2 \xrightarrow{|k_\perp|\to 0} \mathcal{M}_\mu^*(0, r_\perp)\,\frac{k_\perp^\mu k_\perp^\nu}{|k_\perp|^2}\,\mathcal{M}_\nu(0, r_\perp) \xrightarrow{|r_\perp|\to 0} \text{Singular} \times \mathcal{M}_\mu^*(0)\,\frac{k_\perp^\mu k_\perp^\nu}{|k_\perp|^2}\,\mathcal{M}_\nu(0)$$

$$\text{Singular} \times \left|\mathcal{M}(k_\perp - r_\perp)\right|^2 \xrightarrow{|k_\perp|\to 0} \text{Singular} \times \left|\mathcal{M}(-r_\perp)\right|^2 \xrightarrow{|r_\perp|\to 0} \text{Singular} \times \mathcal{M}_\mu^*(0)\,\frac{r_\perp^\mu r_\perp^\nu}{|r_\perp|^2}\,\mathcal{M}_\nu(0)$$

Fortunately, the measure of the problematic phase space vanishes