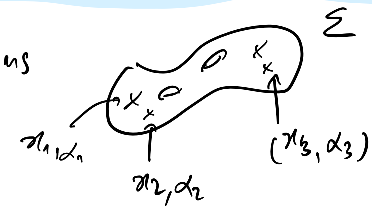
Talk at CEA September 2024
Journées Itzykson.

1) What do we mean by probabilistic construction of CFT in 2d?

Correlations Functions



$x_i \in \Sigma$ Riemann surface
α_i weight (say $\in \mathbb{C}$).

$(\alpha, x) = (\alpha_1 \dots \alpha_n, x_1, \dots x_n)$

$$\langle V_{\alpha_1}(x_1) \cdots V_{\alpha_n}(x_n) \rangle_\Sigma = \mathbb{E}[F_{(\alpha, x)}(X)]$$

Primary Field
Vertex operator

random variable

which satisfy the CFT axioms

in particular if g metric compatible with complex structure
$(g = e^\sigma |dz|^2$ locally$)$

$$\langle \prod_i V_{\alpha_i}(x_i) \rangle_{\Sigma, ge^\omega} = \langle \prod V_{\alpha_i}(x_i) \rangle_{\Sigma, g} \, \underbrace{e^{-\frac{c}{96\pi}\int |d\omega|_g^2 + 2 K_g \omega}}_{\text{Weyl anomaly.}} \times e^{-\sum \Delta_{\alpha_i} \omega(x_i)}$$

c: Central charge

K_g: Scalg.

* physicist have Formulas for certain models

Goal: ① link proba construction to explicit formulas
② Construct probabilistically Hilbert Space $\mathcal{H}$
and a rep. of Virasoro $(L_n)_n$ as operators in $\mathcal{H}$ + decompose $\mathcal{H}$ as sums of rep. of (L_n) (or Kac-Moody).

(3) Prove bootstrap formulas

$$\langle \prod V_{\alpha_i}(x_i) \rangle = \int_{(\mathrm{Spec})^N} \prod C_j(\alpha, p) \, |\mathcal{F}_{\alpha, p}(x)|^2 \, dp$$

3 pt funct°

Conformal blocks

$N = 3g_\Sigma - 3 + n$
moduli space
of complex st
with marked pts.

⚠ Prove convergence of conf blocks

Advantage :
- Crossing symmetry is for free
- Conv of blocks & of formulas in bootstrap should be consequence.
- Concrete objects, can be possibly related to scaling limits
- Application for new formulas (X. Sun & collab)

2) Examples that we know how to construct.

A] Liouville $c_L > 25$

$$\langle \prod_{i=1}^{n} V_{\alpha_i}(x_i) \rangle = \underbrace{\int \prod e^{\alpha_i \phi(x_i)} e^{-S_\Sigma^L(\phi)} D\phi}_{\text{path integral}}$$

with $S_\Sigma^L(\phi) = \frac{1}{4\pi} \int_\Sigma \left(|d\phi|_g^2 + Q K_g \phi + \mu e^{\gamma \phi} \right) dv_g$.

where
$Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $\gamma \in (0,2)$, $\mu > 0$

(DKRV, GRV, DRV)
$\mathbb{S}^2$ Σ $\mathbb{T}^2$

$$\Phi = \underset{\text{0-mode}}{c} + \underset{\text{massless Gaussian Free Field}}{X_g}$$

$$\left\langle \prod V_{\alpha_i}(x_i) \right\rangle_{\Sigma, g} = \int \prod_{i=1}^{n} e^{\alpha_i (c + X_g(x_i))} \, e^{-\frac{1}{4\pi}\int K_g (c+X_g) + \mu e^{\gamma c} \underbrace{e^{\gamma X} dV}_{\text{GMC}}}$$

$$\underbrace{e^{\gamma X} dv}_{\text{Gaussian Multiplicative Chaos}} = \lim_{\varepsilon \to 0} \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\varepsilon} dV_g$$

(Kahane, Robert-Vargas, Duplantier-Sheffield)

$$X_\varepsilon(x) = \int_{C(x,\varepsilon)} X_g(y)\, dy.$$

$C(x,\varepsilon) \to$ geodesic circle of radius $\varepsilon > 0$

Th: Converge if Seiberg bounds $\sum_i \alpha_i - \chi(\Sigma) Q > 0$ & $\alpha_i < Q \ \forall i$

and satisfy Conformal covariance. , $c_L = 1 + 6Q^2$

$\Delta_\alpha = \frac{\alpha}{2}\left(Q - \frac{\alpha}{2}\right)$

B] Compactified Imaginary Liouville

(GRV23).

