Non-perturbative Aspects of Supersymmetric Gauge Theories

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Quantum Field Theories

- Best known mathematical framework for the description of fundamental interactions

- **Gauge theories:** Quantum field theories describing fundamental forces

  Examples: Maxwell theory, QCD, ... $\rightarrow$ Yang-Mills theories
  $\mathcal{L}_{YM} = -\frac{1}{4g^2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$ with g the gauge coupling

Formally encoded in the *partition function*:

$$\mathcal{Z}(g,\mu) = \int \underbrace{\mathcal{D}\phi}_{\text{quantization measure}} \exp\left(i \int d^4x \; \underbrace{\mathcal{L}[\phi(x), g(\mu)]}_{\text{Lagrangian density}} \right) .$$

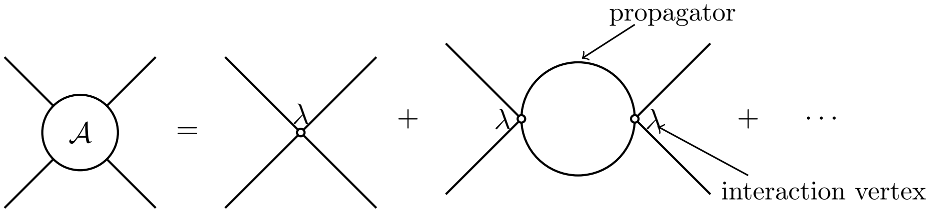
Amplitudes in Quantum Field Theories

Quantum field theories in practice:

Example: - ϕ^4 theory

$$\mathcal{L}[\phi,\lambda] = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + m^2\phi^2}_{\text{free field contribution}} + \frac{\lambda}{4!}\phi^4\,.$$

Amplitude as a perturbative expansion in λ:



Supersymmetric gauge theories

Supersymmetry: Additionnal symmetry relating bosonic and fermionic degrees of freedom.

In practice:

- Extension of Poincaré space-time symmetry $\longrightarrow$ super-space $(\theta, \bar{\theta})$
- Contraint on the matter content of gauge theories and interactions encoded in the *pre-potential* $\mathcal{F}$. Example:

$$\mathcal{N} = 2 \quad \text{super-Yang-Mills (sYM):} \qquad \mathcal{L}[\Psi] = \frac{1}{4\pi}\mathrm{Im}\left(\mathrm{Tr}\int d^4\theta\, \mathcal{F}[\Psi]\right).$$

- Exact evaluation of quantum corrections:

$$\mathcal{F} = \mathcal{F}_{\text{classical}} + \mathcal{F}_{1-\text{loop}} + \underbrace{\mathcal{F}_{\text{instanton}}}_{\text{non-perturbative corrections}} \quad , \quad \mathcal{F}_{\text{classical}} = \frac{1}{2}\tau\Psi^2 .$$

with $\tau = \frac{\theta}{2\pi} + \frac{4i\pi}{g^2}$ the complexified gauge coupling.

Nekrasov Instanton Partition Function

Instantons: solutions of Yang-Mills theory equations of motion satisfying:

$$F_{\mu\nu} = \pm \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

such that $S_{\mathrm{YM}} = -\frac{1}{4g^2}\mathrm{Tr}\int d^4x F_{\mu\nu}F^{\mu\nu} \to \frac{2\pi n}{g^2}$ with $n \in \mathbb{N}$.

