Conformal Symmetry at Finite Temperature

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HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

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HIGHER STRUCTURES, MODULI
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UH

WOLFGANG PAULI CENTRE

DESY.

Based on work with

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Epilogue

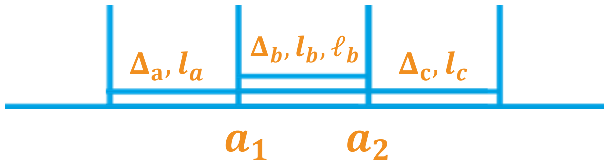
Toward an Introduction

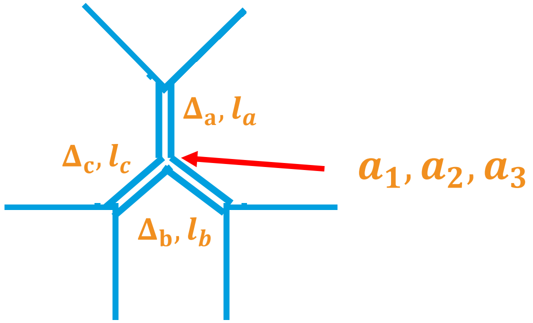
CFT = CPW + OPE data

``blocks“ $(\Delta, \lambda_{123}, \ldots)$

To access OPE data one needs to consider a wide range of correlators: e.g.

e.g. for free fields, N=4 SYM, bootstrap …





Or consider correlators in other geometries, with boundaries, defects etc

Conformal partial waves are wave functions of quantum integrable systems

Multi-point, defects **Gaudin → Calogero-Sutherland**

In this talk I will address thermal correlation functions

Introduction: Thermal Correlators

Definition and some features

Consider multi-point correlators of local fields in thermal geometry $S^1_\beta \times S^{d-1}$

$$\langle \phi_1(x_1)\dots\phi_n(x_n)\rangle_{q,y_a} = \frac{1}{\mathcal{Z}(q,y)}\mathrm{tr}_{\mathcal{H}}\left(\phi_1(x_1)\dots\phi_n(x_n)q^D y_2^{H_2}\dots y_r^{H_r}\right)$$

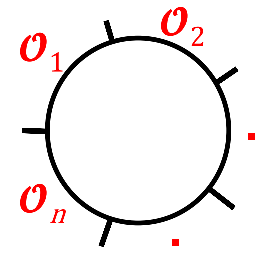
$$\mathcal{Z}(q,y_a) = \mathrm{tr}_{\mathcal{H}}\left(q^D y_2^{H_2}\dots y_r^{H_r}\right) \qquad \phi(x) = e^{-\Delta_\phi \tau}\Phi(\tau,\Omega)$$

Remark 1: For our analysis below non-vanishing fugacities y are essential

Remark 2: Completely determined by data (Δ, λ) of the CFT in Euclidean $\mathbb{R}^d$

e.g. in necklace channel:

$$\langle \phi_1(x_1)\dots\phi_n(x_n)\rangle_{q,y_a} = \sum_{\{\mathcal{O}_i\}} \lambda_{1\mathcal{O}_n\mathcal{O}_1}\lambda_{2\mathcal{O}_1\mathcal{O}_2}\cdots\lambda_{n\mathcal{O}_{n-1}\mathcal{O}_n}$$



Introduction: Thermal Correlators

Comments on the definition

Comment 1: Our thermal correlators are closely related to the correlation functions in the geometry $S^1_\beta \times \mathbb{R}^{d-1} \;\; = \lim_{R\longrightarrow\infty} S^1_\beta \times S^{d-1}_R$

Latter involves new CFT data b_ϕ in the 1-point functions of primary fields

$$\langle \phi(x)\rangle_\beta = \langle \phi(x)\rangle_{S^1_\beta\times\mathbb{R}^{d-1}} = \frac{b_\phi}{\beta^{\Delta_\phi}}$$

(for scalar ϕ)

[Iliesiu,Kologlu,Mahajan, Perlmutter, Simmons-Duffin]