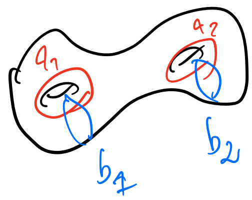
$$c_{IL} = 1 - 6Q^2$$

$$\left\langle \prod_j V_{\alpha_j}(x_j) \right\rangle_{\Sigma, g} = \int_{\text{Maps } \Sigma \to \mathbb{R}/2\pi R \mathbb{Z}} \prod_j e^{i\alpha_j \Phi(x_j)} \, e^{-S^{IL}_\Sigma(\Phi)} D\Phi$$

$$S^{IL}_\Sigma(\Phi) = \frac{1}{4\pi} \int_\Sigma |d\Phi|_g^2 - iQ K_g \Phi - \mu e^{i\beta\Phi} dv$$

$Q = \beta/2 - \frac{2}{\beta}$ with $\beta \in (0, \sqrt{2})$.

We set $\Phi(x) = c + X_g(x) + I_k(x)$ with $k \in \mathbb{Z}^{2g_\Sigma}$



$\in (0, 2\pi R)$ GFF

topological term multi valued but dI_k = harmonic 1-form on Σ with monodromy $2\pi R k_1, \dots, 2\pi R k_{g_\Sigma}$ on $(a_i,b_i)_i$

$(a_i,b_i)_{i=1}^{g_\Sigma}$ basis of $H_1(\Sigma,\mathbb{Z}) = \mathbb{Z}^{2g_\Sigma}$

note: $e^{i\alpha\phi(x)}$ univalued if $\alpha R \in \mathbb{Z}$

Probabilistic definition of correlations:

$$\Big\langle \prod_j V_{\alpha_j}(x_j) \Big\rangle_{\Sigma,g} =$$

$$\sum_{k\in\mathbb{Z}^{2g_\Sigma}} e^{-\frac{\|dI_k\|_2^2}{4\pi}} \int_0^{2\pi R} \mathbb{E}\Big[\prod_j e^{\alpha_j (c+X_g+I_k)(x_j)} - \frac{iQ}{4\pi}\int_\Sigma K_g(c+X_g+I_k) - \mu e^{i\beta(c+X_g+I_k)} dv_g\Big] dc$$

here $\mu \in \mathbb{C}$ cosmological constant

We need $\beta R \in \mathbb{Z}$

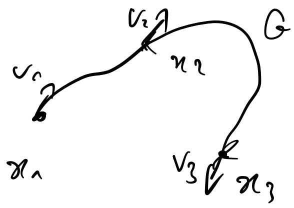
$e^{i\beta X_g} dv_g$ = Imaginary GMC (See: Lacoin - Rhodes - Vargas)

Th: if $\frac{\beta^2}{4} = \frac{p}{q} \in \mathbb{Q}$, $\alpha_j \in \frac{1}{R}\mathbb{Z}$, $\alpha_j > Q$

The correlations converge & it defines a CFT with central charge $c_L = 1 - 6Q^2$, $\Delta_\alpha = \frac{\alpha}{2}\left(\frac{\alpha}{2} - Q\right)$

(Discrete spectrum).

in fact, we can add magnetic charges $m_j \in \mathbb{Z}$ at x_j with $\sum_j m_j = 0$ by setting $\phi = c + X_g + I_k + J_m$



Graph G = cut for multivalued Fct° J_m

$dJ_m = \mathrm{Re}(\mathcal{O}_m)$

meromorphic 1-form with poles at x_j

Vertex op.: $V_{\alpha_i,m_i} \longrightarrow$ Conformal weight $\frac{\alpha_i}{2}\left(\frac{\alpha_i}{2}-Q\right)+\frac{m_i^2R^2}{4}$