Instanton partition function as a non-perturbative expansion:

$$\mathcal{Z}_{\text{instanton}} = \sum_{k\in\mathbb{N}} \underbrace{e^{\frac{2i\pi k}{g^2}}}_{\text{non-perturbative}} Z_k \xrightarrow[\text{SUSY}]{} \sum_{k\in\mathbb{N}} e^{2i\pi\tau k}\widetilde{Z}_k\,,$$

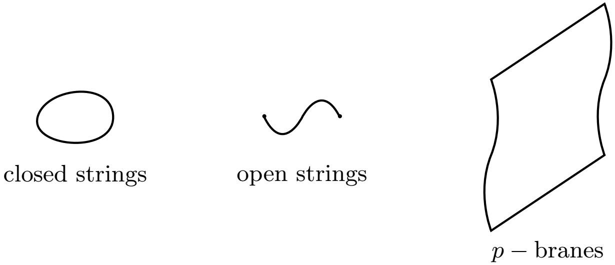
For supersymmetric gauge theories:

$$\mathcal{F}_{\text{instanton}} \sim \log \mathcal{Z}_{\text{instanton}}\,.$$

- $\mathcal{Z}_{\text{instanton}}$ can be computed in the context of *string theory*.

String Theory

- Attempt at unifying gauge theories and gravity at high energy
- Extra-dimensions: 3 spatial + time $\longrightarrow$ 9 spatial + time
- Extended degrees of freedom:



- Five versions of string theory $\longrightarrow$ M-theory

Little String Theories

Little String theories (LSTs): Intermediate class of theories between string theory and supersymmetric gauge theories

- gravity decoupled (closed strings)
- string-like degrees of freedom
- 6-dimensional theory (living on $\mathbb{R}^4 \times \mathbb{T}^2$ space-time)
- supersymmetric gauge theories at low energy

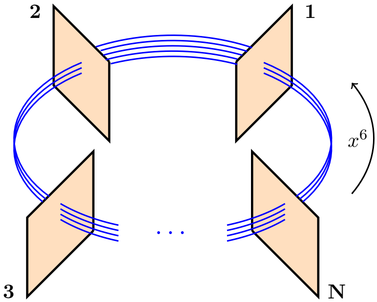
$$\boxed{3+1\text{d sYM}} \xleftarrow[\text{decompactification}]{} \boxed{5+1\text{d LST}} \xleftarrow[\cancel{\text{gravity}}]{} \boxed{9+1\text{d String theory}}$$

M-theory construction of A-type Little String Theories

Rich class of LSTs can be realised through branewebs

[Haghighat, Iqbal, Kozçaz, Lockhart, Vafa '13; Hohenegger, Iqbal '13; ...]

Branewebs: Geometrical arrangement of M-theory branes that preserves supersymmetry

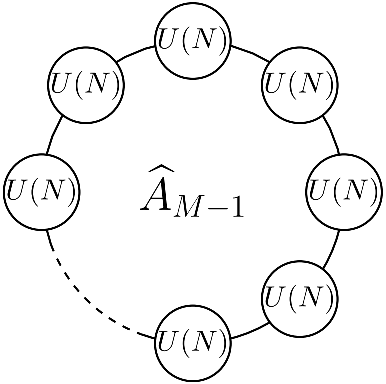


- $(N, 1)$ braneweb: stack of N M5-branes on a circle (can be generalised to (N, M) branewebs)
- LSTs live in the world-volume of the M5-branes $\rightarrow$ 6d theories

Quiver gauge theories

Quiver: Diagram that encodes information about gauge groups and matter content of a theory

Example: (N, M) LSTs:



- $U(N) \longrightarrow \begin{cases} \text{gauge degrees of freedom} \\ \text{gauge parameters } \widehat{a},\ \rho \\ \text{coupling constants } \tau \end{cases}$
- ——— $\longrightarrow \begin{cases} \text{matter degrees of freedom} \\ \text{mass parameter } S \end{cases}$
- Parameters are related to positions of branes and string theory parameters

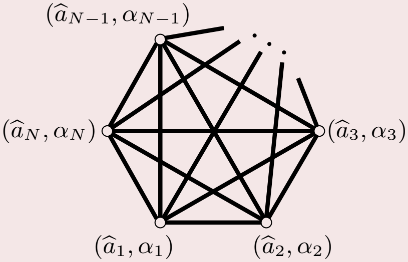
Diagrammatic decomposition of the instanton partition function

For $M = 1$, we can compute the instanton partition function:

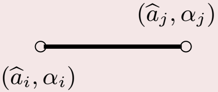
$$\mathcal{Z}_{\text{instanton}}(\widehat{\mathbf{a}}, S, \rho, \tau) = \sum_{\alpha_1,\ldots,\alpha_N} e^{2i\pi\tau\sum|\alpha_i|}\mathcal{P}_{\alpha_1,\ldots,\alpha_N}$$

$\mathcal{P}_{\alpha_1,\ldots,\alpha_N}$ can be organised in a diagrammatic fashion that resembles perturbative expansions [BF, Hohenegger '22]

$\mathcal{P}_{\alpha_1,\ldots,\alpha_N} \propto$



with



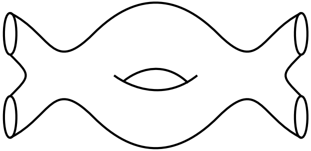
$$= \text{"} \prod_{\square\in\alpha_i} \Omega\,\Omega \prod_{\square\in\alpha_j} \Omega\,\Omega\text{"}$$

scalar propagators

Systematic extension of the diagrammatic expansion found for $\log \mathcal{Z}_{N,1}$ [Hohenegger '19,'20]

Detour: Amplitudes in String theory

- External fields $\longrightarrow$ Insertion of vertex operators: annihilation-creation operators
- Loop/perturbative expansion $\longrightarrow$ Genus expansion
- Integral over physical internal states $\longrightarrow$ Integral over 2d geometries



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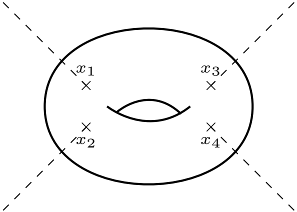


$$= \underbrace{\langle V(x_1)V(x_2)V(x_3)V(x_4)\rangle}_{\text{vertex operators}}$$

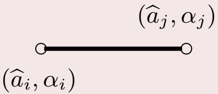
The Partition Function from Vertex Operator Algebras

Similarly, the partition function can be computed through the commutation relations of some operators:

$$\mathcal{Z}_{\text{instanton}} = \langle 0 | S(x_1) \cdots S(x_P) | 0 \rangle \,,$$

[Kimura,Noshita '23,...]

- $S(x)$ are interpreted as annihilation-creation operators of brane states
- $|0\rangle$ is the vacuum of these operators
- Parameters of the gauge theories $(\widehat{a})$ are functions of the positions $x_1, \ldots, x_P$



$$= \langle 0 | S(x_i) S(x_j) | 0 \rangle$$

[BF,Hohenegger,Kimura '23]

Non-perturbative Symmetries

We can construct symmetries that have a linear action on the parameters of the theory:

$$\begin{pmatrix} \widehat{a}_1 \\ \vdots \\ \widehat{a}_N \\ \tau \\ S \\ \rho \end{pmatrix} \underset{\mathfrak{a}}{\longmapsto} \mathcal{A} \cdot \begin{pmatrix} \widehat{a}_1 \\ \vdots \\ \widehat{a}_N \\ \tau \\ S \\ \rho \end{pmatrix} , \qquad \mathcal{Z}_{1-\text{loop}} \mathcal{Z}_{\text{instanton}} \underset{\mathfrak{a}}{\longmapsto} \mathcal{Z}_{1-\text{loop}} \mathcal{Z}_{\text{instanton}} \, .$$

In general, these symmetries do not respect the expansion in $\exp(2i\pi\tau)$:

$$\mathcal{Z}_{\text{instanton}} = \sum_{k=0}^{\infty} e^{2i\pi\tau k} Z_k \, , \qquad Z_k \underset{\mathfrak{a}}{\longmapsto} \underbrace{e^{i\varphi}}_{\text{phase factor}} \cdot Z_{k'} \, ,$$

and are therefore non-perturbative.