Comment 2: Our thermal correlators satisfy Ward identities wrt D, H_i only

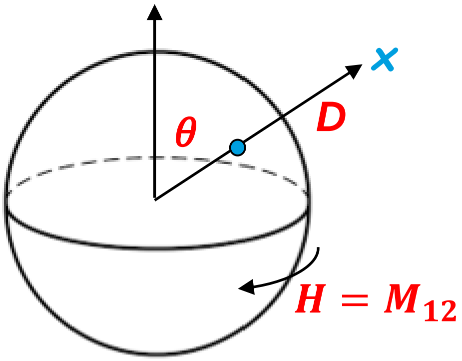
$$\sum_{J=1} \mathcal{D}^{(J)} \langle \phi_1(x_1) \dots \phi_n(x_n)\rangle_{\beta,y} = 0 = \sum_{J=1} \mathcal{H}^{(J)}_i \langle \phi_1(x_1) \dots \phi_n(x_n)\rangle_{\beta,y}$$

$$D^{(J)} = x_J \partial_{x_J} - \Delta_J$$

Introduction: Example of Thermal Correlator

Thermal 1-point function of free boson in 3D

$$\langle \phi^2(x) \rangle_{q,y} = \frac{\sqrt{2}}{r} \sum_{l,m} \frac{(l-m)!}{(l+m)!} \frac{q^{l+1/2} y^m}{1 - q^{l+1/2} y^m} P_l^m(\cos\theta)^2$$



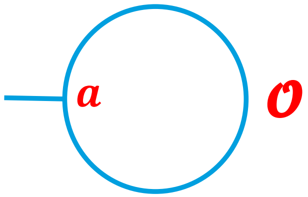
scalar field of weight $\Delta = 1$

Its partial wave expansion gives access to the OPE coefficients $\lambda^a_{\phi^2 \mathcal{O}\mathcal{O}}$

for all operators $\mathcal{O}$

& tensor structures a

$$\langle \phi^2(x) \rangle_{q,\beta} \sim \sum_{\mathcal{O}} \lambda^a_{\phi^2 \mathcal{O}\mathcal{O}}$$



But what is the basis of partial waves $g^{\mathcal{O},a}_{\Phi}(\theta; q, y)$ to expand into ?

Introduction: Main Results and Plan

Claim: Thermal partial waves are obtained from wave functions of elliptic Hitchin integrable system through rational degeneration

Eigenvalue equations for Hitchin Hamiltonian = Casimir equation provide enough control to harvest OPE data $\lambda_{\Phi\mathcal{O}\mathcal{O}}$ from thermal 1-point functions.

here for free field theory

PLAN

I. Thermal Partial Waves and Casimir equations
II. Hitchin integrable systems and thermal PWs
III. Conclusions and Outlook

Thermal Partial Waves
and
Casimir Equations

Thermal Correlations and Partial Waves

A simple but useful observation

$$G_n(x,y) \equiv tr\,(\phi_1(x_1)\dots\phi_n(x_n)q^D y_2^{H_2}\cdots y_r^{H_r})$$

$y_1 = q$

$H_1 = D$

Upon insertion of conformal generators thermal correlators G_n behave as

$$tr\,(\phi_1(x_1)\dots\phi_m(x_m)\,H_j\,\phi_{m+1}(x_{m+1})\dots\,\phi_n(x_n)y_1^{H_1}y_2^{H_2}\cdots y_r^{H_r}) =$$

$$= \left[\,\Sigma_{J=m+1}^{n}\mathcal{H}_j^{(J)} + y_j\partial_{y_j}\right]G_n(x,y)$$

$$tr\,(\phi_1(x_1)\dots\phi_m(x_m)\,E_\alpha\,\phi_{m+1}(x_{m+1})\dots\,\phi_n(x_n)y_1^{H_1}y_2^{H_2}\cdots y_r^{H_r}) =$$

$\alpha \in R_*$ is a root

$$= \frac{1}{1-e^{-\alpha(\mu)}}\left[\sum_{J=1}^{m} e^{-\alpha(\mu)}\ \mathcal{E}_\alpha^{(J)} + \sum_{J=m+1}^{n}\mathcal{E}_\alpha^{(J)}\right]G_n(x,y)$$

$y_1 = e^{i\mu_1}, y_2 = e^{\mu_2}\dots$

$$=: \mathcal{E}_{\alpha|\mu}^{(1,\dots m;m+1\dots n)}$$

$$E_\alpha \prod y_j^{H_j} \quad = \prod y_j^{H_j}\ e^{-\alpha(\mu)}E_\alpha$$

Thermal Casimir Equations

CFT approach

$$G_n^{\mathcal{O}}(x,y) = tr_{\mathcal{H}}\left(P_{\mathcal{O}}\,\phi_1(x_1)\cdots\phi_m(x_m)\,\phi_{m+1}(x_{m+1})\ldots\phi_n(x_n)q^D y_2^{H_2}\cdots y_r^{H_r}\right)$$