Spin: $S_{\alpha,m} = \alpha R m - Q R m.$

Coulomb Gaz formula: $\langle V_{\alpha_1,m_1}(x_1)\, V_{\alpha_2,m_2}(x_2)\, V_{\alpha_3,m_3}(x_3)\rangle_{S^2}$

$$= (2\pi R)\frac{(-\mu)^\ell}{\ell!}\int_{\mathbb{C}^\ell}\prod_{j=1}^{\ell} x_j^{\Delta_1}\bar{x}_j^{\bar{\Delta}_1}(1-x_j)^{\Delta_2}(1-\bar{x}_j)^{\bar{\Delta}_2} \times \prod_{j<j'}|x_j-x_{j'}|^{\beta^2}\, dx_1\dots dx_\ell$$

$\ell = \frac{2Q-\sum_i\alpha_i}{\beta}\in\mathbb{N}$, $\Delta_j = \frac{\beta\alpha_j}{2}+\frac{Rm_j}{2}$, $\bar{\Delta}_j = \frac{\beta\alpha_j}{2}-\frac{Rm_j}{2}$

if $m_j=0 \Rightarrow = 2\pi R(-\mu)^\ell \underbrace{C_\beta^{DOZZ}(\alpha_1,\alpha_2,\alpha_3)}_{\text{Imaginary DOZZ}}$

In general:

$$= 4\pi^2R^2\mu^{\frac{2}{\beta}(2Q-\sum\alpha_i)}\, C_\beta^{DOZZ}(\alpha_1+m_1R,\alpha_2+m_2R,\alpha_3+m_3R)$$
$$\times\, C_\beta^{DOZZ}(\alpha_1-m_1R,\alpha_2-m_2R,\alpha_3-m_3R)$$

3) Cutting, gluing, bootstrap.

The probabilistic approach is well designed to prove Segal axioms, where we cut the path integrals on surfaces into pieces (when the classical action and primary fields are local).

- To S^1, we attach a Hilbert space

$$\boxed{\mathcal{H} = L^2(H^s(S^1), \mu_0)}$$

Sobolev space of order $s<0$

μ_0 is the law of the Gaussian free field on plane $\hat{\mathbb{C}}$ restricted to $S^1 = \{z \in \mathbb{C} \mid |z|=1\}$

GFF on circle:

$$\varphi(\theta) = \sum_{n>0}' \frac{x_n + i y_n}{2\sqrt{n}} e^{in\theta} + \sum_{n>0} \frac{x_n - i y_n}{2\sqrt{n}} e^{-in\theta} + c$$

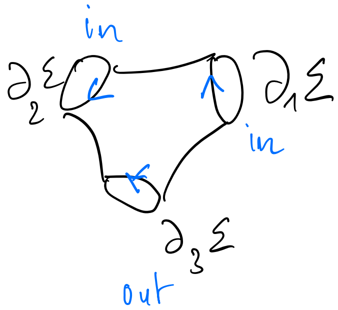
zero mode

with x_n, y_n iid Gaussians $N(0,1)$

$$\mu_0 = dc \otimes \prod_{n \geq 1} \frac{e^{-\frac{x_n^2}{2} - \frac{y_n^2}{2}}}{2\pi} dx_n dy_n .$$

- To surfaces with boundary (Σ, g) and parametrisations of $\partial\Sigma$

$\xi_1 : \mathbb{S}^1 \to \partial_1\Sigma \quad , \dots , \quad \xi_b : \mathbb{S}^1 \to \partial_b\Sigma$



we associates an operator

$$\mathcal{A}_{\Sigma,g,\xi} : \mathcal{H}^{\otimes b_-} \longrightarrow \mathcal{H}^{\otimes b_+}$$

b_- = # of incoming parametrized boundary circles

b_+ = # of outgoing parametrised boundary circles

$\mathcal{A}_{\Sigma,g,\xi}$ is described by some integral kernel on $(H^{-s}(\Sigma))^b$, $b = b_- + b_+$

$$\mathcal{A}_{\Sigma,g,\xi}(\underbrace{\varphi_+,\varphi_-}_{=\varphi\in H^{-s}(\Sigma)^b}) := \left(\det \Delta_{\Sigma,g,D}\right)^{-\frac{1}{2}} \mathcal{A}^0_{\Sigma,g,\xi}(\varphi)$$

$$\times\, \mathbb{E}_\varphi\left[e^{-\frac{Q}{4\pi}\int_\Sigma K_g (X_{g,D}+P\varphi) + \mu e^{\gamma X_{g,D}+\gamma P\varphi} dv} \right]$$