[BF,Hohenegger,Kimura '23]

Little String Theories as doubly elliptic gauge theories

Elliptic: $SL(2,\mathbb{Z})$ symmetry, elliptic parameter $\tau \to \frac{a\tau+b}{c\tau+d}$ with $ad-cb=1,\ a,b,c,d\in\mathbb{Z}$

- rational: $(a-b)(a+b)=a^2-b^2$
- trigonometric: $\sin(a-b)\sin(a+b)=\cos^2(b)-\cos^2(a)$
- elliptic: $\theta_1(a+b)\theta_1(a-b)\theta_4^2=\theta_1^2(a)\theta_4^2(b)-\theta_1^2(b)\theta_4^2(a)$ → Jacobi theta functions

sYM on $\mathbb{R}^4$

$$\mathcal{Z}_{\text{inst.}}=\sum_{k\geq 0} e^{2i\pi\tau k}\cdot \text{"rat."}$$

elliptic in τ, rational function of other parameters

sYM on $\mathbb{R}^4\times\mathbb{S}^1$

$$\mathcal{Z}_{\text{inst.}}=\sum_{k\geq 0} e^{2i\pi\tau k}\cdot \text{"trig."}$$

elliptic in τ, trigonometric function of other parameters

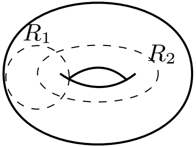
LST on $\mathbb{R}^4\times\mathbb{T}^2$

$$\mathcal{Z}_{\text{inst.}}=\sum_{k\geq 0} e^{2i\pi\tau k}\cdot \text{"ell."}(\rho)$$

elliptic in τ, elliptic function of other parameters in ρ

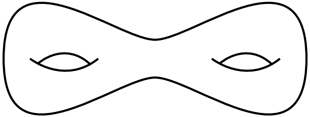
Genus 2 interpretation

LSTs are doubly elliptic in τ and ρ $\rightarrow$ symmetry group $SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho$



$$\tau = R_1 + iR_2$$

$SL(2,\mathbb{Z})_\tau$ is the *homeomorphism group* of $\mathbb{T}^2_\tau$, i.e. the group of transformation that leave the overall form of the torus invariant



$$\Omega = \begin{pmatrix} \tau & S \\ S & \rho \end{pmatrix}$$

The homeomorphism group of the genus 2 torus is $Sp(4,\mathbb{Z})_\Omega$ and has the property:
$Sp(4,\mathbb{Z})_\Omega \underset{S\to 0}{\longrightarrow} SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho$

In the particular case where gauge parameters are identified to τ, S, ρ, the non-perturbative symmetries are some generators $Sp(4,\mathbb{Z})$

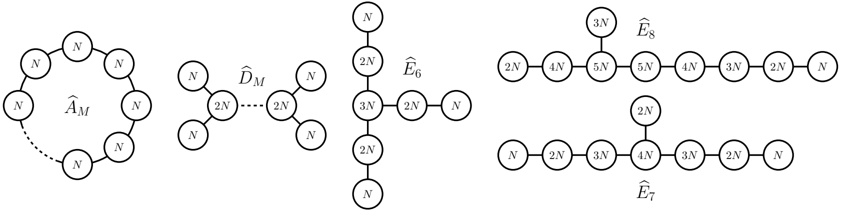
Outlook and Conclusion

Conclusion

- In quantum field theories: perturbative expansions and non-perturbative expansions.
- I have studied the non-perturbative sector of a class of supersymmetric gauge theories that arise in the context of string theory.
- I have uncovered a systematic diagrammatic expansion of the instanton partition function.
- I have constructed symmetries that mixes perturbative and non-perturbative contributions.

Outlook

- Investigate LSTs with different gauge groups or matter content, in particular so-called D and E-type LSTs



- To each supersymmetric gauge theory corresponds an integrable system. In this correspondence, the instanton partition function corresponds to the eigenstate of the Hamiltonian. The Hamiltonian is not known for LSTs but should correpond to a *doubly-elliptic integrable system.*

- Connection to *resurgence*?

Thank you for your attention!