$P_{\mathcal{O}}$ **projects to global conformal family of operator** $\mathcal{O}$

The functions $G_n^{\mathcal{O}}(x,y)$ **can be shown to satisfy following second order Casimir equations**

$$g_n^{\mathcal{O}} = \delta(\mu) G_n^{\mathcal{O}}$$

$$\left[\sum_{j=1}\partial_{\mu_j}^2 - \sum_{\alpha\in R_*}\frac{\mathcal{E}_{\alpha|\mu}^{(1\ldots n)}\mathcal{E}_{-\alpha|\mu}^{(1\ldots n)}}{4\,sinh^2\left(\frac{1}{2}\alpha(\mu)\right)}\right] g_n^{\mathcal{O}_m}(x,\mu) = C_2(\mathcal{O})\, g_n^{\mathcal{O}}(x,\mu)$$

Follows from formulas on previous slide using that

$$Cas_{(2)} = \sum_j H_j^2 + \sum_{\alpha\in R_*} E_\alpha E_{-\alpha} \qquad\qquad Cas_{(2)}P_{\mathcal{O}} = C_2(\mathcal{O})P_{\mathcal{O}}$$

Solving the Casimir equations

Thermal 1-point blocks

The most direct way to construct thermal partial waves is through recursion relations obtained from acting with the Casimir operators on the following

Ansatz for low-temperature expansion: [here for scalar ϕ]

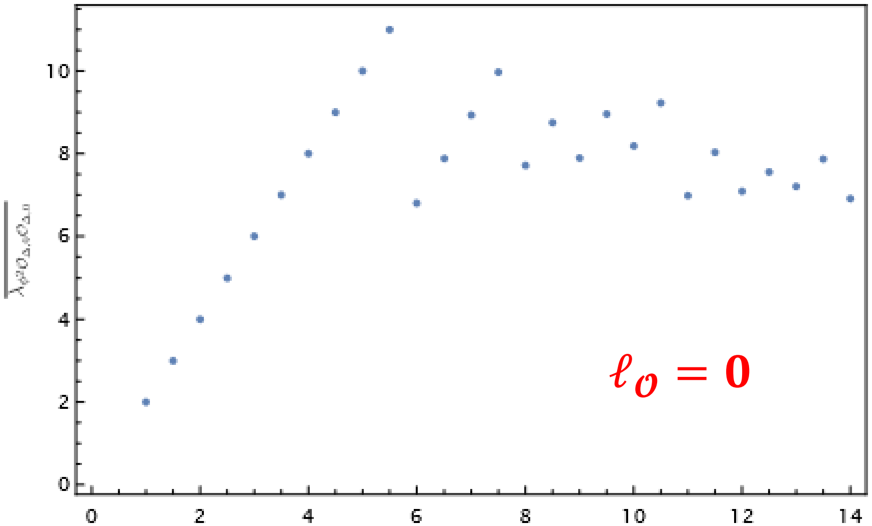
$$g_{\mathcal{O}}^{\Delta_\phi,a}(q,y,\theta) = q^{\Delta_{\mathcal{O}}} \sum_{n_i;\epsilon=0,1} A^{\epsilon}_{n_1 n_2 n_3}\, q^{n_1} y^{n_2} \sin^{n_3}(\theta) \cos^{\epsilon}(\theta)$$