Conditional expectation (φ fixed)

with $P\varphi$ = harmonic extension of φ in Σ

$X_{g,D}$ = Dirichlet GFF.

$$\mathcal{A}^0_{\Sigma,g,\xi}(\varphi) = e^{-\frac{1}{2}\langle (\mathbf{D}_\Sigma - D)\varphi,\varphi\rangle_2}$$

free field amplitude

$$Df(\theta) := \sum_n |n| f_n e^{in\theta}$$ Fourier multiplier

if $f(\theta) = \sum f_n e^{in\theta}$

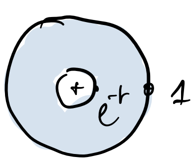
$$D_\Sigma f(\theta) = \partial_n Pf_{|\partial\Sigma}$$ Dirichlet-to-Neumann operator

Note: $D_\Sigma - D$ is smoothing, i.e it maps $\mathcal{D}'(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$

Th [GKRV 22]: Amplitudes are well defined and satisfy Segal gluing/cutting axioms and conformal invariance:

$$\mathcal{A}_{\Sigma, e^{\omega} g, \xi}(\varphi) = e^{-\frac{c_L}{96\pi}\int_\Sigma (|\nabla \omega|^2 + 2K_g\omega)\,dv}\,\mathcal{A}_{\Sigma,g,\xi}(\varphi)$$

Hamiltonian of Liouville CFT:



$\mathbb{A}_{e^{-t}}$ annulus $|z| \in [e^{-t}, 1]$

$g_{\mathbb{A}} = \frac{|dz|^2}{|z|^2}$ $\quad \xi_1^t(e^{i\theta}) = e^{i\theta}$, $\quad \xi_2^t(e^{i\theta}) = e^{-t+i\theta}$

$$t \mapsto \mathcal{A}_{\mathbb{A}_{e^{-t}}, g_{\mathbb{A}}, \xi^t} \in \mathcal{L}(\mathcal{H})$$

is a contraction semigroup.

generator $\partial_t \mathcal{A}_{A_{e^{-t}}, g_A, \xi^t}\Big|_{t=0} = H$

with $H = -\frac{1}{2}\partial_c^2 + \frac{Q^2}{2} + \sum n(\partial_{x_n}^* \partial_{x_n} + \partial_{y_n}^* \partial_{y_n})$

$+ \mu e^{\gamma c} V(\varphi)$ $\in L^p(H^{-s}(\mathbb{S}^1))$ $\forall p < \gamma^2/2$

$$V(\varphi) = \underbrace{\int_0^{2\pi} e^{\gamma \varphi(\theta)} d\theta}_{\text{GMC on circle}}$$

H has continuous spectrum $[\frac{Q^2}{2}, \infty[$.

Th [GKRV21]: $\exists$ an analytic family

$\alpha \mapsto \Psi_{\alpha, \nu, \tilde{\nu}}$, $\forall \nu, \tilde{\nu}$ Young diagram.

$\cap$

$\mathbb{C}$ of eigenstates of H with eigenvalues

$2\Delta_\alpha + |\nu| + |\tilde{\nu}|$ with $(\Psi_{Q+ip, \nu, \tilde{\nu}})_{p \in \mathbb{R}^+}$

$= \alpha(Q - \frac{\alpha}{2})$

being a complete family of eigenfunct° in $\mathcal{H}$

in the sense: $\forall u, u' \in \mathcal{H}$:

$$\langle u, u' \rangle_{\mathcal{H}} = \frac{1}{2\pi} \int_0^\infty \sum_{\substack{\nu, \nu' \\ \tilde{\nu}, \tilde{\nu}'}} \langle u, \Psi_{Q+ip, \nu, \tilde{\nu}} \rangle_{\mathcal{H}} \langle \Psi_{Q+ip, \nu', \tilde{\nu}'}, u' \rangle F_{Q+ip}(\nu, \nu') \times F_{Q+ip}(\tilde{\nu}, \tilde{\nu}')\, dp$$