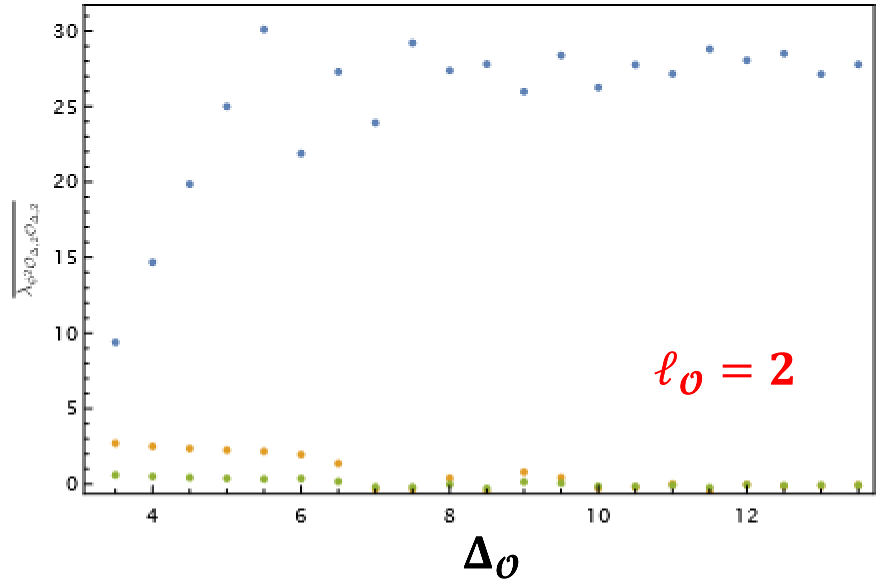
$n_1 = 0,1,2,\ldots,\infty \qquad -n_1 - \ell_{\mathcal{O}} \leq n_2 \leq n_1 + \ell_{\mathcal{O}} \qquad 0 \leq n_3 \leq 2(n_1 + \ell_{\mathcal{O}})$

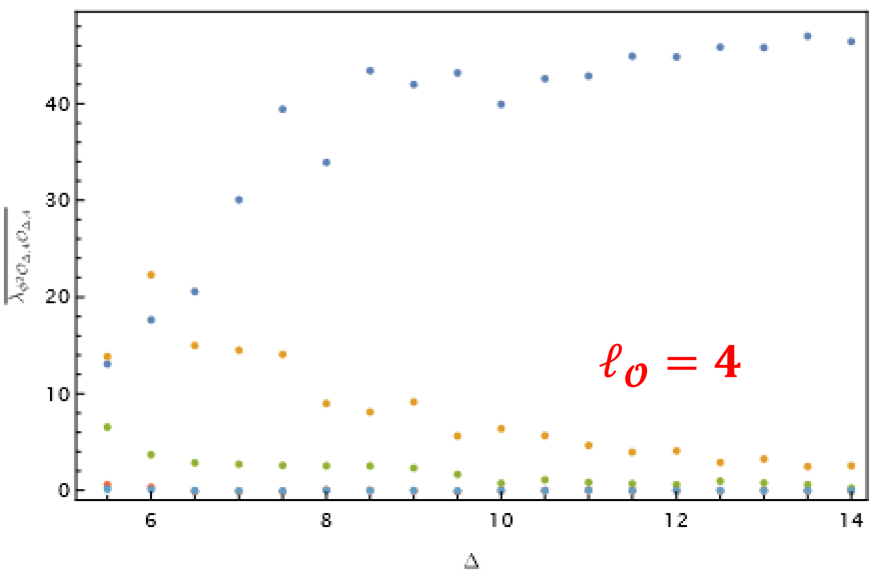
- The extension to spinning external fields ϕ has also been worked out.
  involves one more variable z for polarization & finite summation variable n_4, ϵ'
- Codes to compute $A^{\epsilon,(\epsilon')}_{n_1 n_2 n_{3,}(n_4)}$ @ gitlab.com/russofrancesco1995/thermalblocks
- Extends earlier work by [Gobeil,Maloney,Seng Ng,Wu] ϕ and $\mathcal{O}$ scalar

Application: OPE coefficients of 3D Free Field

From thermal one-point function







<u>Observation</u>: for large weight $\Delta_{\mathcal{O}}$ and for any spin $\ell_{\mathcal{O}} = 0,\ 2,\ 4,\ \ldots$ avarage OPE coefficients $\overline{\lambda^{a}_{\phi\mathcal{O}\mathcal{O}}}$ go to zero for all TS $a \neq a_*(\ell_{\mathcal{O}})$!

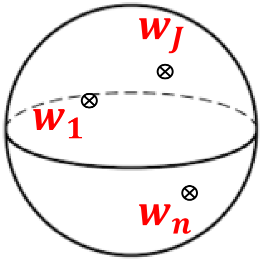
Integrability Approach
to Thermal Partial Waves

Gaudin models and Conformal Partial Waves

Review at zero temperature

[Buric,Lacroix,Mann,Quintavalle,VS]

Lax connection: Introduce following family of matrix valued 1st order DOs



spectral parameter

$$\mathcal{L}(w; w_J) \equiv \sum_{J,\alpha} \frac{T^\alpha \mathcal{T}_\alpha^{(J)}}{w - w_J} = \mathcal{L}_\alpha T^\alpha$$

$\mathcal{T}_\alpha^{(J)} \in \{\mathcal{T}_\alpha^{\Delta}, \mathcal{T}_\alpha^{D_p}\}$

complex parameters

generators of conformal algebra

Note: $\mathcal{L}$ is right hand side of Ward identities for currents in 2D WZW model

$$\mathcal{H}_{(p)}^{Gaudin}(w; w_J) = \kappa_{(p)}^{\alpha_1 \dots \alpha_p} \mathcal{L}_{\alpha_1} \cdots \mathcal{L}_{\alpha_p} + lot$$

Hamiltonians commute among each other & generate commutant of the generators $\mathcal{T}_\alpha = \Sigma\, \mathcal{T}_\alpha^{(J)}$ of conf Ward identities [Feigin,Frenkel,Reshetikhin]

→ Quantum integrable system

depends on parameters ω_J

OPE Channels and Gaudin Limits

Recovering Casimir operators

[Buric, Lacroix, Mann, Quintavalle, VS]

Choice of an OPE channel $\mathcal{C}$ determines a set $\text{Cas}_{\mathcal{C}}$ of Casimir operators

Measure weight and spin of intermediate fields

Demanding $\text{Cas}_{\mathcal{C}} \subset \text{Ham}(w_J)$ fixes w_J to approach a limit w. $w_J^{\mathcal{C}} \in \{0, 1, \infty\}$

$$\mathcal{H}_{(p)}^{[\rho]}(w) = \lim_{\varepsilon\to 0} \varepsilon^{p n_\rho} \mathcal{H}_{(p)}(w = \varepsilon^{n_\rho} w + g_\rho(\varepsilon), w_J = f_J(\varepsilon))$$

$n_\rho = n_\rho^{\mathcal{C}} \in \mathbb{N}$

$g_\rho(w) = g_\rho^{\mathcal{C}}(w)$

$f_J(w) = f_J^{\mathcal{C}}(w)$

Poly-nomial

Functions $\mathcal{H}_{(p)}^{[\rho]}(w)$, $\rho \in$ vertices of $\mathcal{C}$, give *complete* set of Hamiltonians.

<u>Example</u>: n = 4

$w_1 = 0, w_3 = 1, w_4 = \infty$

$$\mathcal{L}_4(w, w_2) = \frac{C_1}{w} + \frac{C_2}{w - w_2} + \frac{C_3}{w - 1}$$

$$C_J = \sum_\alpha T^\alpha \mathcal{T}_\alpha^{(J)}$$

s-channel →

$$\lim_{\varepsilon\to 0} \varepsilon w \mathcal{L}_4(\varepsilon w, \varepsilon^2) = C_1 + C_2$$

t-channel →

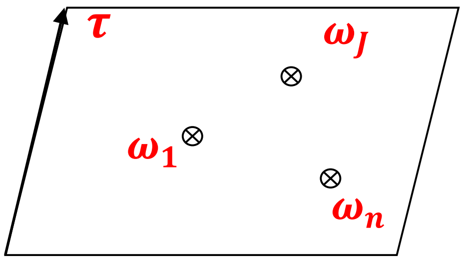
$$\lim_{\varepsilon\to 0} \varepsilon w \mathcal{L}_4(\varepsilon w + 1, \varepsilon^2 + 1) = \; = C_2 + C_3$$

Hitchin systems and Thermal Partial Waves

Going back to finite temperature

Lax connection: Introduce following family of matrix valued 1st order DOs

$$\mathcal{L}(\omega;\omega_J,\tau) = \sum_{i=1} H_i \partial_{\mu_i} + \sum_{J,\alpha} \sigma_{\alpha(\mu)}(\omega-\omega_J;\tau)\, T^\alpha \mathcal{T}_\alpha^{(J)}$$



$$\sigma_\eta(\omega) = \frac{\Theta\left(\frac{i}{2}\eta-\omega;\tau\right)\Theta'(0;\tau)}{\Theta(\omega;\tau)\Theta\left(\frac{i}{2}\eta\right)} \qquad \sigma_0(\omega) = \frac{\theta'(\omega)}{\theta(\omega)}$$