$F_{Q+ip}(\gamma,\nu')$ are Shapovalov coefficients

Moreover for $\alpha<Q$, $\Psi_\alpha = \mathcal{A}_{\underbrace{\mathbb{D},|dz|^2,(0,\alpha)}_{\text{Disc amplitude}}}$

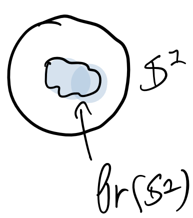
Virasoro representation:

take $v(z)\partial_z = \sum_{n\in\mathbb{Z}} v_n z^{n+1}\partial_z$ holo vector field near $\mathbb{S}^1$

s.t. its flow $f_t(z)$ maps $\mathbb{S}^1$ to $f_t(\mathbb{S}^1)\subset \mathring{\mathbb{D}}$

(we call v Markovian)

Then if $\mathbb{A}_{f_t} = \mathbb{D}\setminus \mathbb{D}_{f_t}$



$\mathbb{D}_{f_t}$ "disc" bounded by $f_t(\mathbb{S}^1)$.

Choosing well the metric g_t on $\mathbb{A}_{f_t}$ we prove with Baverez that

$t\mapsto \mathcal{A}_{\mathbb{A}_{f_t},g_t,\xi^t}$ is a "projective" semi group and

$$\partial_t|_{t=0}\,\mathcal{A}_{\mathbb{A}_{f_t},g_t,\xi^t} = \sum_n v_n L_n + \bar v_n \bar L_n$$

where (L_n), $(\bar L_n)$ are two commuting representat° of Virasoro with central charge $1+6Q^2$

and $\Psi_{Q+ip,\nu,\tilde\nu} = L_{-\nu(1)} \cdots L_{-\nu(k)} \bar{L}_{-\tilde\nu(1)} \cdots \bar{L}_{-\tilde\nu(\ell)} \Psi_{Q+ip}$

if $\nu = (\nu(1), \dots, \nu(k))$ $\tilde\nu = (\tilde\nu(1), \dots, \tilde\nu(\ell))$

Young diagrams

<u>Ward</u>: we can show using this that for a pant (P, g_P)

$$\langle \mathcal{A}_{P,g_P}, \Psi_{Q+ip_1,\nu_1,\tilde\nu_1} \otimes \Psi_{Q+ip_2,\nu_2,\tilde\nu_2} \otimes \Psi_{Q+ip_3,\nu_3,\tilde\nu_3} \rangle$$

$$= w(\nu_1,\nu_2,\nu_3,p_1,p_2,p_3)\, \overline{w(\tilde\nu_1,\tilde\nu_2,\tilde\nu_3,p_1,p_2,p_3)} \times \underbrace{\langle \mathcal{A}_{P,g_P}, \bigotimes_{j=1}^{3} \Psi_{Q+ip_j} \rangle}_{\text{structure constant} \times \text{conformal anomaly } (g_P \text{ depending})}$$

algebraic coefficients

Th. <u>Bootstrap</u>: [GKRV 21+22] $\alpha = (\alpha_1, \dots, \alpha_m)$

$$\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_m}(z_m) \rangle_{\Sigma,g}$$

$$= \int_{(\mathbb{R}^+)^{3g_\Sigma - 3+m}} \rho(p,\alpha) \, |\mathcal{F}_{\alpha,p}(\Sigma, z)|^2 \, dp$$

product of 3-pt functions on $\mathbb{S}^2$

Conformal block

$\Rightarrow$ Shows convergence of Conformal blocks $\forall$ a.e. $p \in (\mathbb{R}^+)^{3g_\Sigma - 3 + m}$

$\Rightarrow$ Shows crossing symmetry.

Proof relies on Segal cutting/gluing + Spectral resolut° + Ward identities.

Remark: (1) blocks do depend on cutting curves.

(2) Changing cutting curve in a fixed homotopy class changes blocks by a cocycle ("Schwarzian cocycle")

(3) blocks are global analytic sections on Teichmüller space

This will lead to unitary representation of Mapping class group in space of Conformal blocks (with Baverez, Kupiainen, Rhodes).

also called Modular Functor.