Is rhs of WI for current in 2D WZW on torus [Bernard]

$$\mathcal{H}_{(p)}^{Hitchin}(\omega,\omega_J;\tau) = \kappa_{(p)}^{\alpha_1\cdots\alpha_p}\mathcal{L}_{\alpha_1}\cdots\mathcal{L}_{\alpha_p} + lot$$

Hamiltonians commute among each other & with generators $\mathcal{H}_i = \Sigma\, \mathcal{H}_i^{(J)}$ of Ward identities for dilaton $D = H_1$ & Cartan elements $H_2, H_3, \ldots$

→ (elliptic) quantum integrable system

depends on parameters ω_J, τ .

Hitchin limits and thermal Casimir equations

Example: 1-point thermal partial waves

$$\mathcal{D}^{(1)}, \mathcal{H}_i^{(1)} = 0$$

For N = 1 and after reduction by Ward identities the Lax connection reads

$$\mathcal{L}(\omega; \omega_1 = 0, \tau) = \sum_{i=1} H_i \partial_{\mu_i} + \sum_{\alpha \in R_*} \sigma_{\alpha(\mu)}(\omega; \tau)\, E_\alpha \mathcal{E}^{(1)}_{-\alpha}$$

$$lim_{\tau\to\infty} \sigma_{\alpha(\mu)}(\omega;\tau) = \frac{sin\left(\frac{i}{2}\alpha(\mu) + \omega\right)}{sin\left(\frac{i}{2}\alpha(\mu)\right) sin\ \omega} \xrightarrow[\varepsilon \to 0]{\omega = \pm i\varepsilon^{-1}} \frac{i\, e^{\pm\frac{1}{2}\alpha(\mu)}}{sinh \frac{1}{2}\alpha(\mu)}$$

In limit $\tau \to i\,\infty$ we recover Casimir operator for 1-point thermal partial wave

$$\lim_{\substack{\tau\to i\infty \\ \omega\to -i\infty}} H^{Hitchin}_{(2)}(\omega, \omega_1 = 0, \tau) \sim \sum_{i=1} \partial^2_{\mu_i} - \sum_{\alpha\in R_*} \frac{\mathcal{E}^{(1)}_\alpha \mathcal{E}^{(1)}_{-\alpha}}{4\, sinh^2\left(\frac{1}{2}\alpha(\mu)\right)} = \delta(\mu) \Delta^{(2)}_{Cas}\, \delta^{-1}(\mu)$$

Remark 1: The results holds for fields $\phi, \mathcal{O}$ of arbitrary spin and for any *d*

Remark 2: Embedding of thermal Casimir equation into Hitchin systems comes with many new tools from integrability and gauge theory.

Conclusions and Open Questions

Thermal partial waves on $S^1_\beta \times S^{d-1}$ emerge from wave functions of multi-particle elliptic Hitchin integrable system upon trigonometric degeneration

Given the relation with elliptic Hitchin, can one levarage insights from study of surface defects in N=2* to better understand the thermal 1-point waves ?

↔ Nekrasov-Shatashvili **cp. e.g. [Hatsuda,Sciarappa,Zakany] [Bourget,Troost] …**

Can we obtain precise results on OPE coefficients $\lambda^a_{\Phi\mathcal{O}\mathcal{O}}$ in free field theory ?

asymptotics for $\Delta_\mathcal{O} \to \infty$, but resolved in $a, \ell_\mathcal{O}$

Would be interesting to compute thermal partial wave expansion for 1-point function of stress tensor in CFT dual of thermal AdS or AdS-Kerr black hole.

AdS-Kerr: not enough to know power series expansion of partial waves in *q*

Conclusions and Open Questions

Is the behavior of (averaged) OPE coefficients $\overline{\lambda^a_{\phi\mathcal{O}\mathcal{O}}}$ for large $\Delta_\mathcal{O}$ universal ?

.. and if so what does it look like ?

LHH

for example $\overline{\lambda^a_{\phi\mathcal{O}\mathcal{O}}}^{(\ell_\mathcal{O},a)} \sim C_\phi \, \Delta_\mathcal{O}^{\Delta_\phi/3}$

Is also seems likely the relation with Hitchin integrable systems extends to the higher dimensional ``genus-2“ conformal blocks that [Benjamin, Lee, Ooguri,Simmons-Duffin] introduced to study asymptotics of HHH OPE.